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ON THE CHARACTERIZATION OF COMPLEX RATIONAL APPROXIMATIONS

BY

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Abstract

An example is constructed showing that best uniform approximation (local or global) from $R_n^m(\mathbf{C})$ can not be characterized by linearization techniques or by alternation properties of the error function.

A class of local best approximations are characterized, and used to demonstrate approximation properties of $R_n^m(\mathbf{C})$.

I. Introduction

Although uniform approximation from $R_n^m(\mathbf{C})$ is a classical area of analysis (see J. Walsh, 1935 and the references there), there are still fundamental unsettled questions. The difficulties come from the lack of an applicable characterization of best approximations. For R_n^m—the rational functions on [0, 1] which have real coefficients—approximations are characterized, both by an extremal alternation property of the error function, and by a linearization technique which reduces the characterization to one for a linear space (definitions will be given below). The characterizations are used, for example, to show that best approximations are unique, that local best approximations are global, and to identify the points of continuity of the best approximation operator (see Cheney [1966]). For the complex rational function, $R_n^m(\mathbf{C})$, no such characterization exists. Even the fact that in $R_1^1(\mathbf{C})$ there are two best approximations to $(x - 1/2)^2$ was only recently discovered (E. Saff and R. Varga [1977]).

Suppose now that f is a real continuous function, and r is a real function in $R_n^m(\mathbf{C})$. Several obvious stratagies to characterize r as a best approximation to f (or a local best approximation) have attracted research. One is to find a linearization characterization. A second, is to find a characterizing extremal alternation property for the error function. Saff and Varga, for example, found two alternation properties—one necessary, the other sufficient—for r to be a best approximation. A third approach is to determine when r being a best approximation from $R_n^m(\mathbf{C})$ implies that r is a best approximation to f from Re $R_n^m(\mathbf{C})$. There are recent characterizations of approximations from Re $R_n^m(\mathbf{C})$; and these, then, would apply to $R_n^m(\mathbf{C})$.

In this paper we will give an example of a real continuous function f and a real rational (in fact, normal) function r in $R_n^m(\mathbf{C})$ such that r is the unique best

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approximation to f, but for $\lambda > 1$, r is not a local best approximation to $\lambda f + (1 - \lambda)r$. Such an example is not compatible with a characterization of the form of any of the three above.

As part of the development of this paper we determine when a real normal function r in $R_n^m(\mathbf{C})$ is a local best approximation to a real continuous function f. A set $T(r)$ is constructed which has the property that r is a local best approximation to f if and only if $f - r$ has zero as a best approximation from $T(r)$. Although $T(r)$ is also nonlinear, for our purposes, it is more tractable than $R_n^m(\mathbf{C})$. For example it shows that if r is a local best approximation to f, then r is a strict local best approximation. This contrasts with the recent example found by A. Ruttan [1977] of a continuum of complex best approximations form $R_n^m(\mathbf{C})$ to a real continuous function.

A final example in this paper shows that, what seemed to be an anomaly in Re $R_n^m(\mathbf{C})$ approximation, appears again in $R_n^m(\mathbf{C})$ approximation. That is, there is a real continuous function f and an $r \in R_n^n \subseteq R_n^n(\mathbf{C})$ such that r is the unique global best approximation to f from $R_n^n(\mathbf{C})$; but, although $(f - 2r) + r$ has the same error function, $f - 2r$ does not have $-r$ as a local best approximation.

Some open problems are listed at the end.

Notation. We use $C[0, 1]$ to represent the Banach space of real valued continuous functions on the unit interval normed with the supremum norm. The real and complex numbers are symbolized by $\mathbf{R}$ and $\mathbf{C}$ respectively. The polynomials (polynomials with complex coefficient, resp.) of degree less than or equal n are represented by $\mathscr{P}_n(\mathscr{P}_n(\mathbf{C})$, resp.). For a polynomial p, ∂p is the degree of p. Put

$$(0.1) \qquad R_n^m = \{p/q \colon p \in \mathscr{P}_m,\ q \in \mathscr{P}_n,\ q(x) \neq 0 \text{ for } 0 \leq x \leq 1\}.$$

We can assume that if $p/q \in R_n^m$, then p and q have no common factors. We may also assume that $\|q\| = 1$. We define $R_n^m(\mathbf{C})$ analogously by replacing $\mathscr{P}_m$ and $\mathscr{P}_n$ with $\mathscr{P}_m(\mathbf{C})$ and $\mathscr{P}_n(\mathbf{C})$. A rational function p/q in either R_n^m or $R_n^m(\mathbf{C})$ is called *normal* if $\partial p = m$ or if $\partial q = n$. For a function f,

$$(0.2) \qquad \operatorname{crit}(f) = \{x \colon |f(x)| = \|f\|\} \quad \text{and} \quad Z(f) = \{x \colon f(x) = 0\}.$$

Let $E \subseteq C[0, 1]$, and $f \in C[0, 1]$. Then

$$(0.3) \qquad \operatorname{dist}(f, E) = \inf\{\|f - m\| \colon m \in E\}.$$

A member m of E is termed a *best approximation* to f if

$$(0.4) \qquad \|f - m\| = \operatorname{dist}(f, E).$$

If there is a neighborhood U of m such that m is a best approximation (the unique best approximation, resp.) to f from $U \cap E$ then m is a *local best approximation* (strict local best approximation, resp.). An *extremal alternation* of length n for a real function f is a set of points $0 \leq x_1 < x_2 < \cdots < x_n \leq 1$ such that

$$\text{(0.5)} \qquad \text{(i) } x_i \in \operatorname{crit}(f) \quad \text{and} \quad \text{(ii) } f(x_i) = -f(x_{i+1}).$$

Best approximations (local best approximations) from E are said to have a *linearization characterization* if for each $m \in E$, there is a convex set $K(m)$ containing zero such that m is a best approximation (local best approximation, resp.) to f if and only if 0 is a best approximation to $f - m$ from $K(m)$.

If g is a complex valued function and E is a set of complex functions, Reg denotes the real part of g, and

$$\text{(0.6)} \qquad \operatorname{Re} E = \{\operatorname{Re} g \colon g \in E\}.$$

The imaginary parts are abbreviated similarly with Im.

Special notational conventions. It will be convenient for us to reserve certain letters for specific meanings. We will use f to be a continuous real valued function on [0, 1]. Let $r_0 = p_0/q_0$ be a normal function in $R_n^m(\mathbf{C})$ which has real coefficients. Put

$$\text{(0.7)} \qquad T = \{[pq_0 - p_0\gamma^2 - iq_0(p_0\gamma - q_0\delta)]/q_0^3\colon \\ p \in \mathscr{P}_{m+n},\ \gamma \in \mathscr{P}_n, \text{ and } \delta \in \mathscr{P}_m\}.$$

We will use t to represent a member of T so it will be written

$$\text{(0.8)} \qquad t = [pq_0 - p_0\gamma^2 - iq_0(p_0\gamma - q_0\delta)]/q_0^3,$$

where p, γ and δ are in the appropriate space of polynomials. Furthermore we may assume that γ is in the orthogonal complement of the span of q_0 in $\mathscr{P}_n$. For λ real,

$$\text{(0.9)} \qquad t_\lambda = \{(\lambda^2 p)q_0 - p_0(\lambda\gamma)^2 - iq_0[p_0(\lambda\gamma) - q_0(\lambda\delta)]\}/q_0^3.$$

Similarly for $\alpha \in \mathscr{P}_m$ and $\beta \in \mathscr{P}_n$, we will write

$$\text{(0.10)} \qquad r = \frac{(p_0 + \alpha) + i\delta}{(q_0 + \beta) + i\gamma} \in R_n^m(\mathbf{C}),$$

and

$$\text{(0.11)} \qquad r_\lambda = \frac{(p_0 + \lambda^2\alpha) + i\lambda\delta}{(q_0 + \lambda^2\beta) + i\lambda\gamma}.$$

Basic computations. We will record below the result of some elementary computations which are needed for reference.

$$\text{(0.12)} \qquad r_0 - r = \frac{(p_0\beta - q_0\alpha)(q_0 + \beta) + p_0\gamma^2 - q_0\delta\gamma}{q_0[(q_0 + \beta)^2 + \gamma^2]} \\ + \frac{i[(p_0\gamma - q_0\delta)(q_0 + \beta) - (p_0\beta - q_0\alpha)\gamma]}{q_0[(q_0 + \beta)^2 + \gamma^2]}.$$

Also if $x \in \operatorname{crit}(f)$ and g is a function on $[0, 1]$, then

$$(0.13) \qquad |f(x) - g(x)| \leq \|f\| \quad \text{if and only if} \quad 2f(x)\operatorname{Re} g(x) \geq |g(x)|^2.$$

The equivalence is also true if both inequalities are strict.

II. Characterization of approximation

1. Lemma. *If r_0 is a local best approximation to f from $R_n^m(\mathbf{C})$, then 0 is a best approximation to $e = f - r_0$ from T.*

Proof. Suppose that $\|e - t\| < \|e\|$. Then on crit (e) there must be an $\varepsilon > 0$ for which (see line 0.13)

$$(1.1) \qquad 2e\left[\frac{pq_0 - p_0\gamma^2}{q_0^3}\right] > \left[\frac{pq_0 - p_0\gamma^2}{q_0^3}\right]^2 + \left[\frac{p_0\gamma - q_0\delta}{q_0^2}\right]^2 + \varepsilon.$$

This inequality must also hold on some neighborhood U of crit (e). If $1 \geq \lambda > 0$ then on this set U,

$$(1.2) \qquad 2e\lambda^2\left[\frac{pq_0 - p_0\gamma^2}{q_0^3}\right] > \lambda^4\left[\frac{pq_0 - p_0\gamma^2}{q_0^3}\right]^2 + \lambda^2\left[\frac{p_0\gamma - q_0\delta}{q_0^2}\right]^2 + \lambda^2\varepsilon.$$

On U we have that for sufficiently small λ,

$$(1.3) \qquad |e - t_\lambda|^2 \leq \|e\|^2 - 2\lambda^2 e\left[\frac{pq_0 - p_0\gamma^2}{q_0^3}\right] + \lambda^4\left[\frac{pq_0 - p_0\gamma^2}{q_0^3}\right]^2 + \lambda^2\left[\frac{p_0\gamma - q_0\delta}{q_0^2}\right]^2$$
$$< \|e\|^2 - \lambda^2\varepsilon < (\|e\| - \lambda^2\mu)^2 \quad \text{where } \mu = \varepsilon/2\|e\|.$$

Now choose α and β so that

$$(1.4) \qquad \beta p_0 - \alpha q_0 = -p - \gamma\delta,$$

and let

$$(1.5) \qquad r_\lambda = \frac{p_0 + \lambda^2\alpha + i\lambda\delta}{q_0 + \lambda^2\beta + i\lambda\delta}.$$

One can compute that

$$(1.6) \qquad \lim_{\lambda \to 0}\left\|\frac{r_0 - r_\lambda + t_\lambda}{\lambda^2}\right\| = 0.$$

Hence from (1.3) and (1.6), we have that for all small λ and all $x \in U$,

$$|(f - r_\lambda)(x)| < \|e\|. \tag{1.7}$$

For $x \notin U$ there is also an $\varepsilon > 0$ for which

$$|(f - r_0)(x)| < \|e\| - \varepsilon. \tag{1.8}$$

Since r_λ converges uniformly to r_0 we again have that for all sufficiently small λ,

$$|(f - r_\lambda)(x)| < \|e\|. \tag{1.9}$$

We then of course have that for all sufficiently small λ,

$$\|f - r_\lambda\| < \|f - r_0\|, \tag{1.10}$$

and the proof is completed. ∎

2. LEMMA. *If f has zero as a local best approximation from T, then f has an extremal alternation of length at least $n + m + 2$.*

Proof. $(1/q_0^2)\mathscr{P}_{m+n} \subseteq T$. ∎

3. LEMMA. *If $t \in T$ has $n + m + 1$ zeros then t is the zero function.*

Proof. If $t(x) = 0$ then at x

$$pq_0 - p_0\gamma^2 = 0 \quad \text{and} \quad p_0\gamma - q_0\delta = 0, \tag{3.1}$$

showing that

$$p(x) = \gamma(x)\delta(x). \tag{3.2}$$

So if t has $n + m + 1$ zeros, $p = \gamma\delta$ and

$$t = -\gamma(p_0\gamma - q_0\delta) - iq_0(p_0\gamma - q_0\delta) \equiv 0. \tag{3.3}$$ ∎

We also record for reference the following obvious fact.

4. LEMMA. *If $t \in T$ and* Im t *has $n + m + 1$ zeros, then t is real.*

5. LEMMA. *If* 0 *is a local best approximation to f from T, then* 0 *is also a best approximation on* crit (f).

Proof. Suppose that on crit (f), $\|f - t\| < \|f\|$. Then

$$2f\,\frac{pq_0 - p_0\gamma^2}{q_0^3} > \left[\frac{pq_0 - p_0\gamma^2}{q_0^3}\right]^2 + \left[\frac{p_0\gamma - q_0\delta}{q_0^2}\right]^2 \tag{5.1}$$

must be valid on some neighborhood U of crit (f). Also there is an $\varepsilon > 0$ such that for $x \notin U$,

$$|f(x)| < \|f\| - \varepsilon. \tag{5.2}$$

For all $0 < \lambda < 1$ such that $\|t_\lambda\| < \varepsilon$ we have

$$\|f - t_\lambda\| < \|f\| \tag{5.3}$$

if $x \notin U$ it is obvious that

$$|f(x) - t_\lambda(x)| < \|f\| \tag{5.4}$$

and for points in U,

$$\begin{aligned} |f - t_\lambda|^2 &= |f|^2 - 2f\lambda^2 \frac{pq_0 - p_0\gamma^2}{q_0^3} \\ &\quad + \lambda^4 \left[\frac{pq_0 - p_0\gamma^2}{q_0^3}\right]^2 + \lambda^2 \left[\frac{p_0\gamma - p_0\delta}{q_0^2}\right]^2 \\ &\leq \|f\| - \lambda^2 \left\{2f \frac{pq_0 - p_0\gamma^2}{q_0^3} - \left[\frac{pq_0 - p_0\gamma^2}{q_0^3}\right]^2 \right. \\ &\quad \left. - \left[\frac{p_0\gamma - q_0\delta}{q_0^2}\right]^2\right\}. \end{aligned} \tag{5.5}$$

By (5.1) this too is less than $\|f\|$. ∎

6. Corollary. *If 0 is a local best approximation to f, from T, then 0 is a global best approximation of f.*

7. Lemma. *Suppose that*

(7.1)

$$1 \leq c_0 = \sup\{c\colon cf \text{ has zero as a local best approximation from } T\}.$$

Then:

(a) *f has zero as a unique global best approximation from T.*
(b) *For each $t \in T$ there is an $x \in \operatorname{crit}(f)$ such that*

$$\text{(i) } t(x) \neq 0 \quad \text{and} \quad \text{(ii) } 2f(x)(\operatorname{Re} t)(x) \leq [(\operatorname{Im} t)(x)]^2.$$

(c) *Zero is not a local best approximation to cf when $c > c_0$.*

Proof. We will first prove part (b). We note that cf has zero as a local best approximation for all $c < c_0$, and in particular for all $c < 1$.
Now suppose there is a $t \in T$ such that

$$2f(x)(\operatorname{Re} t)(x) > (\operatorname{Im} t)(x) \tag{7.2}$$

for all $x \in \operatorname{crit}(f) - Z(t)$. By Lemma 3, t has at most $n + m + 1$ zeros so there is a function g in $q_0\mathscr{P}_{m+n}$ such that for $x \in \operatorname{crit}(f) \cap Z(t)$,

$$\operatorname{sgn} g(x) = \operatorname{sgn} f(x). \tag{7.3}$$

Now consider

$$t^* = \alpha g + t. \tag{7.4}$$

By our choice of g and t we have the existence of an open neighborhood of crit $(f) \cap Z(t)$, U, on which

$$\operatorname{sgn}(\alpha g + t) = \operatorname{sgn} f \tag{7.5}$$

independent of $\alpha > 0$.

Since crit $(f) - U$ is compact, there is an $\varepsilon > 0$ such that on crit $(f) - U$

$$2f(x) \operatorname{Re} t(x) > [\operatorname{Im} t(x)]^2 + \varepsilon. \tag{7.6}$$

So if $\alpha < \varepsilon/\|g\|$, then

$$2f(x) \operatorname{Re} t^*(x) > [\operatorname{Im} t^*(x)]^2. \tag{7.7}$$

Hence there is a $c < 1$ such that

$$2cf(x) \operatorname{Re} t^*(x) > [\operatorname{Im} t^*(x)]^2. \tag{7.8}$$

Furthermore for small λ,

$$2cf(x) \operatorname{Re} t^*(x) > \lambda^2[\operatorname{Re} t^*(x)]^2 + [(\operatorname{Im} t^*)(x)]^2. \tag{7.9}$$

So from (0.13), $\lambda^2 \operatorname{Re} t^* + i\lambda \operatorname{Im} t^* \in T$ is a better approximation to cf on crit (f) than is zero. By Lemma 5, zero is not a local best approximation to cf as hypothesised. This proves part (b).

Part (a) is immediate from (b) and (0.13).

The proof of part (c) follows the construction used in part (a). For let us suppose that $c > c_0$. We must have that there is a $t \in T$ such that

$$2cf(x)(\operatorname{Re} t(x))(x) > [(\operatorname{Im} t)(x)]^2 \tag{7.10}$$

for all $x \in$ crit $(f) - Z(t)$. After all, the denial of this fact says that cf has zero as a unique best approximation on crit (f) (and hence everywhere) contradicting the hypotheses of (7.1). Now we use (7.10) to construct a function which is a better approximation to cf than is zero. This is done exactly as we used (7.2) to produce a better approximation to f than was zero. We then apply Corollary 6. ∎

8. Corollary. *If 0 is a local best approximation to f, from T, then it is the unique global best approximation to f.*

9. Lemma. *If $f - r_0$ has zero as a best approximation from T then r_0 is a strict local best approximation to f from $R_n^m(\mathbf{C})$.*

Proof. Suppose that zero is a best approximation from T, but that also there are $r_j = p_j/q_j$ in $R_n^m(\mathbf{C})$ such that $r_j \to r_0$ and $\|f - r_0\| \le \|f - r_0\|$. We begin by putting r_j in a particular form. Since r_j is bounded and $\|q_j\|$ may be assumed to be equal one, we can find a subsequence (which we assume we already have) with converging numerators and denominators. Since r_0 is a normal function and both q_j and q_0 have norm one, we in fact have

$$p_j \to p_0 \quad \text{and} \quad q_j \to q_0. \tag{9.1}$$

Hence

$$r_j = \frac{(p_0 + \alpha_j) + i\delta_j}{(q_0 + \beta_j) + i\gamma_j} \tag{9.2}$$

where

$$\alpha_j,\ \delta_j \in \mathscr{P}_m; \qquad \beta_j,\ \gamma_j \in \mathscr{P}_n \tag{9.3}$$

and

$$\max\{\|\alpha_j\|, \|\beta_j\|, \|\alpha_j\|, \|\delta_j\|\} \to 0. \tag{9.4}$$

Now let P denote the orthogonal projection of $\mathscr{P}_n$ onto the real span of $\{q_0\}$. Then of course

$$q_0 + \beta_j + i\gamma_j = q_0 + P\beta_j + (I - P)\beta_j + iP\gamma_j + i(I - P)\gamma_j \tag{9.5}$$

so there are constants k_j and c_j and members $\tilde{\beta}_j$ and $\tilde{\gamma}_j$ of $\{q_0\}^\perp$ (the orthogonal complement of real span $\{q_0\}$ in $\mathscr{P}_n$) such that

$$q_0 + \beta_j + i\gamma_j = (1 + k_j + ic_j)q_0 + \tilde{\beta}_j + i\tilde{\gamma}_j. \tag{9.6}$$

and $k_j \to 0$ and $c_j \to 0$. Dividing both the numerator and the denominator by $1 + k_j + ic_j$ we may now assume that the representation of r_j given in (9.2) has β_j and γ_j in $\{q_0\}^\perp$. (We note that now the denominators have norms approaching one—but not necessarily equal one).

Claim. There are constants λ_j such that

$$\frac{(\alpha_j q_0 - \beta_j p_0)(q_0 + \beta_j) + (\gamma_j p_0 - \delta_j q_0)\gamma_j}{\lambda_j^2} + \frac{i\{(p_0\gamma_j - \delta_j q_0)(q_0 + \beta_j) - (p_0\beta_j - q_0\alpha_j)\gamma_j\}}{\lambda_j} \tag{9.7}$$

has a subsequence which converges to the numerator of a nonzero member of $-T$.

Proof of Claim. Let

$$\lambda_j = \max\{\sqrt{\|\alpha_j\|}, \sqrt{\|\beta_j\|}, \|\gamma_j\|, \|\delta_j\|\}. \tag{9.8}$$

We may assume that we already have a subsequence for which each of

$$\frac{\alpha_j}{\lambda_j^2},\ \frac{\beta_j}{\lambda_j^2},\ \frac{\gamma_j}{\lambda_j},\ \frac{\delta_j}{\lambda_j} \tag{9.9}$$

converge to say α, β, γ and δ respectively. The functions of (9.7) then converge to

$$(9.10)\qquad (\alpha q_0 - \beta p_0)q_0 + (p_0\gamma^2 - q_0^{\gamma}\delta) + iq_0(p_0\gamma - q_0\delta).$$

This has the correct form. We have to show that it is not zero. Since r_0 is a normal function and $\gamma \in \{q_0\}^\perp$, either the imaginary part is not equal zero (and we are done) or both γ and δ are zero. If γ and δ are zero the limit function is $(\alpha q_0 - \beta p_0)q_0$. Since β is also in $\{q_0\}^\perp$ either this term is nonzero (and we are done) or α and β are also zero. However from our choice of λ_j, not all the functions α, β, γ and δ are zero, and the limit function in (9.10) is not the zero function.

This completes the proof of the claim and we can now finish the proof of the lemma.

Proof of Lemma continued. From the claim there are λ_j such that

$$(9.11)\qquad \operatorname{Re}\frac{r_j - r_0}{\lambda_j^2} + i\operatorname{Im}\frac{r_j - r_0}{\lambda_j}$$

converges to a nonzero member of T, say h. Hence from Lemma 7 for j large there is an x in the critical set of $e = f - r_0$ for which

$$(9.12)\qquad 2e(x)\operatorname{Re}\left(\frac{r_j - r_0}{\lambda_j^2}\right)(x) < \left[\operatorname{Im}\left(\frac{r_j - r_0}{\lambda_j}\right)(x)\right]^2 + \left[\operatorname{Re}\left(\frac{r_j - r_0}{\lambda_j^2}\right)(x)\right]^2.$$

Hence

$$(9.13)\qquad 2e(x)\operatorname{Re}(r_j - r_0)(x) < [\operatorname{Re}(r_j - r_0)(x)]^2 + [\operatorname{Im}(r_j - r_0)(x)]^2$$

which implies that

$$(9.14)\qquad |(f - r_0 + (r_0 - r_j))(x)|^2 > \|f - r_0\|^2. \quad \blacksquare$$

We now collect the conclusions of the lemmas. We remind the reader that by our notational conventions, f is a real valued function, and r_0 is a real, normal function in $R_n^m(\mathbf{C})$.

10. THEOREM. *The following are equivalent:*

(i) *r_0 is a local best approximation to f from $R_n^m(\mathbf{C})$.*
(ii) *zero is a best approximation to $f - r_0$ from T.*
(iii) *r_0 is a strict local best approximation to f from $R_n^m(\mathbf{C})$.*

Proof. That (i) implies (ii) is Lemma 1. Corollary 8 and Lemma 9 show that (ii) implies (iii), and of course (iii) implies (i). $\blacksquare$

11. LEMMA. *Suppose that f has an extremal alternation of length $n+m+2$, $0 \leq x_0 < x_1 < \cdots < x_{n+m+1} \leq 1$. Then there is a constant c_0 such that for any nonzero function $t \in T$ and any $0 < c < c_0$,*

$$\|cf\| < \sup\{|cf(x_j) - t(x_j)| : 0 \leq j \leq n+m+1\}.$$

Proof. If $t \in T$ is such that

(11.1) $\operatorname{sgn} \operatorname{Re} t(x_j) \neq -\operatorname{sgn} f(x_j)$ and $\operatorname{Re}(t)$ is not identically zero

then $\gamma \neq 0$. Since p_0 and q_0 have no common factors, and since $\gamma \notin \operatorname{span}\{q_0\}$,

$$\|\operatorname{Im}(t)\| > 0. \tag{11.2}$$

Now let

$$\begin{aligned} S = \{s \in T\colon & \operatorname{sgn} \operatorname{Re} s(x_j) \neq -\operatorname{sgn} f(x_j) \\ & \text{for} \quad 0 \leq j \leq n+m+1, \\ & \|\operatorname{Re} s\| = 1 \quad \text{and} \quad \|\operatorname{Im} s\| \leq \sqrt{2\|f\|}\}. \end{aligned} \tag{11.3}$$

Since S is compact, we have from line (11.2) that there is a $0 < c_0 \leq 1$ such that

$$\inf\{\|\operatorname{Im} s\|^2 : s \in S\} > 2c_0\|f\|. \tag{11.4}$$

Suppose, now, that $t \in T$ is such that for some $0 < c < c_0$,

$$\|cf\| \geq \sup\{|cf(x_j) - t(x_j)| : 0 \leq j \leq n+m+1\}. \tag{11.5}$$

Let

$$\lambda = (1/\|\operatorname{Re} t\|)^{1/2}. \tag{11.6}$$

Note that $\|\operatorname{Re} t\| \neq 0$, since that and (11.5) would imply that also $\operatorname{Im} t = 0$ on $\{x_j\}$. By Lemma 3, t would be zero everywhere. We have that for each point x_j,

$$2cf\,\frac{pq_0 - p_0\gamma^2}{q_0^3} \geq \left[\frac{pq_0 - p_0\gamma^2}{q_0^3}\right]^2 + \left[\frac{p_0\gamma - q_0\delta}{q_0^2}\right]^2, \tag{11.7}$$

and so

$$2cf\,\frac{(\lambda^2 p)q_0 - p_0(\gamma\lambda)^2}{q_0^3} \geq \left[\frac{p_0(\lambda\gamma) - q_0(\lambda\delta)}{q_0^2}\right]^2. \tag{11.8}$$

Line (11.8) and our choice of λ show that

$$t_\lambda = \frac{1}{q_0^3}\left[(\lambda^2 p)q_0 - (\lambda\gamma)^2 - iq_0(p_0(\lambda\gamma) - q_0(\lambda\delta))\right] \in S. \tag{11.9}$$

So by (11.4),

$$\left\|\frac{p_0(\lambda\gamma) - q_0(\lambda\delta)}{q_0^2}\right\| > 2c_0\|f\|. \tag{11.10}$$

By our choice of λ and c, (11.8) and (11.10) are not compatible. This contradicts the existence of a $t \in T$ satisfying (11.5). ∎

12. PROPOSITION. *The following are equivalent:*

(i) r_0 *is a best approximation to* f *from* R_n^m.
(ii) $f - r_0$ *has an extremal alternation of length* $m + n + 2$.
(iii) *For all sufficiently small* λ, r_0 *is the unique best approximation to* $\lambda f + (1 - \lambda)r_0$ *from* $R_n^m(\mathrm{C})$.

Proof. Statements (i) and (ii) are equivalent from the classical theory. Since $R_n^m \subseteq R_n^m(\mathrm{C})$, (iii) implies (ii). We have to show that (ii) implies (iii). From Lemma 11, $\lambda f - \lambda r_0$ has zero as the unique best approximation from T for all sufficiently small λ. From Theorem 10, $\lambda f + (1 - \lambda)r_0$ has r as a strict local best approximation from $R_n^m(\mathrm{C})$. By choosing λ smaller yet we can insure that r_0 is, in fact, the unique global best approximation. ∎

Hypothesis. We again remind the reader that by our notational conventions from Section 1, f is a continuous real valued function, r_0 is a real, normal function in $R_m^n(\mathrm{C})$, and all functions are defined on the real interval [0, 1].

III. The nature of approximates

The two examples mentioned in the introduction are presented in this section. The first example (Theorem 14) depends on the previous characterization theorem. The example shows the following in $R_n^m(\mathrm{C})$:

(i) There is no linearization characterization of approximations (local or global).
(ii) Extremal alternations alone can not characterize approximations (local or global) when f and r are real.
(iii) r can be a best approximation to f without being a best approximation from Re $R_n^m(\mathrm{C})$.

The second example (Proposition 23) presents an irregularity phenomenon which occurs in Re $R_n^m(\mathrm{C})$ approximations, but which we had not anticipated for $R_n^m(\mathrm{C})$. The proof uses results from the Re $R_n^m(\mathrm{C})$ theory as well as the characterization of the last section.

13. LEMMA. λf *has zero as a best approximation from* T, *for all* $\lambda > 0$, *if and only if* f *has zero as a best approximation from* Re T.

Proof. The sufficiency is obvious. For the necessity suppose $t \in T$ is such that

$$\|f - \mathrm{Re}\, t\| < \|f\|. \tag{13.1}$$

Then on crit f,

(13.2) $$|\lambda f - t|^2 - \|\lambda f\| = -2\lambda f \operatorname{Re} t + \|t\|^2.$$

Since $f(x) \operatorname{Re} t(x) > 0$ on crit (f), we see that for large λ, t is a better approximation than zero to λf on the domain crit (f). The lemma now follows from Lemma 5. ▮

14. THEOREM. *For any $m > 0$ and $n > 0$ there is a continuous real function f and a real, normal $r \in R_n^m(\mathbf{C})$ such that r is the unique global best approximation to f; but for sufficiently large λ, r is not a local best approximation to $\lambda f + (1 - \lambda)r$.*

Proof. Let $r_0 = x^m$. Let s be any integer bigger than $n/2$ and less than or equal n. Let p be the best approximation to x^{m+2s} from $\mathscr{P}_{m+n}$. Then

(14.1) $$h = pq_0 - p_0\gamma^2$$

has an extremal alternation $\{x_j\}$ of length $m + n + 2$ where $q_0 = 1, p_0 = x^m$, and $\gamma = x^s$.

Put $g = h + r_0$. From Proposition 12, $cg + (1 - c)r_0$ has r_0 as a unique best approximation for all small c. But from Lemma 13, Theorem 10 and our choice of $h \in \operatorname{Re} T$, $cg + (1 - c)r_0$ does not have r_0 as a local best approximation from $R_n^m(\mathbf{C})$. ▮

Let $r_0 = p_0/q^0$ be a normal function in R_n^n. Let

(14.1) $$Z = \begin{cases} Z(q_0) \cap \mathbf{R} & \text{if } \partial a \le \partial b \\ [Z(q_0) \cap \mathbf{R}] \cup \{-\infty, \infty\} & \text{if } \partial b < \partial a. \end{cases}$$

For convenience we write

(14.2) $$g(\infty) \text{ for } \lim_{x\to\infty} g(x) \text{ and } g(-\infty) \text{ for } \lim_{x\to-\infty} g(x),$$

when these limits exist. Now let

(14.3) $$H = \{h \in \mathscr{P}_{2n+\max\{\partial p_0, \partial q_0\}} : \operatorname{sgn} h(x) = -\operatorname{sgn} p_0(x) \quad \text{for} \quad x \in Z\}.$$

15. PROPOSITION. *f has r_0 as a best approximation from* Re $R_n^n(\mathbf{C})$ *if and only if $f - r_0$ has zero as a best approximation from H.*

Proof. This is a variant of a result from [23]. The proof is a multiple case, bookkeeping argument using adaptations of the results Lemma 4-5, Theorem 4-7 and Lemma 4-6 from there. ▮

We will later need to use the fact that best approximations from Re $R_n^n(\mathbf{C})$ are unique [23].

16. LEMMA. *Given a complex, nonreal number ω and a complex number a there is a real quadratic polynomial p such that $[p(\omega)]^2 = a$ and $[p(\bar{\omega})]^2 = \bar{a}$.*

Proof. Let $p(z) = \sqrt{a(z-\omega)} + \sqrt{a(z-\omega)}$. ∎

17. LEMMA. *If $a_0, a_1, \ldots, a_n$ are real numbers with $a_0 > 0$ there is a $p \in \mathscr{P}_n$ and a $g \in \mathscr{P}_{n-1}$ such that $[p(x)]^2 = a_0 x^{2n} + a_1 x^{2n-1} + \cdots + a_n x^n + g(x)$.*

Proof. Let $p(x) = \sum_{i=0}^{n} b_i x^i$; we need to determine the coefficients b_i so that the coefficient of x^{2n-i} in $[p(x)]^2$ is a_i. This is easily done inductively. For example,

$$b_n = \sqrt{a_0}, \quad b_{n-1} = a_1/2\sqrt{a_0}, \quad b_{n-2} = \left[a_2 - \frac{a_1}{4a_0}\right]/2\sqrt{a_0}, \text{ etc. } ∎$$

18. LEMMA. $H \subseteq \mathrm{Re}\; q_0^3 T = \{\mathscr{P}_{2n} q_0 - p_0 \gamma^2 : \gamma \in \mathscr{P}_n\}$

Proof. Let $h \in H$. From Lemma 16 there is a $\gamma_1 \in \mathscr{P}_{\partial q_0}$ such that $(\gamma_1)^2 + h = 0$ on the zero set of q_0. From Lemma 17 there is a γ_2 such that

$$\partial[h + p_0(\gamma_1 \gamma_2)^2] \leq \partial q_0 + 2n, \tag{18.1}$$

where $\partial \gamma_2 \leq n - \partial q_0$. Now put $\gamma = p_0 \gamma_1 \gamma_2$. From our choice of γ_1,

$$h + \gamma p_0^2 = q_0 k \tag{18.2}$$

for some polynomial k. From our choice of γ_2, $\partial k \leq 2n$. So

$$h = q_0 k - p_0 \gamma^2 \in \mathrm{Re}\; (q_0^3 T). \tag{18.3}$$ ∎

19. LEMMA. *Every real continuous function has a best approximation from* $\mathrm{Re}\; q_0^3 T$.

Proof. $\mathrm{Re}\; q_n^3 T$ is, in fact, boundedly compact. This follows from the assumption that the polynomials γ are assumed to be in the orthogonal complement of span $\{q_0\}$ (in $\mathscr{P}_n$). For suppose

$$p_j q_0 - p_0 \gamma_j^2 \tag{19.1}$$

is bounded. Then either γ_j is bounded or

$$(p_j q_0 - p_0 \gamma_j^2)/\|\gamma_j^2\| \tag{19.2}$$

converges to zero. But there is a subsequence so that $\gamma_j/\|\gamma_j\|$ converges to say γ^*, and hence $p_j/\|\gamma_j^2\|$ also converges to say p. Hence $pq_0 - p_0\gamma^2 = 0$ but this is not possible since q_0 is not a factor of γ, and has no zeros in common with p_0. ∎

20. LEMMA. closure $H = \mathrm{Re}\; q_0^3 T$.

Proof. From Lemmas 18 and 19, cl $H \subseteq \mathrm{Re}\; q_0^3 T$. Clearly the set

$$\{\mathscr{P}_{2n} q_0 - p_0 \gamma^2 : \gamma \in \mathscr{P}_{2n}, Z(\gamma) \cap Z(q_0) = \emptyset\} \tag{20.1}$$

is both dense in $\mathrm{Re}\; q_0^3 T$, and contained in H. ∎

21. LEMMA. *f has $t = pq_0 - p_0\gamma^2$ as a best approximation from* Re $q_0^3 T$ *if and only if $f - t$ has zero as a best approximation from $\{pq_0 - p_0\gamma h: p \in \mathscr{P}_{2n}, h \in \mathscr{P}_n\}$.*

Proof. This is a consequence of a general linearization technique. For example see [22, Lemma 15]. ∎

Now let $r(x) = x^n/1 \in R_n^n(\mathrm{C})$. Let c_{2n+1} be the Chebyshev polynomial of degree $2n + 1$.

22. LEMMA. *c_{2n+1} has zero as a best approximation from H.*

Proof. From Lemma 20 we can show zero is a best approximation from Re T. From Lemma 19 there is some member which is a best approximation, say $t = p - x^n\gamma^2$. Since c_{2n+1} has zero as a best approximation from $\mathscr{P}_{2n}$, $\gamma \neq 0$. From Lemma 21, $c_{2n+1} - t$ has zero as a best approximation from

$$\{p - x^n\gamma h: p \in \mathscr{P}_{2n}, h \in \mathscr{P}_n\} = \mathscr{P}_{2n+\partial\gamma}. \tag{22.1}$$

But $c_{2n+1} - t$ itself belongs to this set. So zero could not possibly be the best approximation unless $c_{2n+1} - t \equiv 0$. But this is not possible since the coefficient of x^{2n+1} in c_{n+1} is positive and that of t is not. ∎

23. PROPOSITION. *There is a real continuous function f and a real normal member $r \in R_n^n(\mathrm{C})$ such that*

(1) *r is a unique global best approximation of f, but*
(2) *$-r$ is not a local best approximation to $f - 2r$.*

Proof. Let r, as above, be $x^n/1$, and let

$$f = c_{2n+1} + r. \tag{23.1}$$

From Lemmas 22 and 15, $\lambda f + (1 - \lambda)r$ has zero as a unique global best approximation for all $\lambda > 0$.

However $(f - 2r) + r$ does not have zero as a best approximation from

$$T(-r) = \{p1 - (-x^n)\gamma^2: p \in \mathscr{P}_{2n}, \gamma \in \mathscr{P}_n\}. \tag{23.2}$$

In fact c_{2n+1} belongs to this set. By Lemma 13 there is a λ such that $\lambda(f - 2r) + (1 - \lambda)(-r) + r$ does not have zero as a best approximation from $T(-r)$. So by Theorem 11, $-r$ is not a local best approximation to

$$\lambda(f - 2r) + (1 - \lambda)(-r) = [\lambda f + (1 - \lambda)r] - 2r. \quad ∎ \tag{23.3}$$

Open Problems. Let f be a real continuous function on [0, 1]. Let r be a real (and perhaps-normal) function in $R_n^m(\mathrm{C})$.

(1) If r is a best approximation to f is it the unique best approximation to f? (Saff-Varga)

(2) If r is a local best approximation to f is it a global best approximation?
(3) If r is a local best approximation to $cf + (1 - c)r$ for all $c > 0$, then is r a global best approximation to f?
(4) Suppose $m = n$ and $f - r$ has an extremal alternation of length $2n + 2$, but none of length $2n + 3$. Then there is a λ_0 such that for $0 \leq \lambda \leq \lambda_0$, r is the best approximation to $\lambda f + (1 - \lambda)r = f_\lambda$ from $R_n^n(\mathbf{C})$. But when $\lambda > \lambda_0$, r is not the local best approximation to f_λ. Characterize λ_0.

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