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TRANSFER AND INFINITE LOOP MAPS

BY

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1. Introduction

In what follows an infinite loop space will always mean the zeroth space of a fixed connective Ω-spectrum, while an infinite loop map between two infinite loop spaces is any map which can be de-looped infinitely many times (perhaps in more than one way). All spaces are assumed to be compactly generated.

Let V and W be infinite loop spaces and f: $V \to W$ an H-map (not necessarily an infinite loop map) and suppose that for every finite covering p: $E \to X$ the diagram

$$\begin{array}{ccc} [E;V] & \xrightarrow{f} & [E;W] \\ \Big\downarrow{\scriptstyle p_!^V} & & \Big\downarrow{\scriptstyle p_!^W} \\ [X;V] & \xrightarrow{f} & [X;W] \end{array}$$

is commutative, where $p_!^V$, $p_!^W$ denote the Kahn-Priddy transfer. In such a case we say that f commutes with the transfer. It is well known that an infinite loop map commutes with the transfer. The converse of this is in general false (see [7], [9], [15]) but it holds in a number of special cases (e.g., see [10]). In §2 we show that it holds always when V is an infinite loop space of the form $QX = \operatorname{ind}\lim_i \Omega^i \Sigma^i X$, for some connected X. This seems intuitively obvious but there appears to be no published proof. In §3 we use this result to show that if $V = BU$ and W has only finitely many non-zero homotopy groups, then some positive integer multiple of any transfer commuting map is an infinite loop map. We also show that for certain W's, e.g., products of Eilenberg-Mac Lane spaces, this holds for any H-map (not necessarily transfer commuting).

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2. Transfer commuting maps QX → Y

The principal reference for this section will be [11].

Recall that an E_∞ operad $C = \{C(j)\}_{j \geq 0}$ is a suitably compatible collection of contractible spaces $C(j)$ and maps

$$\gamma_n\colon C(n) \times C(j_1) \times \cdots \times C(j_n) \to C(j_1 + \cdots + j_n).$$

The symmetric group Σ_j acts freely on $C(j)$ so that the orbit spaces $C(j)/\Sigma_j$ are $K(\Sigma_j; 1)$'s. An action of an E_∞ operad C on a space X is a suitably compatible collection of maps $\theta_j\colon C(j) \times_{\Sigma_j} X^j \to X$. Such an action determines the Dyer-Lashof operations in $H_*(X; Z/p)$ and the Kahn-Priddy transfer on the functor $[\ ;X]$.

Now let X be any topological space and C an E_∞ operad. There is a space CX on which the operad C acts, defined as follows:

$$CX = \sqcup_j C(j) \times_{\Sigma_j} X^j/\sim\ ,$$

where $\sqcup$ denotes disjoint union and the equivalence relation $\sim$ is given by certain base point identifications. The action of C on CX is given by the maps $\Psi_n\colon C(n) \times_{\Sigma_n} (CX)^n \to CX$, where

$$\begin{aligned}\Psi_n\big(c_n, [c_{j_1}, x_1, \ldots, x_{j_1}], \ldots, [c_{j_n}, y_1, \ldots, y_{j_n}]\big)\\ = \big[\gamma_n(c_n, c_{j_1}, \ldots, c_{j_n}), x_1, \ldots, x_{j_1}, \ldots, y_1, \ldots, y_{j_n}\big],\end{aligned}$$

with γ_n as above.

Now let $QX = \operatorname{ind\,lim}_i \Omega^i \Sigma^i X$. QX is an infinite loop space, its homotopy groups are the stable homotopy groups of X, and it has the property $\mathrm{Inf}(QX; Y) \simeq \mathrm{Map}(X; Y)$, where Y is any infinite loop space and $\mathrm{Inf}(\ ;\ )$ denotes the set of infinite loop maps.

THEOREM 2.1. *Let X be a connected space and Y an infinite loop space of the homotopy type of a CW-complex. Let $f\colon QX \to Y$ be an H-map such that*

(i)

$$X \xrightarrow{i} QX \xrightarrow{f} Y$$

is null homotopic, where i is the natural inclusion of X in QX and

(ii) *f commutes with the transfer.*
Then $f \simeq 0$.

Remark. It is easy to see that Theorem 2.1 is equivalent to the statement that the transfer commuting maps $QX \to Y$ are precisely the infinite loop maps. For, given such a map f, let $f' = fi$, and let $\tilde{f}'$ be the map

$$QX \xrightarrow{Qf'} QY \xrightarrow{r} Y$$

where r is the natural retraction. Then the map $f - \tilde{f}'$ satisfies the conditions of the theorem and must therefore be null homotopic. Thus f is homotopic to an infinite loop map.

Proof of Theorem 2.1. Let C be the "little cubes" operad. Since C acts on any infinite loop space, it acts on Y. Since X is connected, the natural C map $CX \to QX$ is a weak equivalence. Hence we can replace QX by CX in the above. The space CX is built up by successive cofibrations

$$F_{k-1}CX \to F_kCX \to E_kCX,$$

where F_kCX denotes the image of $\sqcup_{j \le k} C(j) \times_{\Sigma_j} X^j$ in CX with the quotient topology and

$$E_kCX = E\Sigma_k \ltimes_{\Sigma_k} X^{[k]} = E\Sigma_k \times_{\Sigma_k} X^k / E\Sigma_k \times_{\Sigma_k} \mathrm{pt},$$

where $X^{[k]}$ is the k-fold smash product $X \wedge \cdots \wedge X$ and pt stands for the base point in $X^{[k]}$. The map f commutes with the transfer if and only if the indicated diagram is homotopy commutative,

$$\begin{array}{ccc} C(k) \times_{\Sigma_k} (CX)^k & \xrightarrow{1 \times f^k} & C(k) \times_{\Sigma_k} Y^k \\ \Big\downarrow{\scriptstyle \Psi_k} & & \Big\downarrow{\scriptstyle \theta_k} \\ CX & \xrightarrow{f} & Y \end{array},$$

where the vertical maps come from the action of C on CX and Y. This is well known; for example, see Theorem 4.2.1 of [3] or see [15]. Restricting the diagram to

$$X \xrightarrow{i} CX$$

we see that this implies that

$$\begin{tikzcd} C(k) \times_{\Sigma_k} X^k \arrow[r, "1\times(fi)^k"] \arrow[d, "\Psi_k(1\times i^k)"'] & C(k) \times_{\Sigma_k} Y^k \arrow[d, "\theta_k"] \\ CX \arrow[r, "f"] & Y \end{tikzcd}$$

is homotopy commutative. From the formula for the action of C on CX given above, we see that

$$\Psi_k(1 \times i^k)(c_k, x_1, \dots, x_k) = [c_k, x_1, \dots, x_k]$$
$$\text{– the equivalence class of } (c_k, x_1, \dots, x_k)$$

in CX. This is, in other words, the composite of the natural maps

$$C(k) \times_{\Sigma_k} X^k \to \coprod_j C(j) \times_{\Sigma_j} X^j \to \coprod_j C(j) \times_{\Sigma_j} X^j/\sim .$$

We shall denote this map by Φ_k. Since fi is null homotopic (by hypothesis), the homotopy commutativity of the diagram implies that $f\Phi_k$ is null homotopic. Let t denote the composite of the natural projections

$$C(k) \times_{\Sigma_k} X^k \xrightarrow{p'} C(k) \times_{\Sigma_k} X^{[k]} \xrightarrow{p''} C(k) \times_{\Sigma_k} X^{[k]}/C(k) \times \mathrm{pt} = E_k CX.$$

Since Y is an infinite loop space, $[\quad, Y]$ is the zeroth term of a cohomology theory. I shall denote this cohomology theory by $y^*(\quad)$, i.e., $y^i(X) = [X; B^iY]$, where B^iY is the i-th delooping of the given infinite loop structure on Y. We shall need the following result.

LEMMA 2.2. *Let* $y^*(\quad) = [\quad; B^*Y]$ *be as above. Then*

$$t^*: y^*(E_kCX) \to y^*\big(C(k) \times_{\Sigma_k} X^k\big)$$

is a split monomorphism.

A proof of this lemma will be given at the end of this section.

Consider the diagram

$$\begin{tikzcd}
 & C(k)\times_{\Sigma_k}X^k \arrow[d, "p"] \arrow[dr, "t"] & \\
F_{k-1}CX \arrow[r, "k"] & F_kCX \arrow[r, "j"] & E_kCX
\end{tikzcd}$$

where the horizontal line is a cofibration. Apply to this diagram the functor $y^0(\quad) = [\quad ; Y]$ to obtain the diagram

$$\begin{tikzcd}
 & y^0\big(C(k)\times_{\Sigma_k}X^k\big) & \\
y^0(F_{k-1}CX) & y^0(F_kCX) \arrow[l, "k^*"'] \arrow[u, "p^*"] & y^0(E_kCX) \arrow[l, "j^*"'] \arrow[ul, "t^*"']
\end{tikzcd}$$

Since the functor $y^*(\quad)$ takes cofibrations to exact sequences the horizontal line in this diagram is exact. Let $[f] \in y^0(CX)$ denote the homotopy class of f: $CX \to Y$ and let $[f]_k$ denote its image in $y^0(F_kCX)$. Assume inductively that $[f]_{k-1}$ is 0. This means that $k^*([f]_k) = 0$ and therefore there is some $h \in y^0(E_kCX)$ such that $j^*(h) = [f]_k$. Consider $t^*(h) = p^*([f]_k)$. It follows immediately from the definitions that this is $[f\Phi_k]$. But we have shown that this last is 0. Hence $[f]_k = 0$ for all k. Next, observe that since $y^*(\quad)$ is a cohomology theory we actually have a long exact sequence

$$\cdots \leftarrow y^1(F_{k-1}CX) \overset{k^*}{\leftarrow} y^1(F_kCX) \overset{j^*}{\leftarrow} y^1(E_kCX) \leftarrow y^0(F_{k-1}CX)$$
$$\overset{k^*}{\leftarrow} y^0(F_kCX) \overset{j^*}{\leftarrow} y^0(E_kCX) \leftarrow \cdots$$

The above argument shows that all j^* are injective, hence all k^* are surjective. Recall from [1] or [14] that we have the following Milnor exact sequence:

$$0 \to \lim_k{}^1 y^0(F_kCX) \to y^0(CX) \to \varprojlim_k y^0(F_kCX) \to 0.$$

Since all k^* are surjective the $\lim^1$ term is zero. Since each $[f]_k = 0$, $[f] = 0$.

Before proving Lemma 2.2 we state the following:

COROLLARY. *Let r: $QX \to X$ be the natural retraction, where X is an infinite loop space. Let Y be an infinite loop space of the homotopy type of a CW-complex. A map f: $X \to Y$ commutes with the transfer if and only if the composite fr: $QX \to Y$ is an infinite loop map.*

Proof. If f commutes with the transfer then so does fr since r, being an infinite loop map, commutes with the transfer. The result now follows from Theorem 2.1. Conversely, suppose that fr is an infinite loop map. Then it commutes with the transfer. Let $p\colon A \to B$ be a finite covering. Consider the diagram

$$\begin{array}{ccccc} [A; QX] & \xrightarrow{r} & [A; X] & \xrightarrow{f} & [A; Y] \\ \downarrow{\scriptstyle p!} & & \downarrow{\scriptstyle p!} & & \downarrow{\scriptstyle p!} \\ [B; QX] & \xrightarrow{r} & [B; X] & \xrightarrow{f} & [B; Y] \end{array} .$$

The left hand square is commutative since r commutes with the transfer. The rectangle is commutative by hypothesis. Since r is surjective the right hand square commutes completing the proof.

Remark 1. This can be interpreted as, in some sense, showing that commuting with the transfer is the only 'first order' homotopy obstruction to an H-map being an infinite loop map. In particular, a map which commutes with the Kahn-Priddy transfer must also commute with the Becker-Gottlieb transfer [4] for fibre bundles with compact fibre.

Remark 2. Again, let $\tilde{f}$ denote the composite

$$QX \xrightarrow{Qf} QY \xrightarrow{r} Y$$

and let fr be as above (where r denotes the retractions for X and Y). Then the homotopy class $[fr - \tilde{f}] \in [QX; Y]$ is the universal obstruction to f commuting with the transfer.

Proof of Lemma 2.2. For a based space X, let E_kX denote the space $E\Sigma_k \ltimes_{\Sigma_k} X^{[k]}$ for $k \geq 2$, X for $k = 1$ and S^0 for $k = 0$. Let $H\mathscr{S}$ be the stable category defined in [12, II]. J. P. May has proved the following:

THEOREM [13]. *There are functors $E_k\colon H\mathscr{S} \to H\mathscr{S}$ and natural isomorphisms of spectra $Q_\infty E_k X \cong E_k Q_\infty X$ for spaces X, where $Q_\infty Y$ is the free Ω-spectrum generated by the space Y, i.e., $Q_\infty^i Y = Q\Sigma^i Y$.*

Now, $E\Sigma_k \times_{\Sigma_k} X^k = E_k(X_+)$, where $X_+ = X \cup \{\text{base point}\}$ and the natural pointed map $\pi\colon X_+ \to X$ is stably split, in fact there is $s\colon \Sigma X \to \Sigma(X_+)$ such that

$$\Sigma X \xrightarrow{s} \Sigma(X_+) \xrightarrow{\Sigma\pi} \Sigma X$$

is the identity. Thus we have a splitting $Q_\infty(X_+) \leftrightarrows Q_\infty X$; hence

$$E_k Q_\infty(X_+) \leftrightarrows E_k Q_\infty X,$$

and by May's theorem,

$$Q_\infty E_k(X_+) \leftrightarrows Q_\infty E_k X.$$

Since for any space Y we have $y^*(Q_\infty Y) \doteq y^*(Y)$, the natural homomorphism $y^*(E_k X) \to y^*(E_k(X_+))$ is a split monomorphism. It remains to observe that we can take $C(k)$ as a model of $E\Sigma_k$, and the map $C(k) \times_{\Sigma_k} X^k \to E_k CX$ in Lemma 2.2 is just the map $E_k(X_+) \to E_k X$ above.

3. Riemann-Roch formulas

Let i: $CP^\infty \to BU$ be the natural inclusion and let $\tilde{i}$: $QCP^\infty \to BU$ be as before (where the infinite loop structure on BU is that corresponding to reduced connective K-theory). Recall the following results of Segal [16].

THEOREM 3.1. (a) *$\tilde{i}_Q$: $QCP^\infty \to BU_Q$ is a weak homotopy equivalence of infinite loop spaces, where X_Q denotes the rationalization of X.*
(b) *For any space X, the map $[X; QCP^\infty] \to [X; BU]$ is surjective.*

Theorems 2.1 and 3.1 together imply the following "splitting principle".

PROPOSITION 3.2. *Let Y be an infinite loop space of the homotopy type of a CW-complex. Any transfer commuting H-map f: $BU \to Y$ which is null homotopic on CP^∞ is null homotopic on BU.*

This "splitting principle" can also be proved directly using the method of [4], since one can show that any complex vector bundle over a finite complex X is the image under the Becker-Gottlieb transfer of a line bundle over a certain fibre bundle with compact fibre over X [6].

If Y is an Eilenberg-Mac Lane space it is easy to see that the classical splitting principle implies that Proposition 3.2 holds for any H-map (not necessarily transfer commuting). We shall say that an infinite loop space Y has the splitting property if Proposition 3.2 holds for all H-maps $BU \to Y$.

THEOREM 3.3. *Let Y be an infinite loop space of the homotopy type of a CW-complex, such that $\pi_n(Y) = 0$ for all $n \geq N$ for some positive integer N. Let f: $BU \to Y$ be a transfer commuting H-map. Then there is a positive integer M such that Mf: $BU \to Y$ is an infinite loop map. If Y has the splitting property this is true for any H-map.*

Proof. We again work in May's stable category of spectra $H\mathcal{S}$. For any infinite loop space X, let $\mathbf{X}$ denote the corresponding spectrum in $H\mathcal{S}$. Consider the morphism in $H\mathcal{S}$, $\tilde{\mathbf{i}}$: $Q_\infty CP^\infty \to \mathbf{BU}$, determined by the unique delooping of $\tilde{i}$. In $H\mathcal{S}$ we have the following cofibration (or fibration) sequence which extends in both directions since fiberings and cofiberings coincide in H.

$$\cdots \to F \xrightarrow{j} Q_\infty CP^\infty \xrightarrow{\mathbf{i}} \mathbf{BU} \xrightarrow{k} C \to \cdots$$

Applying to this sequence the functor [; Y] we obtain the long exact sequence

$$\cdots \leftarrow [F; \mathbf{Y}] \xleftarrow{j} [Q_\infty CP^\infty; \mathbf{Y}] \xleftarrow{\mathbf{i}} [\mathbf{BU}; \mathbf{Y}] \xleftarrow{k} [C; \mathbf{Y}] \leftarrow \cdots$$

We claim that under our hypothesis $[F; \mathbf{Y}]$ is a torsion group. If X is a finite CW-spectrum, then for any spectrum Y $[X; Y_Q] = [X; Y] \otimes Q$. In our case F is not a finite spectrum but the above still holds because F has finite homotopy groups (and hence can be replaced by a CW-spectrum with finite skeleta) and Y has only finitely many non-zero homotopy groups; more precisely, we can show that $[F; \mathbf{Y}] \simeq [F^N; \mathbf{Y}]$, where F^N is the N-skeleton of F. Now, apply the functor [; $\mathbf{Y}_Q$] to the cofibration sequence. Since $[X; Y_Q] = [X_Q; Y_Q]$ for any spectra X, Y, the map induced by $\mathbf{i}$ is an isomorphism by Theorem 3.1 a). By the exactness of the long exact sequence $[F; \mathbf{Y}_Q] = [F; \mathbf{Y}] \otimes Q = 0$. Hence $[F; \mathbf{Y}]$ is a torsion group. Let f: $BU \to Y$ be an H-map and let f' be the composite

$$CP^\infty \xrightarrow{i} BU \xrightarrow{f} Y.$$

Let $\tilde{f}'$: $QCP^\infty \to Y$ be as before. Since $\tilde{f}'$ is an infinite loop map, its unique delooping determines an element $\alpha \in [Q_\infty CP^\infty; \mathbf{Y}]$. Let $j(\alpha)$ have order M in $[F; \mathbf{Y}]$. Then $j(M\alpha) = 0$, hence there is $\beta \in [\mathbf{BU}; \mathbf{Y}]$ such that $\mathbf{i}(\beta) = M\alpha$. Let g: $BU \to Y$ be the zeroth map of a map of spectra representing β. Then g is an infinite loop map. We claim that $g \simeq Mf$. Clearly, $g \circ i \simeq M(f \circ i)$, i.e., $g \circ i - M(f \circ i) \simeq 0$. The result now follows from Proposition 3.2 if f is transfer commuting, and from the splitting property in the second case.

Remark. We are interested in two special cases which may be viewed as integral analogs of the classical (rational) Riemann-Roch formula. In each case the infinite loop space involved has the splitting property, which follows from the classical splitting principle.

(1) Let Y be an Eilenberg-Mac Lane space $K(Z; 2q)$ and let $f = s_q = q!\, ch_q$, where ch_q is the q-th Chern character. According to Theorem 3.3, there is an integer M_q such that $M_q s_q$ is an infinite loop map.

(2) Let $A = \{A_q\}_{q \geq 0}$ be a commutative graded ring. In [17] a connective cohomology theory $G^*(X; A)$ is constructed, for which $G^0(X; A)$ is the group of multiplicative units of the ring $\prod_{i \geq 0} H^{2i}(X; A_i)$ of the form $1 + a_1 + \cdots + a_i + \cdots$, where $a_i \in H^{2i}(X; A_i)$. Let $A = Z[[x]]$; the graded ring of formal power series in one variable x of degree 2; the total Chern class may be viewed as a natural homomorphism c: $\tilde{K}(X) \to G^0(X; A)$. One may ask if c extends to a transformation of cohomology theories. This is false; in [18] it is shown that c is not transfer commuting. In fact, one can show that there are no non-trivial transfer commuting natural homomorphisms $\tilde{K}(X) \to G^0(X; A)$ [8]. It is natural to consider "truncated" total Chern classes, i.e., natural homomorphisms

$$c_{(n)}: \tilde{K}(X) \to G^0(X; A[n]),$$

where $A[n] = Z[x]/x^n$ viewed as a graded ring. Since the infinite loop space representing $G^0(X; A[n])$ has only finitely many nonzero homotopy groups, Theorem 3.3 shows that for some integers N_n (depending on n), $c_{(n)}^{N_n}$ extends to a transformation of cohomology theories $\tilde{b}u^*(X) \to G^*(X; A[n])$, where $\tilde{b}u^*(X)$ denotes connective (reduced) K-theory. A suggestive way of restating this is to say that each truncated total Chern class can be made an infinite loop map by inverting a finite number of primes; to make the (non-truncated) total Chern class an infinite loop map all primes have to be inverted.

In the above examples the positive integers M_n and N_n can be taken to be the least ones for which the conclusion of Theorem 3.3 holds but they do not necessarily give the least positive integer multiples of s_n and $c_{(n)}$ which are transfer commuting. We can, however, give a similar interpretation to the least such integers $\mathscr{M}_n$ and $\mathscr{N}_n$. Let C be the mapping cone of the map $\tilde{i}$: $QCP^\infty \to BU$; i.e., we have a cofibration

$$QCP^\infty \xrightarrow{\tilde{i}} BU \to C.$$

Let Y be an infinite loop space with the splitting property and again let $y^*(\ )$ denote the cohomology theory given by the infinite loop structure on Y. We have a long exact sequence

$$\cdots \leftarrow y^1(C) \xleftarrow{j} y^0(QCP^\infty) \xleftarrow{\tilde{i}} y^0(BU) \xleftarrow{k} y^0(C) \leftarrow \cdots.$$

The homotopy class of $\tilde{f}'$ is an element α of $y^0(QCP^\infty)$. As above we can show that $y^1(C)$ is a torsion group. Let $\mathscr{M}$ be the order of the image of α in $y^1(C)$. Then there is some $\beta \in y^0(BU)$ with $i(\beta) = \alpha$. By the surjectivity of $\tilde{i}$ (Theorem 3.1(b)) we see, exactly as in the proof of the corollary to Theorem 2.1, that g: $BU \to Y$, representing β, is a transfer commuting H-map. It now follows by the splitting property of Y that $g \simeq \mathscr{M} f$; i.e., $\mathscr{M} f$ is transfer commuting. Clearly $\mathscr{M}$ is the least integer with this property (and a divisor of M of Theorem 3.3).

In the above examples: the positive integer M_n was computed by Adams [1], [2], $\mathscr{M}_n$ by Roush [15] and the present author [7] who has also computed the values of $\mathscr{N}_n$: $\mathscr{N}_1 = \mathscr{N}_2 = 1$, $\mathscr{N}_{n+1} > \mathscr{N}_n > 1$ for $n > 2$. (Strictly speaking the above mentioned articles consider only transformations from un-reduced K-theory, the reduced case is exactly analogous.)

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