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A PROBLEM IN TRIGONOMETRIC APPROXIMATION THEORY

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This note is written to supplement the paper of Seghier [4]. The notation is accordingly chosen to conform to that paper. It is reviewed briefly for the sake of completeness: $e_c = e_c(x) = e^{icx}$ and $H_{(a\,b)}$ denotes the closed linear span of the functions e_t: $a \le t \le b$ in $L^2(R^1, f(x)\,dx)$ where $f(x) = g(x)\bar{g}(x)$, a.e. $x \in R^1$, and $g[\bar{g}]$ is an outer Hardy function of class H^{2+} $[H^{2-}]$; P $[Q]$ denotes the orthogonal projection of $L^2(R^1, dx)$ onto H^{2+} $[H^{2-}]$ and M_c denotes the Hankel operator $M_c = Q(e_{2c}\,g/\bar{g})P$. The symbol $(\ ,\ )$ $[(\ ,\ )_f]$ denotes the standard inner product in $L^2(R^1, dx)$ $[L^2(R^1, f(x)\,dx)]$.

THEOREM. *If f^{-1} is locally summable, then*

$$H_{(-\infty\, a)} \cap H_{(-a\, \infty)} = \bigcap_{\varepsilon>0} H_{(-a-\varepsilon\, a+\varepsilon)} \quad \textit{for every } a \ge 0. \tag{1}$$

If also $\|M_c\| < 1$ for some $c \ge 0$, then

$$H_{(-\infty\, a)} \cap H_{(-a\, \infty)} = H_{(-a\, a)} \quad \textit{for every } a > c. \tag{2}$$

Discussion. Identity (1) was first proved by Levinson–McKean [2] in the special case that f is even and $a = 0$. (2) was first proved by Seghier in an unpublished preliminary version of [4] under the auxiliary assumption that $\|M_0\| < 1$.

Proof of theorem. Identity (1) is an immediate consequence of Theorem 2.1 of Dym [1] and the identification of $\bigcap_{\varepsilon>0} H_{(-a-\varepsilon\, a+\varepsilon)}$ with the space of entire functions of exponential type $\le a$ which are square summable on the line relative to the measure $f(x)\,dx$; see Pitt [3] for a proof of the latter.

Now suppose in addition that $\|M_c\| < 1$ for some $c \ge 0$ and fix $a > c$. Then, in view of (1) and the self-evident inclusion $H_{(-a\,a)} \supset \bigcap_{\varepsilon>0} H_{(-c-\varepsilon\, c+\varepsilon)}$, it follows that

$$\begin{aligned} H^{\perp}_{(-a\,a)} &\subset (H_{(-\infty\, c)} \cap H_{(-c\,\infty)})^{\perp} \\ &= \overline{H^{\perp}_{(-\infty\, c)} + H^{\perp}_{(-c\,\infty)}} \\ &= H^{\perp}_{(-\infty\, c)} + H^{\perp}_{(-c\,\infty)} \\ &= (\overline{e_c g})^{-1} H^{2+} + (e_c g)^{-1} H^{2-}. \end{aligned}$$

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The passage from line 2 to line 3 uses the fact that the cosine of the angle between the indicated two spaces is equal to $\|M_c\|$ and that $\|M_c\| < 1$. The inclusion derived to this point implies that each function $\psi \in H^{\perp}_{(-a\,a)}$ can be expressed in the form

$$\psi = (e_c g)^{-1}\theta_1 + (\overline{e_c g})^{-1}\theta_2$$

for some choice of $\theta_1 \in H^{2-}$ and $\theta_2 \in H^{2+}$. Now

$$\begin{aligned}(\psi, e_u)_f &= ((e_c g)^{-1}\theta_1 + (\overline{e_c g})^{-1}\theta_2, e_u)_f \\ &= (\theta_1, g e_{u+c}) + (\theta_2, \bar{g} e_{u-c}) \\ &= 0\end{aligned}$$

for $|u| \leq a$, and if either $-a \leq u \leq c$ or $-c \leq u \leq a$, then both terms in the next to the last line vanish separately. Therefore $(\theta_1 e_{a-c}, g e_t) = 0$ for $0 \leq t \leq a + c$. In fact the latter equality prevails for all $t \geq 0$ since $\theta_1 \in H^{2-}$ and $g \in H^{2+}$ and so, as g is an outer function, it serves to prove that $e_{a-c}\theta_1 \in H^{2-}$, and hence that

$$(e_c g)^{-1}\theta_1 \in (e_a g)^{-1} H^{2-} = H^{\perp}_{(-a\,\infty)}.$$

Much the same sort of argument serves to prove that $(\overline{e_c g})^{-1}\theta_2 \in H^{\perp}_{(-\infty\,a)}$. Thus

$$H^{\perp}_{(-a\,a)} \subset H^{\perp}_{(-\infty\,a)} + H^{\perp}_{(-a\,\infty)} \subset \overline{H^{\perp}_{(-\infty\,a)} + H^{\perp}_{(-a\,\infty)}} = (H_{(-\infty\,a)} \cap H_{(-a\,\infty)})^{\perp}$$

which is to say that

$$H_{(-a\,a)} \supset H_{(-\infty\,a)} \cap H_{(-a\,\infty)}.$$

Since the opposite inclusion is self-evident the proof is complete.

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