IMPROVING THE INTERSECTION OF POLYHEDRA IN 3-MANIFOLDS

BY

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1. Introduction

Bing showed [10] that for a pair of intersecting 2-spheres S and S' in E^3 such that S' is tame there is a small homeomorphism of E^3 onto itself which adjusts S so that the components of its intersection with S' consist of a finite number of mutually exclusive simple closed curves in the inaccessible part of a Sierpinski curve together with sets of small diameter which fail to intersect the Sierpinski curve. Theorems 6.1 and 6.2 of this paper show that analogous results hold for topological embeddings of polyhedra in 3-manifolds. In order to prove these theorems we will need to extend to the case of polyhedra certain results about tame Sierpinski curves on spheres. These results were developed by Bing in [5]–[9].

In general we follow the definitions employed in [1]–[10]. We include a few important old definitions here as well as introduce a few new terms.

We use the term *complex* to mean *geometric complex* and we allow infinite complexes. Simplexes are closed. That is, they contain their boundaries.

An *n-manifold* is a separable metric space such that each point has a neighborhood which is homeomorphic to Euclidean n-space E^n. An *n-manifold with boundary* is a separable metric space such that each point has a neighborhood which is homeomorphic to either Euclidean n-space or the closed upper half space of Euclidean n-space. We use the term *surface* as a synonym for 2-manifold with boundary.

In a 3-manifold a set X which is homeomorphic to a polyhedron is *tame* if there is a triangulation of the manifold in which X underlies a subcomplex. For triangulated 3-manifolds an equivalent definition is that there is a homeomorphism of the 3-manifold onto itself which carries X onto a polyhedron [1], [22]. A set X in a 3-manifold is *locally tame at a point p* if there is a neighborhood N of p in the 3-manifold and a homeomorphism of Cl (N) into a 3-simplex which takes Cl $(N) \cap X$ onto a polyhedron. A set X in a 3-manifold is *locally tame* if it is locally tame at each of its points. In [1], [22] it is shown that if a closed subset X of a triangulated 3-manifold is locally tame then there is a homeomorphism of the 3-manifold onto itself which carries X onto a polyhedron.

An arc ab in E^3 *pierces* a disk D at a point p of Int (ab) if there is a neighborhood of p in ab which intersects D only at p and a positive number ε such

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that no arc of diameter less than ε connects ap to pb without intersecting D. An equivalent condition is that for any simple closed curve J in E^3 containing a neighborhood of p in ab there are simple closed curves in D arbitrarily near p which link J. Here the linking is homology linking with Z_2 coefficients which is a symmetric form of linking. In [6, Sections 1–3] it is shown that the first condition implies the second, and the other implication follows from arguments like those employed in [7, special case]. An arc ab in a 3-manifold *pierces* a disk D at a point p if there is a neighborhood O of p in the 3-manifold and a homeomorphism h of O into E^3 so that the image under h of some subarc of ab pierces at $h(p)$ the image under h of some subdisk of D.

A *null sequence* or a *null collection* of sets is a sequence or collection of sets such that for each positive number ε at most a finite number of the sets have diameters exceeding ε. A *Sierpinski curve* is any topological space which is homeomorphic to the complement in a 2-sphere of the union of the interiors of a dense null sequence of mutually exclusive disks on the sphere. A point of a Sierpinski curve is an *accessible point* of the curve if there is an embedding of the curve into a 2-sphere so that the image of the point is arcwise accessible from the complement of the image of the curve. If no such embedding exists then the point is called an *inaccessible point* of the curve. Any two Sierpinski curves are homeomorphic; further, the image of an accessible point of a Sierpinski curve under any embedding of the curve into a 2-sphere is arcwise accessible from the complement of the image of the curve [24]. A Sierpinski curve in a 3-manifold is *tame* if it lies on a tame disk.

A disk D is *normally situated* in a surface S if D either lies in Int (S) or intersects Bd (S) in an arc. A Sierpinski curve is *normally situated* in a surface if the closures of the components of its complement in the surface are mutually exclusive normally situated disks. Suppose that S is a surface and suppose that X is a Sierpinski curve which is normally situated in S. Define $A(X, S)$ to be those points of X which are arcwise accessible from $S - X$, and define $I(X, S)$ to be those points of X which are not arcwise accessible from $S - X$. If S is a disk it is possible for points of $I(X, S)$ to be accessible points of the Sierpinski curve X so one should not equate $A(X, S)$ and $I(X, S)$ with respectively the accessible and inaccessible points of X.

A topological space is of *pure dimension two* if it is two-dimensional and has no open subsets of dimension less than two. Suppose that W is a topological space which is homeomorphic to a finite polyhedron of pure dimension two. We call a compact set X in W a *universal curve in* W if W possesses a curvilinear triangulation T_W such that the intersection of X with each 2-simplex of T_W is a Sierpinski curve which is normally situated in that simplex. We say that a universal curve X in W is *normally situated* with respect to a curvilinear triangulation T_W of W if X misses the 0-skeleton of T_W and if the intersection of X with each 2-simplex of T_W is a Sierpinski curve which is normally situated in that 2-simplex. If X is a universal curve in a space W we denote by $A(X, W)$ the set of points of X which are arcwise accessible from $W - X$ and by $I(X, W)$ the set of points of X which are not arcwise accessible from $W - X$. Note that a universal curve in a space W is not connected if W is not connected.

We use the term *general position* in the following senses. A collection of points in Euclidean space E^n is in *general position* if no $k + 2$ of them lie in the same k-plane ($k < n$). A pair of complexes K and L in E^n are in *general position* with respect to each other if for each simplex s of K and each simplex t of L the vertices of s and t miss each other and the combined collection of vertices is in general position. Polyhedra K and L in a piecewise linear n-manifold, or a polyhedron K and a complex L in a piecewise linear n-manifold, or a polyhedron K and a complex L in E^n are in *general position* with respect to each other if for each point p of $K \cap L$ there is a polyhedron $N(p)$ which contains a neighborhood of p in the manifold or E^n, there are polyhedra $K(p)$ and $L(p)$ in $N(p) \cap K$ and $N(p) \cap L$ which contain neighborhoods of p in K and L respectively, and there is a piecewise linear homeomorphism of $N(p)$ into E^n which takes $K(p)$ and $L(p)$ onto complexes in general position with respect to each other.

We denote the join of a pair of joinable simplexes s and t by st. Similarly we denote the join of a pair of joinable complexes K and L by KL. See [26] for a definition of join.

By an *ε-set* in a metric space we mean a set of diameter less than ε. We use the expression *pwl* as an abbreviation for piecewise linear. The letter ρ denotes the metric on a metric space, and the letter I denotes the identity homeomorphism of a space onto itself.

2. Fattening up polyhedra

In this section we prove some lemmas about thickening up topological embeddings of polyhedra in 3-manifolds. These lemmas will later help us to extend certain self homeomorphisms of topological embeddings of polyhedra in 3-manifolds to homeomorphisms of the 3-manifolds onto themselves. In some cases alternate polyhedral versions of lemmas are stated in parentheses. These alternate versions will not be used here but will be used in [11], [12].

The proof of Lemma 2.1 was suggested to the author by Joseph Martin.

LEMMA 2.1. *Suppose that M is a 3-manifold and $D_1, \cdots, D_i, \cdots, D_n$ ($n > 1$) is a collection of disks in M whose pairwise intersections are all the same, an arc A in $\bigcap \mathrm{Bd}(D_i)$. Set $H = \bigcup D_i$ and $B = (\bigcup \mathrm{Bd}(D_i)) - \mathrm{Int}(A)$. Suppose that O is an open set in M containing $H - B$.*

Then there is a reordering $D_{i_1}, \cdots, D_{i_n}, D_{i_{n+1}} = D_{i_1}$ of the disks D_i, and there is a connected open subset U of O which contains $H - B$ such that $U - (H - B)$ has exactly n-components $U_1, \cdots, U_n$ where each $\mathrm{Cl}(U_j) \cap H = D_{i_j} \cup D_{i_{j+1}}$. Further each open disk $\mathrm{Int}(D_{i_j} \cup D_{i_{j+1}})$ separates U into two components U_j and $(\bigcup_{i \neq j} U_i) \cup (\bigcup_{D_i \neq D_{i_j} \text{ or } D_{i_{j+1}}} \mathrm{Int}(D_i))$.

Proof. From [19], H has a neighborhood in M which can be embedded in

E^3 so we might as well assume that $M = E^3$. Further H is an absolute retract so there is a mapping r of Cl (O) onto H such that $r|_H$ is the identity. Let O' denote the subset of O which consists of the points x such that $\rho(x, r(x))$ is less than $\rho(x, B)$. A neighborhood of $H - B$ is contained in O'. Define U to be the component of O' which contains $H - B$.

Let $P(k)$ $(2 \leq k \leq n)$ denote the following proposition. The disks $D_1, \cdots, D_k$ can be reordered $D^k_{i_1}, \cdots, D^k_{i_k}, D^k_{i_{k+1}} = D^k_{i_1}$ so that $U - ((H - B) \cap (\bigcup_{i \leq k} D_i))$ has exactly k components $U^k_1, \cdots, U^k_k$ where each

$$\mathrm{Cl}\ (U^k_j) \cap (\bigcup_{i \leq k} D_i) = D^k_{i_j} \cup D^k_{i_{j+1}}$$

and where U is separated by each Int $(D^k_{i_j} \cup D^k_{i_{j+1}})$ into two components

$$U^k_j \quad \text{and} \quad (\bigcup_{i \neq j} U^k_i) \cup (\bigcup_{i \leq k \text{ and } D_i \neq D^k_{i_j} \text{ or } D^k_{i_{j+1}}} \mathrm{Int}\ (D_i)).$$

If $P(n)$ is valid then the lemma follows by setting each $D_{i_j} = D^n_{i_j}$ and each $U_j = U^n_j$. We show that $P(k)$ is valid for all k $(2 \leq k \leq n)$ by induction.

Proof that $P(2)$ is valid. The open disk Int $(D_1 \cup D_2)$ separates U. To see this use [6, Theorem 5.3] to find an arc t in U which pierces $D_1 \cup D_2$ at a point p and otherwise fails to meet $D_1 \cup D_2$. If Int $(D_1 \cup D_2)$ does not separate U then t can be completed to a simple closed curve J in U whose intersection with $D_1 \cup D_2$ is p. Since t pierces $D_1 \cup D_2$ at p there are simple closed curves on $D_1 \cup D_2$ which link J. Any such simple closed curve is homotopic to Bd $(D_1 \cup D_2)$ missing J so Bd $(D_1 \cup D_2)$ links J. But J can be pushed into $H - B$ missing Bd $(D_1 \cup D_2)$ by pushing each point x of J along the line segment missing B from x to $r(x)$, and then $r(J)$ can be shrunk to a point in the contractible set $H - B$ which does not meet Bd $(D_1 \cup D_2)$. Thus Bd $(D_1 \cup D_2)$ cannot link J, and we conclude from the contradiction that Int $(D_1 \cup D_2)$ separates U.

From [4, Theorem 5] and the fact [25, Theorem 5.35 of Chapter 2] that a 2-sphere in E^3 is locally two sided it follows that a connected surface which is a closed subset of a connected 3-manifold separates the 3-manifold into at most two components and is the boundary of each component. We conclude that there are two components U^2_1 and U^2_2 of U $-$ Int $(D_1 \cup D_2)$. Set $D^2_{i_1} = D_1$, $D^2_{i_2} = D_2$, and $D^2_{i_3} = D_1$. We then have the relation

$$\mathrm{Cl}\ (U^2_j) \cap (D_1 \cup D_2) = D^2_{i_j} \cup D^2_{i_{j+1}}.$$

This verifies that $P(2)$ is valid.

Proof that $P(k)$ implies $P(k + 1)$. Suppose that we have established for an integer k that $P(k)$ is valid. Then we have a reordering $D^k_{i_1}, \cdots, D^k_{i_k}$, $D^k_{i_{k+1}} = D^k_{i_1}$ of the disks $D_1, \cdots, D_k$ and components $U^k_1, \cdots, U^k_k$ of $U - ((H - B) \cap (\bigcup_{i \leq k} D_i)$ such that each

$$\mathrm{Cl}\ (U^k_j) \cap (\bigcup_{i \leq k} D_i) = D^k_{i_j} \cup D^k_{i_{j+1}}.$$

Let U_r^k denote the U_i^k which contains Int (D_{k+1}).

The open disk Int (D_{k+1}) separates U_r^k into two components U_{r1}^k and U_{r2}^k and it is contained in the closure of each. To see this choose an arc s in U_r^k which pierces D_{k+1} at a point q and otherwise fails to meet D_{k+1}. If Int (D_{k+1}) fails to separate U_r^k then there is a simple closed curve J' in U_r^k which contains s and whose intersection with D_{k+1} is q. An argument like the one in the case of $P(2)$ obtains a contradiction by showing that J' both links and fails to link Bd $(D_{k+1} \cup D_{i_r}^k)$. Thus we may conclude that Int (D_{k+1}) separates U_r^k.

Just as in the case of $P(2)$ we conclude that U_r^k − Int (D_{k+1}) has exactly two components U_{r1}^k and U_{r2}^k and that the closure of each of these contains D_{k+1}. Suppose for convenience that Cl $(U_{r1}^k) \cap$ Int $(D_{i_r}^k) \neq \emptyset$. If Cl (U_{r1}^k) shared a point with Cl (U_{r2}^k) in Int $(D_{i_r}^k) \cup$ Int $(D_{i_{r+1}}^k)$ it would follow from [4, Theorem 5] and [25, Theorem 5.35 of Chapter 2] that $U_{r1}^k \cap U_{r2}^k$ would be non empty. Thus we may conclude that

$$\mathrm{Cl}\ (U_{r1}^k) \cap (\textstyle\bigcup_{i \leq k+1} D_i) = D_{i_r}^k \cup D_{k+1}$$

and

$$\mathrm{Cl}\ (U_{r2}^k) \cap (\textstyle\bigcup_{i \leq k+1} D_i) = D_{i_{r+1}}^k \cup D_{k+1}.$$

Set $D_{i_j}^k = D_{i_j}^{k+1}$ $(j \leq r)$, $D_{k+1} = D_{i_{r+1}}^{k+1}$, $D_{i_j}^k = D_{i_{j+1}}^{k+1}$ $(j \geqq r+1)$, $U_j^k = U_j^{k+1}$ $(j < r)$, $U_{r1}^k = U_r^{k+1}$, $U_{r2}^k = U_{r+1}^{k+1}$, and $U_j^k = U_{j+1}^{k+1}$ $(j \geq r+1)$. The reader may use these terms to verify that the first half of the conclusion in $P(k+1)$ is valid. To verify that the second half of the conclusion is valid note that for each integer j the sets

$$U_j^{k+1} \quad \text{and} \quad (\textstyle\bigcup_{i \neq j} U_i^{k+1}) \cup (\bigcup_{i \leq k+1 \text{ and } D_i \neq D_{i_j}^k \text{ or } D_{i_{j+1}}^k} \mathrm{Int}\ (D_i))$$

are mutually exclusive open sets whose sum is U − Int $(D_{i_j}^{k+1} \cup D_{i_{j+1}}^{k+1})$. Since U_j^{k+1} is connected it follows from our remarks in the proof of $P(2)$ that the other open set is connected.

We have established that $P(k)$ implies $P(k+1)$ and that $P(2)$ is valid. By induction we may conclude that $P(k)$ is valid for all k $(2 \leq k \leq n)$ and thus that the lemma is true.

We leave the proof of the following lemma to the reader.

LEMMA 2.2. *Suppose that $v\Delta^{n-1}$ is an n-simplex in E^n which is the join of a point v and an $(n-1)$-simplex Δ^{n-1}. Suppose that O is an open set (open polyhedron) in E^n which contains* Int (Δ^{n-1}) *and which is separated by* Int (Δ^{n-1}) *into two components, O_1 and O_2.*

Suppose O_1 is the component of O − Int (Δ^{n-1}) *such that Δ^{n-1} lies in the closure* of $O_1 \cap$ Int $(v\Delta^{n-1})$.

Then there is a (pwl) homeomorphism ϕ of $v\Delta^{n-1}$ into itself which is the identity on Δ^{n-1} and which sends $v\Delta^{n-1} - \Delta^{n-1}$ into O_1.

LEMMA 2.3. *Suppose that M is a (triangulated) 3-manifold, D is a tame (polyhedral) disk in M, A is an arc on* Bd (D), *and O is an open set (open polyhedron) containing* Int (A).

Then there is a tame (polyhedral) disk D_1 whose boundary contains A such that $D_1 \cap D = A$ and $D_1 - \mathrm{Bd}\,(A) \subset O$.

Proof. If M does not have a triangulation use [3], [20] to give it one. From [1], [22] the triangulation can be chosen so that D is a polyhedron. Let B be a polyhedral cube in M whose interior contains D, and let h_1 be a *pwl* homeomorphism of B into E^3.

From [15, Section 4] there is a pwl homeomorphism h_2 of E^3 onto itself which takes $h_1(D)$ onto a 2-simplex $v\sigma$ where σ is a 1-simplex that is the image of $h_1(A)$. Let Δ_1 be a 2-simplex which has σ as a face, which lies in the same plane as $v\sigma$, and which intersects $v\sigma$ in exactly σ. Let P denote the plane which contains Δ_1 and $v\sigma$. Now $h_2 h_1(O \cap \mathrm{Int}\,(B)) \cap P$ is an open set (open polyhedron) containing $\mathrm{Int}\,(\sigma)$. Thus we may apply the appropriate version of Lemma 2.2 to find a (*pwl*) homeomorphism ϕ of Δ_1 into itself such that

$$\phi(\Delta_1) \cap v\sigma = \sigma, \quad \phi|_\sigma = I \quad \text{and} \quad \phi(\Delta_1) - \mathrm{Bd}\,(\sigma) \subset h_2 h_1(O).$$

For the tame (polyhedral) disk D_1 we take $(h_2 h_1)^{-1}(\phi(\Delta_1))$.

LEMMA 2.4. *Suppose that M is a (triangulated) 3-manifold, D is a tame (polyhedral) disk in M, and O is a connected open set (open polydedron) containing* $\mathrm{Int}\,(D)$ *such that $O - \mathrm{Int}\,(D)$ has two components O_1 and O_2.*

Then there is a (polyhedral) 3-cell C in M such that

$$D \subset \mathrm{Bd}\,(C) \quad \text{and} \quad C - \mathrm{Bd}\,(D) \subset O_1.$$

Proof. As in the proof of Lemma 2.3 we choose a triangulation of M if one has not been provided so that D is a polyhedron in that triangulation; and we find a polyhedral cube B in M whose interior contains D, a *pwl* homeomorphism h_1 of B into E^3, and a *pwl* homeomorphism h_2 of E^3 onto itself which takes $h_1(D)$ onto a 2-simplex Δ.

Let v be a point of E^3 which does not lie in the plane of Δ and which is on the side of Δ such that the points of $v\Delta - \Delta$ near points of $\mathrm{Int}\,(\Delta)$ lie in $h_2 h_1(O_1)$. Use Lemma 2.2 to find a (*pwl*) homeomorphism ϕ of $v\Delta$ into itself such that $\phi|_\Delta = I$ and $\phi(v\Delta - \Delta) \subset O_1$.

For the tame (polydedral) 3-cell take $(h_2 h_1)^{-1}(\phi(v\Delta))$.

3. An engulfing lemma for universal curves

Bing showed [10] how to adjust the inaccessible part of a Sierpinski curve on a 2-sphere so that it engulfs a closed one dimensional subset of the sphere. Here we will prove the same sort of result about universal curves on topological images of polyhedra. Just as in the proof of the engulfing lemma in [10] we will wish to employ the following lemma.

LEMMA 3.1. *Suppose that S is a surface which is either a 2-sphere or a disk and X is a Sierpinski curve which is normally situated in S.*

There is a map g of X onto S which maps $I(X, S)$ homeomorphically onto *the complement of a countable dense set of points in* S *and which maps distinct components of* $A(X, S)$ *onto distinct points in the countable dense set.*

Proof. Let G denote the upper semicontinuous decomposition of S into points and disks where the non-degenerate elements of G consist of the closures of the components of $S - X$. From [23] we find that there is a homeomorphism h of the decomposition space G' associated with G onto S. Let π denote the projection map of S onto G' which sends each element of the decomposition G onto a point of G'. The map g is then given by $h\pi|_X$. Each component of $A(X, S)$ is the boundary of one of the non-degenerate elements of G so g maps $A(X, S)$ onto a countable dense set of points in such a way that distinct components of $A(X, S)$ go onto distinct points. The map g is clearly a homeomorphism on $I(X, S)$.

Here is a construction which we will employ several times in the rest of this paper. We have a normally situated Sierpinski curve X in a surface S, and we wish to construct a certain one dimensional set in $I(X, S)$. We map X onto S by the map g promised in Lemma 3.1, construct a one dimensional set in S, adjust the set slightly so that it misses the image of the components of $A(X, S)$, and bring the adjusted set back under g^{-1} into $I(X, S)$.

LEMMA 3.2. *Suppose that* M *is a* 3-*manifold,* W *is a subset of* M *which is homeomorphic to a finite polyhedron of pure dimension two, and* T_W *is a curvilinear triangulation of* W *with i-skeleton* W_i.

Suppose that X *is a universal curve in* W *which is normally situated with respect to* T_W *and which is such that each component of* $W - X$ *has diameter less than* ε.

Suppose that Y *is a closed one dimensional subset of* W *whose distance from* W_0 *exceeds* ε *and whose intersection with* W_1 *is zero-dimensional. Suppose that* W *is locally tame at each point of the closure of each component of* $W - X$ *whose closure intersects* Y, *and suppose that* Z *is a closed subset of* M *whose intersection with* W *is contained in* $I(X, W)$.

Then there is a homeomorphism h *of* M *onto itself which takes each simplex of* T_W *onto itself, which moves no point by as much as* ε, *which is the identity on both* Z *and the complement of an* ε-*neighborhood of* Y, *and which adjusts* $I(X, W)$ *so that* $h(I(X, W)) = I(h(X), W)$ *contains* Y.

Proof. We will define h first on W and then use the lemmas of Section 2 to extend h to a homeomorphism of all of M. Consider the components of of $W - X$ whose closures intersect Y. Denote the closures of these components by $H_1, \cdots, H_i, \cdots$. This is a null collection of mutually exclusive ε-sets. We use Lemma 3.1 in the manner indicated by the remark following its proof to find a new null collection of mutually exclusive ε-sets $H'_1, \cdots, H'_i, \cdots$ such that (1) each H_i is contained in some H'_j and each H'_i contains some H_j, (2) the intersection of an H'_i with a k-simplex of T_W is either empty or a

k-cell, (3) for each integer i,

$$H'_i \cap \mathrm{Cl}\,(W - H'_i) \subset I(X, W),$$

(4) each H'_i misses Z, and (5) W is locally tame at each point of $\bigcup H'_i$. A similar construction is employed in the proof of the engulfing lemma in [10]. Condition (1) and the fact that each H'_i has diameter less than ε insure that no H'_i intersects W_0.

As in [10] we will define h so that it is the identity except on $\bigcup H'_i$ and so that it moves each $H'_i \cap I(X, W)$ to contain $H'_i \cap Y$.

Each H'_i that does not intersect W_1 is contained in the interior of some 2-simplex of T_W. The intersection of X with H'_i is a Sierpinski curve X'_i such that the boundary of the disk H'_i is contained in $I(X, W)$ and in $I(X'_i, H'_i)$. The proof of the engulfing lemma in [10] shows how to define a homeomorphism h_i on H'_i which is the identity on $\mathrm{Bd}\,(H'_i)$ so that

$$h_i(I(X'_i, H'_i)) = h_i(H'_i \cap I(X, W))$$

contains $Y \cap H'_i$.

For each H'_i that intersects W_1 let $H'_{i1}, \cdots, H'_{ij}, \cdots$ denote the disks which are the intersections of H'_i with the 2-simplexes of T_W. We define a homeomorphism h^1_i of H'_i onto itself which takes each H'_{ij} onto itself, which is the identity on each $\mathrm{Bd}\,(H'_{ij}) - (H'_{ij} \cap W_1)$, and which moves $H'_i \cap I(X, W) \cap W_1$ so that $h^1_i(H'_i \cap I(X, W) \cap W_1)$ contains $Y \cap H'_i \cap W_1$.

For each H'_{ij} let X''_{ij} denote the Sierpinski curve $h^1_i(X \cap H'_{ij})$. Consider the components of $H'_{ij} - X''_{ij}$ whose closures intersect both Y and W_1. Let $H'_{ij1}, \cdots, H'_{ijk}, \cdots$ denote these closures. Define a second homeomorphism h^2_i of H'_i onto itself so that h^2_i sends each H'_{ij} onto itself, is the identity on each $\mathrm{Bd}\,(H'_{ij})$, and adjusts each H'_{ijk} so that $h^2_i(H'_{ijk})$ misses Y. For each H'_{ij} let X'''_{ij} denote the Sierpinski curve $h^2_i(X''_{ij})$.

For each H'_{ij} let H''_{ij} denote the disk which is obtained from H'_{ij} by deleting the components of $H'_{ij} - X'''_{ij}$ that intersect $\mathrm{Bd}\,(H'_{ij})$. Now $Y \cap H'_{ij} \subset H''_{ij}$ and $Y \cap \mathrm{Bd}\,(H''_{ij}) \subset I(X''_{ij}, H'_{ij})$. Thus as before we may employ the construction in [10] to define a homeomorphism h^3_i of H'_i onto itself which is the identity except on each $\mathrm{Int}\,(H''_{ij})$ so that for each X'''_{ij}, $h^3_i(I(X'''_{ij}, H''_{ij}))$ contains $Y \cap H'_{ij}$.

We define a homeomorphism h_i of H'_i onto itself by the rule $h_i = h^3_i h^2_i h^1_i$. The set $h_i(I(X'_i, H'_i))$ contains $Y \cap H'_i$.

The homeomorphism h is defined to be the identity on $W - \bigcup H'_i$ and it is defined to be h_i on each H'_i. Since no H'_i intersects Z and since each H'_i has diameter less than ε we see that the part of h thus far defined is an ε-homeomorphism of W onto itself which is the identity on $Z \cap W$ and which moves each simplex of T_W onto itself.

To extend h to all of M first construct a null collection of mutually exclusive open sets of diameter less than ε, $O_1, \cdots, O_i, \cdots$, so that each O_i misses Z and contains $H'_i - (H'_i \cap \mathrm{Cl}\,(W - H'_i))$. Since W is locally tame at each point of $\bigcup H_i'$ it follows from the two dimensional Schoenflies theorem that arcs in tame disks are tame so each H_i' is locally tame and therefore tame [1], [22]. For each H_i' which is a disk that intersects W_1 use Lemma 2.3 to find a disk H_i''' such that

$$H_i''' \cap W = H_i' \cap W_1 \quad \text{and} \quad H_i''' - \mathrm{Bd}\,(H_i''' \cap W_1) \subset O_i .$$

Let H_i'' denote the disk $H_i' \cup H_i'''$. For the remaining (H_i')'s set $H_i' = H_i''$. Let W' denote the sum $W \cup (\bigcup H_i''')$. Extend h to W' by defining it on each H_i''' so that it takes H_i''' onto itself and is the identity on $\mathrm{Bd}\,(H_i''') - (H_i''' \cap W)$.

If an H_i'' is a disk then h is the identity on $\mathrm{Bd}\,(H_i'')$. Use Lemma 2.4 to find a pair of 3-cells C_{i1} and C_{i2} in M such that

$$C_{i1} \cap C_{i2} = H_i'' \subset \mathrm{Bd}\,(C_{i1}) \cup \mathrm{Bd}\,(C_{i2}) \quad \text{and} \quad (C_{i1} \cup C_{i2}) - \mathrm{Bd}\,(H_i'') \subset O_i .$$

Define h to be the identity on $\mathrm{Bd}\,(C_{ik}) - H_i''$ $(k = 1, 2)$ and then extend h to take each C_{ik} onto itself.

If H_i'' is not a disk then use Lemmas 2.1 and 2.4 to locate a finite collection of 3-cells $C_{i1}, \cdots, C_{ik}, \cdots$ so that the sets $C_{ik} - (C_{ik} \cap H_i'')$ are mutually exclusive sets in O_i and so that each $C_{ik} \cap H_i''$ is a disk on $\mathrm{Bd}\,(C_{ik})$ of the form $H_{ij_1(k)}' \cup H_{ij_2(k)}'$. Define h to be the identity on each $\mathrm{Bd}\,(C_{ik}) - \mathrm{Bd}\,(C_{ik}) \cap H_i''$ and then extend h so that it takes each C_{ik} onto itself.

For each integer i define a 3-cell $C_i = \bigcup C_{ik}$. These 3-cells are mutually exclusive ε-sets which miss Z. The part of h thus far defined moves only points in the interiors of the C_i's. Thus we can define h to be the identity on $M - (W \cup (\bigcup C_i))$ and we have the promised homeomorphism.

If we forget about curvilinear triangulations in the case of a disk we obtain the following corollary to Lemma 3.2.

COROLLARY TO LEMMA 3.2. *Suppose that M is a 3-manifold, D is a disk in M, and X is a Sierpinski curve normally situated in D such that each component of $D - X$ has diameter less than ε.*

Suppose that Y is a closed one dimensional subset of D whose intersection with $\mathrm{Bd}\,(D)$ *is either zero-dimensional or is contained in $I(X, D)$. Suppose that D is locally tame at each point of the closure of each component of $D - X$ whose closure intersects Y, and suppose that Z is a closed subset of M whose intersection with D is contained in $I(X, D)$.*

Then there is a homeomorphism h of M onto itself which takes D onto itself, which moves no point by as much as ε, which is the identity on both Z and the complement of an ε-neighborhood of Y, and which adjusts $I(X, D)$ so that $h(I(X, D)) = I(h(X), D)$ contains Y.

4. Tameness modulo tame sets

LEMMA 4.1. *Suppose that M is a 3-manifold, D is a disk in M, and $\{X_i\}$ is a countable collection each of whose elements is either a tame arc or a tame Sierpinski curve normally situated in D.*

Suppose that D is locally tame modulo $\bigcup X_i$.

Then D is tame.

Proof. As shown in [19] a neighborhood of D can be embedded in E^3 so we might as well assume that M is E^3.

For each X_i that is an arc let $\{X_{ij}\}$ be a collection of arcs in Int (D) such that $\bigcup_j X_{ij} = X_i \cap \text{Int}(D)$.

For each X_i which is a Sierpinski curve let J_i denote that unique simple closed curve in X_i which bounds a disk D_i containing X_i. Use Lemma 3.1 as previously to find for each X_i a collection of simple closed curves $\{J_{ij}\}$ in Int (D_i) such that each J_{ij} lies in $I(X_i, D)$ and such that if D_{ij} denotes the disk which J_{ij} bounds then Int $(D_i) = \bigcup_j D_{ij}$. For each X_i and each D_{ij} let X_{ij} denote the Sierpinski curve $X_i \cap D_{ij}$. We have for each i the relation $X_i \cap \text{Int}(D_i) = \bigcup_j X_{ij}$. For each X_i let $\{A_{ij}\}$ denote the collection of components of $A(X_i, D)$ which lie on J_i. Unless $J_i = \text{Bd}(D)$ each A_{ij} is a spanning arc of Bd (D). For each A_{ij} let $\{A_{ijk}\}$ be a collection of arcs in Int (D) such that $\bigcup_k A_{ijk} = A_{ij} \cap \text{Int}(D)$.

Consider the collections of tame arcs and tame Sierpinski curves $\{A_{ijk}\}$ and $\{X_{ij}\}$ in Int (D). The set $(\bigcup_{i,j} X_{ij}) \cup (\bigcup_{i,j,k} A_{ijk})$ is equal to $(\bigcup X_i) \cap \text{Int}(D)$. Now Int (D) is locally tame modulo $(\bigcup X_{ij}) \cup (\bigcup A_{ijk})$ so from [9, Theorem 3.1], Int (D) is locally tame. From [1], [22] there is a homeomorphism h of E^3 onto itself such that $h(\text{Int}(D))$ is locally polyhedral, and from [16, Lemma 5] there is a 2-sphere S' in E^3 which contains $h(D)$ and which is locally polyhedral modulo $h(D)$. Let S denote the 2-sphere $h^{-1}(S')$. It is locally tame modulo Bd $(D) \cup (\bigcup X_i)$.

From [8, Theorem 8.5] we see that S is locally tame at each point of Bd (D) which misses $\bigcup X_i$. Theorem 3.1 of [9] shows that S is tame since it is locally tame modulo $\bigcup X_i$. Since D is a subset of S it is tame by the two dimensional Schoenflies theorem.

LEMMA 4.2. *Suppose that M is a 3-manifold, W is a subset of M which is homeomorphic to a finite polyhedron of pure dimension two, and T_W is a curvilinear triangulation of W with i-skeleton W_i and 2-simplexes $\Delta_1, \cdots, \Delta_i, \cdots$.*

Suppose that $\{X_j\}$ is a countable collection of sets each of which is either a tame arc which lies in some Δ_i and misses W_0 or a tame Sierpinski curve which is normally situated in some Δ_i and misses W_0.

Suppose that W is locally tame modulo $\bigcup X_j$.

Then W is tame.

Proof. From Lemma 4.1 we find that each 2-simplex Δ_i is tame. This shows that W_1 is locally tame modulo W_0. The fact that W_1 is locally tame at points of W_0 follows from our assumption that W is locally tame at each point of $W - \bigcup X_j$ and that $\bigcup X_j$ fails to meet W_0. Thus W_1 is tame [1], [22].

Theorem 3.1 of [13] says that a set in E^3 which is homeomorphic to a finite polyhedron is tame if it has a curvilinear triangulation whose one skeleton is tame and each of whose 2-simplexes is tame. The proof uses local arguments to show that such a set is locally tame. For this reason the theorem applies equally well to sets in 3-manifolds, and we may thus conclude that W is tame.

5. The existence of tame universal curves

In this section we show that for a given set W in a 3-manifold M where W is homeomorphic to a finite polyhedron of pure dimension two and for a given curvilinear triangulation T_W of W, there are many universal curves which are normally situated with respect to T_W whose intersections with each 2-simplex of T_W are tame Sierpinski curves.

Lemma 5.1 extends a result of Martin [18] that a disk in E^3 contains many tame arcs which reach out to its boundary.

LEMMA 5.1. *Suppose that M is a 3-manifold, D is a disk in M, and ε is a positive number.*

Then there is a tame Sierpinski curve X in D which is normally situated in D such that each component of $D - X$ has diameter less than ε.

Furthermore if $\{X_j\}$ is a finite collection of sets each of which is either a tame arc in D or a tame Sierpinski curve normally situated in D, then X may be chosen so that $\bigcup X_j \subset I(X, D)$.

Proof. We may assume as in the proof of Lemma 4.1 that M is E^3. For convenience we assume there is at least one X_j and ε is so small that each X_j has diameter greater than ε. That we may make the first assumption follows from [5].

Let δ be a positive number such that each 3δ-subset of D is contained in an ε-disk which is normally situated in D.

First we construct a Sierpinski curve X' in D which contains $\mathrm{Bd}\,(D)$ and which lies on a disk that is locally tame modulo $\mathrm{Bd}\,(D)$. Let Q denote the surface $D - ((\bigcup X_j) \cup \mathrm{Bd}\,(D))$. From [9, Theorem 6.1] we find a null sequence of mutually exclusive δ-disks $D_1, \cdots, D_i, \cdots$ which are dense in Q so that the set $X_Q = Q - \bigcup \mathrm{Int}\,(D_i)$ lies on a locally tame surface in M. The proof of Theorem 6.1 of [9] shows that any surface in M is locally tame if it contains X_Q, is locally tame modulo X_Q, and is homeomorphic to Q under a homeomorphism that is the identity on X_Q. We use [2, Theorem 7] to define a homeomorphism g of D into M which is the identity on $X_Q \cup (\bigcup X_j) \cup \mathrm{Bd}\,(D)$ and which replaces each δ-disk D_i by a new δ-disk that is locally tame modulo its boundary. We assume for convenience that g moves points so little that each 3δ-subset of the disk $D' = g(D)$ is contained in an ε-disk that is normally situated in D'. The disk D' is locally tame at each point of $g(Q)$ and is thus locally tame modulo $(\bigcup X_j) \cup \mathrm{Bd}\,(D)$. The proof of Lemma 4.1 shows that D' is locally tame modulo $\mathrm{Bd}\,(D') = \mathrm{Bd}\,(D)$. Let X' denote the Sierpinski curve $X_Q \cup (\bigcup X_j) \cup \mathrm{Bd}\,(D)$.

Just as in the proof of Lemma 4.1 the disk D' lies on a 2-sphere S. Following the construction in the proof of Theorem 9.1 of [8] we find a tame Sierpinski curve X'' in S such that $\bigcup X_j \subset I(X'', S)$ and such that each component of $S - X''$ has diameter less than δ.

It is an inconvenience for us when the closures of two components of $D' - (X'' \cap D')$ intersect in Bd (D') so we will cut away part of X'' to find a Sierpinski curve X''' for which this does not occur. Consider the components of $S - X''$ whose closures intersect Bd (D). Let $E_1, \cdots, E_i, \cdots$ denote these closures. Employ Lemma 3.1 to find an uncountable collection of mutually exclusive simple closed curves $\{J_{1\alpha}\}$ in $I(X'', S)$ so that each $J_{1\alpha}$ bounds a δ-disk $E_{1\alpha}$ on S which misses $\bigcup X_j$ and whose interior contains E_1. The disk D' cannot contain an uncountable collection of mutually exclusive continua which separate it into three or more pieces. Thus we can find a $J_{1\alpha}$ such that each non-degenerate component of $J_{1\alpha} \cap D'$ is an arc which spans Bd (D'). Let E_1' denote the corresponding $E_{1\alpha}$. Let E_{i_2} be the first E_i that is not contained in E_1'. We repeat the step just outlined to find a δ-disk E_2' which fails to meet $E_1' \cup (\bigcup X_j)$ such that Bd $(E_2') \subset I(X'', S)$ and each non-degenerate component of Bd $(E_2') \cap D'$ is an arc which spans Bd (D'). By proceeding in this manner until all the E_i's are covered up we obtain a null collection of mutually exclusive δ-disks $E_1', \cdots, E_j', \cdots$ which miss $\bigcup X_j$ such that each E_i is contained in some Int (E_j'), each Bd $(E_j') \subset I(X'', S)$, and each non-degenerate component of a Bd $(E_j') \cap D'$ is an arc which spans Bd (D').

Let X''' denote the Sierpinski curve $X'' - (X'' \cap (\bigcup \text{Int}\,(E_j')))$. Each component of $S - X'''$ has diameter less than δ. Use the Corollary to Lemma 3.2 to find a δ-homeomorphism h of M onto itself which is the identity on Cl $(S - D') \cup (\bigcup X_j)$, which takes D' onto itself, and which adjusts $I(X', D') = I(X', D)$ so that $h(I(X', D'))$ contains $X''' \cap D'$. The δ-homeomorphism h^{-1} pulls $X''' \cap D'$ back into $I(X', D')$ so we have $h^{-1}(X''' \cap D') \subset I(X', D)$.

Define the promised Sierpinski curve X to be that component of $h^{-1}(X''' \cap D')$ whose diameter exceeds ε. In the next paragraph we show that there is exactly one such component.

Since each X_j has diameter exceeding ε and is contained in $h^{-1}(X''' \cap D')$ there is at least one component of $h^{-1}(X''' \cap D')$ whose diameter exceeds ε. Suppose there were two such components, $X(1)$ and $X(2)$. Let Z be a set in $D' - h^{-1}(X''' - D')$ which is irreducible with respect to separating $X(1)$ from $X(2)$. Since D' is unicoherent Z is connected and is contained in some component of $D' - h^{-1}(X''' \cap D')$ which is the image under h^{-1} of some component of $D' - (X''' \cap D')$. Because of this $h(Z)$ has diameter less than δ so $Z = h^{-1}(h(Z))$ has diameter less than 3δ. Thus Z is contained in a normally situated ε-disk in D'. Such a disk neither separates D' nor contains all of $X(1)$ or of $X(2)$ so Z cannot separate $X(1)$ from $X(2)$. This contradiction comes from our assumption that there was more than one component of $h^{-1}(X''' \cap D')$ with diameter exceeding ε. Thus X is well defined.

We now show that X is a tame Sierpinski curve normally situated in D such that the components of $D - X$ have diameters less than ε and such that

$\bigcup X_j \subset I(X, D)$. The set X is a subset of $h^{-1}(X''')$ so if it is a Sierpinski curve it is a tame one. Note that $\bigcup X_j \subset h^{-1}(I(X''', S))$. Let $O_1, \cdots, O_i, \cdots$ denote the components of $D - X$. For each integer i, $\mathrm{Cl}\,(O_i) \cap X \cap \mathrm{Int}\,(D)$ must be contained in $h^{-1}(A(X''', S))$. Thus if $\mathrm{Cl}\,(O_i)$ is contained in $\mathrm{Int}\,(D)$ then $\mathrm{Bd}\,(\mathrm{Cl}\,(O_i))$ is a simple closed curve of diameter less than 3δ which is a component of $h^{-1}(A(X''', S))$, and hence $\mathrm{Cl}\,(O_i)$ is a disk of diameter less than ε which misses $\bigcup X_j$. Similarly if $\mathrm{Cl}\,(O_i)$ meets $\mathrm{Bd}\,(D)$ then $\mathrm{Cl}\,(O_i) \cap X$ is an arc in $h^{-1}(A(X''', S))$ which misses $\bigcup X_j$, spans $\mathrm{Bd}\,(D)$, and has diameter less than 3δ so $\mathrm{Cl}\,(O_i)$ is a disk of diameter less than ε which meets $\mathrm{Bd}\,(D)$ in exactly an arc whose interior is contained in O_i. It follows from the one dimensionality of X that $\mathrm{Cl}\,(O_1), \cdots, \mathrm{Cl}\,(O_i), \cdots$ is a dense null sequence of mutually exclusive ε-disks which are normally situated in D and miss $\bigcup X_j$. Considered asa subset of S, X is the complement of the union of the interiors of those $\mathrm{Cl}\,(O_i)$'s which are contained in $\mathrm{Int}\,(D)$ and the interior of the big disk which is the sum of $\mathrm{Cl}\,(S - D)$ and all those $\mathrm{Cl}\,(O_i)$'s which meet $\mathrm{Bd}\,(D)$. Thus X is a tame Sierpinski curve normally situated in D such that the components of $D - X$ have diameters less than ε and such that $\bigcup X_j \subset I(X, D)$.

LEMMA 5.2. *Suppose that M is a 3-manifold, and that D_1 and D_2 are disks in M such that $D_1 \cap D_2$ is an arc A on* $\mathrm{Bd}\,(D_1) \cap \mathrm{Bd}\,(D_2)$. *Suppose that X_1 and X_2 are tame Sierpinski curves normally situated in D_1 and D_2 respectivvly so that*

$$\mathrm{Bd}\,(D_k) - \mathrm{Int}\,(A) \subset I(X_k, D_k) \qquad (k = 1, 2).$$

Then there are tame Sierpinski curves X_1' and X_2' normally situated in D_1 and D_2 respectively so that $X_k \subset X_k'$ $(k = 1, 2)$ and so that $X_1' \cap A = X_2' \cap A$.

Furthermore if $(X_2 \cap A) \subset (X_1 \cap A)$ then the curves can be chosen so that $X_1' = X_1$.

Proof. We prove the lemma first for the special case where

$$(X_2 \cap A) \subset (X_1 \cap A).$$

From the proof of Lemma 5.1 there is a Sierpinski curve X_2'' in D_2 which contains $\mathrm{Bd}\,(D_2) \cup X_2$ in $I(X_2'', D_2)$ and which lies on a disk D_2'' that is locally tame modulo $\mathrm{Bd}\,(D_2'') = \mathrm{Bd}\,(D_2)$.

Let $D_{11}, \cdots, D_{1i}, \cdots$ denote the closures of those components of $D_1 - X_1$ whose closures intersect A. Each $D_{1i} \cap A$ is an arc whose interior misses X_2. We employ Lemma 3.1 as we have previously to find a collection of mutually exclusive arcs $t_{21}, \cdots, t_{2i}, \cdots$ in $I(X_2'', D_2)$ which span $\mathrm{Bd}\,(D_2)$ and whose interiors miss X_2 so that for each t_{2i} the closure of one of the components of $D_2 - t_{2i}$ is a disk D_{2i} whose intersection with $\mathrm{Bd}\,(D_2)$ is the same as $D_{1i} \cap A$. These properties insure that the D_{2i}'s are mutually exclusive and that each $D_{2i} - \mathrm{Bd}\,(D_{2i} \cap A)$ misses X_2.

Let X_2' denote the Sierpinski curve $X_2'' - X_2'' \cap (\bigcup \mathrm{Int}\,(D_{1i} \cup D_{2i}))$, and set $X_1' = X_1$. Use [2, Theorem 7] to find a homeomorphism g of $D_1 \cup D_2$ into M which is the identity on $X_1' \cup X_2'$ and which takes $(D_1 \cup D_2) - (X_1' \cup X_2')$ onto a locally tame set. Lemma 4.1 shows that $g(D_1)$ is tame. Thus $g(A)$ is tame. Since $\mathrm{Bd}\,(D_2) - \mathrm{Int}\,(A)$ lies in X_2 it is tame. But $g(\mathrm{Int}\,(D_2))$ is locally tame just as in the proof of Lemma 4.1 so we may conclude from Lemma 4.1 that $g(D_2)$ is tame. This shows that X_2' is tame. By construction $X_2 \subset X_2'$ and $(X_1' \cap A) = (X_2' \cap A)$.

Now we prove the general case of the lemma.

Making use again of the proof of Lemma 5.1 we find a Sierpinski curve X_1'' in D_1 such that $\mathrm{Bd}\,(D_1) \subset I(X_1'', D_1)$ and such that X_1'' lies on a disk D_1'' which is locally tame modulo $\mathrm{Bd}\,(D_1'') = \mathrm{Bd}\,(D_1)$. Then imitating a step in the proof of the special case of this lemma we cut out part of X_1'' to obtain a tame Sierpinski curve X_1''' in D_1 such that

$$\mathrm{Bd}\,(D_1) - \mathrm{Int}\,(A) \subset I(X''', D_1) \quad \text{and} \quad X_1''' \cap A = X_2 \cap A.$$

From Lemma 5.1 we find a tame Sierpinski curve X_1^{iv} in D_1 such that $X_1 \cup X_1''' \subset I(X_1^{\mathrm{iv}}, D_1)$. But now $X_2 \cap A \subset X_1^{\mathrm{iv}} \cap A$ and we can apply the special case of this lemma to find tame Sierpinski surves $X_1' = X_1^{\mathrm{iv}}$ in D_1 and X_2' in D_2 such that $X_k \subset X_k'$ $(k = 1, 2)$ and $(X_1' \cap A) = (X_2' \cap A)$.

LEMMA 5.3. *Suppose that M is a 3-manifold, W is a subset of M which is homeomorphic to a finite polyhedron of pure dimension two, and T_W is a curvilinear triangulation of W with i-skeleton W_i. Let $\Delta_1, \cdots, \Delta_i, \cdots$ denote the 2-simplexes, $\sigma_1, \cdots, \sigma_j, \cdots$ the 1-simplexes, and $v_1, \cdots, v_k, \cdots$ the vertices of T_W. Suppose that ε is a positive number.*

Then there is a universal curve X in W which is normally situated with respect to T_W such that each component of $W - X$ has diameter less than ε and such that each $X \cap \Delta_i$ is a tame Sierpinski curve.

Furthermore if $\{Y_j\}$ is a finite collection of sets in W where each Y_j misses W_0 and is either a tame arc in some Δ_i or a tame Sierpinski curve normally situated in some Δ_i, then X may be chosen so that $\bigcup Y_j \subset I(X, W)$.

Proof. For each 2-simplex Δ_i and each vertex v_k on Δ_i let E_{ik}' be a disk of diameter less than $\varepsilon/2$ such that its intersection with $\mathrm{Bd}\,(\Delta_i)$ is an arc whose interior contains v_k. Choose the (E_{ik}')'s so that they miss $\bigcup Y_j$ and so that two of them intersect only if they are associated with a common vertex v_k. For each E_{ik}' let B_{ik}' denote the arc $\mathrm{Cl}\,(\mathrm{Bd}\,(E_{ik}') \cap \mathrm{Int}\,(\Delta_i))$.

For each 1-simplex σ_j and each vertex v_k on σ_j let s_{jk} be an arc on σ_j with endpoints v_k and p_{jk} where p_{jk} is accessible from some $\mathrm{Int}\,(\Delta_i)$ by a tame arc. Choose s_{jk} so that it does not intersect any B_{ik}'. Lemma 5.1 shows that such arcs s_{jk} can be found. From Lemmas 5.1 and 5.2 a p_{jk} is accessible from an $\mathrm{Int}\,(\Delta_i)$ by a tame arc if σ_j is a face of Δ_i.

Use Lemma 5.1 to find in each Δ_i a tame Sierpinski curve X_i' which is normally situated in Δ_i such that each component of $\Delta_i - X_i'$ has diameter less than $\varepsilon/2$, such that every Y_j that is contained in Δ_i is contained in $I(X_i', \Delta_i)$, such that every p_{jk} that is contained in Bd (Δ_i) is contained in $I(X_i', \Delta_i)$, and such that the closure of no component of $\Delta_i - X_i'$ intersects both some B_{ik}' and some s_{jk}.

Now use Lemma 3.1 to find in each E_{ik}' an arc B_{ik} which spans Bd (Δ_i), lies in $I(X_i', \Delta_i)$, and has p_{jk}'s for its endpoints. The conditions on the choice of X_i' enable us to find such arcs. For each E_{ik}' let E_{ik} denote the subdisk of E_{ik}' which is the closure of that component of $\Delta_i - B_{ik}$ which contains v_k.

Partition each Δ_i into disks as indicated in Figure 5.1 so that Bd (G_i) is contained in $I(X_i', \Delta_i)$. For each F_{ij} let X_{ij}' denote the Sierpinski curve $X_i' \cap F_{ij}$. For each 1-simplex σ_j let $\Delta_{i_1(j)}, \cdots, \Delta_{i_m(j)}$ denote those 2-simplexes which have σ_j as a face.

By repeated applications of Lemma 5.2 we find for each $F_{i_1(j)j}$ a tame Sierpinski curve $X_{i_1(j)j}$ in $F_{i_1(j)j}$ which contains $X_{i_1(j)j}'$ and which is such that $X_{i_1(j)j} \cap \sigma_j$ contains each $X_{i_r(j)j}' \cap \sigma_j$ $(r > 1)$. By further applications of Lemma 5.2 we find in each $F_{i_r(j)j}$ $(r > 1)$ a tame Sierpinski curve $X_{i_r(j)j}$ which contains $X_{i_r(j)j}'$ and whose intersection with σ_j is $X_{i_1(j)j} \cap \sigma_j$.

For each Δ_i let X_i denote the Sierpinski curve $(X_i' \cap G_i) \cup (\bigcup_j X_{ij})$. Let X denote the sum $\bigcup X_i$. Because the $X_{i_r(j)j}$'s match up along the σ_j's, X is a universal curve. By construction no X_i meets W_0 so X is normally situated with respect to T_W. Since each X_i contains the part of X_i' which misses the $\varepsilon/2$-sets E_{ik} we see that each component of a $\Delta_i - X_i$ has diameter less than $\varepsilon/2$ and thus that each component of $W - X$ has diameter less than ε. Further since each $I(X_i', \Delta_i)$ contains the Y_j's that are contained in Δ_i we find that $\bigcup Y_j \subset I(X, W)$. To see that each X_i is tame use [3, Theorem 10] to find a homeomorphism g of W into M which is the identity on X and

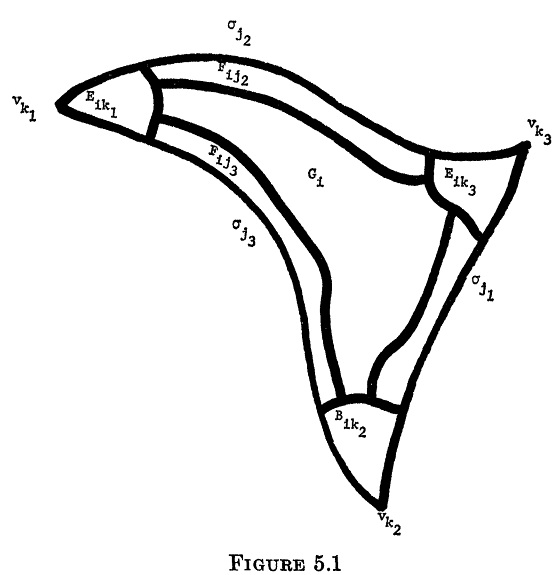

FIGURE 5.1

which replaces $W - X$ by a locally tame set. Since each X_i is the finite sum of tame Sierpinski curves each $g(\Delta_i)$ is tame by Lemma 4.1 and thus each X_i is tame.

6. General position theorems for topological embeddings of polyhedra

The promised theorems follow Lemma 6.1. The two theorems serve different purposes. Theorem 6.2 is the more natural generalization of Bing's result [10]; however, Theorem 6.1 is included because the proofs of the two theorems are so similar. Theorem 6.1 is tailor made to be used in the proofs of major theorems in [11], [12].

LEMMA 6.1. *Suppose that D is a polyhedral disk in E^3, L is a straight line which pierces D at a point p, and ε is a positive number.*

There is a pwl homeomorphism h of E^3 onto itself which is the identity on L and outside an ε-neighborhood of p so that a neighborhood of p in $h(D)$ lies in the plane P through p which is perpendicular to L.

Proof. Let T be a rectilinear triangulation of E^3 of mesh less than $\varepsilon/2$ in which both L and D underlie subcomplexes. Let $v\Delta$ be a 3-simplex in Int $(st(p, T))$ which is the join of a point v on L and a 2-simplex Δ that is pierced by L and misses D so that p is contained in Int $(v\Delta)$. The set $D \cap \text{Bd}\,(v\Delta)$ is a polyhedral simple closed curve J. Since L pierces D at p, J must separate v from Δ on Bd $(v\Delta)$. Let J' denote the simple closed curve $P \cap \text{Bd}\,(v\Delta)$. It also separates v from Δ on Bd $(v\Delta)$.

Define h on Bd $(v\Delta)$ so that it is a *pwl* homeomorphism of Bd $(v\Delta)$ onto itself which is the identity on v and on Δ and so that it takes J onto J'. Extend h to Int $(v\Delta)$ by sending each interval between p and a point x of Bd $(v\Delta)$ linearly onto the interval between p and $h(x)$. Let v' be a point of L in $E^3 - v\Delta$ so that $v\Delta \subset v'\Delta \subset \text{Int}\,(st(p, T))$. Extend h to $v'(v\,\text{Bd}\,(\Delta))$ by sending each interval from v' to a point x of $v\,\text{Bd}\,(\Delta)$ linearly onto the interval from v' to $h(x)$. The homeomorphism h is the identity on Bd $(v'\Delta)$ so we may define h to be the identity on $E^3 - v'\Delta$ and thus to be the identity except in an ε-neighborhood of p.

Since h is the identity on v, v', p, and Δ it is the identity on L. The open 2-cell $h(D) \cap \text{Int}\,(v\Delta)$ is a neighborhood of p in $h(D)$ which lies in P.

THEOREM 6.1. *Suppose that M is a 3-manifold with triangulation T whose i-skeleton is T_i, D is a disk in M, and ε is a positive number.*

Then there is a tame Sierpinski curve X which is normally situated in D, there is an ε-homeomorphism g of D onto a tame disk in M, and there is an ε-homeomorphism h of M onto itself such that

1. *each component of $D - X$ has diameter less than ε,*
2. *g is the identity on X,*
3. *h is the identity except in an ε-neighborhood of D,*

4. *$h(D)$ misses T_0 and $h(D) \cap T_1$ is a finite collection of points in $h(I(X, D))$ where 1-simplexes of T pierce $h(D)$,*
5. *$h(g(D))$ is a polyhedron in general position with respect to T_2, and*
6. *$h(g(D)) \cap T_2 = h(X) \cap T_2 = h(I(X, D)) \cap T_2$.*

Proof. Let O be an ε-neighborhood of D in M. All homeomorphisms h_i constructed in this proof will be assumed to be the identity outside O. In each case the reference used to assert the existence of a homeomorphism h_i permits this assumption.

From [10, Theorem 3] there is an $\varepsilon/20$-homeomorphism h_1 of M onto itself so that $h_1(D)$ misses T_0 and $h_1(D) \cap T_1$ is a finite collection of points $p_1, \cdots, p_k, \cdots$ where 1-simplexes of T pierce $h_1(D)$. From [4, Theorem 5] and [14, Theorem 6] each p_k lies on a tame arc A_k in $h_1(D)$, and thus each $h_1^{-1}(p_k)$ lies on a tame arc $h_1^{-1}(A_k)$ in D.

From Lemma 5.1 we find a tame Sierpinski curve X which is normally situated in D so that each component of $D - X$ has diameter less than $\varepsilon/20$ and so that $\bigcup h_1^{-1}(A_k) \subset I(X, D)$.

Use [3, Theorem 10] to find an $\varepsilon/20$-homeomorphism g of D into M which is the identity on X and which takes $D - X$ onto a locally tame set that misses $h_1^{-1}(T_1)$.

Lemma 4.1 shows that $h_1(g(D))$ is tame. From [1], [22] we find an $\varepsilon/20$-homeomorphism h_2 of M onto itself which is the identity on T_1 so that $h_2 h_1(g(D))$ is locally polyhedral modulo $\bigcup p_k = h_2 h_1(D) \cap T_1$. Then from [21, Theorem 2] we find an $\varepsilon/20$-homeomorphism h_3 of M onto itself which is the identity on T_1 so that $h_3 h_2 h_1(g(D))$ is a polyhedron.

For any 1-simplex s of T the join of s with lk (s, T) can be simplicially embedded in E^3. Thus we can use Lemma 6.1 to find a *pwl* $\varepsilon/20$-homeomorphism h_4 of M onto itself which is the identity on T_1 so that the polyhedral disk $h_4 h_3 h_2 h_1(g(D))$ is in general position with respect to T_2 near the points p_k. Let h_5 be a *pwl* $\varepsilon/20$-homeomorphism of M onto itself which is the identity on a neighborhood of T_1 so that the polyhedron $h_5 h_4 h_3 h_2 h_1(g(D))$ is in general position with respect to T_2.

Each component of $g(D) - X$ has diameter less than $\varepsilon/20 + 2(\varepsilon/20)$. Since each h_i $(i \leq 5)$ is an $\varepsilon/20$-homeomorphism of M each component of $h_5 h_4 h_3 h_2 h_1(g(D)) - h_5 h_4 h_3 h_2 h_1(X)$ has diameter less than $3\varepsilon/20 + 5(2\varepsilon/20) = 13\varepsilon/20$. From the corollary to Lemma 3.2 there is a $13\varepsilon/20$-homeomorphism h_6 of M onto itself which is the identity on T_1 so that h_6 takes $h_5 h_4 h_3 h_2 h_1(g(D))$ onto itself and so that $h_6 h_5 h_4 h_3 h_2 h_1(I(X, D))$ contains $h_5 h_4 h_3 h_2 h_1(g(D)) \cap T_2$.

The promised homeomorphism h is defined to be $h_6 h_5 h_4 h_3 h_2 h_1$. It is a $5(\varepsilon/20) + 13\varepsilon/20$ or ε-homeomorphism of M onto itself. Conditions 1–3 in the conclusion of the theorem are satisfied because of the choice of X and g and the fact that each h_i is the identity except on O. Since $h_1(\mathrm{Bd}\,(D))$ misses T_1 and since h_i $(i > 1)$ is the identity on T_1, $h(\mathrm{Bd}\,(D))$ misses T_1. Theorem 3.4 of [6] shows that each of the points p_k of $h(D) \cap T_1$ is a point where a

1-simplex of T pierces $h(D)$. Thus Condition 4 is satisfied. Since $h_5 h_4 h_3 h_2 h_1(g(D))$ is a polyhedral disk in general position with respect to T_2 and since h_6 moves that disk onto itself we see that Condition 5 is satisfied. Condition 6 is satisfied by the definition of h_6.

In [11], [12] we will make use of Theorem 6.1 in situations where we will want to avoid the tedious restatement of the conclusions of the theorem. To this end we define here a property, *Property* Q, as follows. Suppose that M is a triangulated 3-manifold with triangulation T whose i-skeleton is T_i, D is a disk in M, X is a tame Sierpinski curve normally situated in D, and η is a positive number. If there is an η-homeomorphism g of D onto a polyhedral disk in M such that when η is substituted for ε and the identity homeomorphism for h in the statement of Theorem 6.1 the six conditions are satisfied in the conclusion of the theorem, then we say that the quadruple (D, X, T_2, η) has Property Q.

THEOREM 6.2. *Suppose that M is a (triangulated) 3-manifold, W is a set in M homeomorphic to a finite polyhedron of pure dimension two, T_W is a curvilinear triangulation of W, and V is a tame (polyhedral) subset of M homeomorphic to a finite polyhedron of dimension less than or equal to two.*

Suppose that ε is a positive number.

Then there is a triangulation T of M (there is a subdivision T of the triangulation of M) in which V underlies a subcomplex, there is a universal curve X in W which is normally situated with respect to T_W, there is an ε-homeomorphism g of W onto a tame set in M, and there is an ε-homeomorphism h of M onto itself such that

1. *each component of $W - X$ has diameter less than ε,*
2. *g is the identity on X,*
3. *h is the identity except in an ε-neighborhood of W,*
4. *$h(g(W))$ is a polyhedron in general position with respect to V where V is considered as a polyhedron in T, and*
5. *$h(g(W)) \cap V = h(X) \cap V = h(I(X, W)) \cap V$.*

Proof. As before let O be an ε-neighborhood of W. Each h_i will again be considered to be the identity outside O.

If M is already provided with a triangulation in which V is a polyhedron let T be a subdivision of the triangulation in which V underlies a subcomplex, otherwise use [1], [22] to find a triangulation T of M in which V underlies a subcomplex.

Use Lemma 5.3 and Lemma 4.2 to find a universal curve X in W which is normally situated with respect to T_W so that each component of $W - X$ has diameter less than $\varepsilon/6$ and so that there is an $\varepsilon/6$-homeomorphism g of W onto a tame set in M such that g is the identity on X.

From [1], [22] there is an $\varepsilon/6$-homeomorphism h_1 of M onto itself such that $h_1(g(W))$ is a polyhedron in T. We might as well assume that $h_1(g(W))$ is in general position with respect to V.

Now each component of $g(W) - X$ has diameter less than $3\varepsilon/6$ so each component of $h_1(g(W)) - h_1(X)$ has diameter less than $3\varepsilon/6 + 2(\varepsilon/6) = 5\varepsilon/6$. Thus from Lemma 3.2 there is a $5\varepsilon/6$-homeomorphism h_2 of M onto itself such that for each simplex s of T_W, h takes $h_1(g(s))$ onto itself and such that

$$h_2 h_1(g(W)) \cap V = h_1(g(W)) \cap V \subset h_2 h_1(I(X, W)).$$

Let h denote the $\varepsilon/6 + 5\varepsilon/6$ or ε-homeomorphism $h_2 h_1$. Just as in the proof of Theorem 6.1 the five conditions in the conclusion of the theorem are satisfied.

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