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THE LATTICE OF GROUPS CONTAINING PSL(n,q) AND ACTING ON GRASSMANNIANS

BY

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Section 1

We consider here the set Ω of all subspaces of a fixed dimension inside a vector space. This set is technically called a Grassmannian. The special linear group has a natural representation on Ω, which we will show to be essentially maximal inside the symmetric group on Ω. More precisely, we have the following terminology and result.

Let V be an n-dimensional vector space over a finite field with q elements. Let $\Omega = \Omega(V,k)$ be the set of all k-dimensional subspaces of V. Then $P\Gamma L(n,q)$ has a faithful natural representation on $\Omega(n,k)$, which we will denote by $G_o = G_o(n,k)$. In the case $n = 2k$, (G_o,Ω) is permutation isomorphic to its dual, and we have natural graph automorphisms arising from the inverse transpose transformation. We define $\hat{G}_o = \langle G_o, j \rangle$ where j is any non-trivial graph automorphism of G_o. Observe that G_o has index 2 in $\hat{G}_o$, and all graph automorphisms are contained in $\hat{G}_o$. Let $S_o = S_o(n,k)$ be the representation of $PSL(n,q)$ on Ω. Denote by A_Ω the alternating group on Ω. Finally, let G be any subgroup of S_Ω containing S_o. We will prove:

THEOREM. *Suppose* $1 \leq k \leq n$ *and* $(n,k) \neq (2,1)$.

If $n \neq 2k$, *then* $G \subseteq G_o$ *or* $A_\Omega \subseteq G$.

If $n = 2k$, *then* $G \subseteq \hat{G}_o$ *or* $A_\Omega \subseteq G$.

There are questions concerning what occurs when we represent a Chevalley group on the cosets of a maximal parabolic subgroup. In particular, when is this group maximal in the alternating or symmetric group on these cosets? A maximal parabolic subgroup is maximal as a subgroup of its Chevalley group [9]. In the case of $PSL(n,q)$, the maximal parabolics fix k-dimensional subspaces for $1 \leq k < n$. Therefore the representation of S_o on Ω is primitive. In our case, it's very easy to prove this directly. As the idea of the proof is used in a later lemma, we include it further on in our introduction.

The cases $k = 1$, $n \geq 3$ have already been solved by Kantor and McDonough [7]. Considering the dual space of V, the cases $k = n - 1$ with

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$n \geq 3$ are also done. In particular, all cases when $n = 3$ are completed. This will be used as a starting point for a proof by induction on n.

We will be considering two groups, T and H, where T is generated by a certain group of projective transvections in S_o, and H is the centralizer of T in G. The group H has been introduced to deal with special difficulties arising in the $k = 2$ case. In Section 2, we find key information about the structure of $N_G(T)$ and $N_G(H)$.

Continuing, in Section 3, we show T and H to be almost weakly closed in their Sylow p-subgroups. Finally, in Section 4, G is shown to preserve the relation $\{(\alpha,\beta) \mid \alpha,\beta \in \Omega \text{ and } \dim(\alpha \cap \beta) = k-1\}$. Chow [2] and Dieudonné [3] have used this relation to characterize $P\Gamma L(n,q)$ acting on $\Omega(V,k)$ in such a way as to give a generalization of the fundamental theorem of projective geometry. Using the result [2] or [3], our theorem follows immediately.

A proof of this theorem has been announced by V.A. Ustimenko-Bakumovskiĭ [10], but unfortunately contains serious errors and omissions.

At this point, we present a short, elementary proof on the primitivity of S_o.

Let $\alpha \in \Omega$. Define $\Delta_i(\alpha) = \{\beta \in \Omega \mid \dim(\alpha \cap \beta) = k-i+1\}$, $1 \leq i \leq k+1$. These $\Delta_i(\alpha)$ form the orbits of $(S_o)_\alpha$ on Ω.

LEMMA 1. *$S_o(n,k)$ is primitive on $\Omega(V,k)$ for all $1 \leq k < n$.*

Proof. Clearly $S_o(n,k)$ is transitive. Let Φ be a block of $S_o(n,k)$ with $|\Phi| \geq 2$. Then Φ contains α and β, where $\beta \in \Delta_i(\alpha)$ for some $i > 1$. Thus

$$\{\alpha\} \cup \beta^{(S_o)_\alpha} = \{\alpha\} \cup \Delta_i(\alpha) \subseteq \Phi.$$

By symmetry, $\Delta_i(\beta) \subseteq \Phi$. As an element of the projective geometry $P(V)$,

$$\alpha = \alpha' + (\alpha \cap \beta) \quad \text{and} \quad \beta = \beta' + (\alpha \cap \beta)$$

where $\dim(\alpha') = \dim(\beta') = i-1 \geq 1$.

Let $\alpha_1 \in \Omega(\alpha',1)$ and $\zeta \in \Omega(V,i-2)$, where $\zeta \cap \alpha = \zeta \cap (\alpha_1 + \beta) = 0$. This makes sense as $i-2 \leq k-1$. Then $\gamma = (\alpha \cap \beta) + \alpha_1 + \zeta \in \Delta_i(\beta)$, so $\gamma \in \Phi$. Since $\gamma \in \Delta_{i-1}(\alpha)$ also, $\Delta_{i-1}(\alpha) \subseteq \Phi$.

Suppose $i \neq k+1$. Thus $\dim(\alpha \cap \beta) = k-i+1 \geq 1$. Let

$$\beta_1 \in \Omega(\beta',1), \quad \xi \in \Omega(\alpha \cap \beta, k-i) \quad \text{and} \quad \eta \in \Omega(V,i-1),$$

where $\eta \cap (\alpha + \beta_1) = \eta \cap \beta = 0$. (Here $i-1 \leq k-1$.) Then $\delta = \xi + \beta_1 + \eta \in \Delta_i(\beta)$, so $\delta \in \Phi$. Since $\delta \in \Delta_{i+1}(\alpha)$ also, $\Delta_{i+1}(\alpha) \subseteq \Phi$.

Continuing in this manner, we can show $\Phi = \Omega$.

Notation and terminology. For each element $f \in \mathrm{Hom}(V,F_q)$ and $v \in f^{-1}(0)$ there corresponds a transvection $t_{f,v} : x \to x + f(x)v$, where $x \in V - \{0\}$. Let W be a hyperplane of V, and $T = T(W)$ be the group generated by the projective transvections fixing W. Then T is an elementary abelian p-group stabilizing the chain $V \supset W \supset 0$ for some prime p, and $|T| = |W| = q^{n-1}$.

Let Δ be the support of T on Ω, and Γ the fixed point set of T, so that

$\Gamma = \Omega(W,k)$. For $\omega \in \Omega(W,k-1)$, we define $\psi_\omega = \{\alpha \in \Delta \mid \alpha \cap W = \omega\}$. Clearly, ψ_ω is an orbit of T on $\Omega(V,k)$, and $|\psi_\omega| = |W/\omega| = q^{n-k}$. We shall use $\overline{\Delta}$ for the set $\{\psi_\omega \mid \omega \in \Omega(W,k-1)\}$ of orbits of T on Δ.

We introduce the notation

$$\left[{r \atop s}\right] = \prod_{i=o}^{s-1} \frac{q^{r-i}-1}{q^{i+1}-1} \text{ and } \left[{r \atop o}\right] = 1, \text{ where } r \geq s \geq 0; \left[{r \atop s}\right] = 0 \text{ if } s < 0.$$

Thus

$$|\Omega| = \left[{n \atop k}\right], |\Gamma| = \left[{n-1 \atop k}\right], |\overline{\Delta}| = \left[{n-1 \atop k-1}\right] \text{ and } |\Delta| = q^{n-k}\left[{n-1 \atop k-1}\right].$$

Our proof of the theorem is by induction on n. Therefore, in all the remaining lemmas, we assume that the theorem holds for every vector space of dimension less than n. Because of the result of Kantor and McDonough, we also assume $k \neq 1$, $n-1$ and $n \geq 4$. In addition, we suppose that $A_\Omega \not\subseteq G$.

Section 2

We need the following result in order to prove Lemma 2.1: If $x \geq 25$, then there is a prime r such that $x < r < 6x/5$[5].

LEMMA 2.1. *Let N be any subgroup of G containing $N_{S_o}(T)$ and having Δ and Γ as orbits, with $\overline{\Delta}$ as a set of blocks. Then:*

(i) *either*

$$N^{\overline{\Delta}} \subseteq N^{G_o}(T)^{\overline{\Delta}} \cong P\Gamma L(n-1,q)$$

or

$$n = 2k-1 \quad \textit{and} \quad N^{\overline{\Delta}} \subseteq (N_{\hat{G}_o}(T)^{\overline{\Delta}}),$$

and

(ii) *either*

$$N^{\Gamma} \subseteq N_{G_o}(T)^{\Gamma} \cong P\Gamma L(n-1,q)$$

or

$$n = 2k+1 \quad \textit{and} \quad N^{\Gamma} \subseteq (N_{\hat{G}_o}(T)^{\Gamma}).$$

Proof. Since $N_{S_o}(T) \subseteq N$, it follows that $N_{S_o}(T)^{\overline{\Delta}} \subseteq N^{\overline{\Delta}}$ and $N_{S_o}(T)^{\Gamma} \subseteq N^{\Gamma}$. We observe that $(N_{S_o}(T)^{\overline{\Delta}})'$ acts like $S_o(n-1,k-1)$ on $\Omega(W,k-1)$, and $(N_{S_o}(T)^{\Gamma})'$ like $S_o(n-1,k)$ on $\Omega(W,k)$. Thus we can apply our inductive assumption.

(A) Suppose $A_{\overline{\Delta}} \subseteq N^{\overline{\Delta}}$. We now show that $A_{\overline{\Delta}} \subseteq N_{\Gamma}^{\overline{\Delta}}$ also.

If not, N^{Γ} has $A_{\overline{\Delta}}$ as a composition factor. Using induction to find the possibilities for N^Γ, this can happen only if $|\Gamma| = |\bar{\Delta}|$, that is, $n = 2k$. Then $(N^\Gamma)' \cong (N^{\bar{\Delta}})' \cong (N^{\Gamma\cup\bar{\Delta}})' \cong A_{\bar{\Delta}}$. Let p_1, p_2 be projections of $(N^{\Gamma\cup\bar{\Delta}})'$ onto $(N^\Gamma)'$ and $(N^{\bar{\Delta}})'$, respectively. Thus $\phi = p_2 \circ p_1^{-1} : (N^\Gamma)' \to (N^{\bar{\Delta}})'$ is an isomorphism. As $|\Gamma| > 6$, we know that ϕ is a permutation isomorphism. Therefore, for α in Γ, the subgroup $\phi((N^\Gamma)'_\alpha)$ fixes a point $\bar{\beta}$ in $\bar{\Delta}$. In particular, $\phi(N_{S_0}(T)^\Gamma_\alpha)$ fixes $\bar{\beta}$ also, and we have a contradiction. Hence, $A_{\bar{\Delta}} \subseteq N^{\bar{\Delta}}_\Gamma$. As $|\bar{\Delta}| > 2$, this means that $N^{\bar{\Delta}}_\Gamma$ is transitive.

Since T is transitive on each ψ_ω, and $T \subseteq N_{\Gamma\cup\bar{\Delta}}$, it follows that N_Γ is transitive on Δ. As G is primitive, by 13.1 of [11], G must be doubly-transitive. Therefore 15.1 of [11] holds, i.e.,

$$(*) \qquad m \geq \frac{|\Omega|}{3} - \frac{2\sqrt{|\Omega|}}{3}, \text{ where } m \text{ is the minimal degree of } G.$$

Case (i). $k > 2$. Let h be a p-element of N_Γ whose image in $N^{\bar{\Delta}}_\Gamma$ is a pq^{n-k}-cycle, and let $g = h^{q^{n-k}}$. Then g moves only $(pq^{n-k})q^{n-k}$ points. Since

$$|\sup(g)| < \frac{|\Omega|}{3} - \frac{2\sqrt{|\Omega|}}{3},$$

this contradicts (*).

Case (ii). $k = 2$. Let L be the kernel of the homomorphism $N \to N^{\bar{\Delta}}_\Gamma$.

(a) Assume $q^{n-2} \geq 100$. Let h be an element of N_Γ whose image in $N^{\bar{\Delta}}_\Gamma$ is an r-cycle, where r is a prime such that $|\psi_\omega|/4 < r < 3|\psi_\omega|/10$. If L has no elements of order r, then $|\sup(h^{|L|})| \leq rq^{n-2}$, contradicting (*) as before. Thus, we assume that L has an element of order r. Since $T \subseteq L$, the set of nontrivial orbits of L is $\bar{\Delta}$. Hence each L^{ψ_ω} has an element, say g_ω, of order r. Our g_ω consists of at most 3 r-cycles because $4r > |\psi_\omega|$.

Suppose L^{ψ_ω} is imprimitive. Let θ be a non-trivial block of L^{ψ_ω}. As $|\theta| \mid |\psi_\omega|$, we have $p \leq |\theta| \leq |\psi_\omega|/p$. Choose θ to contain a point α in $\sup(g_\omega)$. Suppose $\theta \not\subseteq \sup(g_\omega)$. Let $\beta \in \theta \cap \mathrm{fix}(g_\omega)$. Clearly $\{\beta\} \cap \alpha^{<g_\omega>} \subseteq \theta$, and so $|\theta| \geq 1 + r$. As $|\psi_\omega| < 4r$, we must have $|\theta| = |\psi_\omega|/p$, where p is 2 or 3. Next, suppose $\theta \subseteq \sup(g_\omega)$. In addition, assume $|\theta| > 3$. Then θ contains two points of an r-cycle of g_ω. As r is a prime, it contains the entire r-cycle, and again $|\theta| = \psi_\omega|/p$. Combining all possibilities, we have $|\theta| = p$ or $|\psi_\omega|/p$, where $p = 2,3$. Hence L^{ψ_ω} is contained in

$$S_p \text{ wr } S_{|\psi_\omega|p} \quad \text{or} \quad S_{|\psi_\omega|p} \text{ wr } S_p,$$

and has a composition factor which acts primitively on a set of degree $|\psi_\omega|/p$ and contains an r-cycle. Since $r < 3\psi_\omega/10$ and $|\psi_\omega| \geq 100$, we have $r + 3 \leq |\theta|$. Then we can use 13.9 of [11] to show that L^{ψ_ω} has $A_{|\psi_\omega|p}$ as a composition factor. Thus $m \leq 6|\bar{\Delta}|$ or $15|\bar{\Delta}|$ for $p = 2$ or 3, respectively. But this contradicts (*). We conclude that L^{ψ_ω} is primitive.

If we consider g_ω again and 13.10 of [11], we must have $A_{\psi_\omega} \subseteq L^{\psi_\omega}$. Since $q^{n-2} \neq 4$ or 6, any non-trivial homomorphism $(L^{\psi_\omega})' \to (L^{\psi_{\omega'}})'$ is a permutation isomorphism. This means if $g|_{\psi_\omega}$ is a 3-cycle, then $g|_{\psi_{\omega'}}$ is a 3-cycle or the identity element. Thus $m \leq 3|\bar{\Delta}|$, a contradiction.

(b) For the cases $q^{n-2} = 25,27,32,49,64$ and 81, let r be 7,11,17,13,17 and 23, respectively. We easily obtain contradictions as above. Now consider the remaining cases: $q^{n-2} = 4,8,9$ and 16. Define s to be 5,13,11 and 19, respectively. Clearly $N_{\Gamma}^{\bar{\Delta}}$ has an element of order s. Let h be a pre-image of this element in N_{Γ}, and $g = h^{|L|}$. Thus Inn(g) acts on each L^{ψ_ω} and has order s. All groups of degrees 4,8,9 and 16 are known, and in every case $s \notin \pi(L^{\psi_\omega})$. Thus g normalizes a Sylow subgroup P_t for each prime $t \in \pi(L^{\psi_\omega})$. Let $t^u \| \ |L^{\psi_\omega}|$ for some integer u, so $\mathrm{Aut}(P_t/\Phi(P_t))$ is a subgroup of $GL(u,t)$. Since $s \nmid |GL(u,t)|$, it follows that g centralizes $P_t/\Phi(P_t)$. By a theorem of Burnside (5.1.4) [4], g centralizes P_t. This is true for all t, thus g centralizes L^{ψ_ω}, for each ω. Hence g centralizes L, and therefore T. We can choose $t \in T$ so that $\sup(g) \neq \sup(t)$, and consequently $t^g \neq t$, a contradiction. Therefore (i) holds.

(B) Suppose $A_\Gamma \subseteq N^\Gamma$. Then N^Γ does not have $PSL(n-1,q)$ as a composition factor, and so $A_\Gamma \subseteq N_{\bar{\Delta}}^{\Gamma}$. As $|\Gamma| > |\psi_\omega|$, we have $A_\Gamma \subseteq N_\Delta^\Gamma$ and a contradiction by 13.5 of [11]. Hence (ii) holds as well.

Let v be a non-zero vector in V-W, and define $T' = T'(v)$ to be the group generated by the projective transvections fixing $<v>$. Then T' is the elementary abelian p-group of order q^{n-1} stabilizing the chain $V \supset <v> \supset 0$.

Further, let $\Delta' = \sup(T')$ and $\Gamma' = \mathrm{fix}(T') = \{\alpha \in \Omega(V,k) \mid <v> \subset \alpha\}$. For each $\alpha \in \Omega(W,k)$, we define $\varrho_\alpha = \alpha^{T'}$. Then $\{\varrho_\alpha \mid \alpha \in \Omega(W,k)\}$ is the set $\overline{\Delta'}$ of orbits of T' on Δ'. Clearly $|\varrho_\alpha| = |\mathrm{Hom}(\alpha, <v>)| = q^k$. Note that

$$|\Gamma'| = [{}^{n-1}_{k-1}],\ |\overline{\Delta'}| = [{}^{n-1}_{k}] \text{ and } |\Delta'| = q^k[{}^{n-1}_{k}].$$

Since $N_{S_0}(T')$ on $\Omega(V,k)$ is permutation isomorphic to $N_{S_0}(T)$ on $\Omega(V,n-k)$, we have the following result.

Corollary 2.2. *Let M be any subgroup of G containing $N_{S_0}(T')$ and having Δ' and Γ' as orbits, with $\overline{\Delta'}$ as a set of blocks. Then*

(i) *either*

$$M^{\overline{\Delta}'} \subseteq N_{G_0}(T')^{\overline{\Delta}'} \cong P\Gamma L(n-1,q)$$

or

$$n = 2k+1 \quad \textit{and} \quad M^{\overline{\Delta}'} \subseteq (N_{\hat{G}_0}(T')^{\overline{\Delta}'}),$$

and

(ii) *either*

$$M^{\Gamma'} \subseteq N_{G_0}(T')^{\Gamma'} \cong P\Gamma L(n-1,q)$$

or

$$n = 2k-1 \quad \textit{and} \quad M^{\Gamma'} \subseteq (N_{\hat{G}_0}(T')^{\Gamma'}).$$

Let X be any group acting on a set Ω, and Λ some subset of Ω. Define $X_{[\Lambda]}$ to be the largest subgroup of X fixing Λ as a set.

LEMMA 2.3. *If J is the largest subgroup of G having Δ' and Γ' as orbits, then*

$$J^{\Gamma'} \subseteq N_{G_0}(T')^{\Gamma'} = G_{0[\Gamma']}^{\Gamma'} \cong P\Gamma L(n-1,q)$$

or

$$n = 2k-1 \quad \textit{and} \quad J^{\Gamma'} \subseteq (G_{0[\Gamma']}^{\hat{}\,\Gamma'}).$$

Proof. Define M as in Corollary 2.2. Then $M \subseteq J$, and so $N_{S_0}(T')^{\Gamma'} \subseteq J^{\Gamma'}$. If the lemma does not hold, then, by induction, $A_{\Gamma'} \subseteq J^{\Gamma'}$.

(A) Suppose $J^{\Delta'}$ is imprimitive. By Corollary 2.2 some α in $\Omega(W,k)$ belongs to a non-trivial block σ distinct from ϱ_α. Since $M \subseteq J$, it follows that σ is a block of $M^{\Delta'}$ also, and so is contained in ϱ_α. As σ is non-trivial, there is a β in $\sigma - \{\alpha\}$. Define $Y = N_{S_0}(T')_\alpha$. Then $Y \subseteq J_\alpha$ and Y maps β to each point in $\varrho_\alpha - \{\alpha\}$. Thus $\sigma \supseteq \varrho_\alpha$, a contradiction.

(B) Suppose $J^{\Delta'}$ is primitive. Define R to be $J_{\Gamma'}$. Since $R \lhd J$ and $T' \subseteq R$, it follows that $R^{\Delta'}$ is transitive.

Case (i). Suppose $R^{\Delta'}$ is imprimitive. Let τ be a non-trivial block of $R^{\Delta'}$ of minimal length. As $R \lhd J$, we know that τ^g is a block of R for each $g \in J$, and, in particular, for each $g \in N_{S_0}(T')$. This means that $\tau \cap \tau^g$ is a block as well. Since τ is non-trivial and of minimal length, $|\tau \cap \tau^g| > 1$ implies that $\tau = \tau^g$.

Now suppose α and β are points of τ in distinct T' orbits on Δ'. We may assume $\alpha \in \Omega(W,k)$.

(a) We claim that we can choose β in $\Omega(W,k)$ also.

Now $\beta \in \varrho_\gamma$ for some $\gamma \neq \alpha$, $\gamma \in \Omega(W,k)$. If $\alpha \cap \gamma = 0$, then $\beta^{T'_\alpha} = \varrho_\gamma$; so in this case, take $\beta = \gamma$. It remains to consider the situation where

$$\ell = \dim(\alpha \cap \gamma) \geq 1.$$

Let

$$\alpha = (\alpha \cap \gamma) + \alpha' \quad \text{and} \quad \gamma = (\alpha \cap \gamma) + \gamma',$$

where $\dim(\alpha') = \dim(\gamma') = k - \ell \geq 1$. Considering $\beta^{T'_\alpha}$ again, we may choose β so that $\gamma' \subseteq \beta$. Suppose $\beta \notin \Omega(W,k)$. Then $\beta = \beta' + \gamma'$, where dim $\beta' = \ell$ and $\beta' \subseteq \alpha \cap \gamma + \langle v \rangle$. Thus $\beta' = \delta + \beta_1$, where $\delta \in \Omega(\alpha \cap \gamma, \ell - 1)$ and $\beta_1 = \langle w + av \rangle$ for some $w \in (\alpha \cap \gamma) - \delta$ and $\alpha \in F_q^{\#}$. We have $\beta = \delta + \beta_1 + \gamma'$.

Let U be the set stabilizer in $N_{S_0}(T')$ of the subspaces $\alpha' + \langle v \rangle$, δ, $\langle w \rangle$ and β_1 of V. As

$$\tau \cap \varrho_\alpha \supseteq S(\gamma) = \{(\alpha \cap \gamma) + \zeta \in \Omega \mid v \notin \zeta \subseteq (\alpha' + \langle v \rangle)\},$$

we have $\tau^u = \tau$ for each $u \in U$. But this means there is an $\epsilon = \beta_1 + \delta + \epsilon'$ in τ, where $\epsilon' \in \Omega(W - (\alpha \cup \gamma),\ k - l)$. Clearly we have a $t \in T' \subseteq J$ such that $\beta^t, \epsilon^t \in \tau^t \cap W$. Replacing (α,β) by (β^t, ϵ^t), we are done.

(b) We claim that $\tau = \Delta'$.

Let α and β be distinct points in $\tau \cap \Omega(W,k)$. Since

$$\tau \cap \varrho_\alpha \supseteq S(\beta) = \{(\alpha \cap \beta) + \zeta \in \Omega \mid v \notin \zeta \subseteq (\alpha' + <v>)\},$$

we have

$$\tau^g = \tau \quad \text{for each } g \in N_{S_0}(T')_{\alpha \cup <v>}$$

and similarly for each $g \in N_{S_0}(T')_{\beta \cup <v>}$. Clearly $\dim(\alpha \cap \beta) = k - i + 1$ for some i, where $2 \leq i \leq k' = \min\{k+1, n/2\}$. Thus

$$\Delta_i(\alpha) \cap \Omega(W,k) \subseteq \tau,$$

and, by symmetry,

$$\Delta_i(\beta) \cap \Omega(W,k) \subseteq \tau$$

also. We proceed as in Lemma 1, to find points

$$\gamma \in \Delta_{i-1}(\alpha) \cap \Delta_i(\beta) \cap \Omega(W,k)$$

and (if $i \neq k'$)

$$\delta \in \Delta_{i+1}(\alpha) \cap \Delta_i(\beta) \cap \Omega(W,k).$$

Continuing, this shows $\Omega(W,k) \subseteq \tau$. Next let α be any point in $\Omega(W,k)$. Choose β in $\Omega(W,k)$ so that $\alpha \cap \beta = 0$. Then $\alpha^{T'_\beta} = \varrho_\alpha \subseteq \tau$, and so $\Delta' \subseteq \tau$, contradicting $R^{\Delta'}$ imprimitive.

We must have $\tau \subseteq \varrho_\alpha$ for some α, hence $|\tau| \leq q^k$. Since $R^{\Delta'} = J_{\Gamma'}^{\Delta'}$ is transitive and G is primitive, by 13.1 of [11], G must be doubly-transitive. Let α and β be distinct elements of Γ'. As $R = J_{\Gamma'}$ is transitive on Δ' and $A_{\Gamma'} \subseteq J^{\Gamma'}$, either $G_{\alpha\beta}$ has orbits of length $|\Gamma'| - 2$ and $|\Delta'|$ or G is triply-transitive. Suppose G_α is imprimitive. If β and γ are distinct points of a non-trivial block σ on $\Omega - \{\alpha\}$, then $\gamma^{G_{\alpha\beta}} \subseteq \sigma$. Since $|\Delta'| > |\Omega|/2$, the G_α blocks have length $|\Gamma'| - 1$. But $R \subseteq G_\alpha$ and $|\Gamma'| - 1 > q^k$. Hence G_α is primitive. We continue with other points in Γ' to obtain the conclusion that G is at least $|\Gamma'| - q^k + 1$ transitive.

Case (ii). $R^{\Gamma'}$ primitive. By an argument as in the above paragraph, G is at least $|\Gamma'|$ – transitive.

We now combine these two cases with a well-known transitivity formula. (See p. 21 of [11].) If we define t to be the degree of transitivity of G, then

$$t < 3 \ln |\Omega|.$$

Observe that $|\Omega| \leq |\Gamma'|^2$. Hence we obtain $|\Gamma'| - q^k + 1 < 6 \ln |\Gamma'|$. This leads to a contradiction in all cases except those when either $(n,k,q) = (5,2,2)$ or $(n,k) = (4,2)$ and $q \leq 43$. Except for the $n/2 = k = q = 2$ case, our transitivity is so large that we can produce a prime s which divides $|G|$ and also satisfies $|\Omega| - t < s < |\Omega| - 2$ unless $n = 4$ and $q = 5,8$ or 13, in which case $|\Omega| - t < 2s < |\Omega| - 2$. For $n/2 = k = q = 2$, if we consider $|\Delta'|$, then G has an element consisting of four 7-cycles. In all cases, we have a contradiction to a well-known theorem (13.10 in [11]) constraining elements of prime order in a primitive group. Thus $A_{\Gamma'} \nsubseteq J^{\Gamma'}$, and our result follows.

Let K be the kernal of the homomorphism $N_G(T) \to N_G(T)^{\overline{\Delta}}$.

LEMMA 2.4. *Let Q be a Sylow p-subgroup of K_Γ. Then, for each ψ_ω in $\overline{\Delta}$, we have $Q^{\psi_\omega} = T^{\psi_\omega}$. If $k > 2$, then $Q = T$.*

Proof. For a non-zero element x of W, set

$$\Gamma_x' = \{\alpha \in \Omega(V,k) \,|\, x \in \alpha\}.$$

Define $Y_x = G_{[\Gamma_x']}$. By Lemma 2.3, using x in place of v, we know that

$$Y_x^{\Gamma_x'} \subseteq G_{o_{[\Gamma_x']}}^{\Gamma_x'} \cong P\Gamma L(n-1,q).$$

Set $\Delta_x = \Delta \cap \Gamma'_x$. Then

$$\Delta_x = \{\alpha \in \Omega - \Omega(W,k) \,|\, x \in \alpha\} = \cup\{\psi_\omega \,|\, \omega \in \Omega(W,k-1) \text{ and } x \in \omega\}.$$

Any subgroup of K (thus K_Γ) leaves each ψ_ω in $\overline{\Delta}$ invariant. Hence $K_\Gamma \subseteq Y_x$. As $T \subseteq K_\Gamma$, we have $Q^{\Delta_x} = T^{\Delta_x}$. In particular, $Q^{\psi_\omega} = T^{\psi_\omega}$ for each ω in $\Omega(W,k-1)$ which contains x.

Now assume $k > 2$. Suppose $Q \supset T$ and take $h \in Q^\#$. Then $h|_{\Delta_x} \in T^{\Delta_x}$ for all $x \in W$. Thus $h|_{\Delta_x} = 1|_{\Delta_x}$ or $h|_{\Delta_x} = t_{g(x)}|_{\Delta_x}$ where $g(x) \in W - \{0\}$ and $t_{g(x)}$ is a projective transvection in T with $(t_{g(x)} - 1)V = \langle g(x)\rangle$. Therefore $h|_{\Delta_x} = 1|_{\Delta_x}$ if and only if $\langle g(x)\rangle = \langle x\rangle$. Now suppose that $h \notin T$. As $h \neq 1$, there is a $u \in W - \{0\}$ such that $h|_{\Delta_u} = t_{g(u)}|_{\Delta_u} \neq 1|_{\Delta_u}$. Replacing h by $ht_{g(u)}^{-1}$, we may assume that $h|_{\Delta_u} = 1|_{\Delta_u}$. Since $h \notin T$, there is a $v \in W - \langle u\rangle$ such that $h|_{\Delta_v} = t_{g(v)}|_{\Delta_v} \neq 1|_{\Delta_v}$. We take $\alpha \in \Delta_u \cap \Delta_v$. It follows that $\alpha = \alpha^h = \alpha^{t_{g(v)}}$, so $g(v) \in \alpha$. Indeed, $g(v)$ belongs to every α in $\Delta_u \cap \Delta_v$. Now,

$$\Delta_u \cap \Delta_v = \{\alpha \in \Omega \,|\, \langle u,v\rangle \subset \alpha \nsubseteq W\},$$

and $\langle u,v\rangle$ is the intersection of the elements of $\Delta_u \cap \Delta_v$ considered as subspaces of V. Thus $g(v) \in \langle u,v\rangle$, and so $g(v) = au + bv$, where $a,b \in F_q$ and $a \neq 0$. Replacing h by $ht_{g(au)}^{-1}$, we may assume $h|_{\Delta_u} = 1|_{\Delta_u}$ and $h|_{\Delta_v} = 1|_{\Delta_v}$.

Since $h \notin T$, there is a $y \in W - (\langle u\rangle \cup \langle v\rangle)$ such that $h|_{\Delta_y} \neq 1|_{\Delta_y}$. Suppose

$$y \in W - \langle u,v\rangle.$$

Using the pairs (u,y) and (v,y) as we did with (u,v), we obtain

$$h|_{\Delta_y} = t_{g(y)}|_{\Delta_y} \quad \text{where } g(y) \in \langle u,y\rangle \cap \langle v,y\rangle = \langle y\rangle.$$

Thus $h|_{\Delta_y} = 1|_{\Delta_y}$, a contradiction. Thus $y \in \langle u,v\rangle - (\langle u\rangle \cup \langle v\rangle)$. We note that u,y and any element in $W - \langle u,y\rangle$ are linearly independent. Therefore $h|_{\Delta_y} = 1|_{\Delta_y}$ here as well. We have a contradiction. Hence $Q = T$.

We define H to be $C_G(T)$. Then $H \lhd N_G(T)$. Observe that $N_G(T)$ satisfies the conditions for N in Lemma 2.1. Thus, if $H^{\bar{\Delta}} \neq 1$, then $(N_{S_o}(T)^{\bar{\Delta}})' \subseteq H^{\bar{\Delta}}$. But then H does not centralize T. Hence $H \subseteq K$. As T^{ψ_ω} is regular and abelian, $H^{\psi_\omega} = T^{\psi_\omega}$ for all $\omega \in \Omega(W,k-1)$.

For the following lemma, we will need another well-known result from number theory:

Let a,b,x and y be positive integers, with $x \neq 2$. There is a prime which divides $a^x - b^x$ and, for every $y < x$, does not divide $a^y - b^y$, with the single exception $2^6 - 1$ [1].

LEMMA 2.5. (i) $K^\Gamma = 1$.
(ii) $Q = H$.
(iii) *If $k > 2$ and P is any Sylow p-subgroup of G normalizing T, then $P \subseteq N_{G_o}(T)$.*

Proof. Let Q be defined as in Lemma 2.4. For all $\omega \in \Omega(W,k-1)$, we have shown $Q^{\psi_\omega} = T^{\psi_\omega}$. Thus Q is an elementary abelian p-group. We note that $N_G(T)$ satisfies the conditions for N in Lemma 2.1, and $K \lhd N_G(T)$. Suppose that (i) does not hold. By Lemma 2.1 (ii), K^Γ contains $PSL(n-1,q)$ as a composition factor.

Now suppose K^Δ has $PSL(n-1,q)$ as a composition factor also. Since $PSL(n-1,q)$ is simple, so does each K^{ψ_ω}. As $K \subseteq N_G(T)$, we must have $T^{\psi_\omega} \lhd K^{\psi_\omega}$. Since T^{ψ_ω} is regular and abelian, it is its own centralizer in K^{ψ_ω}. Thus $K^{\psi_\omega}/T^{\psi_\omega}$ is isomorphic to a subgroup of $GL(n-k)r,p)$, where $q = p^r$. If we assume $(n,q) \neq (7,2)$ or $(4,4)$, there is a prime dividing $q^{n-1} - 1$ and not dividing $p^i - 1$ for $i < r(n-1)$. That is, this prime divides $|PSL(n-1,q)|$ but not $|K^{\psi_\omega}|$, a contradiction. For the two exceptional cases mentioned above, a higher power of 3 divides $|PSL(n-1,q)|$ than $|K^{\psi_\omega}|$, also a contradiction. Hence K^Δ cannot have $PSL(n-1,q)$ as a composition factor.

We conclude $K_\Delta^\Gamma \neq 1$. As $(N_{S_o}(T)^\Gamma)'$ is primitive, so is $N_G(T)^\Gamma$. Thus K_Δ is transitive on Γ. Since $|\Gamma| < \Omega/2$ and G is primitive, we must have $A_\Omega \subseteq G$ by 13.5 of [11], a contradiction. Hence (i) holds.

As $H \subseteq K$, we immediately obtain $Q = H$, so (ii) holds.

If $k > 2$, then $H = T$ and (iii) follows.

Section 3.

If P is a Sylow p-subgroup of G_o, we define W_o to be the 1-dimensional subspace fixed by all elements of P. We let T^* be the group generated by all projective transvections fixing W_o, so that T^* is a contragredient of T. If $n = 2k$, a graph automorphism maps T to T^*.

LEMMA 3.1. *Let $k > 2$. Let P be a Sylow p-subgroup of G containing T and T^*. Suppose g is an element of G such that $T^g \subseteq P$. If $n \neq 2k$, then $T^g = T$, and if $n = 2k$, then $T^g = T$ or T^*.*

Proof. Suppose $T \neq T^g = U$, where g is in G, and T and U in P. By Lemma 2.1 of [8], we may assume that T and U normalize each other. In particular, $U \subseteq N_G(T)$. By Lemma 2.5 (iii), we know $U \subseteq N_{G_o}(T)$.

Since $k > 2$, T is its own centralizer, and consequently $[T,U] \neq 1$. As $[T,U] \subseteq T \cap U$, we must have $T \cap U \neq 1$. Let $t \in (T \cap U)^\#$. Then t is a transvection with $(t-1)V = \langle w \rangle \subseteq W$ for some $w \in W^\#$.

Since T is the sole Sylow p-subgroup of K, it follows that $U \not\subseteq K$. Hence $\sup(U) \cap \Gamma \neq \emptyset$. As $U = T^g$, clearly U has exponent p. Since the Sylow p-subgroups of G_o/S_o are cyclic and $U \subseteq G_o$, we see that U contains a subgroup Y of index at most p in U such that $Y = S_o \cap U$. Both T and U have orbits of length $q^{n-k} > p$, hence $\sup(Y) \cap \Gamma \neq \emptyset$ also. As $Y \subseteq N_{S_o}(T)$, each $g \in Y - T$ acts non-trivially on $W = \mathrm{fix}(T)$. Then

$$|\mathrm{fix}(\langle t,g \rangle)^\Gamma)| = |\mathrm{fix}(g|_\Gamma| \leq \genfrac{[}{]}{0pt}{}{n-2}{k}, \tag{3.1}$$

since g fixes a subspace of W of codimension at least one.

Case (i). As Y normalizes T, Y leaves W invariant. Hence $(g-1)V \subseteq W$ for $g \in Y$. Suppose $(g-1)V \neq (t-1)V$ for some $t \in (T \cap U)^\#$ and $g \in Y - T$. Let

$$\ell = \dim((t-1)V + (g-1)V).$$

Then $\ell \geq 2$, and $\langle t,g \rangle$ is a subgroup of U of order p^2. Each element of Δ which is fixed by $\langle t,g \rangle$ either contains $(t-1)V + (g-1)V$ or contains $(t-1)V$ and no vectors of W moved by g. Since g moves at least one vector in W, we obtain (respectively)

$$\begin{aligned} |\mathrm{fix}(\langle t,g \rangle)^\Delta| &\leq q^{n-k}\left(\genfrac{[}{]}{0pt}{}{n-\ell-1}{k-\ell-1} + \genfrac{[}{]}{0pt}{}{n-3}{k-2}\right) \\ &\leq q^{n-k}\left(\genfrac{[}{]}{0pt}{}{n-3}{k-3} + \genfrac{[}{]}{0pt}{}{n-3}{k-2}\right) \end{aligned} \tag{3.2}$$

Then (3.1) and (3.2) give an upper bound for $|\mathrm{fix}(\langle t,g \rangle|$.

Now, let S be a subgroup of T of order p^2 having two nontrivial elements with distinct supports. Then $|\mathrm{fix}(S)| = q^{n-k}\genfrac{[}{]}{0pt}{}{n-3}{k-3} + \genfrac{[}{]}{0pt}{}{n-1}{k}$. Thus

$$|\mathrm{fix}\langle t,g \rangle| < |\mathrm{fix}(S)|.$$

But as $U = T^g$, U and T are permutation isomorphic. We have a contradiction.

Case (ii). It remains to consider the case that

$$(t-1)V = (g-1)V \text{ for all } t \in (T \cap U)^\# \text{ and } g \in Y.$$

Set $W_o' = (t-1)V$. Then $\dim W_o' = \ell = 1$. Consequently g is a transvection on V. Furthermore, each t in T has the form $t = t_{f,w} : v \to v + f(v)w$ where $w \in W$ and $f \in \mathrm{Hom}(V, F_q)$, $f(w) = 0$. So $T \cap U \subseteq \langle t_{f,w} \mid w \in W_o' \rangle$.

On the other hand, Y consists of transvections of the form t_{f,w_o} where

$f \in \mathrm{Hom}(V,F_q)$, $f(w_o) = 0$ and $<w_o> = W_o'$. Since $|Y| \geq q^{n-1}/p$, we must have $W_o = W_o'$. Also, as $|U/Y| \leq p$ and U is abelian, U cannot contain a field automorphism. Thus $U = Y = <t_{f,w_o} | f(w_o) = 0> = T^*$ as required. In addition, $n = 2k$ since T and T^* are permutation isomorphic.

If we inspect the proof of Lemma 3.1 for the case $k = 2$, we obtain

(i) $T^g \subseteq H$,
(ii) $n = 4$ and $T^g \subseteq T^*H$,
or
(iii) $n > 4$ and $|\mathrm{fix}<g,t>| \leq [{}^{n-2}_{\ 2}] + q^{n-2} < |\Delta|$,
where (iii) leads to a contradiction.

In the next lemma, we deal with the $k = 2$ case, which means using H in place of T. Our analog to T^* is $H^* = C_G(T^*)$.

We note that $N_G(H)$ satisfies the conditions for Lemmas 2.1–2.3. We define L to be the kernel of the homomorphism $N_G(H) \twoheadrightarrow N_G(H)^{\overline{\Delta}}$. Using the first paragraph of the proof of Lemma 2.4 and the proof of Lemma 2.5 (i) and (ii), we find that $L^\Gamma = 1$ and H is the Sylow p-subgroup of L.

LEMMA 3.2. *Assume $k = 2$. Let P be a Sylow p-subgroup of G containing H and H^*, and let $H^g \subseteq P$ for some $g \in G$. If $n \neq 4$, then $H^g = H$, and if $n = 4$, then $H^g = H$ or H^*.*

Proof. Let $U = H^g \subseteq P$ for some $g \in G$, and assume $U \neq H$. By Lemma 2.1 of [8], we may assume $[H,U] \subseteq H \cap U$, and in particular, $U \subseteq N_G(H)$.

Since H is Sylow in $N_G(H)_\Gamma$, we must have $\sup(U) \cap \Gamma \neq \emptyset$. This means $n = 4$ and $T^g \subseteq T^*H$. Thus $U \subseteq T^*H$, and $\sup(U) = \sup(T^*)$ as $|\sup(U)| = |\sup(H)|$.

Clearly $|U^\Gamma| = q^2$, so $U \cap H$ has index q^2 in U and H. Since $U \cap H$ fixes an H-orbit ψ_ω in $\overline{\Delta}$ for some $\omega \in \Omega(W,1)$, we have $U \cap H = H_{\psi_\omega}$. We know that $N_{S_o}(T)$ normalizes H. If $H \supset T$, then $|H_{\psi_\omega}^{\psi_{\omega'}}| = q^2$ for each $\omega' \in \Omega(W,1) - \{\omega\}$. But this means that U has an orbit of length q^3, a contradiction as U and H are permutation isomorphic. Therefore $H = T$, and so $U = T^* = H^*$.

Section 4

We define $\theta_i = \{(\alpha,\beta) | \alpha,\beta \in \Omega \text{ and } \dim(\alpha \cap \beta) = k - i + 1\}$, where $1 \leq i \leq k+1$. These θ_i form the orbits of S_o, G_o and $\hat{G}_o$ on $\Omega \times \Omega$.

Let C be the largest subgroup of S_Ω preserving θ_2. Chow and Dieudonné have shown that $C = G_o$ if $n \neq 2k$ and $C = \hat{G}_o$ if $n = 2k$. Therefore we are essentially done once we show $G \subseteq C$. In the case where $n = 2k$, we define $\overline{G} = <\hat{G}_o, G>$. Clearly if $\overline{G} \subseteq C$, then $G \subseteq C$ also. Thus we can replace G by $\overline{G}$ when $n = 2k$ throughout this section.

LEMMA 4.1. *Assume $k > 2$. Let Λ be an orbit of G on $\Omega \times \Omega$. Then $N_G(T)$ is transitive on $\Lambda \cap (\Gamma \times \Gamma)$.*

Proof. Let $\zeta = (\alpha,\beta) \in \Lambda$. Assume that U is conjugate to T in G, and that T and U are subgroups of G_ζ. We wish to show that U is conjugate to T in G_ζ. There is a $g \in G_\zeta$ such that $T \cup U^g \subseteq R \subseteq P$, where R and P are Sylow p-subgroups of G_ζ and G, respectively. By Lemma 3.1, if $n \neq 2k$, we have $T = U^g$, so T is trivially conjugate to U in G_ζ. If $n = 2k$ and $T \neq U^g$, then there is a graph automorphism $h \in \hat{G}_o \subseteq G$ such that $T = U^{gh}$. We are done if we can choose h to be in G_ζ.

We observe that T and U^g are in S_o. Without loss of generality, we may let $P \cap S_o$ be lower triangular with respect to the basis $\{v_1, v_2, \ldots, v_n\}$, that is,

$$P(<v_i, v_{i+1}, \ldots, v_n>) \subseteq <v_{i+1}, v_{i+2}, \ldots, v_n>) \quad \text{for } i = 1,2,\ldots,n-1.$$

Thus $U^g = T^*$, $v_n \in \alpha \cap \beta$ and $\alpha,\beta \in \Gamma$. Clearly there is a graph automorphism y taking T to T'. (T' is described just before Corollary 2.2.) Furthermore, y can be chosen so that it has a "reverse action" on each point of Ω of the form $<v_{i_1}, \ldots, v_{i_k}>$. That is, y maps

$$<v_{i_1}, \ldots, v_{i_k}> \quad \text{to} \quad <v_{j_1}, \ldots, v_{j_k}>$$

where $v_{j_r} = v_{n+1-i_r}$ and $1 \leq i_r \leq n$. Now let s be the image in S_o of the involution of $SL(n,q)$ exchanging v_ℓ and $-v_{n+1-\ell}$ for $1 \leq \ell \leq n$. Then ys maps T to T^* and fixes all points of Ω of the form $<v_{i_1}, \ldots, v_{i_k}>$. In particular, it fixes two points γ, δ of this form in Γ where $\dim(\gamma \cap \delta) = t$ for each value of t such that $1 \leq t \leq k$. We let $\eta = (\gamma, \delta)$. Then there is an $x \in N_{S_o}(T)$ mapping η to ζ. We set $h^{-1} = (ys)^x$. Then $h \in G_\zeta$, and, as $T^* \lhd N_{S_o}(T)$, we have $T = U^{gh}$. Thus T is conjugate to U in G_ζ again.

We conclude, by 3.5 of [11], that $N_G(T)$ is transitive on $\Lambda \cap \mathrm{fix}(T) = \Lambda \cap (\Gamma \times \Gamma)$.

Using Lemma 3.2 in place of Lemma 3.1, we obtain the following result.

COROLLARY 4.2. *Assume $k = 2$ and $n \neq 4$. If Λ is an orbit of G on $\Omega \times \Omega$, then $N_G(H)$ is transitive on $\Lambda \cap (\Gamma \times \Gamma)$.*

LEMMA 4.3. *If $(n,k) \neq (4,2)$, then $G \subseteq C$.*

Proof. Suppose $G \not\subseteq C$. Since $S_o \subseteq G$, there is an S_o-orbit θ_i, where $i \neq 2$, and a G-orbit Λ such that

$$\theta_2 \cup \theta_i \subseteq \Lambda \subseteq \Omega \times \Omega.$$

Clearly $\theta_2 \cap (\Gamma \times \Gamma) \neq \emptyset$. If $(n,i) \neq (2k,k+1)$, then $\theta_i \cap (\Gamma \times \Gamma) \neq \emptyset$ also. In this case, by Lemma 4.1 and Corollary 4.2, θ_i and θ_1 fuse in $N_G(T)^{\Gamma\times\Gamma}$ if $k > 2$, and in $N^G(H)^{\Gamma\times\Gamma}$ if $k = 2$. Observe that both $N_G(T)$ and $N_G(H)$ satisfy the conditions of N in Lemma 2.1. Hence their constituents on Γ are contained in

$N_{G_o}(T)^{\Gamma}$, or, if $n = 2k+1$, in $(N_{G_o}^{\wedge}(T)^{\Gamma})$. Thus, we have a contradiction as no such fusion takes place. Hence $\Lambda = \theta_2 \cup \theta_{k+1}$ and $n = 2k$. By hypothesis $(n,k) \neq (4,2)$, so we must have $k > 2$.

We introduce Higman intersection numbers here [6]. Let $\alpha, \beta, \gamma \in \Omega$. Recall that $\Delta_i(\alpha) = \{\gamma \in \Omega \mid \dim(\alpha \cap \gamma) = k - i + 1\}$. We define

$$m_{j,r}^{i} = |\Delta_j(\alpha) \cap \Delta_i(\beta)|, \text{ where } \beta \in \Delta_r(\alpha).$$

Let i and r be any numbers except 2 and $k+1$. As $\Lambda = \theta_2 \cup \theta_{k+1}$, we must have $m_{2,r}^{i} = m_{k+1,r}^{i}$. In particular, $m_{2,3}^{3} = m_{k+1,3}^{3}$.

If $k > 4$, then $m_{2,3}^{3} \neq 0$ and $m_{k+1,3}^{3} = 0$, a contradiction.

Suppose $k = 4$. We take any point δ in Ω. Let $\zeta \in \Delta_2(\delta)$ and $\eta \in \Delta_{k+1}(\delta)$. Then

$$m_{2,3}^{3} \geq |D' + Z|_{D',Z} \quad \text{and} \quad m_{k+1,3}^{3} = |D + E|_{D,E},$$

where D', Z, D and E are 2-dimensional subspaces of $\delta \cap \zeta, V - (\delta + \zeta)$, δ and η, respectively. Since $n = 8$, we have $m_{2,3}^{3} > m_{k+1,3}^{3}$, another contradiction.

Suppose $k = 3$. Let $\delta = \langle v_1, v_2, v_3 \rangle$, and let

$$\zeta = \langle v_1, v_2, v_4 \rangle \in \Delta_2(\delta) \quad \text{and} \quad \eta = \langle v_4, v_5, v_6 \rangle \in \Delta_{k+1}(\delta).$$

Thus

$$\Delta_3(\delta) \cap \Delta_3(\zeta) \supseteq \Phi \cup \Xi,$$

where

$$\Phi = \{\langle u, x_1, x_2 \rangle \mid u \in \delta \cap \zeta = \langle v_1, v_2 \rangle, x_i \in \Omega - (\delta + \zeta)\},$$

$$\Xi = \{\langle u_1, u_2, y \rangle \mid u_1 \in \delta - \zeta, u_2 \in \zeta - \delta, y \in \Omega - (\delta + \zeta),$$

and

$$\Delta_3(\delta) \cap \Delta_3(\eta) \subseteq \{\langle y_1, y_2, x \rangle \mid y_1 \in \delta, y_2 \in \eta, x \in \Omega - (\delta \cup \eta \cup \theta)\},$$

where $\theta = \{\langle y_1 + a y_2 \rangle \mid a \in F_q^{\#}\}$. Then

$$m_{2,3}^{3} > \left[{2 \atop 1}\right]\left(\left[{6 \atop 2}\right] - \left[{4 \atop 2}\right]\right) + \left(\left[{3 \atop 1}\right] - \left[{2 \atop 1}\right]\right)^2\left(\left[{6 \atop 1}\right] - \left[{4 \atop 1}\right]\right) > \left[{3 \atop 1}\right]^2\left(\left[{6 \atop 1}\right] - 2\left[{3 \atop 1}\right] - (q-1)\right) \geq m_{k+1,3}^{3}.$$

We have another contradiction. Hence $G \subseteq C$.

LEMMA 4.4. *If $(n,k) = (4,2)$, then $G \subseteq C$.*

Proof. Since S_o is a rank 3 permutation group, if $G \not\subseteq C$, then G is doubly-transitive. Let $\alpha = \langle v_1, v_2 \rangle$, $\beta = \langle v_2, v_4 \rangle$ and $\gamma = \langle v_3, v_4 \rangle$. Since G is 2-transitive, $G_{\alpha\gamma}$ and $G_{\beta\gamma}$ have orbits of the same lengths on Ω. Now we compare the size of $(S_o)_{\alpha\gamma}$-orbits and $(S_o)_{\beta\gamma}$-orbits on Ω. We find that either (i) G is triply-transitive or (ii) $q = 2$ and the one-point stabilizer of G is a rank 3 permutation group with subdegrees 1,9 and 24.

Case (i). G is triply-transitive. By a proof analogous to the one in Lemma 4.1 (with G acting on $\Omega \times \Omega \times \Omega$ this time), we can show that $N_G(H)$ is triply-transitive on $\text{fix}(H) = \Gamma$. But $N_G(H)^\Gamma \subseteq N_{G_o}(T)^\Gamma$, which is not triply-transitive, and we have a contradiction.

Case (ii). Here $q = 2$ and $G_{\beta\gamma}$ has orbits of lengths 9 and 24. We let $\delta = \langle v_2, v_3 \rangle$. Since $N_{G_o}(T)^\Gamma$ is not triply-transitive, if we list our $(S_o)_{\beta\gamma}$-orbits and compare sizes, then we find that δ belongs to the orbit of length 24. As $|G| = |\beta^G|\,|\gamma^{G_\beta}|\,|\delta^{G_{\beta\gamma}}|\,|G_{\beta\gamma\delta}|$, we see that 16 divides $|G : G_{\beta\gamma\delta}|$.

We are assuming that $\hat{G}_o \subseteq G$ by the remark preceding Lemma 4.1. In the proof of Lemma 4.1, we found an element ys of $\hat{G}_o$ that fixes β,γ and δ. Clearly, as this ys maps T to T^*, it maps $\text{fix}(T)$ to $\text{fix}(T^*)$, and so does not fix all points in Γ. Also $(ys)^2 \in N_G(T)$, so $|ys| = 2^\ell b$, where b and ℓ are positive integers, with b odd. Let $g = (ys)^b$ so g is a 2-element. Then $g^2 \in N_G(T)$, and as b is odd, g does not fix Γ also. Since g fixes β, γ and δ, this means $g \in (\hat{G}_o - G_o) \cap (G_{\beta\gamma\delta} - G_\Gamma)$. As $\hat{G}_o \subseteq G$ and $16 \mid |G : G_{\beta\gamma\delta}|$, we have $32 \mid |G : G_\Gamma|$. If we let P be a Sylow p-subgroup of G, as $H \subseteq G_\Gamma$, we see that $32|H| \mid |P|$.

Now we choose our P to contain H and H^*. If we consider the proof of Lemma 4.1 again, we note that P contains a subgroup X of index 2 normalizing H. In the remark preceding Lemma 3.2, we mentioned that H is Sylow in L and $L \subseteq G_\Gamma$. But, as $N_G(H)^{\bar{\Delta}} \cong PGL(3,2)$, we see that $|X| \mid 8|H|$. We have another contradiction. Hence $G \subseteq C$.

Conclusion

THEOREM. *Suppose* $1 \leq k < n$ *and* $(n,k) \neq (2,1)$. *If* $n \neq 2k$, *then* $G \subseteq G_o$ *or* $A_\Omega \subseteq G$. *If* $n = 2k$, *then* $G \subseteq \hat{G}_o$ *or* $A_\Omega \subseteq G$.

Proof. The proof is by induction on n. As we noted in the introduction, the case $n = 3$ is done.

Suppose the theorem holds for all cases $(n-1, i)$, where $1 \leq i \leq n-2$ and $n \geq 4$. If we assume $A_\Omega \not\subseteq G$, then, by Lemmas 4.3 and 4.4, we have $G \subseteq C$. By [2] and [3], it follows that $G \subseteq G_o$ if $n \neq 2k$ and $G \subseteq \hat{G}_o$ if $n = 2k$. Hence the theorem holds for (n,k), and we are done.

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