THE EQUICONTINUOUS STRUCTURE RELATION FOR ERGODIC ABELIAN TRANSFORMATION GROUPS

BY

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I. Introduction

Let $(\tilde{X}, T)$ be a transformation group with compact, metric phase space, $\tilde{X}$, and abelian phase group, T. $(\tilde{X}, T)$ is ergodic if every proper, closed, T-invariant subset is nowhere dense. By [7] this is equivalent to requiring the set of points, X, whose orbits are dense in $\tilde{X}$, to be comeager. $(\tilde{X}, T)$ is weakly mixing if $(\tilde{X} \times \tilde{X}, T)$ is ergodic, where the action of T is given by $(x, x')t = (xt, x't)$.

In [2], it was shown that there exists on $(\tilde{X}, T)$, a least, closed, T-invariant equivalence relation, $\tilde{S}_e$, such that $(\tilde{X}/\tilde{S}_e, T)$ is an equicontinuous transformation group. $\tilde{S}_e$ is called the equicontinuous structure relation on $\tilde{X}$. In [15], Veech made a thorough study of $\tilde{S}_e$ when $(\tilde{X}, T)$ is a minimal set. However, when $(\tilde{X}, T)$ is not minimal, the relation $\tilde{S}_e$ could be quite obscure. Consider, for example, the continuous flow acting on the unit interval with two end points fixed. Then $\tilde{S}_e = \tilde{X} \times \tilde{X}$. If we restrict our attention to the subflow (X, T), where X is the open interval, then there is a faithful homomorphism of (X, T) into the universal almost periodic minimal set. On the other hand, consider the Stepanoff flows on the two torus with one fixed point [13]. In this case, $\tilde{S}_e$ is again equal to $\tilde{X} \times \tilde{X}$, but in some instances, (X, T) cannot be mapped homomorphically into any nontrivial almost periodic minimal flows. The differences between these two examples seem to indicate it is more natural to consider the homomorphisms from (X, T) into almost periodic minimal flows with compact phase space, when $(\tilde{X}, T)$ is ergodic and nonminimal. In this note, we shall prove the existence of a least, closed, invariant equivalence relation, S_e, on (X, T) such that there exists a faithful homomorphism of $(X/S_e, T)$ into a compact, almost periodic, minimal transformation group with a certain universality property. We will demonstrate a condition on $(\tilde{X}, T)$ equivalent to the existence of an invariant, Borel, probability measure on $(\tilde{X}, T)$ with support $\tilde{X}$. Assuming one of these conditions, we will characterize S_e, and show it is contained in the regional proximal relation on (X, T) [2]. Finally, as applications, we will show the eigenfunctions and spatial eigenfunctions of Keynes and Robertson [11] are essentially equal and will give a sufficient condition for $(\tilde{X}, T)$ to be weakly mixing.

II. Construction of an almost periodic, minimal factor of (X, T)

Standing Notation. Throughout this paper $(\tilde{X}, T)$ will denote an ergodic transformation group with compact, metric phase space, $\tilde{X}$, and abelian phase

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group, T. (X, T) will denote the transformation group with phase space $X = \{x \in \tilde{X} \mid \overline{0(X)} = \tilde{X}\}$.

In Section II, (Y, T) will denote an almost periodic, minimal transformation group with compact phase space Y.

We will construct an almost periodic minimal factor of (X, T) which has compact phase space and is also a factor of (Y, T). In the spirit of Theorem 2.1 [16], we will construct invariant equivalence relations $\sim$ and $\approx$ on X and Y so that $(X/\sim, T)$ and $(Y/\approx, T)$ are transformation groups.

LEMMA 2.1. *If $(\tilde{X}, T)$, (X, T), and (Y, T) are as above there exists a closed set $N \subseteq X \times Y$ which is the orbit closure of each of its points, and for which $p_1(N) = X$, $(p_1 : \tilde{X} \times Y \to \tilde{X}$ is the projection onto the first coordinate).*

Proof. Let

$$B = \{Z_\lambda \mid Z_\lambda \subseteq \tilde{X} \times Y,\ Z_\lambda \text{ is closed, invariant, and } p_1(Z_\lambda) = \tilde{X}\}.$$

If $\{Z_\lambda \mid \lambda \in \Lambda\}$ is a subset of B which is totally ordered by inclusion, let $Z = \bigcap_\lambda Z_\lambda$. Z is not empty since $\tilde{X} \times Y$ is compact. Z is closed and invariant. If $p_1(Z) \neq \tilde{X}$, $p_1(Z)$ is compact so $\tilde{X} - p_1(Z) = U$ is open. Pick V, open, such that $V \subseteq \bar{V} \subseteq U$. $(\tilde{X} - \bar{V}) \times Y$ is open in $\tilde{X} \times Y$, $Z \subseteq (\tilde{X} - \bar{V}) \times Y$, and $\tilde{X} \times Y$ compact implies at least one of the Z_λ's is contained in $(\tilde{X} - \bar{V}) \times Y$. Hence

$$\tilde{X} = p_1(Z_\lambda) \subseteq \tilde{X} - \bar{V}$$

a contradiction and $Z \in B$. By Zorn's Lemma there exists a maximal element, Z', in B.

Let $N = Z' \cap (X \times Y)$. N is closed in $X \times Y$, invariant, nonempty, and $(x, y) \in N$ implies $\operatorname{cl} O(x, y) = Z'$ where closure is with respect to $\tilde{X} \times Y$. So

$$\operatorname{cl} O(x, y) = N = Z' \cap (X \times Y)$$

where closure is with respect to $X \times Y$. Since $p_1(Z') = \tilde{X}$, $p_1(N) = X$.

LEMMA 2.2. *If $(\tilde{X}, T)$, (X, T), and (Y, T) are as above then $(X \times Y, T)$ is the disjoint union of closed sets $\{N_j\}$, where N_j is the orbit closure of each of its points and $p_1(N_j) = X$. In fact all such sets are isomorphic.*

Proof. Let $N \subseteq X \times Y$ be the set guaranteed by Lemma 2.1. If $(x_0, y_0) \notin N$, let $\operatorname{cl} O(x_0, y_0) = N_0$. Since $p_1(N) = X$ there exists $y' \in Y$ with $(x_0, y') \in N$. Since Y is a compact abelian group [3, p. 26, Remark 4.6] we may define

$$\alpha : X \times Y \to X \times Y \quad \text{by} \quad \alpha(x, y) = (x, yy'^{-1}y_0).$$

α is an isomorphism and $\alpha(N) = N_0$, has the required properties.

LEMMA 2.3. *Let $N \subseteq X \times Y$ be the orbit closure of each of its points, where $(\tilde{X}, T)$, (X, T) and (Y, T) are as before. Let $x \sim x'$ if there exists $y \in Y$ such that (x, y), $(x', y) \in N$, and $y \approx y'$ if $y = y'$, or if there exists $x \in X$ such that (x, y), $(x, y') \in N$. $\sim$ and $\approx$ are invariant equivalence relations and $\sim$ is closed.*

Proof. $\sim$ and $\approx$ are reflexive, symmetric, and invariant. In order to show they are transitive we will demonstrate that (x, y), (x', y), and $(x', y') \in N$ imply $(x, y') \in N$.

Let $\{t_\lambda\}$ and $\{s_\mu\}$ be nets in T such that

$$\lim_\lambda (x, y)t_\lambda = (x', y') \quad \text{and} \quad \lim_\mu (x', y)s_\mu = (x, y).$$

Since the action of T on Y is equicontinuous [3, p. 25],

$$\lim_\mu\ (\lim_\lambda\ xt_\lambda)s_\mu = \lim_\mu\ x's_\mu = x$$

and

$$\lim_\mu \lim_\lambda yt_\lambda s_\mu = \lim_\mu \lim_\lambda ys_\mu t_\lambda = \lim_\lambda \lim_\mu ys_\mu t_\lambda = \lim_\lambda yt_\lambda = y'.$$

Hence $((x, y)t_\lambda)s_\mu \to (x, y')$ and $(x, y') \in N$.

If $x \sim x'$, $x' \sim x''$ with (x, y), (x', y), (x', y'), and $(x'', y') \in N$, we have shown that $(x, y') \in N$ and hence $x \sim x''$. If $y \approx y'$, $y' \approx y''$, $y \neq y'$, and $y' \neq y''$ then for some $x, x' \in X$, (x, y), (x, y'), (x', y') and $(x', y'') \in N$. If we replace y by y' and y' by y'' in the above paragraph we get $(x, y'') \in N$ and $y \approx y''$. If $y = y'$ or $y' = y''$ the result is obvious.

If $x_\lambda \sim x'_\lambda$, $\lambda \in \Lambda$, with $x_\lambda \to x$, and $x_\lambda' \to x'$, there exist $y_\lambda \in Y$ such that (x_λ, y_λ), $(x_\lambda', y_\lambda) \in N$. Since Y is compact we may assume $y_\lambda \to y \in Y$, so $(x_\lambda, y_\lambda) \to (x, y) \in N$, and $(x_\lambda', y_\lambda) \to (x', y) \in N$, $x \sim x'$ and $\sim$ is closed.

We would like to find a closed, invariant equivalence relation on Y which contains $\approx$.

LEMMA 2.4. *If $N \subseteq X \times Y$, $(\tilde{X}, T)$, (X, T) and (Y, T) are as before, fix a group structure on Y and assume $(x_0, e) \in N$, for some $x_0 \in X$ and e the identity of Y. $H = \{y \mid (x_0, y) \in N\}$ is a closed subgroup of Y and there is a natural action of T on Y/H making $(Y/H, T)$ a compact, almost periodic, minimal transformation group.*

Proof. Let $y_1, y_2 \in H$, so (x_0, e), (x_0, y_1), and $(x_0, y_2) \in N$. Let $\{t_\lambda\}$ be a net in T such that $(x_0, y_2)t_\lambda \to (x_0, e)$. By our identification of T with a dense subgroup of Y [3, p. 26], $\{t_\lambda\}$ is a net in Y and we may assume $t_\lambda \to p \in Y$. Since $y_2 t_\lambda \to e$ we have $p = y_2^{-1}$.

$$\lim\ (x_0, y_1)t_\lambda = (x_0, y_1 y_2^{-1}) \in N \quad \text{and} \quad y_1 y_2^{-1} \in H.$$

N, and hence H, is closed. $(Y/H, T)$ can be made into a compact transformation group. Since T is abelian, (Y, T), and hence $(Y/H, T)$, is almost periodic and minimal [3, p. 26].

LEMMA 2.5. *If* $y \approx y'$ *then* $yy'^{-1} \in H$, $(y, y' \in Y)$.

Proof. If $y \approx y'$ and $y \neq y'$ there exists $x \in X$ such that (x, y), $(x, y') \in N$. Let $\{t_\lambda\}$ be a net in T such that $(x, y')t_\lambda \to (x_0, e) \in N$. As before we may assume $t_\lambda \to p \in Y$ and $y'p = e$ implies $p = y'^{-1}$.

$$(x, y)t_\lambda \to (x_0, yy'^{-1}) \quad \text{and} \quad yy'^{-1} \in H.$$

If $y = y'$ the result is obvious.

LEMMA 2.6. *If* $\approx$, H, (X, T) *and* (Y, T) *are as before and* $\approx^*$ *is the least closed invariant equivalence relation containing* $\approx$, *then* $y \approx^* y'$ *if and only if* $yy'^{-1} \in H$.

Proof. By the above lemma $y \approx y'$ implies $yy'^{-1} \in H$. $H = [e]_\approx$ since $y \in H$ implies (x_0, y), $(x_0, e) \in N$. The proof is completed since H generates a closed, invariant, equivalence relation.

We will now construct the homomorphism of (X, T) into $(Y/H, T)$. Let

$$\pi_1 : (X, T) \to (X/\sim, T), \quad \pi_2 : (Y/\approx, T) \to (Y/H, T),$$

$$\pi_3 : (Y, T) \to (Y/\approx, T),$$

and

$$\beta : (Y, T) \to (Y/H, T)$$

be the natural maps. π_2 is well defined by Lemma 2.5 and $\beta = \pi_2 \circ \pi_3$.

Define the homomorphism

$$\varphi : (X/\sim, T) \to (Y/\approx, T)$$

by $\varphi([x]) = [y]$ where $(x, y) \in N$, $y \in Y$. $\varphi([x]) = [y] = \varphi([x'])$ if and only if (x, y), $(x', y) \in N$ so φ is well defined and one-to-one. Since π_1 is a quotient map, φ is continuous if $\varphi \circ \pi_1$ is continuous. If $x_\lambda \to x$ and (x_λ, y_λ), $(x, y) \in N$ we may assume $y_\lambda \to y' \in Y$ and $(x_\lambda, y_\lambda) \to (x, y') \in N$ so $[y] = [y']$. Q.E.D.

We would like to show

$$\pi_2 \circ \varphi : (X/\sim, T) \to (Y/H, T)$$

is one-to-one. $\pi_2 \circ \varphi([x_1]) = [y]_H = \pi_2 \circ \varphi([x_2])$ implies $\pi_2([y_1]) = \pi_2([y_2])$ for $y_1, y_2 \in Y$ such that (x_1, y_1), $(x_2, y_2) \in N$. $y_1 y_2^{-1} \in H$ and $(x_0, y_1 y_2^{-1}) \in N$. There exists a net $\{s_\mu\}$ in T such that $(x_0, e)s_\mu \to (x_2, y_2)$. We may assume $s_\mu \to y_2$, so $(x_0, y_1 y_2^{-1})s_\mu \to (x_2, y_1) \in N$ and $x_1 \sim x_2$. Q.E.D.

We have the following commutative diagram with $\pi_2 \circ \varphi$ one-to-one.

$$
\begin{tikzcd}
(X, T) \arrow[dr, "\pi_1"] \arrow[dddrr, "\alpha"'] & & & & (Y, T) \arrow[ddll, "\pi_3"'] \arrow[dddll, "\beta"] \\
& (X/\sim, T) \arrow[dr, "\varphi"] & & & \\
& & (Y/\approx, T) \arrow[d, "\pi_2"] & & \\
& & (Y/H, T) & &
\end{tikzcd}
$$

The following example shows that $\pi_2 \circ \varphi$ need not be onto and hence $(X/\sim, T)$ need not be isomorphic to $(Y/H, T)$.

Example 2.1. Let $(\tilde{X}, T)$ be the one point compactification of the transformation group (R, R), with the group action. Let (Y, T) be the two torus with the irrational flow. $(\tilde{X}, T)$ is ergodic and (Y, T) is almost periodic, compact and minimal. (X, T) is just the transformation group (R, R). Fix $(x_0, y_0) \in Y$ and let $N = \{(t, (x_0, y_0)t) \mid t \in T\}$. $N \subseteq X \times Y$, is the orbit closure of each of its points and we may assume $(x_0, y_0) = e$, the identity of Y. $(0, (x_0, y_0)) \in N$ and we can form $H = \{(x, y) \mid (0, (x, y)) \in N\}$. $H = \{(x_0, y_0)\} = \{e\}$.

If $\pi_2 \circ \varphi : (X/\sim, T) \to (Y/H, T)$ is onto then for each $(x, y) \in Y$ there exists

$$[(x', y')] \in \varphi(X/\sim)$$

such that $(x, y)(x', y')^{-1} \in H = \{e\}$. Hence φ must be onto. If $p_2 : X \times Y \to Y$,

$$p_2(N) = O((x_0, y_0)) \subsetneq Y.$$

Pick $(x'', y'') \notin O((x_0, y_0))$. We have $[(x'', y'')] \notin \varphi(X/\sim)$. Q.E.D.

THEOREM 2.1. *Let $(\tilde{X}, T)$ be an ergodic transformation group with compact metric phase space $\tilde{X}$, and abelian phase group T, (Y, T) a compact, almost periodic, minimal transformation group and (X, T) the transformation group on*

$$X = \{x \in \tilde{X} \mid \operatorname{cl}(O(x)) = \tilde{X}\}.$$

There exists a closed, invariant, equivalence relation, $\sim$, on X such that $(X/\sim, T)$ can be immersed in a one-to-one fashion into a compact, almost periodic, minimal factor $(Y/H, T)$ of (Y, T).

III. Certain universal almost periodic minimal transformation groups

DEFINITION 3.1. $(B(T), T)$ is a universal almost periodic minimal transformation group if $(B(T), T)$ is almost periodic and minimal, $B(T)$ is compact, T is abelian, there exists a continuous homomorphism, π, from T onto a dense subgroup of $B(T)$, and $(B(T), T)$ has the following universality property: given any compact, almost periodic, minimal transformation group (Y, T), there exists a continuous homomorphism $\theta' : (B(T), T) \to (Y, T)$ extending the natural homomorphism, θ, of T into Y.

Consider $\{(Z_\lambda, T) \mid (Z_\lambda, T)$ is compact, almost periodic, and minimal, and $\pi_\lambda : T \to Z_\lambda$ is the natural homomorphism$\}$. Define $\pi : T \to \prod_\lambda Z_\lambda$ by $\pi(t) = \{\pi_\lambda(t)\}$ and let $Z = \mathrm{cl}\,(\pi(T)) \subseteq \prod_\lambda Z_\lambda$. (Z, T) is the universal almost periodic minimal transformation group and $\theta' : (Z, T) \to (Z_\lambda, T)$ is defined by $\theta'(\{z_\lambda\}) = z_\lambda$.

DEFINITION 3.2. If X is Hausdorff, and T is abelian, then θ is an almost periodic immersion of (X, T) if θ is a homomorphism from (X, T) onto a dense subset of a compact, almost periodic, minimal transformation group (Y, T). We say that

$$\theta : (X, T) \to (Y, T) \quad \text{and} \quad \theta' : (X, T) \to (Y, T)$$

are equivalent if there exists a homomorphism $\varphi : (Y, T) \to (Y, T)$ such that $\theta = \varphi \circ \theta'$.

Remark 3.1. A continuous automorphism of a compact almost periodic minimal transformation group is an isomorphism, [1, p. 12], so the above relation is an equivalence relation.

DEFINITION 3.3. $\theta : (X, T) \to (Y, T)$ is the universal almost periodic immersion of (X, T) if, given any other almost periodic immersion $\theta' : (X, T) \to (Y', T)$, there exists a homomorphism $\varphi : (Y, T) \to (Y', T)$ such that $\varphi \circ \theta = \theta'$.

If $A = \{\theta_\lambda \mid \theta_\lambda : (X, T) \to (Y_\lambda, T)$ is an almost periodic immersion$\}$, fix $x_0 \in X$ and let $y_\lambda = \theta_\lambda(x_0)$. Let $Y = \mathrm{cl}\,\{y_\lambda\} T \subseteq \prod_\lambda Y_\lambda$ and $\theta : (X, T) \to (Y, T)$ be defined by $\theta(x) = \{\theta_\lambda(x)\}_\lambda$. (Y, T) is almost periodic and minimal since $\prod_\lambda Y_\lambda$, and hence Y, is a compact topological group with a homomorphic image of T as a dense subgroup. If $\theta_\lambda : (X, T) \to (Y_\lambda, T)$ is any almost periodic immersion there exists $\pi_\lambda : (\prod_\mu Y_\mu, T) \to (Y_\lambda, T)$ since $\theta_\lambda \in A$. π_λ restricted to (Y, T) is the required homomorphism.

The material in Section II allows us to give the following representation of the universal almost periodic immersion of (X, T).

THEOREM 3.1. *If $(\tilde{X}, T)$ is ergodic, $\tilde{X}$ compact metric, T abelian, and*

$$X = \{x \in \tilde{X} \mid \mathrm{cl}\, O(x) = \tilde{X}\},$$

let $(B(T), T)$ be the universal almost periodic minimal transformation group.

Choose N and H as before. If $\alpha : (X, T) \to (B(T)/H, T)$ is defined as in Section II, *it is the universal almost periodic immersion of (X, T).*

Proof. If $\gamma : (X, T) \to (Y, T)$ is an almost periodic immersion let

$$\delta : (B(T), T) \to (Y, T)$$

be the induced homomorphism. We may assume that $\gamma(X)$ contains, $\bar{e}$, the identity of Y and $\delta(e) = \bar{e}$ where e is the identity of $B(T)$.

$A = \{(x, y) \in X \times B(T) \mid \gamma(x) = \delta(y)\}$ is closed and invariant. Let x_0 be some point in $\gamma^{-1}(\bar{e})$ and form

$$N = \mathrm{cl}\, O(x_0, e) \subseteq X \times B(T) \quad \text{and} \quad H = \{y \in B(T) \mid (x_0, y) \in N\}.$$

We have (from the last diagram, with Y replaced by $B(T)$)

$$\alpha = \pi_2 \circ \varphi \circ \pi_1 : (X, T) \to (B(T)/H, T)$$

$$\text{and} \quad \beta = \pi_2 \circ \pi_3 : (B(T), T) \to (B(T)/H, T).$$

If $H \subseteq \ker(\delta)$, we can define $\epsilon : (B(T)/H, T) \to (Y, T)$ such that the following diagram commutes:

$$\begin{tikzcd}
(X, T) \arrow[dr, "\alpha"] \arrow[ddr, bend right, "\gamma"'] & & (B(T), T) \arrow[dl, "\beta"'] \arrow[ddl, bend left, "\delta"] \\
& (B(T)/H, T) \arrow[d, "\epsilon"] & \\
& (Y, T) &
\end{tikzcd}$$

$(x_0, e) \in A$ since $\gamma(x_0) = \bar{e} = \delta(e)$, and $N = \mathrm{cl}\, O(x_0, e) \subseteq A$. For each $y \in H$, $\bar{e} = \gamma(x_0) = \delta(y)$ and $y \in \ker(\delta)$ or $H \subseteq \ker(\delta)$.

COROLLARY 3.1. *The equivalence relation S_e discussed in the introduction is the one induced by $N = \mathrm{cl}\, O(x_0, e) \subseteq X \times B(T)$.*

IV. An equivalent condition for the existence of an invariant, Borel, probability measure on $(\tilde{X}, T)$ with support $\tilde{X}$

We will follow the notation built up in Sections II and III. If E is a subset of T, let f_E denote the characteristic function of E. Let g be a real-valued function of T and g^t denote the function that has values $g^t(s) = g(st)$.

DEFINITION 4.1. If $f(t)$ is a bounded real-valued function on an abelian group T, let $A = \{\langle t_1, \cdots, t_n; \alpha_1, \cdots, \alpha_n\rangle \mid t_i \in T, \alpha_i > 0, \sum_{i=1}^n \alpha_i = 1\}$. The upper mean of f, $\bar{M}(f)$, is defined by

$$\bar{M}(f) = \bar{M}_x(f(x)) = \inf_A \sup_t \textstyle\sum_{i=1}^n \alpha_i f(tt_i).$$

The upper mean of $E \subseteq T$ is $\bar{M}(f_E)$.

DEFINITION 4.2. If $x \in X$ and U is a neighborhood of x, let

$$[U, x] = \{t \mid xt \in U\}.$$

We will write $f_{[U,x]}$ as $f[U, x]$ for convenience.

$[U; x]$, $f[U; x]$, and $\bar{M}(g)$ satisfy the following relations ($x \in X$; U and W are neighborhoods of x; s and $t \in T$; g, and h are bounded real-valued functions on T; and $\alpha > 0$), [4, p. 8]:

(I) $([U, x])t^{-1} = [U, xt]$,
(II) $[Ut, xt] = [U, x]$,
(III) $f[U, x](ts) = f_{([U,x])t^{-1}}(s) = f[U, xt](s)$,
(IV) $U \subseteq W$ implies $f[U, x] \leq f[W, x]$,
(V) $\bar{M}_s(g(st)) = \bar{M}(g^t) = \bar{M}(g) = \bar{M}_s(g(s))$,
(VI) $\bar{M}_s(f[U; x](s)) = \bar{M}_s(f[U, x](st)) = \bar{M}_s(f[U, xt](s))$,
(VII) $\bar{M}(g + h) \leq \bar{M}(g) + \bar{M}(h)$,
(VIII) $\bar{M}(\alpha g) = \alpha \bar{M}(g); \alpha \geq 0$,
(IX) $\bar{M}_s(g(s) - g(st)) = \bar{M}(g - g^t) = 0$,
(X) $g \leq h$ implies $\bar{M}(g) \leq \bar{M}(h)$,
(XI) $U \subseteq W$ implies $\bar{M}(f[U, x]) \leq \bar{M}(f[W, x])$.

DEFINITION 4.3. If (Z, T) is any transformation group, with T abelian, we will say that (Z, T) is strongly ergodic at $z_0 \in Z$, if given any neighborhood U of z_0, $\bar{M}(f[U, z_0]) > 0$. (Z, T) is said to be strongly ergodic if it is strongly ergodic at each of its points. (Note that the terminology "strongly ergodic" is not standard.)

THEOREM 4.1. *If $(\tilde{X}, T)$ is an ergodic transformation group with compact metric space $\tilde{X}$, and abelian phase group, T, then the following are equivalent:*

(a) *there exists an invariant, Borel, probability measure on $(\tilde{X}, T)$ with support $\tilde{X}$,*

(b) *$(\tilde{X}, T)$ is strongly ergodic,*

(c) *$(\tilde{X}, T)$ is strongly ergodic at some point in X.*

Proof. (a) implies (b). Let U be an open set containing $x_0 \in X$, and $\langle t_1, \cdots, t_n; \alpha_1, \cdots, \alpha_n\rangle \in A$.

$$0 < \int_{\tilde{X}} f_U \, d\mu = \textstyle\sum_{i=1}^n \alpha_i \displaystyle\int_{\tilde{X}} f_U(x) \, d\mu(x) = \textstyle\sum_{i=1}^n \alpha_i \displaystyle\int_{\tilde{X}} f_U(xt_i) \, d\mu(x)$$

$$= \int_{\tilde{X}} \textstyle\sum_{i=1}^n \alpha_i f_U(xt_i) \, d\mu$$

$$\leq \sup_{x \in X} \left(\sum_{i=1}^{n} \alpha_i f_U(xt_i)\right),$$

where the second equality follows since $\mu(x)$ is invariant.

For every $\varepsilon > 0$, there exists an $x' \in \tilde{X}$ such that

$$\sup_{x \in \tilde{X}} \left(\sum_{i=1}^{n} \alpha_i f_U(xt_i)\right) - \varepsilon \leq \sum_{i=1}^{n} \alpha_i f_U(x't_i).$$

There exists a $\delta > 0$ such that $d(x'', x') < \delta$ implies $x''t_i \in U$ whenever $x't_i$ are. Since $O(x_0)$ is dense in $\tilde{X}$ there exists an $s \in T$ such that

$$d(x_0 s, x') < \delta \quad \text{and} \quad f_U(x_0 st_i) \geq f_U(x't_i).$$

We have

$$\sup_{x \in \tilde{X}} \left(\sum_{i=1}^{n} \alpha_i f_U(xt_i)\right) - \varepsilon \leq \sum_{i=1}^{n} \alpha_i f_U(x't_i) \leq \sum_{i=1}^{n} \alpha_i f_U(x_0 st_i),$$

which holds for some s given any $\varepsilon > 0$, and

$$0 < \int_{\tilde{X}} f_U \, d\mu \leq \sup_{x \in \tilde{X}} \left(\sum_{i=1}^{n} \alpha_i f_U(xt_i)\right) \leq \sup_s \sum_{i=1}^{n} \alpha_i f_U(x_0 st_i)$$

for all $\langle t_1, \cdots, t_n; \alpha_1, \cdots, \alpha_n \rangle$. Finally,

$$0 < \int_{\tilde{X}} f_U \, d\mu \leq \inf_A \sup_s \sum_{i=1}^{n} \alpha_i f_U(x_0 st_i) = \bar{M}(f[U, x_0]).$$

(b) implies (c). Obvious.

(c) implies (a). Let $\{W_i\}_{i=1}^{\infty}$ be a countable basis of $\tilde{X}$ made up of compact sets. Fix one of the W_i's and call it W. We will produce an invariant, Borel, probability measure, η^*, on $\tilde{X}$ such that $\eta^*(W) > 0$. Let $(\tilde{X}, T)$ be strongly ergodic at $x_0 \in X$.

Let L be the linear space generated by $\{f[U, x_0] \mid U \subseteq \tilde{X}\}$, and let H be the subspace generated by the identically one function. $\bar{M}$ is a positive, subadditive function on L with the invariance property V above.

If we define M on H by $M(n1) = n$ then $M(h) = \bar{M}(h)$ $(h \in H)$. $f[W, x_0]$ is an element of $L - H$ and

$$\sup\{-\bar{M}(-h - f[W, x_0]) - M(h) \mid h \in H\}$$
$$\leq \bar{M}(f[W, x_0]) \leq \inf\{\bar{M}(h + f[W, x_0]) - M(h) \mid h \in H\}.$$

By the Hahn-Banach Theorem [10, p. 454–455] we may extend M to a linear functional, $\tilde{M}$, on all of L such that

$$\tilde{M}(f[W, x_0]) = \bar{M}(f[W, x_0]) \quad \text{and} \quad \tilde{M}(g) \leq \bar{M}(g) \quad (g \in L).$$

$\tilde{M}$ has the following properties, (see [4, p. 8] for (i) and (iii)):

(i) $\inf_{t \in T} f(t) \leq M_-(f) \leq \tilde{M}(f) \leq \bar{M}(f) \leq \sup_{t \in T} f(t)$ where $M_-(f) = -\bar{M}(-f)$,

(ii) $\tilde{M}$ is a positive linear functional on L,

(iii) $\tilde{M}_t(f(st)) = \tilde{M}_t(f(t))$.

If U is an open or closed subset of $\tilde{X}$ define $\eta(U) = M(f[U, x_0])$. If

$S \subseteq \tilde{X}$ define $\eta^*(S) = \inf\{\sum_{i=1}^{\infty} \eta(U_i) \mid$ the U_i's are open and $S \subseteq \bigcup_{i=1}^{\infty} U_i\}$. η^* is a Carateodory outer measure and hence defines a Borel measure. $\eta^*(\tilde{X}) = 1$ so η^* is bounded. η, and hence η^*, is T-invariant, vis

$$\eta(At) = \tilde{M}_s(f[At, x_0](s)) = \tilde{M}_s(f[At, x_0](st)) = \tilde{M}_s(f[At, x_0t](s))$$
$$= \tilde{M}_s(f[A, x_0](s)) = \eta(A)$$

(A open or closed in $\tilde{X}$). Since W is compact, $\eta^*(W) = \eta(W)$ and

$$\eta^*(W) = \eta(W) = \tilde{M}(f[W, x_0]) = \bar{M}(f[W, x_0]).$$

Since $(\tilde{X}, T)$ is strongly ergodic at x_0, $\eta^*(W) > 0$.

Let η_i^* denote the normalized measure associated with W_i and define

$$\mu = \sum_{i=1}^{\infty} 1/2^i \eta_i^*.$$

μ is the required measure.

DEFINITION 4.3. A subset S of T is (left) syndetic if there exists a compact subset, K, of T such that $SK = T$.

DEFINITION 4.4. A transformation group (Y, T) is regionally almost periodic if for each open set U in Y there exists a syndetic subset S of T such that $Us \cap U \neq \emptyset$ $(s \in S)$.

LEMMA 4.1. [5, p. 61]. *If $\bar{M}(f_E) > 0$ then EE^{-1} is a syndetic subset of T.*

THEOREM 4.2. *If $(\tilde{X}, T)$ is an ergodic and strongly ergodic transformation group with $\tilde{X}$ compact metric, and T abelian then it is regionally almost periodic.*

Proof. If U is open and $[U, x_0] = E$ then there exists a $t \in T$ with $x_0 t \in U$. $(\tilde{X}, T)$ is strongly ergodic at $x_0 t$ so

$$\bar{M}(f[U, x_0]) = \bar{M}(f[U, x_0 t]) > 0$$

and EE^{-1} is syndetic.

If $s, s' \in E$, $x_0 s$, $x_0 s' \in U$. $Uss'^{-1} \cap U$ is nonempty since $x_0 \in Us'^{-1}$ implies $x_0 s \in Us'^{-1} s \cap U$.

V. A Characterization of S_e

In this section we will retain the notation built up in the first four sections and give the characterization of S_e mentioned in the introduction.

LEMMA 5.1. *If (X, T) is strongly ergodic, (Y, T) and $N \subseteq X \times Y$ are as in Section* II, *then (N, T) is strongly ergodic.*

Proof. Given $(x_1, y) \in N$, we will show that there exists $(x_1, y') \in N$ such that for each neighborhood $V \times W$ of (x_1, y'),

$$\bar{M}(f[(V \times W) \cap N, (x_1, y)]) > 0.$$

If not, for each $(x_1, y') \in N$, there exists $P_{y'} = (V_{y'} \times W_{y'}) \cap N$ such that

$$\bar{M}(f[P_{y'}, (x_1, y)]) = 0.$$

$Q_{x_1} = \{y \in Y \mid (x_1, y) \in N\}$ is compact since N is closed in $X \times Y$. Pick $y_1, \cdots, y_n$ such that $W = W_{y_1} \cup \cdots \cup W_{y_n} \supseteq Q_{x_1}$. Let $V = \bigcap_{i=1}^n V_{y_i}$.

$C = \{v \in V \mid (v, y) \in N$ and $y \in W^c$, for some $y \in Y\}$ is closed in X since Y is compact. $V' = V - C$ is open in X and contains x_1. If $x_1 s \in V'$ then $(x_1, y) \in N$ implies $(x_1 s, ys) \in N$ and hence $ys \in W$. Hence,

$$f[V', x_1](s) \leq f[(V' \times W) \cap N, (x_1, y)](s) \quad \text{for all} \quad s \in T.$$

Since $V' \times W \subseteq V \times W \subseteq \bigcup_{i=1}^n V_{y_i} \times W_{y_i}$ we have

$$(V' \times W) \cap N \subseteq \bigcup_{i=1}^n (V_{y_i} \times W_{y_i}) \cap N = \bigcup_{i=1}^n P_{y_i},$$

as well as

$$f[\bigcup_{i=1}^n P_{y_i}, (x_1, y)] \leq \sum_{i=1}^n f[P_{y_i}, (x_1, y)].$$

$$\begin{aligned} 0 < \bar{M}(f[V', x_1]) &\leq \bar{M}(f[(V' \times W) \cap N, (x_1, y)]) \\ &\leq \bar{M}(f[\bigcup_{i=1}^n P_{y_i}, (x_1, y)]) \\ &\leq \bar{M}(\sum_{i=1}^n f[P_{y_i}, (x_1, y)]) \\ &\leq \sum_{i=1}^n \bar{M}(f[P_{y_i}, (x_1, y)]) = 0 \end{aligned}$$

and we have a contradiction.

If A is any neighborhood of (x_1, y) in N we have (x_1, y') as above and $(x_1, y')t \in A$ for some $t \in T$. Hence there exists a neighborhood B of (x_1, y') such that $Bt \subseteq A$. Choose $E \times F$ a neighborhood of (x_1, y') such that $(E \times F) \cap N \subseteq B$.

$$\begin{aligned} 0 < \bar{M}(f[(E \times F) \cap N, (x_1, y)]) \leq \bar{M}(f[B, (x_1, y)]) &= \bar{M}(f[Bt, (x_1, y)t]) \\ \leq \bar{M}(f[A, (x_1, y)t]) &= \bar{M}(f[A, (x_1, y)]) \end{aligned}$$

and N is strongly ergodic.

In [6] Følner proved the following useful theorem:

THEOREM 5.1. *Let V have upper mean greater than zero and let S be an arbitrary neighborhood of the identity of T. There exist continuous characters $\chi_1, \cdots, \chi_n$ such that the set of*

$$t \in F(\chi_1, \cdots, \chi_n) = \{t \mid \operatorname{Re} \chi_j(t) > 0, j = 1, 2, \cdots, n\}$$

not expressible as $t_1 t_2^{-1} s$, $t_1, t_2 \in V$, $s \in S$, has upper mean equal to zero.

Following [15, Theorem 1.1] we shall characterize $\sim$, defined from $N \subseteq X \times B(T)$, in the same way as Veech characterized the equicontinuous structure relation of a minimal transformation group.

THEOREM 5.2. *If $(\tilde{X}, T)$ is ergodic and strongly ergodic, $\tilde{X}$ is compact metric, T is abelian, and (X, T), $(B(T), T)$ and N are as before, then $x_1 \sim x_2$ if and only if there exist nets $\{t_\lambda\}$ and $\{t'_\lambda\}$ in T such that*

$$\lim_\lambda x_1 t_\lambda = x_1, \quad \lim_\lambda x_1 t'_\lambda = x_1 \quad \textit{and} \quad \lim_\lambda x_1 t_\lambda t'^{-1}_\mu = x_2.$$

Proof. Consider the following commutative diagram:

$$\begin{tikzcd}
(X, T) \arrow[dr, "\pi_1"] \arrow[dddrr, "\alpha"'] & & & & (B(T), T) \arrow[ddll, "\pi_3"'] \arrow[dddll, "\beta"] \\
& (X/\sim, T) \arrow[dr, "\varphi"] & & & \\
& & (B(T)/\approx, T) \arrow[d, "\pi_2"] & & \\
& & (B(T)/H, T) & &
\end{tikzcd}$$

If $x_1 \sim x_2$ then $\varphi([x_1]) = [y_0]_\approx = \varphi([x_2])$ for some $y_0 \in B(T)$ with (x_1, y_0), $(x_2, y_0) \in N$. Let y_0 be the identity of $B(T)$ and choose U, and S neighborhoods of x_1 and y_0 respectively. Since $\bar{M}(f[U, x_1]) > 0$, we have the characters $\chi_1, \cdots, \chi_n$ generated by Følner's Theorem. $\{\chi_i\}_{i=1}^n$ may be considered as the restrictions to T of continuous characters on $B(T)$. Since $y_0 \in F(\chi_1, \cdots, \chi_n)$ and the χ_i are continuous on $B(T)$ there exists a neighborhood V of y_0 such that $y_0 t \in V$ implies $t \in F(\chi_1, \cdots, \chi_n)$.

Let W be any neighborhood of x_2. Since (x_1, y_0), $(x_2, y_0) \in N$, there exists a $t \in T$ and a neighborhood $W' \times V'$ of (x_1, y_0) such that $(W' \times V')t \subseteq W \times V$.

$$\begin{aligned}
0 < \bar{M}(f[(W' \times V') \cap N, (x_1, y_0)]) &\leq \bar{M}(f[W' \times V', (x_1, y_0)]) \\
&= \bar{M}(f[(W' \times V')t, (x_1, y_0)t]) \leq \bar{M}(f[W \times V, (x_1, y_0)t]) \\
&= \bar{M}(f[W \times V, (x_1, y_0)]).
\end{aligned}$$

The first inequality follows since N is strongly ergodic and the others by relations II, III and XI in Section IV.

$$[W \times V, (x_1, y_0)] \subseteq F \quad \text{for } (x_1, y_0)t \in W \times V$$

implies $y_0 t \in V$ and $t \in F$. Since $\bar{M}(f[W \times V, (x_1, y_0)]) > 0$ we can find a

$$t'_{(U,W,S)} \in [W \times V, (x_1, y_0)]$$

such that

$$t'_{(U,W,S)} = \bar{t}_{1,(U,W,S)} \, t_{2,((U,W,S)}^{-1} \, s_{(U,W,S)}$$

with $\bar{t}_{1,(U,W,S)}$, $t_{2,(U,W,S)} \in [U, x_1]$ and $s_{(U,W,S)} \in S$.

Choose a neighborhood, Q, of $t'_{(U,W,S)}$ such that $t \in Q$ implies $x_1 t \in W$. Choose

$$t_{(U,W,S)} \in T \cap t_{2,(U,W,S)} \, \bar{t}_{1,(U,W,S)}^{-1} \, Q \cap S,$$

and let $t_{1,(U,W,S)} = \bar{t}_{1,(U,W,S)}\, t_{(U,W,S)}$. $t_{(U,W,S)} \to e$ in T and hence

$$x_1\bar{t}_{1,(U,W,S)}\, t_{(U,W,S)} \to x_1 .$$

$x_1\, t_{2,(U,W,S)} \to x_1$, $x_1\, t_{1,(U,W,S)}\, t^{-1}_{2,(U,W,S)} \to x_2$, and $\{t_{1,(U,W,S)}\}$, $\{t_{2,(U,W,S)}\}$ are the required nets.

Conversely, if the condition holds, then

$$\lim_\lambda \alpha(x_1 t_\lambda) = \alpha(x_1), \quad \lim_\lambda \alpha(x_1 t'_\lambda) = \alpha(x_1)$$

and $\{t_\lambda\}$, $\{t'_\lambda\}$ converge to the identity, e, of $B(T)/H$. Since $(B(T)/H, T)$ is equicontinuous we have

$$\begin{aligned}\alpha(x_2) &= \lim_\lambda \alpha(x_1\, t_\lambda\, t'^{-1}_\lambda) \\ &= \lim_\lambda \alpha(x_1)\,(t_\lambda\, t'^{-1}_\lambda) = \lim_\lambda\, (\lim_\lambda \alpha(x_1) t_\lambda) t'^{-1}_\lambda = \lim_\lambda \alpha(x_1) t'^{-1}_\lambda = \alpha(x_1).\end{aligned}$$

If $\varphi \circ \pi_1(x_1) = [y_1]$, and $\varphi \circ \pi_1(x_2) = [y_2]$, then

$$y_1\, y_2^{-1} \in H \quad \text{or} \quad (x_0\, ,\, y_1\, y_2^{-1}) \in N.$$

Let $\{s_\mu\}$ be a net in T such that $\lim_\mu (x_0\, ,\, e)s_\mu = (x_2\, ,\, y_2)$. We may assume $\{s_\mu\}$ converges to y_2 as a net in $B(T)/H$, so

$$(x_0\, ,\, y_1\, y_2^{-1})s_\mu \to (x_2\, ,\, y_1\, y_2^{-1} y_2) = (x_2\, ,\, y_1) \in N$$

which gives $x_1 \sim x_2$.

Corollary 5.1. *If $(\tilde{X}, T)$ is as in Section II and if $\tilde{X} = X$ then Veech's result follows* [15, p. 723, Theorem 1.1].

Proof. See [11, p. 365, Theorem 2.10].

Corollary 5.2. *If $(\tilde{X}, T)$ is as in the above theorem we can characterize the relation $S_e = \sim$ of (X, T) and we know that $S_e \subseteq Q(X)$, the regional proximal relation on (X, T).*

Proof. $Q(X) = \bigcap \{\mathrm{cl}\,(\alpha T) \mid \alpha \text{ is an index of } X\}$. If $x_1 \sim x_2$ and $\{t_\lambda\}$, $\{t'_\lambda\}$ satisfy the conditions in the above theorem then

$$(x_1\, ,\, x_1\, t_\lambda\, t'^{-1}_\lambda)t'_\lambda \to (x_1\, ,\, x_1) \quad \text{and} \quad ((x_1\, ,\, x_1\, t_\lambda\, t'^{-1}_\lambda)t'_\lambda)t'^{-1}_\lambda \to (x_1\, ,\, x_2).$$

If α is an index of X, we may assume $(x_1\, ,\, x_1\, t_\lambda\, t'^{-1}_\lambda)t'_\lambda \in \alpha$ for all λ. We have

$$((x_1\, ,\, x_1\, t_\lambda\, t'^{-1}_\lambda)t'_\lambda)t'^{-1}_\lambda \in \alpha T \quad \text{for all } \lambda,$$

$(x_1\, ,\, x_2) \in \mathrm{cl}\ (\alpha \mathrm{T})$ and the proof is completed.

VI. Eigenfunctions and the weakly mixing property

Let $(\tilde{X}, T)$ be an ergodic transformation group with Baire phase space $\tilde{X}$. Consider, $\bar{B}(\tilde{X})$, the algebra of all bounded complex-valued functions on $\tilde{X}$ whose restriction to X is continuous. If f and g are elements of $\bar{B}(\tilde{X})$ we will say they are equal if $\{x \in \tilde{X} \mid f(x) = g(x)\}$ is comeager.

$f \in \bar{B}(\tilde{X})$ is a topological eigenfunction of $(\tilde{X}, T)$ with eigenvalue χ, if f is not equal to the zero function and $\chi : T \to S^1$ is a continuous character of T such that $f(xt) = f(x)\chi(t)$ for all $t \in T$ and a comeager subset of $x \in \tilde{X}$. A topological eigenfunction is invariant if its eigenvalue is the trivial character.

A topological eigenfunction, f, is a spatial topological eigenfunction of $(\tilde{X}, T)$ if $f(xt) = f(x)\chi(t)$ for all $t \in T$ and $x \in \tilde{X}$.

As in [11] let $B(\tilde{X})$ be the algebra of all bounded complex-valued functions, f, on $\tilde{X}$ such that $c(f) = \{x \mid f \text{ is continuous at } x\}$ is comeager. Again, f and g elements of $B(\tilde{X})$ are said to be equal if $\{x \in \tilde{X} \mid f(x) = g(x)\}$ is comeager.

$f \in B(\tilde{X})$ is an eigenfunction of $(\tilde{X}, T)$ with eigenvalue χ, if f is not equal to the zero function and $\chi : T \to S^1$ is a character (not necessarily continuous) such that $f(xt) = f(x)\chi(t)$ for all $t \in T$ and a comeager subset of $x \in \tilde{X}$. An eigenfunction is invariant if its eigenvalue is the trivial character.

An eigenfunction, f, is a spatial eigenfunction if $f(xt) = f(x)\chi(t)$ for all $x \in \tilde{X}$ and $t \in T$.

Remark 6.1. Let (Z, T) be a point transitive transformation group with Baire phase space, Z. Let $W = \{z \in Z \mid \operatorname{cl} O(z) = Z\}$ and let $f : Z \to \mathbf{C}$ be a spatial eigenfunction on (Z, T) with eigenvalue $\chi : T \to S^1$. By a theorem due to Kakutani (cf. [8, p. 506]), $c(f)$ contains W. Since the eigenvalues of a spatial eigenfunction are always continuous each spatial eigenfunction $f : Z \to \mathbf{C}$ is also a (spatial) topological eigenfunction.

Remark 6.2. If f is a topological eigenfunction of $(\tilde{X}, T)$ then

$$X \subseteq \{x \mid f(xt) = f(x)\chi(t) \text{ for all } t \in T\}.$$

Remark 6.3. If we give T the compact open topology, $\mathfrak{I}$, then $(T, \mathfrak{I})$ is second countable and $\mathfrak{I}$ is the smallest topology on T making $\tilde{X} \times T \to \tilde{X}$ continuous $((x, t) \to xt)$. The eigenvalue, χ, of each eigenfunction, f, of $(\tilde{X}, (T, \mathfrak{I}))$ is sequentially continuous and hence continuous on $(T, \mathfrak{I})$. If $\mathfrak{S}$ is the original topology on T, $\chi : (T, \mathfrak{S}) \to (T, \mathfrak{I}) \to S^1$ is continuous and all eigenvalues are continuous.

Given a topological eigenfunction $f : \tilde{X} \to \mathbf{C}$ with eigenvalue $\chi : T \to S^1$ we would like to construct a spatial eigenfunction which equals f on the comeager subset X and has the same eigenvalue.

Let $f : \tilde{X} \to \mathbf{C}$ be a topological eigenfunction with eigenvalue $\chi : T \to S^1$. Fix $x_0 \in X$ and define $\tilde{F} : \tilde{X} \to \mathbf{C}$ by $\tilde{F}(x) = f(x)/|f(x_0)|$. (Note: $f(x_0) \neq 0$ for $f(x_0) = 0$ implies $f(x_0 t) = 0$ and hence $f/X = 0$). $\tilde{F}$ is a topological eigenfunction with eigenvalue χ. Let $F : X \to S^1$ be the restriction of $\tilde{F}$ to X.

If we define an action of T on S^1 by $st = s\chi(t)$ $(s \in S^1, t \in T)$ then (S^1, T) is an equicontinuous transformation group and $F : (X, T) \to (S^1, T)$ is a homomorphism, (cf. Remark 6.2). Let $Z = \operatorname{cl} F(X) \subseteq S^1$. (Note that if $\chi(t)$ is incommensurable with π for any $t \in T$, then $Z = S^1$.) (Z, T) is point transitive, compact and equicontinuous and hence is minimal and almost periodic. $F : (X, T) \to (Z, T)$ is an almost periodic immersion of (X, T).

If $N(x_0) = \operatorname{cl} O(x_0, F(x_0)) \subseteq X \times Z$ then $N(x_0)$ is the orbit closure of each of its points and defines an almost periodic immersion

$$G : (X, T) \to (Z/H(x_0), T)$$

where $H(x_0) = \{y \in Z \mid (x_0, y), (x_0, y_0) \in N(x_0)\}$ and $y_0 = F(x_0)$ is the identity of Z.

LEMMA 6.1. *If*

$$F : (X, T) \to (Z, T), \qquad N(x_0) = \text{cl}\, O(x_0, F(x_0)) \subseteq X \times Z, H(x_0),$$

and

$$G : (X, T) \to (Z/H(x_0), T)$$

are as above then $H(x_0) = \{e\}$, $F \equiv G$ *and* $N(x_0)$ *is a "graph" in* $X \times Z$, *i.e.*,

$$\{y \mid (x, y) \in N(x_0)\}$$

is a singleton $(x \in X)$.

Proof. $N(x_0) = \text{cl}\, O(x_0, y_0) = \text{cl}\,\{(x_0 t, F(x_0 t)) \mid t \in T\} \subseteq X \times Z$ and hence

$$N(x_0) = \{(x, F(x)) \mid x \in X\}$$

and is a graph. $H(x_0) = \{y \in Z \mid (x_0, y) \in N(x_0)\} = \{y_0\}$. $G(x_0) = [y_0]_{H(x_0)} = F(x_0)$ so $F \equiv G$.

COROLLARY 6.1. $\{y \mid (x, y) \in \text{cl}\, O(x_0, y_0)$ *where closure is in* $\tilde{X} \times Z\}$ *is a singleton for each* $x \in X$.

We would like to extend our almost periodic immersion

$$F : (X, T) \to (Z, T)$$

to a spatial eigenfunction, $h : \tilde{X} \to C$. To do so we first extend it to an open subset of $\tilde{X}$ which contains X.

Partition $\tilde{X}$ into the disjoint union

$$\tilde{X} = \bigcup \{X_x \times \mid x \in \hat{X}\}$$

where $X_x = \{x' \in \tilde{X} \mid \text{cl}\, O(x') = \text{cl}\, O(x)\}$. X is such a set and will be denoted by X_{x_0} for $x_0 \in \hat{X}$. (Notice that we can pick $\hat{X}$ so that this x_0 is the one we used to define $F : X \to S^1$.)

We have already constructed an almost periodic immersion

$$F : (X, T) \to (Z, T)$$

of the set X_{x_0}. We will construct an almost periodic immersion

$$\lambda_x : (X_x, T) \to (Y, T)$$

for each $x \in \hat{X} - \{x_0\}$. $(\bar{X}_x, T)$ satisfies the hypotheses of Lemmas 2.1, and 2.2, where $X_x = \{x' \mid \text{cl}\ O(x) = \bar{X}_x\}$. Hence $X_x \times Z$ is the disjoint union of sets $\{N_j\}$ of the type described. Choose an element $N(x)$ of $\{N_j\}$ so that

$$\text{cl}\, O(x_0, y_0) \cap N(x) \neq \emptyset$$

(here closure is in $\tilde{X} \times Z$). Let (x, y_x) be an element of

$$N(x) \subseteq \text{cl}\, O(x_0, y_0) \subseteq \tilde{X} \times Z$$

and define $\beta_{y_x} : \tilde{X} \times Z \to \tilde{X} \times Z$ by $\beta_{y_x}(x', y') = (x', y_x^{-1}y')$. β_{y_x} is an isomorphism. $N'(x) = \beta_{y_x}(N(x))$ is also the orbit closure of each of its points and $(x, y_0) \in N'(x)$. If

$$H(x) = \{y \in Z \mid (x, y) \in N'(x)\}$$

we have by Lemma 2.4 that $H(x)$ is a closed subgroup of Z and $(Z/H(x), T)$ is an almost periodic, minimal transformation group. Following the method of Section II we define the almost periodic immersion

$$\lambda_x : (X_x, T) \xrightarrow{\pi_1} (X_x/\sim_x, T) \xrightarrow{\varphi} (Z/\approx_x, T) \xrightarrow{\pi_2} (Z/H(x), T)$$

where $\sim_x$, and $\approx_x$ are the equivalence relations defined in Lemma 2.3.

Let

$$A = \{x \mid \{x\} \times Z \subseteq \text{cl}\, O(x_0, y_0)\}$$

and

$$B = \{x' \mid H(x) \neq Z, \text{ and } H(x) \neq \{y_0\} \text{ where } x' \in X_x\}.$$

A is closed and invariant. Since $\{y \mid (x, y) \in \text{cl}\, O(x_0, y_0)\}$ is a singleton for all $x \in X$ we have $A \cap X = \emptyset$.

LEMMA 6.2. *$\bar{B} \cap X = \emptyset$ and $\bar{B}$ is invariant.*

Proof. If $x^* \in \bar{B}$ there exists a net $\{x'_\lambda\}_{\lambda \in \Lambda}$ in B such that $x'_\lambda \to x^*$. Let $x'_\lambda \in X_{x_\lambda}$ and choose $y'_\lambda \in Z$ such that

$$(x'_\lambda, y'_\lambda) \in N(x_\lambda) \subseteq \text{cl}\, O(x_0, y_0) \subseteq \tilde{X} \times Z \quad (\lambda \in \Lambda).$$

(Remember that $p_1(N(x_\lambda)) = X_{x_\lambda}$.) Since $H(x_\lambda) \neq S^1$ it must be finite cyclic and we can choose the generating element, n_λ, from each $H(x_\lambda)$.

$$H(x_\lambda) = \{y \mid (x_\lambda, y) \in N'(x_\lambda)\}$$

implies $(x_\lambda, n_\lambda) \in N'(x_\lambda)$ and hence $(x_\lambda, y_{x_\lambda} n_\lambda) \in N(x_\lambda)$. If

$$(x_\lambda, y_{x_\lambda})t_{\lambda,\mu} \to (x'_\lambda, y'_\lambda) \quad \text{and} \quad t_{\lambda,\mu} \xrightarrow{\mu} p_\lambda$$

in Z then $(x'_\lambda, y_{x_\lambda} p_\lambda) = (x'_\lambda, y'_\lambda) \in \text{cl}\, O(x_0, y_0)$.

$$(x_\lambda, y_{x_\lambda} n_\lambda)t_{\lambda,\mu} \xrightarrow{\mu} (x'_\lambda, y_{x_\lambda} n_\lambda p_\lambda) = (x'_\lambda, y_{x_\lambda} p_\lambda n_\lambda) \in \text{cl}\, O(x_0, y_0).$$

If $(x'_\lambda, y'_\lambda) \to (x^*, y_1)$ and $(x'_\lambda, y'_\lambda n_\lambda) \to (x^*, y_1 q)$, where $x'_\lambda \to x^*$ and $n_\lambda \to q$, then $x^* \in X$ implies $y_1 = y_1 q$ or $y_0 = q = \lim_\lambda n_\lambda$. The cardinality of $\{n_\lambda^r \mid r$ is an integer$\}$ will go to infinity in λ. Hence

$$\{(x'_\lambda, y'_\lambda (n_\lambda)^r) \mid r \text{ is an integer}\} \subseteq \text{cl}\, O(x_0, y_0)$$

must have each point in $\{x^*\} \times Z$ as a cluster point, and $\{x^*\} \times Z \subseteq \text{cl}\, O(x_0, y_0)$, a contradiction to Corollary 6.1. B, and hence $\bar{B}$, is invariant.

LEMMA 6.3. *Let* $f : \tilde{X} \to \mathbf{C}$ *be a topological eigenfunction with eigenvalue* $\chi : T \to S^1$ *and* $\lambda_{x_0} = F : (X, T) \to (Z, T)$ *be defined as before. If we define*

$$\lambda : \tilde{X} - (A \cup \bar{B}) \to Z$$

by

$$\lambda / X_x = \lambda_x : (X_x, T) \to (Z, T) \quad (x \in \hat{X} - (A \cup \bar{B})),$$

and

$$\lambda(x) = 0 \quad (x \in A \cup \bar{B}),$$

then λ *is continuous at each point in* X.

Proof. Let $\{x'_\mu\}$ be a net in $\tilde{X} - (A \cup B)$ which converges to $x \in X$. Assume $x'_\mu \in X_{x_\mu}$. Since $H(z) = \{e\}$ if $z' \in X_z \cap [\tilde{X} - (A \cup \bar{B})]$ there exists but one $y'_\mu \in Z$ such that $(x'_\mu, y'_\mu) \in N(x_\mu)$. If $\{y'_\mu\}$ has subnets $\{y'_{\mu,1}\}$ and $\{y'_{\mu,2}\}$ converging to y_1 and y_2 respectively then

$$\lim (x'_{\mu,1}, y'_{\mu,1}) = (x, y_1) \quad \text{and} \quad \lim (x'_{\mu,2}, y'_{\mu,2}) = (x, y_2)$$

and (x, y_1), $(x, y_2) \in \mathrm{cl}\, O(x_0, y_0)$. By Corollary 6.1, $y_1 = y_2$ and $\{y'_\mu\}$ converges to $y \in Z$. (x, y), $(x, \lambda_{x_0}(x)) \in \mathrm{cl}\, O(x_0, y_0)$ implies that $\lambda_{x_0}(x) = [y]_{H(x_0)} = \{y\}$. Q.E.D.

THEOREM 6.1. *Let* $(\tilde{X}, T)$ *be ergodic,* $\tilde{X}$ *compact metric and* T *abelian. There exists a (spatial) topological eigenfunction,* f, *of* $(\tilde{X}, T)$ *if and only if there exists a spatial eigenfunction,* g, *of* $(\tilde{X}, T)$ *which is equal to* f, *i.e.* $\{x \mid f(x) = g(x)\}$ *is comeager.*

Proof. The "if" portion follows from Remark 6.1.

If $f : \tilde{X} \to \mathbf{C}$ is a (spatial) topological eigenfunction, define

$$F : (X, T) \to (Z, T) \quad \text{and} \quad \lambda : X - (A \cup B) \to Z$$

as in the above lemma. Define $h : \tilde{X} \to \mathbf{C}$ by $h(x) = \lambda(x)\, |f(x_0)|$ $(x \in \tilde{X})$. h is the required spatial eigenfunction.

LEMMA 6.4. *Let* $(\tilde{X}, T)$ *be ergodic,* $\tilde{X}$ *compact metric,* T *abelian and countable. There exists an eigenfunction,* $f : \tilde{X} \to \mathbf{C}$, *of* $(\tilde{X}, T)$ *if and only if there exists a spatial eigenfunction,* $g : \tilde{X} \to \mathbf{C}$, *which is equal to* f, *i.e* $\{x \mid f(x) = g(x)\}$ *is comeager.*

Proof. The "if" portion is obvious.

If $f : \tilde{X} \to \mathbf{C}$ is an eigenfunction of $(\tilde{X}, T)$ let $c(f) = \{x \mid f \text{ is continuous at } x\}$ and $D = \{x \mid f(xt) = f(x)\chi(t) \ (t \in T)\}$. We will find a spatial eigenfunction which equals f on $c(f)$.

If for each $x \in X \cap D$ there exists a $t \in T$ such that $xt \notin c(f)$ then

$$((X \cap D) - c(f))T = X \cap D$$

is meager since $(X \cap D) - c(f)$ is meager and T is countable. Hence

$$(X \cap D)^c \cup (X \cap D) = \tilde{X}$$

is meager, a contradiction. Hence there exists $x_0 \in X \cap D \cap c(f)$ with $O(x_0) \subseteq X \cap D \cap c(f)$.

Define $F' : O(x_0) \to S^1$ by

$$F'(x_0 t) = f(x_0 t)/|f(x_0)|,$$

and let $Z = \mathrm{cl}\, F'(O(x_0)) \subseteq S^1$. $F' : (O(x_0), T) \to (Z, T)$ is an almost periodic immersion if we define the action of T on Z as at the beginning of this section.

If $\gamma : (B(T), T) \to (Z, T)$ is the induced homomorphism let

$$y \in \gamma^{-1}(F(x_0))$$

and form $N = \mathrm{cl}\, O(x_0, y) \subseteq X \times B(T)$, and $H = \{y \in B(T) \mid (x_0, y) \in N\}$. If

$$\theta : (X, T) \to (B(T)/H, T)$$

is the universal almost periodic immersion induced by N and H then

$$\theta/_{O(x_0)} : (O(x_0), T) \to (B(T)/H, T)$$

is the universal almost periodic immersion of $(O(x_0), T)$ induced by

$N' = \mathrm{cl}\, O(x_0, y) \subseteq O(x_0) \times B(T)$ and $H' = \{y \in B(T) \mid (x_0, y) \in N'\} = H$. We have a homomorphism $\epsilon : (B(T)/H, T) \to (Z, T)$ such that the following commutes:

$$\begin{tikzcd}
(O(x_0), T) \arrow[dd, "F'"'] \arrow[dr, "\theta"] & \subseteq & (X, T) \arrow[dl, "\theta"] \\
& (B(T)/H, T) \arrow[dl, "\epsilon"] & \\
(Z, T) & &
\end{tikzcd}$$

$F = \epsilon \circ \theta : (X, T) \to (Z, T)$ is an almost periodic immersion of (X, T) which extends F' and $c(f) \subseteq \{x \mid F'(x) = F(x)\}$. By Lemma 6.3 and Theorem 6.1 we can extend F' to an eigenfunction $\lambda : \tilde{X} \to S'_1$. $g : \tilde{X} \to S'_1$ defined by $g(x) = \lambda(x)\, |f(x_0)|$ is a spatial eigenfunction and equals f on the comeager set $c(f)$.

THEOREM 6.2. *Let $(\tilde{X}, T)$ be ergodic, X compact metric and T abelian. There exists an eigenfunction, $f : \tilde{X} \to \mathbf{C}$, of $(\tilde{X}, T)$ if and only if there exists a spatial eigenfunction, $h : \tilde{X} \to \mathbf{C}$, which is equal to f, i.e. $\{x \mid f(x) = h(x)\}$ is comeager.*

Proof. The "if" portion is obvious.

If $f : \tilde{X} \to \mathbf{C}$ is an eigenfunction of $(\tilde{X}, T)$ let $c(f) = \{x \mid f \text{ is continuous at } x\}$ and $D = \{x \mid f(xt) = f(x)\chi(t) \ (t \in T)\}$. Give T the compact-open topology, $\mathfrak{J}$, and choose a dense subgroup S in T. $f : \tilde{X} \to \mathbf{C}$ is an eigenfunction of $(\tilde{X}, S)$ and by Lemma 6.4 there exists a spatial eigenfunction, $g : \tilde{X} \to \mathbf{C}$, of $(\tilde{X}, S)$ which equals f. By a theorem due to Kakutani (cf. [8, p. 506]) the set of points with dense orbit (with respect to S) are contained in $c(g)$. If $\operatorname{cl} xS = \tilde{X}$ then $x \in c(g)$ and $t \in T$ implies

$$\operatorname{cl} xtS = \operatorname{cl} xSt = \tilde{X} \text{ so } xT \subset c(g).$$

$g : \tilde{X} \to \mathbf{C}$ is also an eigenfunction of $(\tilde{X}, T)$ and there exists an $x_0 \in X$ with $x_0 T \subseteq X \cap D \cap c(f)$. We may now use the proof of Lemma 6.4 to construct the spatial eigenfunction, $h : \tilde{X} \to C$, which is equal to g and f.

THEOREM 6.3. *If $(\tilde{X}, T)$ is ergodic, strongly ergodic, $\tilde{X}$ compact metric and T abelian then the following are equivalent*:

(a) $(\tilde{X}, T)$ *is weakly mixing,*
(b) *there exists no nontrivial almost periodic immersion of* (X, T),
(c) *there exist no nonconstant* (*spatial*) *topological eigenfunctions of* $(\tilde{X}, T)$,
(d) *there exist no nonconstant* (*spatial*) *eigenfunctions of* $(\tilde{X}, T)$,
(e) *for every* $x \in X$ *there exists no nontrivial almost periodic immersion of* $(O(x), T)$.

Proof. (a) implies (b). Let $\theta : (X, T) \to (Y, T)$ be an almost periodic immersion and $(\tilde{X}, T)$ be weakly mixing. $(X \times X, T)$ is point transitive and since $(\theta \times \theta((X \times X)$ is a dense subset of $Y \times Y$, $(Y \times Y, T)$ is point transitive. Since $(Y \times Y, T)$ is equicontinuous it is minimal and hence trivial.

(b) implies (a). If there exist no nontrivial almost periodic immersions, $\sim = X \times X$. Let A be a closed invariant subset of $\tilde{X} \times \tilde{X}$ with nonempty interior. We would like to show $A = \tilde{X} \times \tilde{X}$. Let $p : \tilde{X} \times \tilde{X} \to \tilde{X}$ be the projection onto the first coordinate. p is open so $p(A^\circ)$ is open and nonempty. Pick $x \in p_1(A^\circ) \cap X$. Since A° is open we can pick an open set V with $\{x\} \times V \subseteq A^\circ$. Since $x \in X$ there exists a $t \in T$ with $(x, xt) \in \{x\} \times V \subseteq A^\circ$. Consider the homorphism

$$\theta_t : \tilde{X} \times \tilde{X} \to \tilde{X} \times \tilde{X}$$

defined by $\theta_t(x_1, x_2) = (x_1, x_2 t^{-1})$. $B = \theta_t(A^\circ)$ is open and contains (x, x). $\bar{B}$ is closed, invariant, has nonempty interior and $\bar{B}^\circ$ contains $O(x, x)$ for some $x \in X$. Since $A = \tilde{X} \times \tilde{X}$ if and only if $\bar{B} = \tilde{X} \times \tilde{X}$ we may assume that $O(x, x) \subseteq A^\circ$ for some $x \in X$.

If $x_1 \sim x_2$ and $x_1 \in O(x)$ we have by Theorem 5.2 two nets $\{t_\lambda\}$, $\{t'_\lambda\}$ in T with the given properties

$$(x_1, x_1(t_\lambda t_\lambda'^{-1})) \to (x_1, x_2),$$

$$(x_1, x_1(t_\lambda t_\lambda'^{-1}))t_\lambda' \to (x_1, x_1) \in A^\circ$$

and we may assume $(x_1, x_1(t_\lambda t_\lambda'^{-1}))t_\lambda' \in A^\circ$ for all λ,

$$((x_1, x_1(t_\lambda t_\lambda'^{-1}))t_\lambda')t_\lambda'^{-1} = (x_1, x_1(t_\lambda t_\lambda'^{-1})) \in A^\circ T$$

and $(x_1, x_2) \in \operatorname{cl} A^\circ T = A$.

We have shown that $O(x) \times X \subseteq A$ and hence

$$\operatorname{cl}\{O(x) \times X\} = \tilde{X} \times \tilde{X} \subseteq A \subseteq \tilde{X} \times \tilde{X}.$$

(b) implies (c) and (c)-spatial. If $f : X \to \mathbf{C}$ is a (spatial) topological eigenfunction we can define the almost periodic immersion $F : (X, T) \to (Z, T)$ as in the discussion following Remark 6.3.

(c) or (c)-spatial implies (b). If $\theta : (X, T) \to (Y, T)$ is an almost periodic immersion, let χ be a nontrivial continuous character of the compact, abelian, topological group Y. $\chi/_T : T \to S^1$ is a nontrivial continuous character since T is dense in Y. $\chi \circ \theta : X \to S^1$ is continuous and can be extended to $\tilde{X}$ by defining $\chi \circ \theta(x) = 0 \quad (x \in X^c)$. The extension is a (spatial) topological eigenfunction with eigenvalue χ.

(c) or (c)-spatial if and only if (d)-spatial. See Theorem 6.1.

(d) if and only if (d)-spatial. See Theorem 6.2.

(b) if and only if (e). If $x \in X$, (Y, T) is a compact almost periodic minimal transformation group and $X \times Y = \bigcup_j N_j$ is the partition of $X \times Y$ discussed in Section II, let $N_j' = (O(x) \times Y) \cap N_j$. $\{N_j'\}$ is a partition of $O(x) \times Y$ and the method of section II can be applied to produce an almost periodic immersion of $(O(x), T)$. If $Y = B(T)$, we get a universal almost periodic immersion of $(O(x), T)$ which is defined by

$$\theta : (O(x), T) \to (B(T)/H', T),$$

$(H' = \{y \in B(T) \mid (x, y) \in N'\}, N' = N \cap (O(x) \times B(T)))$.

If $H = \{y \in B(T) \mid (x, y) \in N\}$ then $H' = H$ and $H = B(T)$ if and only if $H' = B(T)$ which yields our conclusion.

As a corollary we have the following result by Peterson [12].

Corollary 6.2. *If (X, T) is a minimal transformation group with X compact metric, T abelian and $S_e = X \times X$, then (X, T) is weakly mixing.*

Proof. Let $X = \tilde{X}$, and use Theorem 6.3 with Corollary 3.1, and [11, Theorem 2.10].

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