ILLINOIS JOURNAL OF MATHEMATICS
Volume 38, Number 2, Summer 1994

CODIMENSION REDUCTION THEOREMS IN CONFORMAL GEOMETRY

GIULIANO ROMANI[1]

Introduction

We give the proof of the analogue, in conformal geometry, of the local version of the Theorem of Erbacher [E], and of theorems related to it.

The theorems obtained can also be viewed as the extension of classical theorems on reduction of codimension of a submanifold of a space of constant curvature to the case of a submanifold of a space locally conformal to a space of constant curvature that is locally conformally flat.

We start from the investigation of the geometric meaning of the nullity of the Willmore conformal forms, $\overset{r}{w}_M$, of a submanifold M. These conformal forms, introduced in [R], are invariant under conformal changes of the metric of the ambient space $\overline{M}$.

We prove:

THEOREM. *Let $\overline{M}$ be a locally conformally flat manifold. If M is a submanifold of $\overline{M}$ conformally nicely curved in $\overline{M}$, then $\overset{r}{w}_M$ is zero if and only if M is locally contained in a totally unbilical submanifold of $\overline{M}$ of dimension $p = \dim \overset{r+1}{\Omega}_x M$ ($\overset{r+1}{\Omega}_x M$, $(r+1)$-conformal osculating space).*

From the theorem, we deduce the local conformal version of the Erbacher Theorem (recently proved by Okomura [O] in the case of $\overline{M}$ of constant curvature).

COROLLARY III. *Let $\overline{M}$ be locally conformally flat and M a submanifold of $\overline{M}$. If $\overset{1}{W}_x M$ has constant dimension and is parallel in the normal bundle of M, then M is locally contained in a totally umbilical submanifold of $\overline{M}$ of dimension $p = \dim M + \dim \overset{1}{W}_x M$ ($\overset{1}{W}_x M$ first Willmore space).*

Introducing the notion of *conformally parallel distribution along a submanifold*, the analogue in conformal geometry of the notion of parallel distribution along a submanifold in Riemannian geometry, we are able to prove the

Received March 25, 1992.
1991 Mathematics Subject Classification. Primary 53A30; Secondary 53B25.
[1]Work supported by the M.U.R.S.T. 60% and 40%.

local conformal version of a well known theorem related with the theorem of Erbacher:

COROLLARY I. *Let $\overline{M}$ be locally conformally flat. If Δ is a distribution of $\overline{M}$, conformally parallel along a conformally nicely curved submanifold M of $\overline{M}$, then M is locally contained in a totally umbilical submanifold of $\overline{M}$ of dimension* $p \leq \dim \Delta$.

Corollary II is the following.

COROLLARY II. *Let M be a submanifold of a locally conformally flat manifold $\overline{M}$. If* $\dim \overset{k}{W}_x(M) = \text{const.}$, *for* $k = 1, \ldots, r$ *and if* $\overset{1}{W}M \oplus \cdots \oplus \overset{r}{W}M$ *is parallel in the normal bundle of M, then M is locally contained in a totally umbilical submanifold of $\overline{M}$ of dimension* $p = \dim \overset{r+1}{\Omega}_x M$, $x \in M$.

Obviously Corollary III is only Corollary II in the case $r = 1$, but we have pointed out Corollary III because it is the analog of the Theorem of Erbacher in its local version.

The results of [E] and [O] are global. Our results are local. Since all the spaces of constant curvature are locally conformally flat we have been able to obtain a unitary proof for the three different cases of positive, negative, zero curvature. These cases are treated separately in [E] and [O].

Acknowledgement. The definition of "Conformally parallel distribution," more geometric than my original one, was proposed by the referee. Using it I was able to simplify the proof of Lemma 5 of Section III. I want to thank the referee for this and other suggestions.

I. Preliminaries

Let $\overline{M}$ be a Riemannian manifold endowed with the metric $\bar{g}$.

If $\sigma: \overline{M} \to \mathbf{R}$ is a function of $\overline{M}$ the manifold $\overline{M}$, endowed with the metric $\bar{g}^* = e^{2\sigma}\bar{g}$, conformal with $\bar{g}$, will be denoted by $\overline{M}^*$. In a similar way the Levi-Civita connections of $\bar{g}$ and $\bar{g}^*$ will be denoted, respectively, by $\overline{\nabla}$ and $\overline{\nabla}^*$.

It is well known that the two connections $\overline{\nabla}$ and $\overline{\nabla}^*$ are related by the equality

$$(1) \qquad \overline{\nabla}^*_{X_x} Y = \overline{\nabla}_{X_x} Y + X_x(\sigma) Y_x + Y_x(\sigma) X_x - \bar{g}(X_x, Y_x)\left(\operatorname{grad}_{\bar{g}} \sigma\right)_x$$

where, $\operatorname{grad}_{\bar{g}}\sigma$, denote the gradient of σ taken with respect to the metric $\bar{g}$.

If M is a submanifold of $\overline{M}$, the second fundamental form of M will be denoted by $\overset{0}{s}_M$ and the mean curvature vector at $x \in M$ by $H_x(M)$. If M is considered as submanifold of $\overline{M}^*$ we shall denote M by M^* its second fundamental form by $\overset{0}{s}_{M^*}$ and the mean curvature vector at $x \in M^*$ by $H_x(M^*)$.

We denote by $\overset{k}{N}_x M$ $(k = 0, 1 \dots)$ the k-normal space of M at x (with the convention $\overset{0}{N}_x M = T_x M$) and by $\overset{k}{s}_M$ the k-fundamental form of M (for these notions and for related theorems see [S], volume IV, Chapter 7).

In [R] we have defined the *k-Willmore space*, $\overset{k}{W}_x M$, and the *k-Willmore form*, $\overset{k}{w}_M$, the analogue, in conformal geometry of submanifolds, of $\overset{k}{N}_M$ and $\overset{k}{s}_M$ respectively. Both are invariant under conformal changes of the metric $\bar{g}$.

The forms $\overset{k}{w}_M$ are defined by

$$\text{(2)} \qquad \overset{0}{w}_M(X_x Y_x) = \overset{0}{s}_M(X_x Y_x) - \bar{g}(X_x Y_x) H_x(M), \quad X_x, Y_x \in T_x M,$$

$$\text{(3)} \qquad \overset{k}{w}_M\left(X_x, \overset{k}{W}_x\right) = P_{(T_x M \oplus \overset{1}{W}_x M \oplus \cdots \oplus \overset{k}{W}_x M)} \perp \left(\overline{\nabla}_{X_x} \overset{k}{W}\right), \quad X_x \in T_x M,$$

The spaces $\overset{k}{W}_x M$ are those generated by the forms $\overset{k-1}{w}_M$ in $x \in M$.

Using the notion of *r-conformal osculating space* of M at $x, \overset{r}{\Omega}_x M = T_x M \oplus \overset{1}{W}_x M \oplus \cdots \oplus \overset{r-1}{W}_x M$ $(r = 1, \dots)$, we can write

$$\text{(4)} \qquad \overset{k}{w}_M\left(X_x, \overset{k}{W}_x\right) = P_{(\overset{k}{\Omega}_x M)} \perp \left(\overline{\nabla}_{X_x} \overset{k}{W}\right), \quad X_x \in T_x M, \quad \overset{k}{W} \in \Gamma\left(\overset{k}{W} M\right), \quad k > 0.$$

If for each $k = 1, \dots,$ the dimension of $\overset{k}{W}_x M$ is constant we say that M is *conformally nicely curved* in $\overline{M}$. Obviously that property is invariant under conformal changes of the metric in $\overline{M}$.

If M is conformally nicely curved we have proved [R] the so called *conformal Frenet equations of M*. Here we recall the property

$$\text{(5)} \quad \overline{\nabla}_{X_x} \overset{k}{W} \in T_x M + \overset{k-1}{W_x} M \oplus \overset{k}{W}_x M \oplus \overset{k-1}{W_x} M, \quad X_x \in T_x M, \overset{k}{W} \in \Gamma\left(\overset{k}{W} M\right)$$

(we assume $\overset{0}{W}_x M = T_x M$).

6. *Remark*. If, for an $x \in M$, $H_x(M) = 0$ then for each $k = 0, 1, \dots, \overset{k}{N}_x M = \overset{k}{W}_x M$ and $\overset{k}{s}_m = \overset{k}{w}_M$.

7. *Remark*. By definition a point $x \in M$ is umbilical if $\overset{0}{w}_M$ is zero in x.

The conformal invariance of $\overset{0}{w}_M$ implies that each totally umbilical submanifold M of $\overline{M}$ is changed in a totally umbilical submanifold by any conformal change of the metric $\bar{g}$.

8. PROPOSITION [S.S]. *For each point z of any submanifold M of $\overline{M}$ it is always possible to change the metric $\bar{g}$, around z, in a convenient one, $\bar{g}^*$, conformal to $\bar{g}$, in such a way that M, around z, is minimal with respect to $\bar{g}^*$.*

In particular from Remark 7 and Proposition 8 we can deduce:

9. PROPOSITION. *It is always possible, with a convenient conformal change of the metric $\bar{g}$, to change locally a totally umbilical submanifold M of $\overline{M}$, in a totally geodesic submanifold.*

Let l such that $\overset{l-1}{W} M \neq 0$ and $\overset{l}{W} M = 0$ or, equivalent, such that $\overset{l-2}{w}_M \neq 0$ and $\overset{l-1}{w}_M = 0$. The number $s = \dim(T_x M \oplus \overset{1}{w}_x M \oplus \cdots \oplus \overset{l-1}{W}_x M) = \dim(\overset{l}{\Omega}_x M)$ is called the *conformal number of the immersion $M \to \overline{M}$.*

Finally we recall the two theorems that we generalize to the case of conformal geometry.

10. THEOREM (Erbacher [E]). *Let $\overline{M}$ be a manifold of constant curvature. If M is a submanifold of $\overline{M}$ and the first normal space of M, $\overset{1}{N}_x M$, is of constant dimension, and is parallel in the normal bundle of M, then M is contained in a totally geodesic submanifold of $\overline{M}$ of dimension $p = \dim M + \dim \overset{1}{N}_x M$.*

11. THEOREM [S]. *Let $\overline{M}$ be a manifold of constant curvature. If Δ is a distribution of $\overline{M}$ defined along M, containing the tangent space of M and parallel along M, then M is contained in a totally geodesic submanifold of $\overline{M}$ of dimension $p = \dim \Delta$.*

Theorem 11 is stated also in [E] in a quite different version related with the higher normal spaces of M in $\overline{M}$.

II. Inverse part of the theorem

INVERSE PART OF THE THEOREM. *If $\overset{r}{w}_M = 0$ then M is locally contained in a totally umbilical submanifold of $\overline{M}$ of dimension $p = \dim \overset{r+1}{\Omega}_x M$.*

Before doing the proof we prove some lemmas and propositions.

For any fixed point of $z \in M$, Proposition 8 of Section I asserts that we can choose a metric $\bar{g}^* = e^{2\sigma}\bar{g}$ in $\overline{M}$ and a neighborhood U_z of z in M such that

$$H_x(M^*) = 0, \quad x \in U_z. \tag{1}$$

Denote $\overset{1}{W}_x M \oplus \cdots \oplus \overset{r}{W}_x M$ by $W_x^{(r)}M$.

Let $\overset{r}{M}{}^*$ be a submanifold of $\overline{M}{}^*$ which is a tubular neighborhood of U_z in the set of the points

$$y = \exp_x^* w_x, \quad x \in U_z,$$

where $\exp_x^*$ is the exponential of $\overline{M}{}^*$ at $x \in M$ and w_x has to be chosen in some neighborhood of the origin of $W_x^{(r)}M$. It is clear that for $x \in U_z$,

$$T_x\overset{r}{M}{}^* = T_x M \oplus W_x^{(r)}M = T_x M \oplus \overset{1}{W}_x M \oplus \cdots \oplus \overset{r}{W}_x M = \overset{r+1}{\Omega}_x M$$

2. Lemma. $\overset{0}{w}_{\overset{r}{M}{}^*_x} = 0$, $x \in U_z$.

Proof of Lemma 2. For each $x \in U_z$, $W_x^{(r)}M \perp T_x M$; moreover each vector of $W_x^{(r)}M$ can be extended in a tangent vector field of $\overset{r}{M}{}^*$, parallel in $\overline{M}{}^*$, along its direction. Consequently, denoting with X_α^* an orthonormal basis in $T_x M^*$ and with w_β^* an orthonormal basis in $W_x^{(r)}M^*$, for $x \in U_z$ we have

$$\begin{aligned}
H_x(\overset{r}{M}{}^*) &= \frac{1}{\dim \overset{r}{M}{}^*}\left[\sum_\alpha \overset{0}{s}_{\overset{r}{M}{}^*}(X_\alpha^*, X_\alpha^*) + \sum_\beta \overset{0}{s}_{\overset{r}{M}{}^*}(w_\beta^*, w_\beta^*)\right] \\
&= \frac{1}{\dim \overset{r}{M}{}^*}\sum_\alpha \overset{0}{s}_{\overset{r}{M}{}^*}(X_\alpha^*, X_\alpha^*) \\
&= \frac{1}{\dim \overset{r}{M}{}^*}\sum_\alpha P_{\perp_x \overset{r}{M}{}^*}\left(\overline{\nabla}{}^*_{X_\alpha^*} X_\alpha^*\right) \\
&= \frac{1}{\dim \overset{r}{M}{}^*} P_{\perp_x \overset{r}{M}{}^*}\left(\sum_\alpha \overset{M^*}{\nabla}_{X_\alpha^*} X_\alpha^* + \sum_\alpha \overset{0}{s}_{M^*}(X_\alpha^*, X_\alpha^*)\right) = 0
\end{aligned}$$

($\overset{M^*}{\nabla}$ connection on M^* induced by the connection $\overline{\nabla}{}^*$). In particular

$$\overset{0}{w}_{\overset{r}{M}{}^*} = \overset{0}{s}_{\overset{r}{M}{}^*} \quad \text{for } x \in U_z \tag{3}$$

and we can compute $\overset{0}{w}_{\overset{r}{M}{}^*}$ at the points $x \in U_z$, computing $\overset{0}{s}_{\overset{r}{M}{}^*}$.

As observed before, for $w_x \in W_x^{(r)}M$, $\overset{0}{s}_{\overset{r}{M}{}^*}(w_x, w_x) = 0$. Then since $\overset{0}{s}_{\overset{r}{M}{}^*}$ is bilinear and symmetric for $w_x, w_x' \in W_x^{(r)}M$,

$$\overset{0}{s}_{\overset{r}{M}{}^*}(w_x, w_x') = 0. \tag{4}$$

For $X_x \in T_xM$ and $w \in \Gamma(W^{(r)}M)$ from conformal Frenet equations of M^* in $\overline{M}^*$, (5) of Section I, and by hypothesis $\overset{r}{w}_M = 0$,

$$\overline{\nabla}^*_{X_x} w \in T_xM \oplus W^{(r)}_xM = T_x\overset{r}{M}{}^*$$

and then

$$\overset{0}{s}_{\overset{r}{M}{}^*}(X_x, w_x) = P_{\perp_x \overset{r}{M}{}^*}\left(\overline{\nabla}^*_{X_x} w\right) = 0 \tag{5}$$

If $X_x, Y_x \in T_xM$ from (1) and Remark 6 of Section I,

$$\overset{1}{N}_xM^* = \overset{1}{W}_xM^* = \overset{1}{W}_xM \subset T_x\overset{r}{M}{}^*$$

and then

$$\overset{0}{s}_{\overset{r}{M}{}^*}(X_x, Y_x) = P_{\perp_x \overset{r}{M}{}^*}\left(\overline{\nabla}^*_{X_x} Y\right) = P_{\perp_x \overset{r}{M}{}^*}\left(\overset{M^*}{\nabla}_{X_x} Y + \overset{0}{s}_{M^*}(X_x, Y_x)\right) = 0 \tag{6}$$

and Lemma 2 follows from (3), (4), (5), (6). □

Since $\overline{M}$ is locally conformally flat, it is locally conformal with the Euclidean space $\mathbf{E}^{\overline{m}}$, $\overline{m} = \dim \overline{M}$, with the canonic metric $\langle\ ,\ \rangle$.

We shall denote with $\overset{r}{M}{}^e$ and M^e, respectively, the submanifolds $\overset{r}{M}{}^*$ and M^* as submanifolds of $\overline{M}$ endowed with the metric $\langle\ ,\ \rangle$ of $\mathbf{E}^{\overline{m}}$. Lemma 2 and conformal invariance of the form $\overset{0}{w}$ allow us to assert the next result.

7. Proposition. *The points of U_z are umbilical points of $\overset{r}{M}{}^e$:*

$$\overset{0}{s}_{\overset{r}{M}{}^e}\left(\overset{r}{X}_x, \overset{r}{Y}_x\right) = \langle \overset{r}{X}_x, \overset{r}{Y}_x \rangle H_x(\overset{r}{M}{}^e), \quad x \in U_z. \qquad \square$$

For $m = \dim M \geq 2$, the proof of Lemma 25 of Volume IV of Spivak [S], works to prove the next result if we denote by D the standard connection of $\mathbf{E}^{\overline{m}}$:

8. Proposition. *$|H_x(\overset{r}{M}{}^e)|$ is constant on M and for $x \in U_z, X_x \in T_x\overset{r}{M}$,*

$$D_{\overset{r}{X}_x} H(\overset{r}{M}{}^e) = -\left|H_x(\overset{r}{M}{}^e)\right|^2 \overset{r}{X}_x \qquad \square$$

(If $\dim M = 1$ our theorems are obviously true because M is always totally umbilical.)

As an immediate consequence of Proposition 8 we have that $H_x(\overset{r}{M}{}^e)$ is always zero, or is always different from zero along U_z and then:

9. LEMMA. *The distribution along U_z,*

$$\Delta_x = T_x\overset{r}{M}{}^e \oplus \{H_x(\overset{r}{M}{}^e)\}\left(= \overset{r}{\Omega}_x M \oplus \{H_x(\overset{r}{M}{}^e)\}\right), \quad x \in U_z,$$

is of constant dimension.

Moreover we have

10. PROPOSITION. *The distribution along U_z,*

$$\Delta_x = T_x\overset{r}{M}{}^e \oplus \{H_x(\overset{r}{M}{}^e)\}\left(= \overset{r}{\Omega}_x M \oplus \{H(\overset{r}{M}{}^e)\}\right), \quad x \in U_z,$$

is parallel in $\mathbf{E}^{\overline{m}}$. In particular M is contained in a totally geodesic submanifold of $\mathbf{E}^{\overline{m}}$ with dimension $q = \dim \Delta_x$.

Proof of Proposition 10. Immediate from Proposition 7, Proposition 8, and Theorem 11 of Section I. □

And now we are able to prove the inverse part of the theorem.

If $H_x(\overset{r}{M}{}^e)$ is identically zero for $x \in U_z$ we deduce from Proposition 10 that U_z is contained in a totally geodesic submanifold of $\mathbf{E}^{\overline{m}}$ of dimension $p = \dim \overset{r}{M}{}^e = \dim \overset{r+1}{\Omega}_x$ and then U_z is contained in a totally umbilical submanifold of $\overline{M}$ of the same dimension and the inverse part of the theorem is proved in that case.

If $H_x(\overset{r}{M}{}^e) \neq 0$ we consider a curve $x(t)$ of U_z and we denote with $X_{x(t)}$ its tangent vector at the point $x(t)$. Using Proposition 8 we obtain

$$\begin{aligned}
\frac{d}{dt}\left(x(t) + \frac{1}{\left|H_{x(t)}(\overset{r}{M}{}^e)\right|^2} H_{x(t)}(\overset{r}{M}{}^e)\right) &= X_{x(t)} + \frac{1}{\left|H_{x(t)}(\overset{r}{M}{}^e)\right|^2} D_{X_{x(t)}} H_{x(t)}(\overset{r}{M}{}^e) \\
&= X_{x(t)} - \frac{\left|H_{x(t)}(\overset{r}{M}{}^e)\right|^2}{\left|H_{x(t)}(\overset{r}{M}{}^e)\right|^2} X_{x(t)} \\
&= 0.
\end{aligned}$$

Then the point

$$y = x + \frac{H_x(\overset{r}{M}{}^e)}{|H_x(\overset{r}{M}{}^e)|^2}$$

does not depend on the choice of x on U_z; moreover, from Prop. 8,

$$|\vec{yx}| = \frac{|H_x(\overset{r}{M}{}^e)|}{|H_x(\overset{r}{M}{}^e)|^2} = \frac{1}{|H_x(\overset{r}{M}{}^e)|} = \text{const. on } U_z$$

and then U_z is contained in a sphere of $\mathbf{E}^{\overline{m}}$. But Proposition 10 says that it is also contained in an affine space of $\mathbf{E}^{\overline{m}}$ of dimension $q = \dim \overset{r}{M} + 1 = \dim \overset{r+1}{\Omega}_x M + 1$, and then in a totally umbilical submanifold of $\overline{M}$ of dimension $p = q - 1 = \overset{r+1}{\Omega}_x M$, so the inverse part is proved also in that case. □

III. Direct part of the Theorem

Direct Part of the Theorem. *If M is contained in a totally umbilical submanifold of $\overline{M}$ of dimension $p = \overset{r+1}{\Omega}_x M$ then $\overset{r}{w}_M = 0$.*

We start proving some lemmas and propositions.

1. Definition. *Let Δ be a distribution along the submanifold M of $\overline{M}$. Δ is called conformally parallel if*

(i) $T_x M \oplus \overset{1}{W}_x M \subset \Delta_x, \quad x \in M$,

(ii) *the normal part of Δ is parallel in the normal bundle.*

From (1) of Section I we can deduce that Definition 1 is conformally invariant.

The following lemma give a relation between parallel and conformally parallel distributions.

2. Lemma. *If Δ is a parallel distribution along a submanifold $\tilde{M}$ of $\overline{M}$ such that for each $x \in \overline{M}$,*

$$T_x \tilde{M} \subset \Delta_x,$$

then Δ is also conformally parallel.

Proof. For $X_x \in T_x\tilde{M}$ and $Y \in \Gamma(T\tilde{M})$ we have $\overset{\tilde{M}}{\nabla}_{X_x} Y \in T_x\tilde{M} \subset \Delta_x$. From the parallelism of Δ_x we also have $\overline{\nabla}_{X_x} Y \in \Delta_x$. Then

$$\overset{0}{s}_{\tilde{M}}(X_x Y_x) = \overline{\nabla}_{X_x} Y - \overset{\tilde{M}}{\nabla}_{X_x} Y \in \Delta_x$$

and $\overset{1}{N}_x\tilde{M} \subset \Delta_x$. But $\overset{1}{W}_x\tilde{M} \subset \overset{1}{N}_x\tilde{M}$ and condition (i) follows.

Take $X_x \in T_x\tilde{M}$ and $\overset{d}{Y} \in \Gamma(\Delta \cap \perp \tilde{M})$. The parallelism of Δ implies $\overline{\nabla}_{X_x}\overset{d}{Y} \in \Delta_x$ and then $P_{\perp \tilde{M}}(\overline{\nabla}_{X_x}\overset{d}{Y}) \in \Delta_x \cap \perp_x \tilde{M}$, which is condition (ii). □

3. *Remark*. Lemma 2 implies that the tangent space to a totally geodesic submanifold $\tilde{M}$ is a conformally parallel distribution along $\tilde{M}$ and along each submanifold M of $\tilde{M}$. Using Proposition 9 of Section I we can extend Remark 3 to the following lemma.

4. Lemma. *The tangent space to a totally umbilical submanifold $\tilde{M}$ is a conformally parallel distribution along $\tilde{M}$ and along each submanifold M of $\tilde{M}$.* □

We recall that we have denoted by l the integer such that $\overset{l-2}{w}_M \neq 0$ and $\overset{l-1}{w}_M = 0$, and by $s = \dim \overset{l}{\Omega}_x M$ the conformal number of immersion $M \to \overline{M}$.

By induction from Definition 1, using (4) and (5) of Section I we deduce:

5. Lemma. *If Δ is a conformally parallel distribution along M then for each $x \in M$ and $i = 0, 1, \ldots, l$, $\overset{l}{W}_x M \subset \Delta_x$; in particular $\overset{l}{\Omega}_x M \subset \Delta_x$ and $\dim \Delta_x \geq s$.* □

Suppose M contained in a totally umbilical submanifold $\tilde{M}$ of $\overline{M}$. By Lemma 4, $T\tilde{M}$ is conformally parallel along M. By Lemma 5, $\overset{l}{\Omega} M \subset T\tilde{M}$. If $\dim \tilde{M} = \dim \overset{r+1}{\Omega} M$ we have $\overset{r-1}{\Omega} M = \overset{l}{\Omega} M = T\tilde{M}$; in particular $\overset{r}{w}_M = 0$ and the direct part of the theorem is proved.

IV. Corollaries

From Lemma 5 of Section III and the inverse part of the theorem, we have:

1. Corollary I. *Let $\overline{M}$ be locally conformally flat. If Δ is a distribution of $\overline{M}$, conformally parallel along a conformally nicely curved submanifold M of $\overline{M}$, then M is locally contained in a totally umbilical submanifold of $\overline{M}$ of dimension $p \leq \dim \Delta$.* □

Suppose that, for a given submanifold M of $\overline{M}$, for each $i = 1, \ldots, r$, we have $\overset{i}{W}_x M = c(i) = \text{const}$. Moreover suppose $\overset{1}{W}M \oplus \cdots \oplus \overset{r}{W}M$ is parallel in the normal bundle of M.

By property (5) of Section I it follows that $\overset{r-1}{W}_x M = 0$ for $x \in M$. In particular M is conformally nicely curved in $\overline{M}$ and $\overset{r}{w}_M = 0$. Using the inverse part of the theorem we obtain:

2. Corollary II. *Let M be a submanifold of a manifold $\overline{M}$ locally conformally flat. If* $\dim \overset{k}{W}_x(M) = \text{const.}$ *for $k = 1, \ldots, r$ and if $\overset{1}{W}M \oplus \cdots \oplus \overset{r}{W}M$ is parallel in the normal bundle of M, then M is locally contained in a totally umbilical submanifold of $\overline{M}$ of dimension* $p = \dim \overset{r+1}{\Omega}_x M$, $x \in M$. □

As observed in the introduction, for $r = 1$, Corollary II gives:

Corollary III. *Let $\overline{M}$ be locally conformally flat and M a submanifold of $\overline{M}$. If $\overset{1}{W}_x M$ has constant dimension and is parallel in the normal bundle of M, then M is contained in a totally umbilical submanifold of $\overline{M}$ of dimension* $p = \dim M + \dim \overset{1}{W}_x M$. □

References

[E] J. Erbacher, *Reduction of the codimension of an isometric immersion*, J. Differential Geometry, **5** (1971), 333–340

[O] M. Okumura, *Conformal first normal space of a submanifold*, preprint.

[R] G. Romani, *Conformally invariant forms for submanifold*. Seminario Matematico dell'Universitá e del Politecnico di Torino, to appear.

[S] M. Spivak, *A comprehensive introduction to differential geometry*, vol. IV, Chapter 7, Publish or Perish. Houston, Texas, 1979.

[S.S] Schouten and Struik, *Un théorème sur la transformation conforme dans la géométrie différentielle à n dimensions*, Comptes Rendus Acad. Paris, **176** (1923), 1597–1600)

Università di Roma "La Sapienza"
Rome