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SOME COHOMOLOGY INVARIANTS FOR DEFORMATIONS OF FOLIATIONS

BY

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In this paper we examine some new cohomology invariants for deformations of foliations, or what we call n-foliations. An n-foliation of codimension q on M is a codimension $n + q$ foliation on $M \times I^n$ (where I^n is the unit n-cube $[0, 1]^n$) which intersects each slice $M \times \{x\}$, $x \in I^n$, as a codimension q foliation. Roughly speaking, these invariants are obtained by integrating the differential forms in the image of the map $WO_{q+n} \to \bigwedge^* (M \times I^n, \mathbf{R})$ over the fiber I^n. If certain conditions are satisfied, the resulting forms determine cohomology classes in $H^*(M, \mathbf{R})$. These classes have been examined in the case of 1-foliations in [8], and the construction given there uses Gelfand-Fuks cohomology. Their primary interest was the classes for 1-foliations which are the derivatives of deformable classes in $H^*(WOq)$. However, there are also many other classes for 1-foliations (and n-foliations) which cannot be interpreted in this way. A discussion of characteristic classes for deformations of foliations can also be found in [13, Section 8.7].

In Section 1 we give the constructions of these classes for C^∞ n-foliations and for complex holomorphic n-foliations. This construction also has a local form where, instead of integrating over the fiber I^n, one takes the interior product of a form from WO_{q+n} with the volume element $\partial/\partial t_1 \wedge \cdots \wedge \partial/\partial t_n$ on I^n. In this way one gets characteristic classes at each point $x \in I^n$ which contain local information about the deformation at that point.

Section 2 is concerned with the construction of non-trivial examples of these invariants. The basic idea is to look at cross products of n-foliations and m-foliations, obtaining $n + m$-foliations. The classes for the product then factor into products of classes for each of the factors. This fact was first pointed out in the case of (undeformed) foliations in [14]. However, it seems to be a much richer source of non-trivial examples for n-foliations than it is in the undeformed case. It is also worth noting that, in the examples constructed at the end of Section 2, the usual invariants for undeformed foliations all vanish on the individual foliations which make up the deformation.

If $B\Gamma q$ is the classifying space for codimension q Haefliger structures, a codimension q n-foliation on M determines a family of maps $F_x\colon M \to B\Gamma q$, parameterized by points $x \in I^n$. If the n-foliation has a non-trivial characteristic

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class, it is natural to ask what can be said about the homotopy or homology types of the maps F_x as x varies. This question seems to be very difficult.

If we ask instead for information about diffeomorphism type (or, for complex analytic foliations, the biholomorphic equivalence type) of the foliations on M as we vary the deformation parameter, then we are led very naturally to the deformation theory of Kodaira and Spencer (see [12]). In this theory, given a foliation on M, one constructs a subsheaf θ of the sheaf of germs of sections in the tangent bundle to M, and is led to view the sheaf cohomology group $H^1(M, \theta)$ as the tangent space to the space of all non-trivial deformations of the foliation on M. Thus an n-foliation on M determines at each point $x \in I^n$ a subspace of $H^1(M, \theta_x)$, where θ_x is the sheaf associated to the foliation at x, and this subspace should be viewed as the tangent space to the deformation at x. This point of view is also taken in [10] where the author shows how to compute the derivative of a class in $H^*(WO_q)$ given only a codimension q foliation on M and an element of $H^1(M, \theta)$.

In Section 3 we state two theorems (Theorems 3.3 and 3.6) which can be viewed as generalizations of the results in [10]. They show that the cohomology invariants for n-foliations at a point $x \in I^n$ are determined by the associated subspace of $H^1(M, \theta_x)$ and the structure of the foliation on M at the point x.

Such a theorem is of interest for two reasons. First, the sheaf cohomology $H^1(M, \theta)$ is in general very difficult to compute. An n-foliation on M has non-trivial cohomology invariants at x only if the subspace of $H^1(M, \theta_x)$ it determines has dimension n. Thus the characteristic classes for n-foliations at x give information about the size of $H^1(M, \theta_x)$. Second, it shows that these characteristic classes are actually defined as homomorphisms $\bigwedge^n (H^1(M, \theta)) \to H^*(M, \mathbf{R})$ for arbitrary undeformed foliations on M. The element of $\bigwedge^n (H^1(M, \theta))$ here plays the role of the tangent space to an n-parameter deformation of the foliation.

Because the proofs of Theorems 3.3 and 3.6 are very long and tedious, we do not give them here. Instead we state a corollary which we will need (Corollary 3.4) and give a short independent proof of this fact.

In Section 4 we prove the following non-triviality theorem about n-foliations (Theorem 4.1): Given a complex holomorphic n-foliation on a complex manifold M, suppose that for some value z_0 of the deformation parameter some characteristic class at z_0 is non-trivial. Then there is an open set $z_0 \in U$ in the parameter space and the set of $z \in U$ for which the foliation at z is biholomorphically equivalent to the foliation at z_0 is at most countable. Since these invariants are continuous functions of the parameter space, if some characteristic class is non-trivial at z_0, it is non-trivial in some open neighborhood of z_0 (in fact this is the neighborhood U in the above theorem). It then follows that any open neighborhood of z_0 in the parameter space contains an uncountable number of biholomorphically non-equivalent foliations.

This theorem is proven using techniques from the paper [7] on deformations of analytic structure.

We do not know if such a theorem is valid for C^∞ n-foliations. At the end of Section 4 we make some comments on where our proof breaks down in the C^∞ case.

I. Definition of the characteristic classes

By an n-foliation of codimension q on a manifold M we will mean a codimension $n+q$ foliation on $M \times I^n$, where $I^n = [0, 1]^n$ is the unit n-cube, which intersects each slice $M \times \{x\}$, $x \in I^n$, as a codimension q foliation. Clearly n-foliations can be thought of as deformations of the foliate structure on M parameterized by I^n. The n-foliations we will consider will either be C^∞ or complex holomorphic. In the case of complex holomorphic foliations we will use an open subset $O^n \subset \mathbf{C}^n$ as the parameter space of the deformation, instead of I^n. We will use the word foliation or 0-foliation to mean a foliation on M (undeformed).

In this section we will generalize the definition of characteristic classes for foliations given in [1] to obtain characteristic classes for n-foliations. We will assume that the reader is already familiar with the constructions in [1] as well as the TP forms in [5], so that we shall be brief on these subjects.

Given an n-foliation of codimension q on M, let $\nu_M \to M \times I^n$ be the normal bundle to the codimension $n+q$ foliation on $M \times I^n$. If $X \in T(M \times I^n)$, the tangent bundle to $M \times I^n$, let $\tilde{X}$ be its image in the normal bundle ν_M under the canonical projection. Call a connection ∇ on ν_M a basic connection if $\nabla_X \tilde{Y} = \widetilde{[X, Y]}$ whenever X is a vector field tangent to leaves of the foliation. In [1] it is shown that this definition is well defined, and that a choice of a Riemannian and a basic connection on ν_M lead to a map $\Phi\colon WO_{n+q} \to \bigwedge^* (M \times I^n, \mathbf{R})$ where $\bigwedge^* (M \times I^n, \mathbf{R})$ is the complex of differential forms on $M \times I^n$. We will give the construction of characteristic classes for n-foliations using this map Φ.

Let $c_I h_J \in WO_{n+q}$ be a monomial where $I = i_1 \leq \cdots \leq i_k$ and $J = j_1 < \cdots < j_l$ are multi-indices and $c_I = c_{i_1}, \ldots, c_{i_k}$, $h_J = h_{j_1}, \ldots, h_{j_l}$. Let $w(c_I h_J) = i_1 + \cdots + i_k$ and let ZO_{n+q} be the cycles in WO_{n+q} generated by monomials α with $w(\alpha) = n+q$. For $\alpha \in ZO_{n+q}$ let $\tilde{\alpha} \in \bigwedge^* (M, \mathbf{R})$ be the form obtained by integrating $\Phi(\alpha) \in \bigwedge^* (M \times I^n, \mathbf{R})$ over the fiber I^n (see [1]).

LEMMA 1.1. *$\tilde{\alpha}$ determines a characteristic class $[\tilde{\alpha}] \in H^*(M, \mathbf{R})$, i.e. $\tilde{\alpha}$ is a closed form whose cohomology class is independent of the choice of basic connection and Riemannian connection which determined Φ.*

Proof. For Y a smooth simplex in M,

$$\int_{\partial Y} \tilde{\alpha} = \int_{\partial Y \times I^n} \Phi(\alpha) = \int_{Y \times I^n} d\Phi(\alpha) \pm \int_{Y \times \partial I^n} \Phi(\alpha) = \pm \int_{Y \times \partial I^n} \Phi(\alpha)$$

(since $\Phi(\alpha)$ is closed). Now the restriction of the foliation to $Y \times \partial I^n$ has codimension $\leq n + q - 1$. Since $w(\alpha) = n + q$, $\int_{Y \times \partial I^n} \Phi(\alpha) = 0$ (this is essentially Bott's vanishing theorem for basic connections, see [1]). Thus $\tilde{\alpha}$ is closed.

The proof that $[\tilde{\alpha}]$ is independent of the choice of connections is standard. Let θ_0 and θ_1 be two choices of basic connections and ω_0, ω_1 two choices of Riemannian connections. We can pull the n-foliation on M back to one on $M \times I$ using projection $M \times I \to M$. On $M \times I$ we can define a basic connection θ whose restriction to each slice $M \times \{t\}$ is given by $t\theta_1 + (1-t)\theta_0$. Similarly choose a Riemannian connection ω whose restriction to each slice $M \times \{t\}$ is given by $t\omega_1 + (1-t)\omega_0$. The resulting form $\tilde{\alpha} \in \bigwedge^* (M \times I, \mathbf{R})$ is closed so that its restriction to $M \times \{0\}$ differs from its restriction to $M \times \{1\}$ by an exact amount. At $M \times \{0\}$, $\theta = \theta_0$ and $\omega = \omega_0$, and at $M \times \{1\}$, $\theta = \theta_1$ and $\omega = \omega_1$. This proves the lemma.

At this point one should note that some of the classes from ZO_{n+q} will always vanish on codimension q n-foliations.

LEMMA 1.2. *For $\alpha \in ZO_{n+q}$, write $\alpha = c_I h_J$ where $I = i_1 \leq \cdots \leq i_k$, $J = j_1 < \cdots < j_l$. Then if either $i_k > q$ or $j_l > q$ the class $[\tilde{\alpha}] \in H^*(M, \mathbf{R})$ will vanish.*

Proof. If $(t_1, \ldots, t_n)$ are the standard coordinates on I^n, then one has n linearly independent sections in ν_M: $\widetilde{\partial/\partial t_1}, \ldots, \widetilde{\partial/\partial t_n}$. Since the leaves of the n-foliation are contained in the slices $M \times \{x\}$ of $M \times I^n$, if X is a vector field tangent to the leaves then $[X, \partial/\partial t_i]$ must be tangent to the slices $M \times \{x\}$. If one chooses a basis of local sections for ν_M, $\tilde{Y}_1, \ldots, \tilde{Y}q, \widetilde{\partial/\partial t_1}, \ldots, \widetilde{\partial/\partial t_n}$, where $Y_1, \ldots, Yq$ are local vector fields on M, then one can choose a basic (and Riemannian) connection to satisfy $\nabla\tilde{Y}_j = \sum_i \theta_{ij} \tilde{Y}_i$ and $\nabla\widetilde{\partial/\partial t_j} = \sum_i \theta_{i,q+j} \tilde{Y}_i$ for certain one-forms θ_{ij}. It follows that the connection form θ takes values in the Lie subalgebra of $\mathfrak{gl}(n+q, \mathbf{R})$ whose last n rows are zero. Since the Weil polynomials for the Chern classes c_k, $k > q$, vanish on this subalgebra, so will $\Phi(h_k)$ and $\Phi(c_k) \in \bigwedge^* (M \times I^n, \mathbf{R})$. Q.E.D.

Remarks. In the next section we will give many examples of non-trivial classes from ZO_{n+q}. The reason we do not redefine ZO_{n+q} to exclude the classes in Lemma 1.2 is that in what follows it will be notationally more convenient not to.

If the normal bundle ν_M has a basis of global sections, then one gets a map

$$\Phi: W_{n+q} \to \bigwedge^* (M \times I^n, \mathbf{R}).$$

Let Z_{n+q} be the cycles in W_{n+q} generated by monomials α with $w(\alpha) = n+q$. Then a construction analogous to the one just given yields characteristic classes from Z_{n+q}.

The classes $[\tilde{\alpha}]$ have a local formulation which will at times be more convenient. Let $V = \partial/\partial t_1 \wedge \cdots \wedge \partial/\partial t_n$ be the volume element on I^n. Then for each point $x \in I^n$ we get a closed form $\tilde{\alpha}_x \in \bigwedge^* (M, \mathbf{R})$ given by $\tilde{\alpha}_x = i_V \Phi(\alpha)|_{M \times \{x\}}$ where i_V denotes interior multiplication by V and we restrict the form to the slice $M \times \{x\}$. The form $\tilde{\alpha}_x$ can be viewed as a derivative of $\tilde{\alpha}$ as we evaluate $\tilde{\alpha}$ on smaller and smaller cubes centered at x. Thus $\tilde{\alpha}_x$ determines a characteristic class in $H^*(M, \mathbf{R})$ which we will denote as α_x and which contains local information about the deformation at the point $x \in I^n$.

The construction of characteristic classes for complex holomorphic n-foliations is similar to the C^∞ case, but there are some differences and in fact there seem to be a number of different approaches one can take, resulting in different characteristic classes. We give here one construction of a group of classes that will contain many non-trivial examples, as will be seen in the next section.

Let M be a complex manifold with a complex holomorphic n-foliation of codimension q defined on it. The normal bundle ν_M of the codimension $n + q$ foliation on $M \times O^n$ is then the quotient of the complexified tangent bundle $T(M \times O^n)$ by the subbundle of vectors which are either tangent to leaves of the foliation or vectors of type (0, 1), (A tangent vector is of type (0, 1) if, in local coordinates, it is a linear combination of the $\partial/\partial \bar{z}_i$.) As in the real case, if $X \in T(M \times O^n)$, let $\tilde{X}$ denote its image in ν_M. A connection ∇ on ν_M is called basic if $\nabla_X \tilde{Y} = [X, Y]^\sim$ whenever X is a vector field which is either tangent to leaves of the foliation or of type (0, 1). This is well defined and a choice of a basic and a Hermitian connection on ν_M determine a map $\Phi\colon WU_{n+q} \to \bigwedge^*$ $(M \times O^n, \mathbf{C})$, (see [3]). If ν_M has a basis of global sections then we get a map

$$\Phi\colon W_{n+q} \otimes \overline{W}_{n+q} \to \textstyle\bigwedge^* (M \times O^n, \mathbf{C})$$

and this is the case we shall examine.

The complex $Z_{n+q} \subset W_{n+q} \otimes 1$ will be the source of characteristic classes, and in analogy with the C^∞ case one might here try to define a class for each $\alpha \in Z_{n+q}$ by integrating $\Phi(\alpha)$ over the fiber O^n of $M \times O^n$. However, in order to insure convergence of this integral, some additional assumptions about O^n would be needed, so it seems more convenient to use the local formulation and take the interior product $i_V \Phi(\alpha)$ where V is the volume element on O^n. It will turn out when looking at examples in the next section that this is still not quite what we want. Instead let $V = \partial/\partial z_1 \wedge \cdots \wedge \partial/\partial z_n$ where $(z_1, \ldots, z_n)$ are the standard complex coordinates on $O^n \subset \mathbf{C}^n$.

LEMMA 1.3. *For each $z \in O^n$, the form $i_V \Phi(\alpha)$ restricted to the slice $M \times \{z\}$ is a closed form and determines a characteristic class for the n-foliation.*

Proof. The proof uses the formula $i_X\, d + di_X = L_X$ where X is some complex tangent vector, i_X is interior product, and L_X is Lie derivative.

Suppose $n = 1$. Then

$$i_{\partial/\partial z_1}\, d\Phi(\alpha) + di_{\partial/\partial z_1}\Phi(\alpha) = di_{\partial/\partial z_1}\Phi(\alpha) = L_{\partial/\partial z_1}\Phi(\alpha).$$

Since $w(\alpha) = n + q$, and on each slice $M \times \{z\}$ the foliation has codimension q, it follows that $\Phi(\alpha)$ vanishes on the slices $M \times \{z\}$, so that $L_{\partial/\partial z_1}\Phi(\alpha)$ must also vanish on $M \times \{z\}$. Thus, when $n = 1$, $i_{\partial/\partial z_1}\Phi(\alpha)$ is a closed form on $M \times \{z\}$.

Assume inductively that $i_V \Phi(\alpha)$ is closed for $(n - 1)$-foliations. By viewing O^n

as $O^1 \times O^{n-1}$, we can view an n-foliation on M as an $(n-1)$-foliation on $M \times O^1$. Let z_1 be the O^1 coordinate and let $(z_2, \ldots, z_n)$ be coordinates on O^{n-1}. Let $V' = \partial/\partial z_2 \wedge \cdots \wedge \partial/\partial z_n$. Then

$$i_{\partial/\partial z_1} di_{V'} \Phi(\alpha) + di_{\partial/\partial z_1} i_{V'} \Phi(\alpha) = L_{\partial/\partial z_1} i_{V'} \Phi(\alpha)$$

By the induction hypothesis the first term on the left must vanish on any slice $M \times O^1 \times \{z'\}$. Thus to prove $i_V \Phi(\alpha)$ is closed on $M \times \{z\}$ we must show that $i_{V'} \Phi(\alpha) = 0$ on the slices $M \times \{z\}$, so that $L_{\partial/\partial z_1} i_{V'} \Phi(\alpha) = 0$ as well.

Recall that, for a basic connection, the curvature Ω_{ij} is a sum of forms $\omega \wedge \eta$ where ω is of type $(1, 0)$ and vanishes on the leaves of the foliation (see [2]). It follows that, since $w(\alpha) = n + q$, $\Phi(\alpha)$ is a sum of forms $\omega \wedge \eta$ where ω is of type $(n + q, 0)$ and $\omega(X_1, \ldots, X_{n+q}) = 0$ whenever any of $X_1, \ldots, X_{n+q}$ are tangent to a leaf. Since the normal space to a leaf in $M \times O^n$ has complex dimension $n + q$, and $\partial/\partial z_1, \ldots, \partial/\partial z_n$ lie in this normal space it follows that the form ω must contain a factor $dz_1 \wedge \cdots \wedge dz_n$, hence so does $\Phi(\alpha)$. From this it is clear that $i_{\partial/\partial z_2 \wedge \cdots \wedge \partial/\partial z_n} \Phi(\alpha) = 0$ on any slice $M \times \{z\}$.

The proof that the cohomology class of $i_V \Phi(x)$ in $H^*(M, \mathbf{C})$ is independent of the choice of basic connection is exactly the same as in Lemma 1.1.

It is inconvenient to require trivializations of ν_M in order to get characteristic classes, and we would like to eliminate the need for this condition. Perhaps the easiest way to do this is to regard Φ as a map

$$\Phi\colon W_{n+q} \otimes \overline{W}_{n+q} \to \textstyle\bigwedge^* (N_M, \mathbf{C})$$

where N_M is the bundle of bases associated to ν_M. This can be done without a trivialization, and gives us a characteristic map $S\colon Z_{n+q} \to H^*(N_M, \mathbf{C})$.

The following is an explicit expression for the map Φ:

$$\Phi(c_k \otimes 1) = C_k(\Omega)$$

where C_k is the Weil polynomial for the kth Chern class and Ω is the curvature of a basic connection θ on N_M;

$$\Phi(1 \otimes \bar{C}_k) = C_k(\bar{\Omega})$$

where $\bar{\Omega}$ is the conjugate form to Ω;

$$\Phi(h_k \otimes 1) = TC_k(\theta)$$

where

$$TC_k(\theta) = k \int_0^1 C_k(\theta \wedge \psi_t^{k-1})\, dt$$

for

$$\psi_t = t\Omega + \tfrac{1}{2}(t^2 - t)[\theta, \theta].$$

$TC_k(\theta)$ is the TP form associated to C_k (see [5]);

$$\Phi(1 \otimes \bar{h}_k) = \overline{TC_k(\theta)},$$

the conjugate form to $TC_k(\theta)$.

Remarks. Note that some of the classes in Z_{n+q} will live in $H^*(M, \mathbf{C})$ even if ν_M isn't trivialized. They are the classes of the form c_I, involving no h_J's. Let $P_{n+q} \subset Z_{n+q}$ be the subspace spanned by these elements. Then for arbitrary n-foliations, at every point $z \in O^n$ there is a characteristic map

$$\tilde{S}: P_{n+q} \to H^{2q+n}(M, \mathbf{C})$$

defined. Theorems 2.3, 3.3, and 4.1 in the sequel remain valid if we replace the map S defined above with this map $\tilde{S}$.

There are actually a number of variations of the above construction which lead to other classes. For example let

$$\tilde{w}(\alpha) = w(c_{I_1}) + w(c_{I_2}) \quad \text{for} \quad \alpha = c_{I_1}\bar{c}_{I_2}h_J \in WU_{n+q}.$$

Then by looking at cycles $\alpha \in WU_{n+q}$ with $\tilde{w}(\alpha) = 2(n+q)$ and by taking interior products with $\partial/\partial z_1 \wedge \cdots \wedge \partial/\partial z_n \wedge \partial/\partial \bar{z}_1 \wedge \cdots \wedge \partial/\partial \bar{z}_n$, we would obtain a different set of classes. We have made the choices we did because in Section 2 we are able to construct many non-trivial examples of the resulting classes.

II. Examples

Characteristic classes for 1-foliations have been examined in [8], where the authors were primarily interested in those classes which are the derivative of deformable classes in $H^*(WO_q)$. Their construction of these classes uses Gelfand-Fuks cohomology and is different from the construction given here.

For example consider the Godbillon-Vey class $c_1^q h_1 \in H^{2q+1}(WO_q)$. Under the natural inclusion $WO_q \subset WO_{q+1}$, $c_1^q h_1$ is no longer a cycle and

$$d(c_1^q h_1) = c_1^{q+1} \in ZO_{q+1}.$$

Given a 1-foliation on M, suppose that Z is a smooth singular $2q+1$-cycle in M. Then

$$\int_{Z \times I} \Phi(c_1^{q+1}) = \int_{Z \times I} d\Phi(c_1^q h_1) = \pm \int_{Z \times \partial I} \Phi(c_1^q h_1).$$

Thus integrating the form $\tilde{c}_1^{q+1}$ over Z gives the difference of the Godbillon-Vey class evaluated at time 0 and time 1 on the deformation. For any point $x \in I$, $(c_1^{q+1})_x$ is the derivative of the Godbillon-Vey class at the point x in the deformation.

Similarly, given a complex holomorphic 1-foliation of codimension q on a complex manifold M, and a smooth singular $2q+1$-cycle y in M,

$$\begin{aligned}\int_{y\times\{z\}} i_{\partial/\partial z}\Phi(c_1^{q+1}\otimes 1) &= \int_{y\times\{z\}} i_{\partial/\partial z}\, d\Phi(c_1^q h_1\otimes 1)\\ &= \int_{y\times\{z\}} L_{\partial/\partial z}\Phi(c_1^q h_1\otimes 1)\\ &\quad - \int_{y\times\{z\}} d i_{\partial/\partial z}\Phi(c_1^q h_1\otimes 1)\\ &= \frac{\partial}{\partial z}\left[\int_{y\times\{z\}}\Phi(c_1^q h_1\otimes 1)\right] \quad (\text{since } \partial y = 0).\end{aligned}$$

By now there are many examples of non-trivial deformations of classes in $H^*(WO_q)$ and $H^*(\overline{W}_q\otimes W_q)$, see [4], [9]. It follows that the corresponding classes for 1-foliations are also non-trivial. However, most of the classes in ZO_{q+1} cannot be interpreted as derivatives of other classes. For example $c_1^{q+1}h_1 \in ZO_{q+1}$ is not exact in WO_{q+1}, so it cannot be the derivative of a class in $H^*(WO_q)$.

Suppose M is foliated in codimension p, and N is foliated in codimension q. Then the cross product foliation on $M\times N$ has codimension $p+q$, and in [14] an algorithm is given for computing the characteristic classes for the foliation on $M\times N$ in terms of the values of characteristic classes for the foliations on M and N. We shall use the same principle to construct many new examples of non-triviality for characteristic classes of n-foliations.

Suppose $\mathscr{F}_0$ is an n-foliation of codimension p on M and $\mathscr{F}_1$ is an m-foliation of codimension q on N. Then $\mathscr{F}_0\times\mathscr{F}_1$ is an $n+m$-foliation of codimension $p+q$ on $M\times N$. We require $n+m>0$.

Define a map $\psi\colon WO_{p+q+n+m}\to WO_{p+n}\otimes WO_{q+m}$ by

$$\psi(c_k) = \sum_{i+j=k} c_i\otimes c_j \quad (\text{where } c_0 = 1)$$

and

$$\psi(h_k) = \sum_{i+j=k} h_i\otimes c_j + c_i\otimes h_j \quad (\text{where } h_s = 0 \text{ for } s \text{ even}).$$

The map ψ is extended as an anti-derivation. Now if $w(c_I h_J) = p+q+n+m$, then $\psi(c_I h_J)$ will be a sum of terms $\alpha\otimes\beta$ where $w(\alpha)=p+n$, $w(\beta)=q+m$. It follows that ψ restricts to a map $\rho\colon ZO_{p+q+n+m}\to ZO_{p+n}\otimes ZO_{q+m}$.

THEOREM 2.1. *There is a commutative diagram*

$$\begin{array}{ccc} ZO_{p+q+n+m} & \longrightarrow & H^*(M\times N,\mathbf{R})\\ \Big\downarrow{\scriptstyle\rho} & & \Big\uparrow{\scriptstyle\wr}\\ ZO_{p+n}\otimes ZO_{q+m} & \longrightarrow & H^*(M,\mathbf{R})\otimes H^*(N,\mathbf{R})\end{array}$$

where the top map is the characteristic map for $\mathscr{F}_0 \times \mathscr{F}_1$ and the bottom map is the tensor product of the characteristic maps for $\mathscr{F}_0$ and $\mathscr{F}_1$.

Proof. Note that the normal bundle of $\mathscr{F}_0 \times \mathscr{F}_1$ splits as a direct sum of the pull-backs over $M \times N$ of the normal bundles for $\mathscr{F}_0$ and $\mathscr{F}_1$. One can choose basic and Riemannian connections to respect this splitting and obtain a diagram

$$\begin{tikzcd}
WO_{p+q+n+m} \arrow[r, "\Phi_{\mathscr{F}_0 \times \mathscr{F}_1}"] \arrow[d, "\psi"'] \arrow[dr, "\Delta"] & \bigwedge^* (M \times N \times I^{n+m}, \mathbf{R}) \\
WO_{p+n} \otimes WO_{q+m} \arrow[r, "\Phi_{\mathscr{F}_0} \otimes \Phi_{\mathscr{F}_1}"] & \bigwedge^* (M \times I^n, \mathbf{R}) \otimes \bigwedge^* (N \times I^m, \mathbf{R}) \arrow[u, "E"']
\end{tikzcd}$$

The map E is given by exterior product, and the diagonal map Δ arises because the connections for $\mathscr{F}_0 \times \mathscr{F}_1$ are Whitney sums of the pull backs of corresponding connections for $\mathscr{F}_0$ and $\mathscr{F}_1$. The upper triangle commutes. If the lower triangle commuted as well the theorem would be proven, but unfortunately it is only cochain homotopy commutative. Since we are dealing with cycles in $WO_{p+q+n+m}$ and not cohomology classes, some care must be taken.

From the Whitney sum formula for the Weil polynomials for Chern classes, it follows that

$$\Phi_{\mathscr{F}_0} \otimes \Phi_{\mathscr{F}_1} \circ \psi(c_k) = \Delta(c_k).$$

Unfortunately $\Phi_{\mathscr{F}_0} \otimes \Phi_{\mathscr{F}_1} \circ \psi(h_k) = \Delta(h_k) + \text{exact}$ (see [5]). Thus to prove Theorem 2.1, we need the following lemma:

LEMMA 2.2. *Suppose $\bar{\Phi}$: $WO_{p+q+n+m} \to \bigwedge^* (M \times N \times I^{n+m}, \mathbf{R})$ satisfies*

$$\bar{\Phi}(c_k) = \Phi_{\mathscr{F}_0 \times \mathscr{F}_1}(c_k), \quad \bar{\Phi}(h_k) = \Phi_{\mathscr{F}_0 \times \mathscr{F}_1}(h_k) + \text{exact}.$$

Then the maps $\bar{\Phi}$ and $\Phi_{\mathscr{F}_0 \times \mathscr{F}_1}$ will induce the same characteristic map on cohomology for $n + m$-foliations.

Theorem 2.1 now follows by letting $\bar{\Phi} = E \circ \Phi_{\mathscr{F}_0} \otimes \Phi_{\mathscr{F}_1} \circ \psi$

Proof of Lemma 2.2. We can assume inductively that $\bar{\Phi}(h_k) = \Phi_{\mathscr{F}_0 \times \mathscr{F}_1}(h_k)$ for all k except one, say k_0, and $\bar{\Phi}(h_{k_0}) = \Phi_{\mathscr{F}_0 \times \mathscr{F}_1}(h_{k_0}) + d\xi$.

Recall that, for $x \in I^{n+m}$ and $c_I h_J \in ZO_{p+q+n+m}$, the form

$$\widetilde{(c_I h_J)}_x = i_V(\Phi_{\mathscr{F}_0 \times \mathscr{F}_1}(c_I h_J))\,|_{M \times \{x\}} \quad \text{where } V = \partial/\partial t_1 \wedge \cdots \wedge \partial/\partial t_{n+m}$$

is the volume element on I^{n+m}. A little thought shows that

$$i_V(\Phi_{\mathscr{F}_0 \times \mathscr{F}_1}(c_I h_J)) = i_V(\Phi_{\mathscr{F}_0 \times \mathscr{F}_1}(c_I)) \wedge \Phi_{\mathscr{F}_0 \times \mathscr{F}_1}(h_J) + \text{other terms}$$

and the other terms will vanish when restricted to $M \times N \times \{x\}$. This is because

$$w(c_I) = p + q + n + m,$$

so for dimension reasons $\Phi_{\mathscr{F}_0 \times \mathscr{F}_1}(c_I)$ must contain a factor $dt_1 \wedge \cdots \wedge dt_{n+m}$. Thus

$$\begin{aligned}\widetilde{(c_I h_J)}_x &= i_V(\Phi_{\mathscr{F}_0 \times \mathscr{F}_1}(c_I)) \wedge \Phi_{\mathscr{F}_0 \times \mathscr{F}_1}(h_J)\,|_{M \times N \times \{x\}} \\ &= i_V(\bar{\Phi}(c_I)) \wedge \bar{\Phi}(h_J)\,|_{M \times N \times \{x\}}\end{aligned}$$

if the multi-index J doesn't contain k_0. If J does contain k_0 let J' be the same multi-index but with k_0 removed. Then

$$\begin{aligned}\widetilde{(c_I h_J)}_x &= \pm i_V(\bar{\Phi}(c_I)) \wedge \bar{\Phi}(h_{J'}) \wedge (\bar{\Phi}(h_{k_0}) - d\xi) \\ &= i_V(\bar{\Phi}(c_I h_J)) \pm d[i_V \bar{\Phi}(c_I h_{J'}) \wedge \xi]\end{aligned}$$

and this proves the lemma.

Theorem 2.1 has an obvious analogue for complex holomorphic n-foliations. There is the map ψ: $W_{p+q+n+m} \to W_{p+n} \otimes W_{q+m}$,

$$\psi(c_k) = \sum_{i+j=k} c_i \otimes c_j \quad \text{and} \quad \psi(h_k) = \frac{1}{2} \sum_{i+j=k} h_i \otimes c_j + c_i \otimes h_j,$$

which restricts to ρ: $Z_{p+q+n+m} \to Z_{p+n} \otimes Z_{q+m}$. Suppose $\mathscr{F}_0$ is a complex holomorphic n-foliation of codimension p on a complex manifold M and $\mathscr{F}_1$ is a complex holomorphic m-foliation of codimension q on N, where $n + m > 0$.

THEOREM 2.3. *There is a commutative diagram*

$$\begin{array}{ccc} Z_{p+q+n+m} & \longrightarrow & H^*(N_{M \times N}, \mathbf{C}) \\ \big\downarrow{\scriptstyle\rho} & & \big\downarrow \\ Z_{p+n} \otimes Z_{q+m} & \longrightarrow & H^*(N_M, \mathbf{C}) \otimes H^*(N_N, \mathbf{C}) \end{array}$$

where the top map is the characteristic map for $\mathscr{F}_0 \times \mathscr{F}_1$, the bottom map is the tensor product of the characteristic maps for $\mathscr{F}_0$ and $\mathscr{F}_1$, and the right hand vertical arrow is induced by the inclusion $N_M \times N_N \hookrightarrow N_{M \times N}$.

Example 1. Thurston has constructed a deformation of the Godbillon-Vey class $[c_1 h_1]$ on S^3 (see [15]), i.e., a 1-foliation on S^3 where $[\tilde{c}_1^2] \in H^3(S^3, \mathbf{R})$ is non-zero. By taking a slice $S^3 \times \{x\}$ we also have a 0-foliation on S^3 with non-trivial $[c_1 h_1]$. The cross product of the 0-foliation and the 1-foliation yields a 1-foliation of codimension 2 on $S^3 \times S^3$. The map

$$\rho\colon ZO_3 \to ZO_1 \otimes ZO_2$$

sends $c_1^3 h_1$ to $3c_1 h_1 \otimes c_1^2 + 3c_1 \otimes c_1^2 h_1$. Thus $\widetilde{[c_1^3 h_1]} \in H^6(S^3 \times S^3, \mathbf{R})$ is non-zero. One can continue taking cross products with the 1-foliation on S^3 obtaining n-foliations of codimension $n+1$ on $S^3 \times \cdots \times S^3 (n+1$ times) and the class $\widetilde{[c_1^{2n+1} h_1]} \in H^{3n+3}(S^3 \times \cdots \times S^3, \mathbf{R})$ will be non-zero. Note that the usual invariants for 0-foliations, coming from $H^*(WO_{n+1}, \mathbf{R})$, must vanish on the slices $S^3 \times \cdots \times S^3 \times \{x\}$, $x \in I^n$, in these examples.

Example 2. In [1] an example is given of a complex analytic 1-foliation on $\mathbf{C}^2 - \{0\}$ which deforms the complex Godbillon-Vey class $[c_1 h_1]$. Note that the value of $[c_1 h_1] \in H^3(\mathbf{C}^2 - \{0\}, \mathbf{C})$ is a holomorphic function of the deformation parameter z. Thus $i_{\partial/\partial z}\Phi(c_1^2)$ will be non-zero, and the same procedure that was used in Example 1 can be used here to construct non-trivial invariants for the n-foliation of codimension $n+1$ on $(\mathbf{C}^2 - \{0\}) \times \cdots \times (\mathbf{C}^2 - \{0\})$ ($n+1$ times).

For all of the deformations constructed in [4], the values of the deformed classes are holomorphic functions of the deformation parameters, so that Theorem 2.3 can be used to generate non-trivial classes on cross products of these foliations.

III. Kodaira-Spencer deformation theory

There is a connection between the previously described invariants and the sheaf cohomology invariants of Kodaira-Spencer deformation theory; see [11] and [12]. The connection we will develop fits into the context of both C^∞ n-foliations and complex holomorphic n-foliations. However, since in the next section we shall be interested in the complex holomorphic case, we shall do this case explicitly, and only remark briefly on the modifications needed for the C^∞ case.

Let M be a complex manifold with a complex holomorphic n-foliation of codimension q defined on it. As before, let $O^n \subset \mathbf{C}^n$ be the parameter space of the deformation so that the n-foliation is actually a codimension $n+q$ foliation on $M \times O^n$. Let $(z_1, \ldots, z_n)$ denote the standard complex coordinate system on O^n, and let $\pi\colon M \times O^n \to O^n$ be projection.

A coordinate chart on some open set $U_\alpha \subset M \times O^n$ will be called distinguished if the coordinates $(x_1^\alpha, \ldots, x_r^\alpha, y_1^\alpha, \ldots, y_q^\alpha, z_1^\alpha, \ldots, z_n^\alpha)$ satisfy the following properties:

(a) the slices $y_1^\alpha, \ldots, y_q^\alpha, z_1^\alpha, \ldots, z_n^\alpha$ held constant are the intersection of the leaves of the n-foliation with U_α and

(b) in these local coordinates, $\pi\colon M \times O^n \to O^n$ has the form

$$\pi(x_1^\alpha, \ldots, x_r^\alpha, y_1^\alpha, \ldots, y_q^\alpha, z_1^\alpha, \ldots, z_n^\alpha) = (z_1^\alpha, \ldots, z_n^\alpha).$$

Note that for 0-foliations this definition still makes sense except that there are no z coordinates and condition (b) becomes vacuous. Given a 0-foliation on M, we can define a subsheaf θ of the sheaf of germs of holomorphic vector fields on M as follows: A holomorphic vector field X on M is a section of θ if and only if, using any distinguished coordinates, $X = \sum f_i\, \partial/\partial x_i^\alpha + \sum g_i\, \partial/\partial y_j^\alpha$ where $\partial g_j/\partial x_i^\alpha = 0$ for all $i = 1, \ldots, r$ and $j = 1, \ldots, q$.

Let TM be the complex tangent bundle for M, ν the complex normal bundle for the foliation, and $p\colon TM \to \nu$ the canonical projection. Then a holomorphic vector field X is a section of θ if and only if $p(X)$ is a holomorphic section of ν which is covariant constant along leaves of the foliation with respect to a basic connection. Let $\tilde{\theta}$ be the sheaf of germs of such sections in v, so that the projection p induces a map of sheaves $\tilde{p}: \theta \to \tilde{\theta}$.

For an n-foliation on M and any value $z = (z_1, \ldots, z_n) \in O^n$, let M_z be the slice $M \times \{z\}$ of $M \times O^n$ and let θ_z be the sheaf associated to the 0-foliation on M_z which is the restriction of the n-foliation on $M \times O^n$. Let O^n_z be the complex tangent space to O^n at z. Kodaira and Spencer define a sheaf cohomology deformation invariant

$$\rho_z: O^n_z \to H^1(M_z, \theta_z)$$

as follows: Choose an open set V with $z \in V \subset O^n$. For U_α, U_β open sets in M choose distinguished coordinates $(x^\alpha_i, y^\alpha_j, z^\alpha_k)$ on $U_\alpha \times V$ and $(x^\beta_i, y^\beta_j, z^\beta_k)$ on $U_\beta \times V$. Then a one cocycle for $\rho_z(\partial/\partial z_k)$ on $U_\alpha \cap U_\beta$ is given by the vector field $\partial/\partial z^\alpha_k - \partial/\partial z^\beta_k$. We must check that this is actually a cocycle and that it lies in θ_z.

On $U_\alpha \cap U_\beta \cap U_\gamma$, $(\partial/\partial z^\alpha_k - \partial/\partial z^\beta_k) + (\partial/\partial z^\beta_k - \partial/\partial z^\gamma_k) + (\partial/\partial z^\gamma_k - \partial/\partial z^\alpha_k) = 0$, so this is a one cocycle.

Note that, since $(x^\alpha_i, y^\alpha_j, z^\alpha_k)$ and $(x^\beta_i, y^\beta_j, z^\beta_k)$ are distinguished coordinates, the transition functions must satisfy $\partial y^\beta_j/\partial x^\alpha_i = 0$ and $z^\alpha_k = z^\beta_k$, It then follows that

$$\partial/\partial x^\beta_i = \sum a_{il}\, \partial/\partial x^\alpha_l$$

for some holomorphic functions a_{il}. Then

$$(3.1) \qquad \partial^2 y^\beta_j/\partial x^\beta_i\, \partial z^\alpha_k = \sum a_{il}\, \partial^2 y^\beta_j/\partial x^\alpha_l\, \partial z^\alpha_k = \sum a_{il}\, \partial^2 y^\beta_j/\partial z^\alpha_k\, \partial x^\alpha_l = 0$$

since $\partial y^\beta_j/\partial x^\alpha_l = 0$. Thus

$$\partial/\partial z^\alpha_k = \sum \partial x^\beta_i/\partial z^\alpha_k\, \partial/\partial x^\beta_i + \sum \partial y^\beta_j/\partial z^\alpha_k\, \partial/\partial y^\beta_j + \partial/\partial z^\beta_k$$

and

$$\partial/\partial z^\alpha_k - \partial/\partial z^\beta_k = \sum \partial x^\beta_i/\partial x^\alpha_k\, \partial/\partial x^\beta_i + \sum \partial y^\beta_j/\partial z^\alpha_k\, \partial/\partial y^\beta_j.$$

This vector field lies in θ_z by equation (3.1).

One must check that a different choice of distinguished coordinates changes the value of this one cocycle by a coboundary and this is done in [12].

Call an n-foliation $\mathscr{F}$ trivial if there is a biholomorphic equivalence of $M \times O^n$ with itself, commuting with projection $\pi: M \times O^n \to O^n$, which carries $\mathscr{F}$ onto a cross product of a fixed foliation on M with O^n.

LEMMA 3.2 (see [12]). *If an n-foliation is trivial then $\rho_z = 0$ for all $z \in O^n$.*

Proof. For the cross product foliation, one can choose distinguished coordinates on $M \times O^n$ which are cross products of distinguished coordinates for the fixed foliation on M with the usual coordinates on O^n. For two such choices of coordinates it will follow that $\partial/\partial z^\alpha_k = \partial/\partial z^\beta_k$ so $\rho_z(\partial/\partial z_k) = 0$.

The biholomorphic equivalence in the definition of trivial n-foliations must carry distinguished coordinates to distinguished coordinates. Thus $\rho_z(\partial/\partial z_k) = 0$ for any trivial foliation, which proves the lemma.

THEOREM 3.3. *Given a 0-foliation of codimension q on M, there is a map*

$$\psi\colon \bigwedge^n (H^1(M, \tilde{\theta})) \otimes Z_{n+q} \to H^*(N_M, \mathbf{C})$$

with the property that, for an n-foliation of codimension q on M and $z \in O^n$, there is a commutative diagram

$$\begin{tikzcd} \bigwedge^n (H^1(M_z, \tilde{\theta}_z)) \otimes Z_{n+q} \arrow[r, "\psi"] & H^*(N_M, \mathbf{C}) \\ Z_{n+q} \arrow[u, "\xi"] \arrow[ur] & \end{tikzcd}$$

where $\xi(\alpha) = \tilde{p}^*(\rho_z(\partial/\partial z_1)) \wedge \cdots \wedge \tilde{p}^*(\rho_z(\partial/\partial z_n)) \otimes \alpha$. Here $\tilde{p}^*$ is the map on cohomology induced by $\tilde{p}\colon \theta \to \tilde{\theta}$, and the diagonal map is the characteristic map for the n-foliation at $z \in O^n$.

Remarks. Suppose $n = 1$ and $\alpha \in Z_{n+q}$ represents the derivative of a characteristic class for (C^∞ or holomorphic) 0-foliations. Then an explicit algorithm for computing $\psi(x, \alpha)$, $x \in H^1(M, \theta)$, has been given by J. Heitsch and can be found in [10].

The map ψ is determined completely by the foliation on M, and is an invariant of the foliation. Thus if the dimension of the image of ψ is different for two different foliations on M, then they cannot be biholomorphically equivalent. The examples of the previous section show that the map ψ is non-trivial for many foliations.

The proof of Theorem 3.3 is rather long and involves many sheaf theoretic constructions. We omit the proof and here give an independent proof of the following corollary:

COROLLARY 3.4. *Given an n-foliation of codimension q on M, suppose the characteristic map $Z_{n+q} \to H^*(N_M, \mathbf{C})$ is non-trivial at some point $z \in O^n$. Then the Kodaira-Spencer map $\rho_z\colon O_z^n \to H^1(M_z, \theta_z)$ is an injection.*

Proof. Call a tangent vector X in $T(M \times O^n)$ a leaf vector if X is tangent to a leaf of the foliation or if X is type (0, 1). Recall that a basic connection for the normal bundle of a foliation is defined uniquely only in the directions of leaf vectors, and extended in other directions in an arbitrary fashion.

Suppose ρ_z isn't injective. The idea of the proof is to choose an extension of a basic connection in such a way that the characteristic map

$$\Phi\colon Z_{n+q} \to \bigwedge^* (N_M, \mathbf{C})$$

will be 0 over all points in M_z, and from this the corollary will follow. We can assume, for example, that $\rho_z(\partial/\partial z_1) = 0$. Then, for the one-cocycle for $\rho_z(\partial/\partial z_1)$ defined in terms of distinguished coordinates, $\partial/\partial z_1^\alpha - \partial/\partial z_1^\beta = X^\alpha - X^\beta$ where $\{X^\alpha\}$ is a zero cochain in θ_z. Thus $\partial/\partial z_1^\alpha - X^\alpha = \partial/\partial z_1^\beta - X^\beta$ and there is a holomorphic vector field Z in $T(M \times O^n)$ defined only over M_z, which locally has the form $Z = \partial/\partial z_1^\alpha - X^\alpha$. Z also has the property that $[X, Z]$ is a leaf vector field whenever X is a leaf vector field.

For Y a tangent vector in $T(M \times O^n)$, let $\tilde{Y}$ be its image in the normal bundle to the foliation. Now choose an extension of a basic connection so that, at points in M_z, $\nabla_Z \tilde{Y} = \widetilde{[Z, Y]}$. Note that, since $[X, Z]$ is a leaf vector field if X is, this is well defined.

It now follows that, for any leaf vector X,

$$
\begin{aligned}
(3.5) \qquad R(X, Z)Y &= \nabla_X \nabla_Z \tilde{Y} - \nabla_Z \nabla_X \tilde{Y} - \nabla_{[X,Z]} \tilde{Y} \\
&= \widetilde{[X, [Z, Y]]} - \widetilde{[Z, [X, Y]]} - \widetilde{[[X, Z], Y]} \\
&= 0
\end{aligned}
$$

by the Jacobi identity. The rest of the argument is standard (see [1]). Using distinguished coordinates and the basis $\{\widetilde{\partial/\partial y_j^\alpha}, \widetilde{\partial/\partial z_k^\alpha}\}$ for the normal bundle of the foliation, one finds that the curvature form Ω_{il} is a sum of forms of the type $\omega \wedge dz_k^\alpha$ or $\omega \wedge dy_j^\alpha$ for arbitrary forms ω. Note that $dz_1^\alpha(Z) = 1$ so that, by equation (3.5), the only terms in Ω_{il} involving dz_1^α at points in M_z must be linear combinations of $dz_k^\alpha \wedge dz_1^\alpha$ and $dy_j^\alpha \wedge dz_1^\alpha$. From this it follows that any Chern polynomial in Ω of total degree $2(n + q)$ must vanish at points in M_z, which proves the corollary.

Finally we indicate briefly the modifications necessary to prove analogous theorems for C^∞ foliations. Distinguished coordinates are defined in the same way except of course they will be real valued instead of complex valued. For a 0-foliation on M, one defines a subsheaf θ of the sheaf of germs of C^∞ vector fields on M in an analogous fashion to the holomorphic case. Let $\tilde{\theta}$ be germs of sections in the normal bundle to the foliation on M which are covariant constant along leaves of the foliation with respect to a basic connection, so that, as in the complex holomorphic case, there is the map of sheaves $\tilde{p}\colon \theta \to \tilde{\theta}$. The kernel of $\tilde{p}$ is just germs of C^∞ vector fields on M which are tangent to leaves, and this is a fine sheaf. It follows that the sheaf cohomology groups $H^k(M, \theta)$ equal $H^k(M, \tilde{\theta})$ for $k > 0$.

For $x \in I^n$ let I_x^n be the tangent bundle to I^n at x. Then the Kodaira-Spencer invariant $\rho_x\colon I_x^n \to H^1(M, \theta_x) = H^1(M, \tilde{\theta}_x)$ is defined in exactly the same way as in the holomorphic case (see [12]). The fact that one can view the image as $H^1(M, \tilde{\theta}_x)$ instead of $H^1(M, \theta_x)$ is different from the holomorphic case where the kernel of the map $\tilde{p}\colon \theta \to \tilde{\theta}$ is the sheaf of germs of holomorphic vector fields tangent to leaves, not a fine sheaf. This corresponds to the fact that, in the complex analytic case, it might be possible to non-trivially deform a foliation by altering the complex analytic structure of the leaves, leaving the normal structure unchanged.

Returning to the C^∞ case, one still has the subcomplex ZO_{n+q} and the characteristic map for n-foliations.

THEOREM 3.6. *Given a 0-foliation of codimension q on M, there is a map*

$$\psi\colon \bigwedge^n (H^1(M, \tilde{\theta})) \otimes ZO_{n+q} \to H^*(M, \mathbf{R})$$

with the property that, for an n-foliation of codimension q on M and $x \in I^n$, there is a commutative diagram

$$\begin{tikzcd} \bigwedge^n (H^1(M, \tilde{\theta}_x)) \otimes ZO_{n+q} \arrow[r, "\psi"] & H^*(M, \mathbf{R}) \\ ZO_{n+q} \arrow[u, "\xi"] \arrow[ur] & \end{tikzcd}$$

where $\xi(\alpha) = \rho_x(\partial/\partial t_1) \wedge \cdots \wedge \rho_x(\partial/\partial t_n) \otimes \alpha$ and the diagonal map is the characteristic map for the n-foliation at $x \in I^n$ (using its local formulation).

As in the complex holomorphic case, the map ψ is an invariant of the foliation on M, and, if two foliations on M are diffeomorphic, then the dimension of the image of ψ must be equal for these foliations.

IV. Non-triviality of deformations

Lemma 3.2 shows that for trivial n-foliations the Kodaira-Spencer map (and hence all characteristic classes for n-foliations) must be zero. This characterization of triviality (or non-triviality) still leaves something to be desired. It does not exclude the possibility of non-trivial invariants for an n-foliation on M where, for each $z \in O^n$, there is a biholomorphic equivalence of M_z onto a fixed M_{z_0}, carrying the foliation on M_z to the foliation on M_{z_0}, and varying *discontinuously* in the parameter z.

This same question first arose in the context of deformations of analytic structure on a complex manifold. A theorem of Fischer and Grauert, [7], states that if all the compact manifolds M_z are biholomorphically equivalent, then the Kodaira-Spencer invariant must vanish identically. It then follows from [11] that the deformation is locally an analytic cross product.

The following theorem about non-triviality for n-foliations is proven using a modification of the techniques appearing in [7]. It is valid for non-compact as well as compact M.

THEOREM 4.1. *Given a complex analytic n-foliation of codimension q on M, suppose that for some $z_0 \in O^n$ the characteristic map $Z_{n+q} \to H^*(N_M, \mathbf{C})$ is non-trivial. Then there is an open neighborhood $z_0 \in U \subset O^n$ and the set of $z \in U$ for which the foliations at M_z are biholomorphically equivalent to the foliation at M_{z_0} is at most countable.*

Remark. Clearly if the characteristic map at z_0 is non-trivial it will be non-trivial in some open neighborhood $z_0 \in U \subset O^n$, and this is the neighborhood referred to in the theorem. We reiterate, as in the introduction, that

Theorem 4.1 implies that any open neighborhood of $z_0 \in O^n$ contains an uncountable number of non-biholomorphically equivalent foliations on M.

Proof. Let A be the set of $z \in U$ for which the foliation at M_z is biholomorphically equivalent to the foliation at M_{z_0}. If A is uncountable we will show that, for some $z_1 \in A$, the characteristic map at z_1 is zero, a contradiction. This is done by showing that there is a holomorphic vector field in $T(M \times O^n)$, defined only over M_{z_1}, having the same properties as the vector field Z in the proof of Corollary 3.4. Specifically $[X, Z]$ is a leaf vector field whenever X is, and there are no points in M_{z_1} at which Z is a leaf vector. From this it follows as in the proof of Corollary 3.4 that the characteristic map at z_1 is trivial.

For each $z \in A$, let $f_z: M_{z_0} \to M_z$ be a biholomorphic equivalence preserving foliations. We show that there is a point $z_1 \in A$ and a sequence of points $\{\omega_s\}$ for which

$$f_{\omega_s}: M_{z_0} \to M_{\omega_s}$$

converge uniformly on compact subsets to $f_{z_1}: M_{z_0} \to M_{z_1}$ (it follows of course that $\{\omega_s\}$ converges to z_1 in O^n).

Let $\mathscr{M}(M_{z_0}, M \times O^n)$ be the space of analytic maps from M_{z_0} to $M \times O^n$, topologized with the compact open topology. It suffices to find a sequence $\{f_{\omega_s}\}$ converging to f_{z_1} in $\mathscr{M}(M_{z_0}, M \times O^n)$. Since M_{z_0} is locally compact and the topologies on M_{z_0} and $M \times O^n$ are second countable, it follows that the topology on $\mathscr{M}(M_{z_0}, M \times O^n)$ is also second countable (see [6], p. 265]). If

$$\{f_z: M_{z_0} \to M_z \mid z \in A\}$$

is uncountable then this set must have a cluster point in $\mathscr{M}(M_{z_0}, M \times O^n)$. This cluster point is the map f_{z_1} and clearly the sequence $\{f_{\omega_s}\}$ must also exist.

Let $g_{\omega_s} = f_{\omega_s} \circ f_{z_1}^{-1}: M_{z_1} \to M_{\omega_s}$. Clearly g_{ω_s} is a biholomorphic equivalence preserving foliations, and the sequence $\{g_{\omega_s}\}$ converges uniformly on compact sets to the identity id: $M_{z_1} \to M_{z_1}$. The idea now is to try to take the derivative of the sequence $\{g_{\omega_s}\}$ and let this be the vector field Z on M_{z_1}. The problem is that each g_{ω_s} can be altered by composing it with a foliation preserving biholomorphic equivalence of M_{z_1} with itself, and this ambiguity means that there is no reason to expect a derivative to exist. Thus we must introduce certain corrections to the sequence $\{g_{\omega_s}\}$ to rectify the situation, and this turns out to be a somewhat technical procedure.

To render what follows more digestable, we give a brief outline of the rest of the proof. Our first concern is to modify the sequence $\{g_{\omega_s}\}$ so that we can take its derivative on M_{z_1}, obtaining a holomorphic vector field Z. This modified sequence, $\{h_{\omega_s}\}$, is finally constructed in Lemma 4.4. Because the functions h_{ω_s} preserve the foliation, it will follow that $[Z, X]$ is a leaf vector field whenever X is. The vector field Z must also be nowhere tangent to the leaves of the foliation on M_{z_1}, and in fact we show in Lemma 4.6 that, for $\pi: M \times O^n \to O^n$ projection, $d\pi(Z) \neq 0$. It is in the proof of Lemma 4.6 that the technical condition imposed on the sequence $\{h_{\omega_s}\}$ in Lemma 4.4 is used. This completes the outline.

For each s choose a compact set $K_s \subset M_{z_1}$ so that K_1 is the closure of an open set in M_{z_1}, K_s is contained in the interior of K_{s+1}, and $\bigcup_1^\infty K_s = M_{z_1}$. Choose a countable dense subset $\{x_s\}$ in the interior of K_1. Let

$$\mathscr{G}_0 = H^0(M_{z_1}, \theta_{z_1}), \quad \mathscr{G}_s = \{\eta \in H^0(M_{z_1}, \theta_{z_1}) \,|\, \eta(x_s) = 0\},$$

and let $\mathscr{H}_s$ be a complementary subspace to $\mathscr{G}_0 \cap \cdots \cap \mathscr{G}_s$ in $\mathscr{G}_0 \cap \cdots \cap \mathscr{G}_{s-1}$.

Let T_s be the tangent space to $M \times O^n$ at x_s, and choose a splitting $T_s = V_s \oplus W_s$ where V_s is the subspace of T_s spanned by the vector fields in $\mathscr{H}_s$. One can choose coordinates in $M \times O^n$ about each point x_s, and it will be convenient to think of them as maps

$$\Phi_s\colon U_s \to T_s = V_s \oplus W_s$$

for some open set $x_s \in U_s \subset M \times O^n$. In this way we can require that $\Phi_s(x_s) = 0$ and that $d\Phi_{s|x_s}$ be the identity map.

Given $X \in H^0(M_{z_1}, \theta_{z_1})$, X may not generate a one parameter family of biholomorphic equivalences on M_{z_1}, since M_{z_1} isn't compact. However, for any K_s, X generates a one parameter family of holomorphic embeddings of K_s in M_{z_1}, and these embeddings preserve the leaves of the foliation on M_{z_1} (this is most easily seen by looking at the expression for X in distinguished coordinates). Furthermore, for $X \in \mathscr{H}_s$, this one parameter family leaves $x_1, \ldots, x_{s-1}$ fixed. Let

$$G_0^t = \{f \in \mathscr{M}(K_t, M_{z_1}) \,|\, f \text{ preserves the foliation}\}$$

and let

$$G_s^t = \{f \in G_0^t \,|\, f(x_s) = x_s\}.$$

Choose a metric d on $M \times O^n$ and, for $f, g \in \mathscr{M}(K_t, M \times O^n)$ let

$$d_t(f, g) = \sup_{x \in K_t}\{d(f(x), g(x))\}.$$

LEMMA 4.2. *Fix ε, s, and t. Then there is a δ such that for*

$$g \in \mathscr{M}(K_{t+1}, M \times O^n)$$

with $d_{t+1}(g, \mathrm{id}) < \delta$ there is a $\beta \in \bigcap_{0 \le r \le s-1} G_r^t$ with composition $g \circ \beta$ defined on K_t, $d_t(g \circ \beta, \mathrm{id}) < \varepsilon$ and $\Phi_s \circ g \circ \beta(x_s) \in W_s$, i.e., *using the chart Φ_s, $g \circ \beta(x_s)$ has no non-trivial V_s coordinates.*

Proof. Since the vector fields in $\mathscr{H}_s$ span V_s, an integral curve construction gives an open neighborhood $0 \in B \subset \mathbf{C}^\alpha$, $\alpha = \dim V_s$, and a holomorphic map $\psi\colon B \times K_t \to M_{z_1}$ with $\psi(0, k) = k$ and $d\psi|_{(0,x_s)}$ mapping the tangent space to B onto V_s. Let $B_s = \psi(B \times \{x_s\})$ and let $H_s = \Phi_s^{-1}(W_s)$. Then B_s and H_s are clearly transverse to each other at x_s.

Note that ψ induces a map $\psi\colon B \to \bigcap_{0\le r\le s-1} G_r^t$. Now if $d_{t+1}(g, \mathrm{id})$ is small, then $g^{-1}(H_s)$ will still intersect B_s and there will be a $b \in B$ such that

$$g \circ \psi(b, x_s) \in H_s.$$

Clearly b approaches 0 as g approaches id. Choose δ small enough so that the map $\psi(b, \cdot)$ maps K_t into K_{t+1} (so composition with g is defined) and so that

$$d_t(\psi(b, \cdot), \mathrm{id}) < (\varepsilon - \delta).$$

Then for $\beta = \psi(b, \cdot)$,

$$d_t(g \circ \beta, \mathrm{id}) \le d_t(g \circ \beta, \beta) + d_t(\beta, \mathrm{id}) \le d_{t+1}(g, \mathrm{id}) + d_t(\beta, \mathrm{id}) < \varepsilon,$$

Q.E.D.

LEMMA 4.3. *Fix ε and s. Then there is a δ such that if $d_{2s}(g, \mathrm{id}) < \delta$, there are $\beta_t \in \bigcap_{0\le r\le t-1} G_r^{2s-t}$, $1 \le t \le s$, with the composition*

$$h = g \circ \beta_1 \circ \cdots \circ \beta_s$$

defined on K_s, $d_s(h, \mathrm{id}) < \varepsilon$ and $\Phi_r \circ h(x_r) \in W_r$ for $r \le s$. In particular if g preserves the foliation on $M \times O^n$, so will h.

Proof. This is an easy induction on Lemma 4.2, noting that $\beta_t(x_r) = x_r$ if $r < t$.

LEMMA 4.4. *There is a sequence of maps $h_{\omega_s} \in \mathscr{M}(K_s, M \times O^n)$ preserving the foliation, converging uniformly to the identity on compact sets, and satisfying $\Phi_r \circ h_{\omega_s}(x_r) \in W_r$ for $r \le s$.*

Proof. Apply Lemma 4.3 to a subsequence of $\{g_{\omega_s}\}$. Note that $h_{\omega_s}(K_s) \subset M_{\omega_s}$.

Now fix a number t and a finite number of open sets $\{U_\alpha\}$ in M_{z_1} so that

(a) $K_t \subset \bigcup_\alpha U_\alpha$ and

(b) for each α the closure $\bar{U}_\alpha$ is compact with $\bar{U}_\alpha \subset V_\alpha$ for some open $V_\alpha \subset M_{z_1}$, and there is a distinguished coordinate chart on some neighborhood of V_α in $M \times O^n$.

It will facilitate the following computation to write the distinguished coordinate functions as $(x_1^\alpha, \ldots, x_{n+m}^\alpha)$, making no distinction between coordinates which vary in the leaf or in the parameter space. Since $\{h_{\omega_s}\}$ converges uniformly to the identity on $\bar{U}_\alpha$, we can assume that $h_{\omega_s}(\bar{U}_\alpha)$ is contained in the domain of our chart. In this way we get a Čech zero cochain on the covering $\{U_\alpha\}$, $\xi_s \in \check{C}^0(\{U_\alpha\}, \mathcal{O}^{n+m})$ where $n + m = \dim M \times O^n$, $\mathcal{O}^{n+m}$ is the sheaf of germs of $\mathbf{C}^{n+m}$-valued holomorphic functions, and ξ_s^α is defined to be the map $h_{\omega_s} - \mathrm{id}$ written in local coordinates on U_α.

Define

$$\|\xi_s\|_t = \sup_{\alpha,\, x \in U_\alpha,\, 1 \le i \le m+n} \{|\xi_s^{i,\alpha}(x)|\} \text{ where } \xi_s^\alpha = (\xi_s^{1,\alpha}, \ldots, \xi_s^{n+m,\alpha}).$$

Note that $0 < \|\xi_s\|_t < \infty$ so we can define $\zeta_s \in \check{C}^0(\{U_\alpha\}, \mathcal{O}^{n+m})$ by

$$\zeta_s^\alpha = \|\xi_s\|_t^{-1} \cdot \xi_s^\alpha.$$

It follows that $\|\zeta_s\|_t = 1$, so that, by Vitali's Theorem, there is a subsequence of $\{\zeta_s\}$ (which we will also call $\{\zeta_s\}$) with ζ_s^α converging uniformly to some ζ^α on $\bar{U}_\alpha$. Let $\zeta \in \check{C}^0(\{U_\alpha\}, \mathcal{O}^{n+m})$ be this limit cochain.

Now the ith coordinate of ζ_s^α is given at some point $x \in U_\alpha \cap U_\beta$ by

$$\begin{aligned}\zeta_s^{i,\alpha}(x) &= [x_\alpha^i(h_{\omega_s}(x)) - x_i^\alpha(x)]\|\xi_s\|_t^{-1} \\ &= \|\xi_s\|_t^{-1} \\ &\quad \times \left\{\sum_j \partial x_i^\alpha/\partial x_j^\beta \,|_x[x_j^\beta(h_{\omega_s}(x)) - x_j^\beta(x)] + \text{higher order terms}\right\}\end{aligned}$$

where the higher order terms involve products of the $[x_j^\beta(h_{\omega_s}(x)) - x_j^\beta(x)]$. Since

$$\|\xi_s\|_t \geq |x_j^\beta(h_{\omega_s}(x)) - x_j^\beta(x)|$$

it follows that the higher order terms vanish in the limit and that $\zeta^\alpha = (\zeta^{1,\alpha}, \ldots, \zeta^{n+m,\alpha})$ where

$$\zeta^{i,\alpha} = \sum \partial x_i^\alpha/\partial x_j^\beta \zeta^{j,\beta}.$$

Thus the zero cochain ζ transforms like a vector field and determines a holomorphic vector field Z in $T(M \times O^n)$ defined over K_t.

We claim that $Z \neq 0$. For each ζ_s there is a point $y_s \in \bigcup_\alpha \bar{U}_\alpha$ with $|\zeta_s^{i,\alpha}(y_s)| = 1$ for some value of i and α. (Note that ζ_s^α is actually defined on $\bar{U}_\alpha$, not just U_α.) By the compactness of $\bigcup_\alpha \bar{U}_\alpha$ there is a subsequence of $\{y_s\}$ converging to $y \in \bigcup_\alpha \bar{U}_\alpha$. It follows that $\zeta^\alpha(y) \neq 0$ for all α with $y \in \bar{U}_\alpha$, hence $Z(y) \neq 0$.

Furthermore we claim that $[Z, X]$ is a leaf vector field whenever X is. To show this we relabel our distinguished coordinates as $(x_i^\alpha, y_j^\alpha, z_k^\alpha)$ using the convention given in the last section. Since h_{ω_s} preserves the foliation and $z_k^\alpha \circ h_{\omega_s}$ is the kth coordinate of $\omega_s - z_1$,

$$\partial/\partial x_i^\alpha(y_j^\alpha \circ h_{\omega_s}) = 0 \quad \text{and} \quad \partial/\partial x_i^\alpha(z_k^\alpha \circ h_{\omega_s}) = \partial/\partial y_j^\alpha(z_k^\alpha \circ h_{\omega_s}) = 0.$$

It follows that $Z = \sum a_i \,\partial/\partial x_i^\alpha + \sum b_j \,\partial/\partial y_j^\alpha + \sum c_k \,\partial/\partial z_k^\alpha$ where the holomorphic functions b_j are constant on leaves of the foliation and c_k are constant on all of U_α. This is sufficient to prove the claim.

LEMMA 4.5. *The vector field Z extends to a holomorphic vector field on all of M_{z_1} satisfying $[Z, X]$ is a leaf vector field whenever X is.*

Proof. The construction we have given for Z on K_t works as well on K_{t+1}.

On U_α

$$\begin{aligned} Z_{t+1} &= \lim_{s\to\infty} \|\xi_s\|_{t+1}^{-1} \xi_s^\alpha \\ &= \lim_{s\to\infty} \left(\frac{\|\xi_s\|_t}{\|\xi_s\|_{t+1}}\right) \|\xi_s\|_t^{-1} \xi_s^\alpha \\ &= \left(\lim_{s\to\infty} \frac{\|\xi_s\|_t}{\|\xi_s\|_{t+1}}\right) Z_t \end{aligned}$$

and since both Z_{t+1} and Z_t are non-zero they must be multiples of each other. This shows that Z extends to K_{t+1} satisfying the lemma. By induction we can extend Z to all of $M_{z_1} = \bigcup_1^\infty K_t$.

It remains only to show that Z is nowhere tangent to the leaves of the foliation on M_{z_1}. For this it suffices to show that $d\pi(Z) \neq 0$ for $\pi\colon M \times O^n \to O^n$. Note that

$$d\pi(Z) = \sum c_k \,\partial/\partial z_k = \lim_{s\to\infty} (\omega_s - z_1)\|\xi_s\|_t^{-1}.$$

Recall the sequence $\{x_s\}$ used in the construction of $\{h_{\omega_s}\}$.

LEMMA 4.6. $d\pi(Z) \neq 0$.

Proof. We first show that if $d\pi(Z) = 0$, then $Z(x_s) = 0$ for all s.

If $d\pi(Z) = 0$ then $Z \in H^0(M_{z_1}, \theta_{z_1})$. Recall that

$$\mathscr{G}_s = \{\eta \in H^0(M_{z_1}, \theta_{z_1}) \mid \eta(x_s) = 0\}$$

and $\mathscr{H}_s$ is a complement to $\mathscr{G}_0 \cap \cdots \cap \mathscr{G}_s$ in $\mathscr{G}_0 \cap \cdots \cap \mathscr{G}_{s-1}$.

Note that $Z(x_t)$ is tangent to the sequence of points $\{h_{\omega_s}(x_t)\}$ converging to x_t. Since h_{ω_s} was constructed so that $\Phi_s \circ h_{\omega_s}(x_t) \in W_t$ for $t \leq s$, it follows that $Z(x_t) \in W_t$ for all t. Thus $Z(x_t) \in V_t$ if and only if $Z(x_t) = 0$, by Lemma 4.4.

Now $Z \in H^0(M_{z_1}, \theta_{z_1}) = \mathscr{G}_0$ so assume inductively that $Z \in \mathscr{G}_0 \cap \cdots \cap \mathscr{G}_{s-1}$. Then $Z = X + Y$ for $X \in \mathscr{G}_0 \cap \cdots \cap \mathscr{G}_s$ and $Y \in \mathscr{H}_s$. We have

$$Z(x_s) = X(x_s) + Y(x_s) = 0 + Y(x_s) \in V_s$$

since $Y \in \mathscr{H}_s$. Thus $Z(x_s) = 0$, and $Z \in \mathscr{G}_s$ for all s.

Since the set $\{x_s\}$ is dense in the interior of K_1, and Z is holomorphic, if Z vanishes on $\{x_s\}$, Z must vanish identically on all of M_{z_1}. But as noted in the construction of Z, $Z \neq 0$, so $Z(x_s) \neq 0$ and hence $d\pi(Z) \neq 0$, Q.E.D.

This completes the construction of Z and all its properties and so completes the proof of Theorem 4.1.

Remarks. Actually we have shown that the Kodaira-Spencer invariant $\rho_{z_1}(d\pi(Z)) = 0$, since the zero cochain

$$X^\alpha = \sum c_k \,\partial/\partial z_k^\alpha - Z,$$

for $d\pi(Z) = \sum c_k \, \partial/\partial z_k$, will bound the one cocycle for $\rho_{z_1}(d\pi(Z))$. The only place where Theorem 4.1 uses the non-vanishing of characteristic classes, as opposed to the Kodaira-Spencer invariant, is in the fact that characteristic classes vary continuously and are non-zero on open subsets of the parameter space.

The proof of Theorem 4.1 does not generalize to the case of C^∞ n-foliations because Vitali's Theorem that locally bounded families of holomorphic functions are normal does not hold for C^∞ functions. Thus, in the C^∞ case, there is no way to extract a convergent subsequence of the zero cochains $\{\zeta_s\}$. At this time we do not know if a C^∞ version of this theorem is true.

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