EXTENSIONS AND RETRACTIONS OF ALGEBRAS OF CONTINUOUS FUNCTIONS

BY

C. W. KOHLS AND L. J. LARDY[1]

1. Introduction

A partial duality for homomorphisms between matrix algebras of continuous functions into the real numbers, the complex numbers, or the integers modulo a prime is obtained in §2. This leads to results on retractions and splitting extensions of algebras of continuous functions. We then present, for homomorphisms between matrix algebras of continuous real- or complex-valued functions, a representation by means of a continuous mapping between the underlying spaces and an automorphism. This is used in §3 to obtain a natural bijection from the family of equivalence classes of extensions of a matrix algebra of continuous functions by another to a subset of the product set of continuous mappings of certain related spaces and quotient groups of groups of units in related algebras. Finally, we give conditions under which an extension of an algebra of real- or complex-valued functions by another consists of functions that are continuous with respect to a topology obtained naturally from the topologies on the original spaces (cf. [9, Corollary 1]).

For background the reader is referred to [3] and [10]. We shall use the notation, terminology, and results of [3] and [10] freely.

In this paper, the real field, complex field and field of integers modulo a prime p are denoted by R, K, and J_p, respectively. The letter $\mathcal{K}$ will be a generic symbol for R, K, or J_p; after Corollary 1, it will indicate R or K only. If the $\mathcal{K}$-algebra A has an identity, the element $1E_{ij}$ of $L_n(A)$ is written simply E_{ij}, and the identity matrix is designated by I. We do *not* assume that a $\mathcal{K}$-homomorphism between $\mathcal{K}$-algebras with identity necessarily maps one identity to the other.

All topological spaces are assumed to be completely regular Hausdorff spaces. The algebra of all continuous functions from a space X into $L_n(\mathcal{K})$ will be denoted by $C_n(X, \mathcal{K})$, the subalgebra of bounded functions by $C_n^*(X, \mathcal{K})$, and the subalgebra of functions vanishing at infinity by $C_{n0}(X, \mathcal{K})$. The symbol $\mathcal{K}$ will be omitted from these three expressions when it seems appropriate, and the subscript n will not be included when $n = 1$. Note that $L_n(C(X, \mathcal{K}))$ is a $\mathcal{K}$-algebra, and that there is a natural $\mathcal{K}$-isomorphism of $C_n(X, \mathcal{K})$ onto $L_n(C(X, \mathcal{K}))$. We shall use whichever $\mathcal{K}$-algebra is convenient at any point in the discussion. The same symbol will be used for a constant function and the matrix that is its value. For $F \in C_n(X)$, the ma-

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trix obtained by extending each element of F to υX or to βX will be written F^{υ} or F^{β}, respectively.

Since $C_A = 0$ for any algebra A of $\mathcal{K}$-valued continuous functions, by [10, Proposition 2] we may and shall represent any extension of A by Λ with the graph Γ of the regular homomorphism $\theta : \Lambda \to P_A$ that it induces. If we choose $\sigma_f \in \theta(f)$ so that [10, (4)] holds, then we have $\theta(f) = \sigma_f + \nu[A]$, for all $f \in \Lambda$. Thus the elements (f, σ) of Γ may be written in the form $(f, \sigma_f + \nu g)$, where $g \in A$.

2. Duality and retractions

The first step in studying retractions is to generalize the duality theorem given in [3, 10.8]. To prove Proposition 1 for the cases $\mathcal{K} = K$ and J_p, we require two lemmas.

LEMMA 1. *If X is a realcompact space, and χ is a K-algebra homomorphism from $C(X, K)$ onto K, then there exists a point x of X such that $\chi(f) = f(x)$ for all $f \in C(X, K)$.*

Proof. If $\chi(f)$ were not real when $f \in C(X, R)$, then $f - \chi(f)\mathbf{1}$ would be invertible, which is impossible. Thus, χ restricted to $C(X, R)$ is a homomorphism onto R, so the result follows from [3, 10.5(c)].

LEMMA 2. *Let I be any proper ideal in $C(X, J_p)$, and set*

$$\mathbf{Z}(f) = \{x \in X : f(x) = 0\}.$$

Then the family $\{\mathbf{Z}(f) : f \in I\}$ has the finite intersection property.

Proof. This follows from the identity

$$\mathbf{Z}(f) \cap \mathbf{Z}(g) = \mathbf{Z}(f^{p-1} + g^{p-1} - (fg)^{p-1}).$$

PROPOSITION 1. *Let X and Y be completely regular spaces, with X realcompact. When $\mathcal{K}$ is finite, assume in addition that Y is locally compact and zero-dimensional and that X is compact and zero-dimensional. Then every homomorphism*

$$T : C_n(X, \mathcal{K}) \to C_n(Y, \mathcal{K})$$

induces a homomorphism

$$T_0 : C(X, \mathcal{K}) \to C(Y, \mathcal{K})$$

and a unique continuous mapping ϕ from the set $D = \{y \in Y : (TI)(y) \neq 0\}$ into X such that for $g \in C(X, \mathcal{K})$, $(T_0 g)(y) = g(\phi y)$ for all $y \in D$, and $(T_0 g)(y) = 0$ for all $y \in Y - D$. The set D is open-and-closed in Y.

Proof. For each $y \in D$, we consider the mapping from $C_n(X, \mathcal{K})$ into $L_n(\mathcal{K})$ defined by $F \to TF(y)$. Its restriction to the constant functions induces a $\mathcal{K}$-algebra endomorphism t_y of $L_n(\mathcal{K})$. Since $(TI)(y) \neq 0$, $t_y \neq 0$; and because $L_n(\mathcal{K})$ is a simple algebra, t_y is a monomorphism, and hence a nonsingular linear transformation of $L_n(\mathcal{K})$. Thus t_y is onto $L_n(\mathcal{K})$, whence so is the mapping $F \to TF(y)$.

Let $f \in C(X, \mathcal{K})$. For $y \in D$, $T(S(f))(y) \in L_n(\mathcal{K})$, and because $S(f)$ is in the center of $C_n(X, \mathcal{K})$, $T(S(f))(y)$ is in the center of $L_n(\mathcal{K})$. Thus, $T(S(f))(y) = S(k)$ for some $k \in \mathcal{K}$. Writing $k = g(y)$, we obtain in this way a function $g \in C(Y, \mathcal{K})$ such that $T(S(f)) = S(g)$. Note that $g(y) = 0$ for $y \in Y - D$. The $\mathcal{K}$-algebra homomorphism $T_0 : C(X, \mathcal{K}) \to C(Y, \mathcal{K})$ is now defined by $S(T_0 f) = T(S(f))$.

We first prove the result about ϕ under the assumption that $T_0 \mathbf{1} = \mathbf{1}$ (so that $D = Y$). For each $y \in Y$, the mapping $g \to (T_0 g)(y)$ is a $\mathcal{K}$-homomorphism of $C(X, \mathcal{K})$ into $\mathcal{K}$; and since $(T_0 \mathbf{1})(y) = \mathbf{1}(y)$, it is onto $\mathcal{K}$. By Lemmas 1 and 2 and the hypotheses on Y, the kernel of this homomorphism is a fixed maximal ideal M in $C(X, \mathcal{K})$. Because $C(X, \mathcal{K})$ separates the points of X, M has the form $\{g \in C(X, \mathcal{K}) : g(p) = 0\}$ for a unique $p \in X$. We define $\phi(y) = $ p. Clearly $(T_0 g)(y) = g(\phi(y))$ for all $g \in C(X, \mathcal{K})$ and $y \in Y$ (cf. [3, 10.5(c)]). Thus, $T_0 (g) = g\phi$. Since $C(X, \mathcal{K})$ determines the topology of X, the proof of [3, 3.8] with the real field R replaced by $\mathcal{K}$ shows that ϕ is continuous. By [3, 10.2], ϕ is unique, because $C(X, \mathcal{K})$ separates points.

The proof for general homomorphisms T_0 is obtained from the result above in the same manner that [3, 10.8] is obtained from [3, 10.6]. We need only observe that D is locally compact zero-dimensional if Y is, and that the algebra of functions in $C(Y, \mathcal{K})$ that are zero on $Y - D$ may be identified with $C(D, \mathcal{K})$, to which it is isomorphic under restriction.

The duality exhibited in Proposition 1 will now be used to obtain a duality for retracts. The proof is complicated slightly by the fact that a retract of an algebra might be a retract of a direct summand.

THEOREM 1. *Let X and Y be realcompact spaces, and assume they are compact and zero-dimensional when $\mathcal{K}$ is finite. Then $C_n(Y, \mathcal{K})$ is a retract of $C_n(X, \mathcal{K})$ if and only if Y is a retract of X.*

Proof. Assume that $C_n(Y, \mathcal{K})$ is a retract of $C_n(X, \mathcal{K})$. For homomorphisms

$$T : C_n(X, \mathcal{K}) \to C_n(Y, \mathcal{K}) \quad \text{and} \quad T' : C_n(Y, \mathcal{K}) \to C_n(X, \mathcal{K})$$

such that TT' is the identity on $C_n(Y, \mathcal{K})$, we have $T'(I) = e$ and $T(e) = I$, where e is an idempotent in $C_n(X, \mathcal{K})$. Let D be the open-and-closed set

$$\{x \in X : e(x) \neq 0\},$$

and identify the subalgebra of functions in $C_n(X, \mathcal{K})$ that are zero on $X - D$ with $C_n(D, \mathcal{K})$. By Proposition 1, there exist continuous maps $\phi : Y \to D$ and $\phi' : D \to Y$ such that $T_0(f) = f\phi$ for $f \in C(D, \mathcal{K})$ and $T_0'(g) = g\phi'$ for

$g \in C(Y, \mathcal{K})$. Now for $g \in C(Y, \mathcal{K})$ and $y \in Y$,

$$(g\phi'\phi)(y) = ((g\phi')\phi)(y) = T_0(g\phi')(y) = (T_0(T_0 g'))(y) = g(y);$$

since $C(Y, \mathcal{K})$ separates points, $\phi'\phi$ is the identity on Y. Hence Y is a retract of D. But D is trivially a retract of X, and retraction is transitive; so Y is a retract of X.

Conversely, if Y is a retract of X, and $\phi : Y \to X$, $\phi' : X \to Y$ are maps such that $\phi'\phi$ is the identity on Y, we define homomorphisms

$$T : C_n(X, \mathcal{K}) \to C_n(Y, \mathcal{K}), \qquad T' : C_n(Y, \mathcal{K}) \to C_n(X, \mathcal{K})$$

by $T(F) = F\phi$, $T'(G) = G\phi'$. Then for all $G \in C_n(Y, \mathcal{K})$, $(TT')(G) = G\phi'\phi = G$, whence $C_n(Y, \mathcal{K})$ is a retract of $C_n(X, \mathcal{K})$.

COROLLARY 1. *Let Y be a closed subset of a realcompact space X, and assume that X is compact and zero-dimensional when $\mathcal{K}$ is finite. Set*

$$A = \{G \in C_n(X, \mathcal{K}) : G[Y] = \{0\}\}.$$

Then $C_n(X, \mathcal{K})$ is equivalent to a splitting extension of A by $C_n(Y, \mathcal{K})$ if and only if Y is a retract of X.

Proof. If X is realcompact, then Y is realcompact [3, 8.10]. Hence, by Theorem 1, Y is a retract of X if and only if $C_n(Y, \mathcal{K})$ is a retract of $C_n(X, \mathcal{K})$. By [10, Theorem 2], this occurs exactly when $C_n(X, \mathcal{K})$ is equivalent to a splitting extension of A by $C_n(Y, \mathcal{K})$.

As a special case of Corollary 1, we have a precise characterization of those compactifications X of locally compact spaces Z such that the extension $C(X)$ of $C_0(Z)$ by $C(X - Z)$ is a splitting extension.

We note that by [7, Thm. 2, p. 215], Proposition 1, Theorem 1, and Corollary 1 apply to Boolean rings and p-rings.

Example 1. Let $\mathbf{N}$ be the discrete space of positive integers. The structure space of the Boolean algebra E of all subsets of $\mathbf{N}$ is $\beta\mathbf{N}$ [4, p. 76]. Since E is isomorphic to $C(\mathbf{N}, J_2)$, and every function in $C(\mathbf{N}, J_2)$ can be extended to $\beta\mathbf{N}$, one sees quickly that the intersection A of all the free maximal ideals in E is the ring of all finite subsets of $\mathbf{N}$. The structure space of A is $\mathbf{N}$. The structure space of the algebra $\Lambda = E/A$ is $\beta\mathbf{N} - \mathbf{N}$ [7, Prop. 1, p. 205]. Since $\beta\mathbf{N} - \mathbf{N}$ is not a retract of $\beta\mathbf{N}$ [2, Cor. 2.8], E is not equivalent to a splitting extension of A by Λ. Of course, Corollary 1 yields more; we can get similar information about $C(\beta\mathbf{N}, \mathcal{K})$ when $\mathcal{K}$ is R, K, or any J_p.

Most of the results presented in the rest of the paper could be obtained for J_p, but to simplify the statements we now restrict $\mathcal{K}$ to the real and complex fields.

The set of all nonsingular matrices U over the algebra of $\mathcal{K}$-valued functions on Y satisfying the condition $U^{-1}FU \in C_n(Y, \mathcal{K})$ for all $F \in C_n(Y, \mathcal{K})$ will be denoted by $G_n(Y, \mathcal{K})$ or $G_n(Y)$. We give some characterizations of $G_n(Y)$.

PROPOSITION 2. *Let $U = [u_{ij}]$ be a nonsingular n by n matrix over the algebra of $\mathcal{K}$-valued functions on Y, and write $U^{-1} = [v_{ij}]$. The following are equivalent.*

(1) $U \in G_n(Y)$.

(2) *For all i and j, $U^{-1}E_{ij}\, U \in C_n(Y)$.*

(3) *For all i, j, k, and l, $v_{ki}\, u_{jl} \in C(Y)$.*

(4) *The product of $|\, U\,|^{-1}$ and n (not necessarily distinct) elements of U is in $C(Y)$.*

Proof. The equivalence of (1) and (2) follows from the fact that each $F \in C_n(Y)$ can be written in the form $F = \sum_{i,j} f_{ij}\, E_{ij}$ with $f_{ij} \in C(Y)$. Also, the element of $U^{-1}E_{ij}\, U$ in the kl-th place is $v_{ki}\, u_{jl}$, so it is immediate that (2) and (3) are equivalent.

(3) implies (4). Since $(U^{-1})^{-1} = U$, every element of U is the product of $|\, U\,|$ and a finite sum of terms which, apart from sign, are products of $n - 1$ elements of U^{-1}. Hence the product of $|\, U\,|^{-1}$ and n elements of U may be written as a finite sum of products, each having $n - 1$ factors of the form $v_{ki}\, u_{jl}$. By assumption each $v_{ki}\, u_{jl} \in C(Y)$, so the product is in $C(Y)$.

(4) implies (3). Each $v_{ki}\, u_{jl}$ is a finite sum of terms which, apart from sign, are products of the type described in (4).

From the definition of $G_n(Y)$ and the equivalence of (1) and (3), it follows easily that $G_n(Y)$ is a group of units. The following example shows that in general $G_n(Y)$ properly contains the units of $C_n(Y)$. Indeed, there can be elements in $G_n(Y)$ such that no nonsingular scalar multiple is in $C_n(Y)$.

Example 2. Let Y be the unit circle, and let U be the 2 by 2 matrix of functions on Y such that

$$u_{11} = u_{22} = \sin(\theta/2), \qquad u_{12} = -u_{21} = \cos(\theta/2), \qquad 0 \leq \theta < 2\pi.$$

Applying Proposition 2, we see easily that $U \in G_2(Y, R)$. Now suppose r is a nonzero scalar such that $rU \in C_2(Y, R)$. Then r is continuous except possibly at 0, since u_{11} is continuous and nonzero except at 0; also, r is continuous from the right at 0 because u_{12} has this property and $u_{12}(0) \neq 0$. It follows that r does not change sign, whence ru_{12} is not continuous at 0, a contradiction.

The next theorem is an improvement on Proposition 1 for $\mathcal{K} = R$ or K, and is related to [8, Theorem 1].

THEOREM 2. *Let X and Y be completely regular Hausdorff spaces, and let $T : C_n(X, \mathcal{K}) \to C_n(Y, \mathcal{K})$ be a $\mathcal{K}$-algebra homomorphism. Set*

$$D = \{y \in Y : (TI)(y) \neq 0\}.$$

Then there exists a continuous mapping ϕ of D into vX, and a unit $U \in G_n(D, \mathcal{K})$, satisfying

$$\begin{aligned} TF(y) &= U^{-1}(y)F^{v}(\phi(y))U(y), && \text{if } y \in D, \\ &= 0, && \text{if } y \in Y - D, \qquad F \in C_n(X, \mathcal{K}). \end{aligned}$$

Furthermore, the mapping $(F, y) \to U^{-1}(y)FU(y)$ from $L_n(\mathcal{K}) \times D$ into $L_n(\mathcal{K})$ is continuous, D is an open-and-closed subset of Y, and $\phi[D]$ is dense in vX if and only if T is one-to-one.

If we assume in addition that T maps $C_n(X, \mathcal{K})$ onto $C_n(Y, \mathcal{K})$, then ϕ is a homeomorphism of Y onto a C-embedded subset of vX.

Proof. As observed in the proof of Proposition 1, for each $y \in D$, the endomorphism t_y is a linear automorphism of $L_n(\mathcal{K})$, so it is the identity on the center. By [6, p. 237], there exists a nonsingular matrix $U(y)$ such that $t_y(F) = U^{-1}(y)FU(y)$ for each $F \in L_n(\mathcal{K})$. This defines U on D. Now the restriction of TE_{ij} to D is $U^{-1}E_{ij}\, U$, so $U^{-1}E_{ij}\, U \in C_n(D)$. By Proposition 2, $U \in G_n(D)$.

By Proposition 1, there exists a continuous mapping $\phi : D \to vX$ such that $T_0 f(y) = f^v(\phi(y))$ if $y \in D$ and $T_0 f(y) = 0$ if $y \in Y - D$. Thus for $y \in D$,

$$T(S(f))(y) = S(T_0 f(y)) = S(f^v\phi(y)).$$

Now given any $F = [f_{ij}] \in C_n(X)$, we write $F = \sum_{i,j} S(f_{ij})E_{ij}$. Then

$$\begin{aligned} TF(y) &= \textstyle\sum_{i,j} T(S(f_{ij}))(y)TE_{ij}(y) \\ &= \textstyle\sum_{i,j} S(f_{ij}^v(\phi(y)))U^{-1}(y)E_{ij}\, U(y) \\ &= U^{-1}(y)(\textstyle\sum_{i,j} S(f_{ij}^v\phi(y))E_{ij})U(y) \\ &= U^{-1}(y)F^v(\phi(y))U(y), \qquad\qquad y \in D. \end{aligned}$$

It is obvious that $TF(y) = 0$ for $y \in Y - D$.

For a fixed constant function $F = [c_{ij}]$, if we write $U = [u_{ij}]$, $U^{-1} = [v_{ij}]$, the kl-th element of $U^{-1}FU$ is

$$\textstyle\sum_{i,j} v_{ki}\, c_{ij}\, u_{jl} = \sum_{i,j} c_{ij}\, v_{ki}\, u_{jl}\,.$$

By Proposition 2, each $v_{ki}\, u_{jl} \in C(D)$, so each element of $U^{-1}FU$ is a linear combination of continuous functions on D, with coefficients c_{kl}. From this one can show without difficulty that the function $(F, y) \to U^{-1}(y)FU(y)$ from $L_n(\mathcal{K}) \times D$ into $L_n(\mathcal{K})$ is jointly continuous.

Proposition 1 implies that D is an open-and-closed subset of Y. The remaining statements of the theorem may be obtained by modifying the proof of [8, Theorem 1], replacing $\omega(\cdot, y)$ by $U^{-1}(y)(\cdot)U(y)$ and $\alpha(\cdot, y)$ by $U(y)(\cdot)U^{-1}(y)$. In this case, the same method can be used to prove the joint continuity of both functions.

3. Extensions of algebras of continuous functions

We consider only extensions of $C_{n0}(Z, \mathcal{K})$, where Z is locally compact, by $C_n(Y, \mathcal{K})$, where Y is realcompact, and $\mathcal{K} = R$ or K. We shall be concerned with spaces that are, as sets, the union of the given sets Z and Y. The various topologies on $Z \cup Y$ that will arise are obtained as follows (cf. [5, p. 121]).

DEFINITION. Let W be a subspace of βZ that contains Z, with $W - Z$ closed in W, and let g be a continuous mapping from $W - Z$ into Y. The *adjunction space* obtained by adjoining W to Y by means of g is defined to be the quotient space of the topological sum of W and Y in which each point in the range of g is identified with all of the points in $W - Z$ in its preimage under g, while points in Z, and in Y but not in the range of g, remain distinct.

We allow the possibility that $W = Z$ and g be the empty function, that is, that the adjunction space be the topological sum of Z and Y. Also, it may turn out that distinct adjunction spaces going with the same spaces Z and Y are homeomorphic; but they are essentially different, relative to the embedding of Z and Y in them.

PROPOSITION 3. *Let Z be a locally compact space. Then $M_{C_{n0}(Z)}$ is $\mathcal{K}$-isomorphic to $C_n^*(Z)$, and $P_{C_{n0}(Z)}$ is $\mathcal{K}$-isomorphic to $C_n(\beta Z - Z)$.*

Proof. By [10, Theorem 3], $M_{C_0(Z)}$ is $\mathcal{K}$-isomorphic to a subalgebra of $C(Z)$, and it is clear that the image of $M_{C_0(Z)}$ in $C(Z)$ contains $C^*(Z)$. Let $f \in C(Z)$ be unbounded, and let $\{z_j\}$ be a sequence of points in Z obtained by selecting one point from the preimage of each point of a sequence in the range of f that approaches infinity. Define a function g on the compact set $\{z_j\} \cup (\beta Z - Z)$ by $g(z_j) = 1/f(z_j)$, and $g[\beta Z - Z] = \{0\}$. Then g has an extension to a function $g' \in C(\beta Z)$ such that $h = g' \mid Z \in C_0(Z)$. Obviously, $fh \notin C_0(Z)$. Thus, by [10, Theorem 3], f is not in the image of $M_{C_0(Z)}$. Hence $M_{C_0(Z)}$ is $\mathcal{K}$-isomorphic to $C^*(Z)$.

Now the $\mathcal{K}$-algebra $C_{n0}(Z)$ is $\mathcal{K}$-isomorphic to $L_n(C_0(Z))$, and $C_{C_0(Z)} = 0$; so by [10, Theorem 4], $M_{C_{n0}(Z)}$ is $\mathcal{K}$-isomorphic to $L_n(M_{C_0(Z)})$. It follows that $L_n(M_{C_0(Z)})$ is $\mathcal{K}$-isomorphic to $L_n(C^*(Z))$, which is $\mathcal{K}$-isomorphic to $C_n^*(Z)$.

Since $C_n^*(Z)$ is $\mathcal{K}$-isomorphic to $C_n(\beta Z)$, the second conclusion follows easily.

THEOREM 3. *Let Z be a locally compact space and Y a realcompact space. There is a natural bijection from the family of equivalence classes of extensions of $C_{n0}(Z)$ by $C_n(Y)$ onto the set of ordered pairs $(\phi, \bar{U})$, where ϕ is a continuous mapping of an open-and-closed subset D of $\beta Z - Z$ into Y and $\bar{U}$ is an element of the group $G_n(D)$ modulo its center.*

Proof. By [10, Proposition 2], there is a natural bijection from the family of equivalence classes of extensions of $C_{n0}(Z)$ by $C_n(Y)$ onto the family of $\mathcal{K}$-homomorphisms of $C_n(Y)$ into $P_{C_{n0}(Z)}$; and by Proposition 3, $P_{C_{n0}(Z)}$ is $\mathcal{K}$-isomorphic to $C_n(\beta Z - Z)$. Thus, we need only show that there is a natural bijection from the family of $\mathcal{K}$-homomorphisms of $C_n(Y)$ into $C_n(\beta Z - Z)$ onto the set of ordered pairs $(\phi, \bar{U})$ of the type described.

Now given any pair $(\phi, \bar{U})$, let D be the domain of ϕ and U any element in the class $\bar{U}$. We associate with $(\phi, \bar{U})$ the $\mathcal{K}$-homomorphism

$$T : C_n(Y) \to C_n(\beta Z - Z)$$

defined by

$$TF(z) = U^{-1}(z)F(\phi(z))U(z) \quad \text{if} \quad z \in D,$$

and

$$TF(z) = 0 \quad \text{if} \quad z \in (\beta Z - Z) - D.$$

This is clearly well defined. By Theorem 2, the mapping $(\phi, \bar{U}) \to T$ thus defined is onto. One can show without difficulty that it is one-to-one, by first disposing of the case $T = 0$, then considering the set of constant matrices on D, and finally choosing F to be a suitable scalar matrix.

Extensions of C^*-algebras have been studied in [1]. Theorem 3 gives more precise information about a special kind of C^*-algebra than may be found in [1], where ordinary equivalence classes are not computed. Also, Proposition 3 could be deduced from [1, Theorem 3.15]; our approach is quite different.

An extension of $C_0(Z)$ by $C(Y)$ will be said to be equivalent to a subalgebra of $C(X)$ if it is equivalent to an extension of the form

$$0 \to C_0(Z) \xrightarrow{\alpha'} E' \xrightarrow{\beta'} C(Y) \to 0,$$

where E' is a subalgebra of $C(X)$, α' is the monomorphism naturally induced by an embedding of Z into X, and β' is the epimorphism naturally induced by restriction to a subspace of X homeomorphic with Y.

THEOREM 4. *Let Z be a locally compact space, Y a realcompact space, and X a space containing complementary subspaces homeomorphic with Z and Y, such that Y is closed in X. An extension of $C_0(Z)$ by $C(Y)$ is equivalent to a subalgebra of $C(X)$ if and only if X is the adjunction space obtained by adjoining a subspace of βZ to Y by means of a restriction of the corresponding continuous mapping of an open-and-closed subset of $\beta Z - Z$ into Y.*

Proof. Let E be an extension of $C_0(Z)$ by $C(Y)$, let θ' be the corresponding continuous mapping of an open-and-closed subset V of $\beta Z - Z$ into Y, and let X be the adjunction space obtained by adjoining the subspace $Z \cup U$ of βZ to Y by means of $\theta' \mid U$, where $U \subset V$. For each $f \in C(Y)$, $\theta(f)$ may be identified with $\sigma_f^\beta \mid (\beta Z - Z)$. But also $\theta(f) = f\theta'$ on V, whence $\sigma_f^\beta = f\theta'$ on V. Thus the function h defined on the topological sum of Y and $Z \cup U$ by $h \mid Y = f$, $h \mid Z = \sigma_f$, and $h \mid U = f\theta'$, is continuous. It follows from the definition of adjunction space that h induces a function $\bar{f} \in C(X)$ such that $\bar{f} \mid Y = f$ and $\bar{f} \mid Z = \sigma_f$. For any $g \in C_0(Z)$, the function $\bar{g}$ defined by $\bar{g} \mid Y = 0$, $\bar{g} \mid Z = g$, is clearly in $C(X)$. We define a mapping ρ of the extension E onto the subalgebra

$$\{\bar{f} + \bar{g} : f \in C(Y), g \in C_0(Z)\}$$

of $C(X)$ by $\rho(f, \sigma_f + \nu g) = \bar{f} + \bar{g}$. It is easy to verify that ρ is a homomorphism, if one observes that $^-$ is additive on $C_0(Z)$, and $\overline{fg'} = \overline{\sigma_f g'}$ for $f \in C(Y)$, $g' \in C_0(Z)$. It then follows that E is equivalent to a subalgebra of $C(X)$.

Conversely, assume that

$$0 \to C_0(Z) \xrightarrow{\alpha} E \xrightarrow{\beta} C(Y) \to 0$$

is equivalent under ψ to

$$0 \to C_0(Z) \xrightarrow{\alpha'} E' \xrightarrow{\beta'} C(Y) \to 0,$$

where E' is a subalgebra of $C(X)$, and X is a space that contains complementary subspaces homeomorphic with Z and Y, such that Y is closed in X. We suppose, as we may, that E is the graph of a regular homomorphism. For any $z \in Z$, choose $g_z \in C_0(Z)$ so that $g_z(z) = 1$. Note that for any $g \in C_0(Z)$, $\psi(0, \nu g) = \psi\alpha(g) = \alpha'(g) = g$. Now for any $f \in C(Y)$,

$$\begin{aligned} \psi(f, \sigma_f)(z) &= [\psi(f, \sigma_f)\psi(0, \nu g_z)](z) \\ &= [\psi((f, \sigma_f)(0, \nu g_z))](z) \\ &= \psi(0, \nu(\sigma_f\, g_z))(z) \\ &= \sigma_f\, g_z(z) \\ &= \sigma_f(z) \end{aligned}$$

Thus, $\psi(f, \sigma_f) \mid Z = \sigma_f$. We also have

$$f = \beta(f, \sigma_f) = \beta'\psi(f, \sigma_f) = \psi(f, \sigma_f) \mid Y.$$

Let θ' be the continuous mapping of an open-and-closed subset V of $\beta Z - Z$ into Y corresponding to the class of extensions containing E. Let i' denote the continuous extension to βZ of the embedding map of Z into βX, and let W be the preimage of X under i'. Now $W - Z$ is the preimage of Y in W [3, 6.11], so $W - Z$ is a closed subset of W. Also $i' \mid W$ is a closed mapping [3, 10.13], so X is the adjunction space obtained by adjoining W to Y by means of $i' \mid (W - Z)$. Next, let $q \in W - Z$, and let $\{z_\delta\}$ be a net in Z converging to q. Then $\{i'(z_\delta)\}$ converges to $i'(q)$. Since

$$\psi(\mathbf{1}, \sigma_{\mathbf{1}})(i'(q)) = 1,$$

$\{\psi(\mathbf{1}, \sigma_{\mathbf{1}})(i'(z_\delta))\} = \{\sigma_{\mathbf{1}}(z_\delta)\}$ converges to 1. Hence $\sigma_{\mathbf{1}}^\beta(q) = 1$, or

$$\theta(\mathbf{1})(q) = 1,$$

whence $q \in V$. This implies that $W - Z \subset V$. If we show that $\theta' \mid (W - Z) = i' \mid (W - Z)$, then it will follow that X is the adjunction space obtained by adjoining W to Y by means of $\theta' \mid (W - Z)$. Suppose, on the contrary, that there exists $p \in W - Z$ such that $\theta'(p) \neq i'(p)$. Let $\{z_\gamma\}$ be a net in Z converging to p. Choose $f \in C(Y)$ so that $f(\theta'(p)) = 1$ and $f(i'(p)) = 0$. Since $\sigma_f^\beta \mid V = f\theta'$, we have $\sigma_f^\beta(p) = 1$, so that $\{\sigma_f(z_\gamma)\}$ converges to 1, whence $\{\sigma_f(i'(z_\gamma))\}$ converges to 1. Hence $\{i'(z_\gamma)\}$ converges to $i'(p)$, $\{\psi(f, \sigma_f)(i'(z_\gamma))\}$ converges to 1, and $\psi(f, \sigma_f)(i'(p)) = 0$; this contradicts the continuity of $\psi(f, \sigma_f)$ at $i'(p)$. It follows that

$$\theta' \mid (W - Z) = i' \mid (W - Z).$$

Example 3. Let $Z = R$, and let Y be a one-point space, so that $C(Y, R) = R$. Since $\beta Z - Z$ consists of two connected open-and-closed subspaces [3, 6.10], there are exactly four continuous mappings of an open-and-closed subset of $\beta Z - Z$ into Y, namely, the mappings whose domains are $\beta Z - Z$, the two components of $\beta Z - Z$, and the empty set. Thus there are exactly four equivalence classes of extensions of $C_0(R, R)$ by R. The second and third classes contain extensions that are isomorphic, although inequivalent. The last class contains the direct sum of $C_0(R, R)$ and R. In the first three cases, there are infinitely many nonhomeomorphic adjunction spaces X such that some subalgebra of $C(X, R)$ is in the equivalence class; in the last case, there is only one, namely, the topological sum of Z and Y.

Now let Y be a two-point space. Similar considerations show that there are exactly nine possible continuous mappings, and hence nine equivalence classes of extensions.

COROLLARY 2. *Let X be a locally compact space, Y a compact subset of X, and $Z = X - Y$. There is an equivalence class of extensions of $C_0(Z)$ by $C(Y)$ containing a subalgebra of $C(X)$.*

Proof. The mapping e that embeds the topological sum of Z and Y into βX is continuous. It has a continuous extension e' to the topological sum of βZ and Y. Since X is locally compact, it is open in βX [3, 3.15(d)]. Let W denote the preimage of X under e'. Then $W \cap \beta Z$ is open in βZ, and hence $W \cap (\beta Z - Z)$ is open in $\beta Z - Z$. Also $W - Z$ is the preimage of Y under e', and since Y is compact it is closed in βX; thus

$$(W - Z) \cap \beta Z = W \cap (\beta Z - Z)$$

is closed in $\beta Z - Z$. It is easy to verify that X is the adjunction space obtained by adjoining the subspace $W \cap \beta Z$ of βZ to Y by means of the mapping e' restricted to the open-and-closed subset $W \cap (\beta Z - Z)$ of $\beta Z - Z$. Hence the equivalence class of extensions corresponding to this mapping contains a subalgebra of $C(X)$.

Compactness of Y is not necessary in Corollary 2, as one sees easily by taking X to be R and Y to be $\{x \in R : x \leq 0\}$. Now it is clearly necessary that Y be C-embedded in X, so one would like to assume only that Y is a closed, C-embedded realcompact subset of X. But the following example shows that this is not sufficient.

Example 4. Let $X = R$, $Y = \mathbf{N}$, and $Z = X - Y$. Then X is a locally compact space and Y is a closed, C-embedded realcompact subset of X. But no extension of $C_0(Z)$ by $C(Y)$ can be equivalent to a subalgebra of $C(X)$. For, as shown in the proof of Theorem 4, if an extension E is equivalent to a subalgebra E' of $C(X)$ then each (f, σ_f) in E maps to a function in E' that coincides with σ_f on Z and f on Y. Choose $f \in C(Y)$ to be the characteristic function of the even integers, and let g be the image of (f, σ_f) in E'. Then $f^2 = f$, so

$$(g \mid Z)^2 - (g \mid Z) = (\sigma_f)^2 - \sigma_f = (\sigma_f)^2 - \sigma_{f^2} \in C_0(Z).$$

But this is impossible, since there is a noncompact set $A \subset Z$ such that $g \mid A = \mathbf{1/2}$.

Alternatively, one can apply Theorem 4 directly. It is not hard to see that X cannot be the adjunction space obtained by adjoining a subspace of βZ to Y by means of the restriction of *any* continuous mapping of an open-and-closed subset of $\beta Z - Z$ into Y.

There seems to be no nice analogue of Theorem 4 for matrix-valued functions. The following example shows that the only adjunction space on which the functions will, in general, be continuous is the trivial one—the topological sum of Z and Y.

Example 5. Let $Z = R$, Y be a one-point space, and

$$\theta : C_2(Y) \rightarrow C_2(\beta Z - Z)$$

be the monomorphism onto the constant functions defined by $\theta F(z) = U^{-1}FU$ for all $z \in \beta Z - Z$, where U is a unit in $C_2(Y) = L_2(\mathcal{K})$ that is not in the center. Then the induced mapping $\phi : \beta Z - Z \rightarrow Y$ is onto Y. Let $F \in C_2(Y)$ be a matrix that does not commute with U. Now (F, σ_F) is an element of the graph of θ, which is in the equivalence class of extensions of $C_{20}(Z)$ by $C_2(Y)$ corresponding to $(\phi, \bar{U})$. Since $\sigma_F^\beta \mid (\beta Z - Z) = \theta(F)$, we have

$$\lim_{z \to \pm\infty} \sigma_F(z) = \theta(F) = U^{-1}FU \neq F.$$

It follows that (F, σ_F) cannot represent a continuous function on any adjunction space except the topological sum of Z and Y.

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Syracuse University
Syracuse, New York