ILLINOIS JOURNAL OF MATHEMATICS
Volume 39, Number 1, Spring 1995

THE GENERALIZED MCSHANE INTEGRAL

D.H. FREMLIN

Introduction

An interesting definition of the Lebesgue integral on $[0,1]$, as a limit of suitable Riemann sums, was developed by E.J. McShane in a series of papers; see [16] for a full account of this, including extensions to such spaces as $\mathbf{R}^n$ and $\mathbf{R}^{\mathbf{N}}$, as indicated in 1G below. It is characteristic of such definitions of the integral that they are readily adaptable to provide a theory of integration for vector-valued functions, and this was done for the McShane integral on $[0,1]$ by R.A. Gordon [13]. McShane was primarily concerned to provide an intuitively and technically straightforward construction of the Lebesgue interval, and made no attempt to push his method to the most general case. My aim in this paper is to show that, with a little effort, a successful generalization can be found, which can deal with functions from any of a wide variety of topological measure spaces to a Banach space, is related in interesting ways to other known integrals, and has a satisfying number of properties of its own.

The context in which I work is that of 'σ-finite outer regular quasi-Radon measure spaces' (see 1Ba–c below); this covers most of the important topological measure spaces which have been described. The paper has four sections.

1. I begin by defining the integral (1A–1B) and showing that it does indeed agree with Gordon's version when the domain space is $[0,1]$, and with McShane's versions when the range space is $\mathbf{R}$ and the domain space is one of those he considers (1C–1G). I continue by showing that the McShane integral lies between the Bochner and Pettis integrals (1K, 1Q), and in particular always agrees with the ordinary integral when the range space is $\mathbf{R}$ (1O).

2. In the second section I give some results of a technical type, showing that 'lim sup' in the definition of the integral may be replaced by a simple limit (2D) and that the two natural definitions of $\int_E \phi$ agree for measurable sets E(2E–2F).

3. I then describe the relationship between the McShane and Talagrand integrals; this follows the lines established in [10] for the case in which the domain space is $[0,1]$. If the unit ball of X^* is w^*-separable, then an

Received September 2, 1992.
1991 Mathematics Subject Classification. Primary 28B05; Secondary 46G10.

X-valued McShane integrable function is properly measurable, so is Talagrand integrable if its norm has finite upper integral (3D). (However, a Talagrand integrable function need not be McShane integrable, as shown already by an example in [10].)

4. In the last section I give two theorems concerning the integral of a weak limit of a sequence of McShane integrable functions (4A, 4E), with corollaries. I conclude with three open questions (4G).

Some of the results here have been circulated in University of Essex Mathematics Department Research Report 92-4.

1. The McShane integral

I propose to use this name for a method of integrating vector-valued functions which is adapted from the integration process described in [16]. As I wish to make rather a large step (from real-valued functions defined on $\mathbf{R}^n$ or $\mathbf{R}^{\mathbf{N}}$ to vector-valued functions defined on σ-finite outer regular quasi-Radon measure spaces), I give a full list of the definitions and theorems in the elementary theory as I develop it, even though most of the proofs will not involve any new ideas.

1A Definitions. Let $(S, \mathfrak{T}, \Sigma, \mu)$ be a non-empty σ-finite quasi-Radon measure space which is *outer regular*, that is, such that $\mu E = \inf\{\mu G: E \subseteq G \in \mathfrak{T}\}$ for every $E \in \Sigma$. A *generalized McShane partition* of S is a sequence $\langle (E_i, t_i)\rangle_{i \in \mathbf{N}}$ such that $\langle E_i\rangle_{i \in \mathbf{N}}$ is a disjoint family of measurable sets of finite measure, $\mu(S \setminus \bigcup_{i \in \mathbf{N}} E_i) = 0$ and $t_i \in S$ for each i. A *gauge* on S is a function $\Delta: S \to \mathfrak{T}$ such that $s \in \Delta(s)$ for every $s \in S$. A generalized McShane partition $\langle (E_i, t_i)\rangle_{i \in \mathbf{N}}$ is *subordinate* to a gauge Δ if $E_i \subseteq \Delta(t_i)$ for every $i \in \mathbf{N}$.

Now let X be a Banach space. I will say that a function $\phi: S \to X$ is *McShane integrable*, with *McShane integral* w, if for every $\varepsilon > 0$ there is a gauge $\Delta: S \to \mathfrak{T}$ such that

$$\limsup_{n \to \infty} \left\| w - \sum_{i \le n} \mu E_i . \phi(t_i) \right\| \le \varepsilon$$

for every generalized McShane partition $\langle (E_i, t_i)\rangle_{i \in \mathbf{N}}$ of S subordinate to Δ.

1B *Remarks.* (a) For the elementary theory of quasi-Radon measure spaces see [4], [6] and [7]; the same idea, expressed in a more general context, underlies the 'Radon spaces of type $(\mathscr{H})$' of B. Rodriguez-Salinas [18], [17]. The principal examples of σ-finite outer regular quasi-Radon measure spaces are

((i) all totally finite Radon and quasi-Radon measure spaces;

(ii) all Lindelöf Radon measure spaces (e.g., Lebesgue measure on $\mathbf{R}^n$);

(iii) all subspaces of such spaces (1L below);
(iv) finite products of such spaces ([6], 4C, or [7], A7Ea);
(v) all products of probability spaces of these types ([6], 4F, or [7], A7Eb).

(b) The essential facts I shall need here are that a quasi-Radon measure μ is inner regular for the closed sets (that is, $\mu E = \sup\{\mu F\colon F \subseteq E,\ F$ is closed$\}$ for every measurable E) and τ-smooth (that is, $\mu(\bigcup \mathscr{G}) = \sup_{G \in \mathscr{G}} \mu G$ for every non-empty upwards-directed family $\mathscr{G}$ of open sets).

(c) In addition, we shall need to know that an outer regular quasi-Radon measure is locally finite (that is, every point belongs to an open set of finite measure). If it is σ-finite, it has the following property, stronger than what is declared by the definition of 'outer regular' given in 1A: if E is any measurable set, and $\varepsilon > 0$, there is an open set $G \supseteq E$ such that $\mu(G \setminus E) \leq \varepsilon$. Another elementary fact about outer regular measures is that if μ is an outer regular measure on S, and $f\colon S \to [0, \infty[$ is an integrable function, then for any $\varepsilon > 0$ there is a lower semi-continuous function $h\colon S \to [0, \infty]$ such that $f(t) < h(t)$ for every $t \in S$ and $\int h \leq \varepsilon + \int f$.

(d) I had better remark straight away that my version of the McShane integral is well-defined, in the sense that any given function has at most one value of the integral. Of course this is just because there are enough generalized McShane partitions: if $S \neq \emptyset$ and $\Delta\colon S \to \mathfrak{T}$ is any gauge, there is a generalized McShane partition subordinate to it. To see this, observe that

$$\mathscr{G} = \{G\colon G \in \mathfrak{T},\ \mu G < \infty,\ \exists s \in S,\ G \subseteq \Delta(s)\}$$

is an open cover of S, so that (because μ is τ-smooth) we have

$$\mu H = \sup\{\mu(H \cap \bigcup \mathscr{G}_0)\colon \mathscr{G}_0 \subseteq \mathscr{G} \text{ is finite}\}$$

for every open $H \subseteq S$; now, because μ is σ-finite, there is a sequence $\langle G_i \rangle_{i \in \mathbf{N}}$ in $\mathscr{G}$ such that $\mu(S \setminus \bigcup_{i \in \mathbf{N}} G_i) = 0$. If we choose for each i a $t_i \in S$ such that $G_i \subseteq \Delta(t_i)$, and write $E_i = G_i \setminus \bigcup_{j<i} G_j$ for $i \in \mathbf{N}$, we shall have a generalized McShane partition $\langle (E_i, t_i) \rangle_{i \in \mathbf{N}}$ subordinate to Δ.

Now because the family of gauges on S is directed downwards (if Δ_0 and Δ_1 are gauges, so is $s \mapsto \Delta_0(s) \cap \Delta_1(s)$) this shows that for any particular ϕ there will be at most one w satisfying the definition above.

(e) It will be convenient later to say that if $(S, \mathfrak{T}, \Sigma, \mu)$ is a σ-finite outer regular quasi-Radon measure space, then a *partial McShane partition* of S is a countable family $\langle (E_i, t_i) \rangle_{i \in I}$ where $\langle E_i \rangle_{i \in I}$ is a disjoint family of sets of finite measure, and $t_i \in S$ for each i; and that it is *subordinate* to a gauge Δ if $E_i \subseteq \Delta(t_i)$ for every i.

(f) There is a technical fault in the definition of the McShane integral above. It ignores the case $S = \emptyset$. On the other hand, I certainly wish to count the empty set as a quasi-Radon measure space, and to accept the empty function as McShane integrable, with integral zero. Of course this is a triviality, and in the proofs below I shall systematically pass the case $S = \emptyset$ by, though I do wish it to be included in the statements of the results.

(g) It is in fact possible to define a McShane integral on outer regular quasi-Radon measure spaces which are not σ-finite. As however such a space must consist of a σ-finite part together with a family of closed sets, of strictly positive measure, on each of which the topology is indiscrete (see [12], §13), the McShane integral outside the σ-finite part corresponds just to unconditional summability of appropriate families in X; and the extra technical complications (we have to use uncountable families $\langle(E_i, t_i)\rangle_{i\in I}$ instead of sequences) seem more trouble than they're worth. It might however be right to consider such an elaboration if one wished to extend these ideas to the context of linear topological spaces.

(h) A slightly simpler alternative definition of the McShane integral, which some readers may prefer, may be found in 2D below.

1C We are now ready for some elementary facts about the McShane integral. I give no proofs as the arguments are of a type familiar from [16].

PROPOSITION. *Let $(S, \mathfrak{T}, \Sigma, \mu)$ be a σ-finite outer regular quasi-Radon space and X a Banach space.*

(a) *If ϕ, ψ: $S \to X$ are McShane integrable functions with McShane integrals w, z respectively then $\phi + \psi$ is McShane integrable, with integral $w + z$.*

(b) *Let Y be another Banach space and T: $X \to Y$ a bounded linear operator. If ϕ: $S \to X$ is McShane integrable, with McShane integral w, then $T\phi$: $S \to Y$ is McShane integrable, with McShane integral Tw.*

(c) *If $C \subseteq X$ is a closed cone and ϕ: $S \to C$ is a McShane integrable function, then its McShane integral belongs to C.*

Remark. Of course the principal use of (b) is with $Y = \mathbf{R}$, and the principal use of (c) is with $X = \mathbf{R}$, $C = [0, \infty[$.

1D Readers familiar with [16] will already have observed that my definition of the McShane integral is significantly different from (and more complex than) the most natural generalisations of the work in [16]; a much simpler expression is used in [10] and [8]. The extra elaboration of my definition here is necessary to deal with the wider context in which I operate. However I must of course justify my terminology by showing that in the limited contexts considered in [16] and [13] my formulations agree with the simpler ones. The first point is that for compact spaces S there is no need to take infinite McShane partitions. Let us say that a *finite strict generalized McShane partition* of S is a family $\langle(E_i, t_i)\rangle_{i\leq n}$ such that $E_0, \ldots, E_n$ is a finite disjoint cover of S by measurable sets of finite measure (I find it convenient still to allow $E_i = \emptyset$ for some i) and $t_i \in S$ for each $i \leq n$. Now we have the following:

1E Proposition. *Let* $(S, \mathfrak{T}, \Sigma, \mu)$ *be a compact Radon measure space and* X *a Banach space; let* ϕ: $S \to X$ *be a function. Then* ϕ *is McShane integrable, with McShane integral* w, *if and only if for every* $\varepsilon > 0$ *there is a gauge* Δ: $S \to \mathfrak{T}$ *such that whenever* $\langle (E_i, t_i)\rangle_{i \le n}$ *is a finite strict generalized McShane partition of* S *subordinate to* Δ *then* $\|w - \sum_{i \le n} \mu E_i \phi(t_i)\| \le \varepsilon$.

Remark. I follow [4] in taking a Radon measure space to be a Hausdorff locally finite quasi-Radon measure space in which the measure is inner regular for the compact sets.

Proof. Evidently any McShane integrable function ϕ: $S \to X$ must satisfy the condition offered, as this merely restricts the class of partitions considered (of course a finite McShane partition can be extended to an infinite one by adding empty E_i.) For the reverse implication, suppose that ϕ, w satisfy the condition. Let $\varepsilon > 0$ and let Δ: $S \to \mathfrak{T}$ be a gauge such that

$$\Big\| w - \sum_{i \le n} \mu E_i \phi(t_i) \Big\| \le \varepsilon$$

for every finite strict generalized McShane partition $\langle (E_i, t_i)\rangle_{i \le n}$ subordinate to Δ. Now let $\langle (E_i, t_i)\rangle_{i \in \mathbf{N}}$ be an infinite generalized McShane partition subordinate to Δ. Because S is compact, we can find a finite cover of it by sets of the form $\Delta(t)$; accordingly, adding finitely many negligible sets E_i to the beginning of the sequence if necessary, we may take it that $S = \bigcup_{i \in \mathbf{N}} E_i$. For each $i \in \mathbf{N}$ choose an open set G_i such that

$$E_i \subseteq G_i \subseteq \Delta(t_i) \quad \text{and} \quad \mu(G_i \setminus E_i)\|\phi(t_i)\| \le 2^{-i}\varepsilon.$$

There is a $k \in \mathbf{N}$ such that $S = \bigcup_{i \le k} G_i$. Now if $n \ge k$, we have $S = \bigcup_{i \le n} G_i$, so there is a disjoint family $\langle E_i' \rangle_{i \le n}$ of measurable sets such that $E_i \subseteq E_i' \subseteq G_i$ for every $i \le n$ and $S = \bigcup_{i \le n} E_i'$. But in this case $\langle (E_i', t_i)\rangle_{i \le n}$ is a finite strict generalized McShane partition of S subordinate to Δ, so we must have

$$\Big\| w - \sum_{i \le n} \mu E_i' . \phi(t_i) \Big\| \le \varepsilon.$$

On the other hand, we also have

$$\Big\| \sum_{i \le n} \mu E_i' \phi(t_i) - \sum_{i \le n} \mu E_i \phi(t_i) \Big\|$$
$$\le \sum_{i \le n} (\mu E_i' - \mu E_i)\|\phi(t_i)\| \le \sum_{i \le n} 2^{-i}\varepsilon \le 2\varepsilon.$$

So

$$\left\| w - \sum_{i \le n} \mu E_i \phi(t_i) \right\| \le 3\varepsilon$$

for all $n \ge k$; as ε is arbitrary, ϕ is McShane integrable with integral w.

1F The definitions of [16] do not as a rule refer to partitions into arbitrary measurable sets; instead they use various types of 'interval' for the E_i—e.g., half-open intervals in $\mathbf{R}$. I can give a general criterion for the applicability of such methods, as follows.

PROPOSITION. *Let $(S, \mathfrak{T}, \Sigma, \mu)$ be a compact Radon measure space and X a Banach space. Let $\mathscr{A} \subseteq \Sigma$ be a subalgebra of Σ such that whenever $F \subseteq G \subseteq S$, F is closed and G is open there is an $A \in \mathscr{A}$ such that $F \subseteq A \subseteq G$; let $\mathscr{C} \subseteq \mathscr{A}$ be such that every member of $\mathscr{A}$ is a finite disjoint union of members of $\mathscr{C}$. Then a function ϕ: $S \to X$ is McShane integrable, with McShane integral w, iff for every $\varepsilon > 0$ there is a gauge Δ: $S \to \mathfrak{T}$ such that*

$$\left\| w - \sum_{i \le n} \mu C_i \phi(t_i) \right\| \le \varepsilon$$

for every finite strict generalized McShane partition $\langle (C_i, t_i) \rangle_{i \le n}$ of S, subordinate to Δ, such that $C_i \in \mathscr{C}$ for every $i \le n$.

Proof. (a) Of course a McShane integrable function (as I have defined it) must satisfy the condition.

(b) For the converse, I use the following facts.

(i) If $E \in \Sigma$ and $E \subseteq G \in \mathfrak{T}$ and $\eta > 0$ there is an $A \in \mathscr{A}$ such that $A \subseteq G$ and $\mu(E \Delta A) \le \eta$. For take any closed set $F \subseteq E$, open set $H \supseteq E$ such that $\mu(H \setminus F) \le \eta$, and take A such that $F \subseteq A \subseteq G \cap H$.

(ii) Suppose that Δ: $S \to \mathfrak{T}$ is a gauge and that $\langle (E_i, t_i) \rangle_{i \le n}$ is a strict finite generalized McShane partition of S subordinate to Δ. Then for any $\varepsilon > 0$ there are $A_0, \dots, A_n \in \mathscr{A}$ such that $\langle (A_i, t_i) \rangle_{i \le n}$ is a strict finite generalized McShane partition of S, subordinate to Δ, and $\sum_{i \le n} \mu(A_i \Delta E_i) \|\phi(t_i)\| < \varepsilon$. To see this, take $\eta > 0$ so small that $2(n+1)^2 \eta \max_{i \le n} \|\phi(t_i)\| \le \varepsilon$. Now for each $i \le n$ take $A'_i \in \mathscr{A}$ such that $A'_i \subseteq \Delta(t_i)$ and $\mu(E_i \Delta A'_i) \le \eta$. Set

$$A = S \setminus \bigcup_{i \le n} A'_i \in \mathscr{A}.$$

Because S is compact and Hausdorff and $S = \bigcup_{i \le n} \Delta(t_i)$, the set

$$\{G : G \in \mathfrak{T}, \exists i \le n, \overline{G} \subseteq \Delta(t_i)\}$$

is an open cover of S and has a finite subcover, and there are closed sets $F_0, \dots, F_n$ such that $F_i \subseteq \Delta(t_i)$ for each i and $\bigcup_{i \le n} F_i = S$; consequently there are $A''_0, \dots, A''_n \in \mathscr{A}$ such that $A''_i \subseteq \Delta(t_i)$ for each i and $\bigcup_{i \le n} A''_i = S$ (take A''_i such that $F_i \subseteq A''_i \subseteq \Delta(t_i)$ for each i). Now set

$$A_i = (A'_i \cup (A \cap A''_i)) \setminus \bigcup_{j<i} A_j$$

for each $i \le n$. Evidently $A_0, \dots, A_n$ are disjoint, belong to $\mathscr{A}$ and cover S, and $A_i \subseteq \Delta(t_i)$ for each i. Also

$$\begin{aligned}\mu(E_i \Delta A_i) &\le \mu(E_i \Delta A'_i) + \mu A + \sum_{j<i} \mu(E_i \cap A'_j) \\ &\le \eta + (n+1)\eta + i\eta \le (2n+2)\eta\end{aligned}$$

for each i. So

$$\sum_{i \le n} \mu(E_i \Delta A_i) \| \phi(t_i) \| \le 2(n+1)^2 \eta \max_{i \le n} \| \phi(t_i) \| \le \varepsilon,$$

as required.

(c) Now suppose that ϕ satisfies the condition. Let $\varepsilon > 0$ and let Δ: $S \to \mathfrak{T}$ be a gauge such that $\|w - \sum_{i \le n} \mu C_i \phi(t_i)\| \le \varepsilon$ whenever $\langle (C_i, t_i) \rangle_{i \le n}$ is a strict finite generalized McShane cover of S by members of $\mathscr{C}$ subordinate to Δ. Let $\langle (E_i, t_i) \rangle_{i \le n}$ be any strict finite generalized McShane cover of S subordinate to Δ. By (b), there are disjoint $A_0, \dots, A_n \in \mathscr{A}$ such that $\bigcup_{i \le n} A_i = S$, $A_i \subseteq \Delta(t_i)$ for each i and $\sum_{i \le n} \mu(E_i \Delta A_i) \|\phi(t_i)\| \le \varepsilon$. By the hypothesis on $\mathscr{C}$, we can express each A_i as a disjoint union $C_{i0} \cup \cdots \cup C_{i,k(i)}$ of members of $\mathscr{C}$. Now write $t_{ij} = t_i$ for $j \le k(i)$; we see that $\langle (C_{ij}, t_{ij}) \rangle_{i \le n, j \le k(i)}$ is a strict finite generalized McShane cover of S subordinate to Δ, so

$$\begin{aligned}\left\| w - \sum_{i \le n} \mu E_i \phi(t_i) \right\| &\le \left\| w - \sum_{i \le n} \mu A_i \phi(t_i) \right\| + \sum_{i \le n} |\mu E_i - \mu A_i| \, \|\phi(t_i)\| \\ &\le \left\| w - \sum_{i \le n, j \le k(i)} \mu C_{ij} \phi(t_i) \right\| + \varepsilon \\ &\le 2\varepsilon.\end{aligned}$$

As ε is arbitrary, the criterion of 1E shows that ϕ is McShane integrable.

1G *Examples.* Examples relevant to the work of [16] are

(i) $S = [a, b]$, $\mathscr{C} = \{[c, d[\colon a \le c < d \le b\} \cup \{\{b\}, \emptyset\}$

(ii) $S = \prod_{i \le n} [a_i, b_i]$, $\mathscr{C} = \{\prod_{i \le n} C_i\colon C_i \in \mathscr{C}_i \ \forall i \le n\}$ where $\mathscr{C}_i$ consists of intervals, as in (i).

For infinite products, if each S_i is a compact Radon probability space with an associated family $\mathscr{C}_i$, then the corresponding cylinder sets in $S = \Pi_i S_i$, of the form $\Pi_i C_i$ where each C_i belongs to $\mathscr{C}_i \cup \{S_i\}$ and $\{i: C_i \neq S_i\}$ is finite, do the same for S.

Of course [16] uses gauge functions of the form $\delta: S \to]0, \infty[$ rather than of the form $\Delta: S \to \mathfrak{T}$; but the translation from one to the other, in a metric space (S, ρ), is trivial, if we match $\delta(s)$ to the open set $\Delta(s) = \{t: \rho(t, s) < \delta(s)\}$.

In [13], [10] and [8], 'partitions' into non-overlapping closed intervals are used systematically; but of course these could be read throughout as half-open intervals without it making any difference.

1H The next step is to show that my version of the McShane integral agrees with the ordinary integral in the case $X = \mathbf{R}$. For the case $S = [0, 1]$, this is already covered by 1F and the results of [13]; for other S we still have some work to do. In fact I show a more general result in one direction: for any Banach space X, if $\phi: S \to X$ is Bochner integrable, with Bochner integral w, then it is McShane integrable, with McShane integral w. (For the definition and elementary properties of the Bochner integral, see [2].)

We need two fairly straightforward lemmas.

1I Lemma. *Let $(S, \mathfrak{T}, \Sigma, \mu)$ be a σ-finite outer regular quasi-Radon measure space and X a Banach space. Let $E \subseteq X$ be a set of finite measure and $x \in X$; let $\phi: S \to X$ be the function which takes the value x on E, 0 elsewhere. Then ϕ is McShane integrable, with integral $w = \mu E.x$.*

Proof. Let $\varepsilon > 0$. Let F be a closed set and G an open set such that $F \subseteq E \subseteq G$ and $\mu(G \setminus F) \leq \varepsilon$. Set $\Delta(s) = G$ if $s \in F$, $G \setminus F$ if $s \in G \setminus F$, $S \setminus F$ if $s \in S \setminus G$. Then an easy calculation shows that $\lim_{n\to\infty} \|w - \sum_{i \leq n} \mu E_i \phi(t_i)\| \leq \varepsilon \|x\|$ whenever $\langle (E_i, t_i) \rangle_{i \in \mathbf{N}}$ is a generalized McShane partition of S subordinate to Δ.

1J Lemma. *Let $(S, \mathfrak{T}, \Sigma, \mu)$ be a σ-finite outer regular quasi-Radon measure space and X a Banach space. Let $\phi: S \to X$ be a function and $\varepsilon > 0$. Then there is a gauge $\Delta: S \to \mathfrak{T}$ such that*

$$\sum_{i \in \mathbf{N}} \mu E_i \|\phi(t_i)\| \leq \overline{\int} \|\phi(t)\| \mu(dt) + \varepsilon$$

whenever $\langle (E_i, t_i) \rangle_{i \in \mathbf{N}}$ is a generalized McShane partition of S subordinate to ϕ.

Proof. If $\overline{\int} \|\phi(t)\| \mu(dt) = \infty$, this is trivial. Otherwise, let $g: S \to \mathbf{R}$ be a function such that $g(t) \geq \|\phi(t)\|$ for every t and $\int g = \overline{\int} \|\phi\|$. Now let h be a lower semi-continuous function such that $g(t) < h(t)$ for every t and $\int h \leq \int g + \varepsilon$ (see 1Bc). Set $\Delta(t) = \{s\colon h(s) > \|\phi(t)\|\}$ for each t; this works.

1K THEOREM. *Let $(S, \mathfrak{T}, \Sigma, \mu)$ be a σ-finite outer regular quasi-Radon measure space and X a Banach space. Let $\phi\colon S \to X$ be a Bochner integrable function with Bochner integral w. Then ϕ is McShane integrable with McShane integral w.*

Proof. Let $\varepsilon > 0$. Then there is a 'simple' function $\psi\colon S \to X$, of the form

$$\psi(s) = x_i \text{ when } s \in F_i, 0 \text{ if } s \notin \bigcup_{i \leq n} F_i,$$

where $F_0, \ldots, F_n$ are disjoint sets of finite measure and each $x_i \in X$, such that

$$\int \|\phi(s) - \psi(s)\| \mu(ds) \leq \varepsilon.$$

Set $w_0 = \sum_{i \leq n} \mu F_i . x_i$; then $\|w - w_0\| \leq \varepsilon$. Now Lemma 1I tells us that ψ is McShane integrable, with McShane integral w_0; let Δ_0 be a gauge such that

$$\limsup_{n \to \infty} \left\| w_0 - \sum_{i \leq n} \mu E_i \psi(t_i) \right\| \leq \varepsilon$$

whenever $\langle (E_i, t_i) \rangle_{i \in \mathbf{N}}$ is a generalized McShane partition of S subordinate to Δ_0. Also Lemma 1J tells us that there is a gauge Δ_1 on S such that

$$\sum_{i \in \mathbf{N}} \mu E_i \| \phi(t_i) - \psi(t_i) \| \leq 2\varepsilon$$

whenever $\langle (E_i, t_i) \rangle_{i \in \mathbf{N}}$ is a generalized McShane partition of S subordinate to Δ_1.

If we now take $\Delta(s) = \Delta_0(s) \cap \Delta_1(s)$ for each $s \in S$, we see that Δ is a gauge on S and that

$$\limsup_{n \to \infty} \left\| w - \sum_{i \leq n} \mu E_i \phi(t_i) \right\| \leq 4\varepsilon$$

for every generalized McShane partition $\langle (E_i, t_i) \rangle_{i \in \mathbf{N}}$ of S subordinate to Δ. As ε is arbitrary, ϕ is McShane integrable with McShane integral w.

1L My next objective is to prove a result in the opposite direction: if $\phi\colon S \to \mathbf{R}$ is McShane integrable, it is integrable in the usual sense. This will lead directly to a more general result: if $\phi\colon S \to X$ is McShane integrable, it is Pettis integrable. My route to this takes us past some further useful facts.

Recall that if $(S, \mathfrak{T}, \Sigma, \mu)$ is any quasi-Radon space, and $A \subseteq S$ is any set (not necessarily measurable), then $(A, \mathfrak{T}_A, \Sigma_A, \mu_A)$ is a quasi-Radon measure space, where $\mathfrak{T}_A$ is the induced topology on A, $\Sigma_A = \{E \cap A: E \in \Sigma\}$, and

$$\mu_A(B) = \min\{\mu E: B = A \cap E\} \quad \text{for } B \in \Sigma_A.$$

(See [6], 5B and [7], A7D.) It is easy to see that if $(S, \mathfrak{T}, \Sigma, \mu)$ is σ-finite or outer regular, so is $(A, \mathfrak{T}_A, \Sigma_A, \mu_A)$. Accordingly, if X is a Banach space and $\phi: S \to X$ is a function, we may discuss the McShane integrability of $\phi \restriction A$: $A \to X$. Now we have the following results. The first is an elementary lemma.

1M Lemma. *Let $(S, \mathfrak{T}, \Sigma, \mu)$ be a non-empty σ-finite outer regular quasi-Radon measure space and X a Banach space. Suppose that $\phi: S \to X$ has the property that for every $\varepsilon > 0$ there is a gauge $\Delta: S \to \mathfrak{T}$ such that*

$$\limsup_{n\to\infty} \left\| \sum_{i\le n} \mu E_i \phi(t_i) - \sum_{i\le n} \mu F_i \phi(u_i) \right\| \le \varepsilon$$

whenever $\langle (E_i, t_i) \rangle_{i\in\mathbf{N}}$ and $\langle (F_i, u_i) \rangle_{i\in\mathbf{N}}$ are generalized McShane partitions of X subordinate to Δ. Then ϕ is McShane integrable.

Proof. Take ε, Δ as above. The point is that if $\langle (F_i, u_i) \rangle_{i\in\mathbf{N}}$ is a generalized McShane partition of S subordinate to Δ, and $\pi: \mathbf{N} \to \mathbf{N}$ is any bijection, than $\langle (F_{\pi(i)}, u_{\pi(i)}) \rangle_{i\in\mathbf{N}}$ is also a generalized McShane partition of S subordinate to Δ, so that

$$\limsup_{n\to\infty} \left\| \sum_{i\le n} \mu F_i \phi(u_i) - \sum_{i\le n} \mu F_{\pi(i)} \phi(u_{\pi(i)}) \right\| \le \varepsilon.$$

It follows at once that there is some $k \in \mathbf{N}$ such that

$$\sup_{n\ge k} \left\| w - \sum_{i\le n} \mu F_i \phi(u_i) \right\| \le 2\varepsilon,$$

where $w = \sum_{i\le k} \mu F_i \phi(t_i)$. Now

$$\limsup_{n\to\infty} \left\| w - \sum_{i\le n} \mu E_i \phi(t_i) \right\| \le 3\varepsilon$$

whenever $\langle (E_i, t_i) \rangle_{i\in\mathbf{N}}$ is a generalized McShane partition of S subordinate to Δ.

If for each $\varepsilon > 0$ we use the method above to find a gauge Δ_ε and a vector w_ε, we see that $\|w_\varepsilon - w_\eta\| \le 3(\varepsilon + \eta)$ for all $\varepsilon, \eta > 0$; so that $w = \lim_{\varepsilon\downarrow 0} w_\varepsilon$ is defined in X (this is one of the few points where we need X to be complete), and of course w will be the McShane integral of ϕ.

1N THEOREM. *Let $(S, \mathfrak{T}, \Sigma, \mu)$ be a σ-finite outer regular quasi-Radon measure space and X a Banach space. If $\phi: S \to X$ is McShane integrable, then $\phi \restriction A$ is McShane integrable for every $A \subseteq S$.*

Proof. Let w be the McShane integral of ϕ, and $\varepsilon > 0$. Let $\Delta: S \to \mathscr{T}$ be a gauge such that $\limsup_{n\to\infty} \|w - \sum_{i \le n} \mu E_i \phi(t_i)\| \le \varepsilon$ whenever $\langle (E_i, t_i)\rangle_{i \in \mathbf{N}}$ is a generalized McShane partition of S subordinate to Δ.

Let $\Delta_A(s) = A \cap \Delta(s)$ for $s \in A$; then Δ_A is a gauge on A. Let $\langle (E_i, t_i)\rangle_{i \in \mathbf{N}}$ and $\langle (F_i, u_i)\rangle_{i \in \mathbf{N}}$ be generalized McShane partitions of A subordinate to Δ_A. For each $i \in \mathbf{N}$ choose $\tilde{E}_i, \tilde{F}_i \in \Sigma$ such that $E_i = \tilde{E}_i \cap A$, $\mu_A E_i = \mu \tilde{E}_i$, $F_i = \tilde{F}_i \cap A$ and $\mu_A F_i = \mu \tilde{F}_i$. Set

$$H = \bigcup_{i \in \mathbf{N}} \big(\tilde{E}_i \cap \Delta(t_i)\big) \cap \bigcup_{i \in \mathbf{N}} \big(\tilde{F}_i \cap \Delta(u_i)\big).$$

For $i \in \mathbf{N}$ set

$$E_i^* = H \cap \tilde{E}_i \cap \Delta(t_i) \setminus \bigcup_{j<i} E_j^*,$$

$$F_i^* = H \cap \tilde{F}_i \cap \Delta(u_i) \setminus \bigcup_{j<i} F_j^*.$$

Then $\bigcup_{i \in \mathbf{N}} E_i^* = \bigcup_{i \in \mathbf{N}} F_i^* = H$; moreover, $\mu E_i^* = \mu_A E_i$ and $\mu F_i^* = \mu_A F_i$ for each i.

Fix any generalized McShane partition $\langle (H_i, v_i)\rangle_{i \in \mathbf{N}}$ of S subordinate to Δ. Define H_i', v_i', H_i'', v_i'' by writing

$$H_{2i}' = E_i^*,\ v_{2i}' = t_i,\ H_{2i+1}' = H_i \setminus H,\ v_{2i+1}' = v_i,$$
$$H_{2i}'' = F_i^*,\ v_{2i}'' = u_i,\ H_{2i+1}'' = H_i \setminus H,\ v_{2i+1}'' = v_i$$

for each $i \in \mathbf{N}$. Then $\langle (H_i', v_i')\rangle_{i \in \mathbf{N}}$ and $\langle (H_i'', v_i'')\rangle_{i \in \mathbf{N}}$ are both generalized McShane partitions of S subordinate to Δ. So

$$\limsup_{n \to \infty} \left\| \sum_{i \le n} \mu H_i' \phi(v_i') - \sum_{i \le n} \mu H_i'' \phi(v_i'') \right\| \le 2\varepsilon.$$

But on translating this through the definitions above, we see that

$$\limsup_{n \to \infty} \left\| \sum_{i \le n} \mu_A E_i \phi(t_i) - \sum_{i \le n} \mu_A F_i \phi(u_i) \right\| \le 2\varepsilon.$$

So the criterion of Lemma 1M is satisfied, and $\phi \restriction A$ is McShane integrable.

Remark. If A is such that $\mu_*(S \setminus A) = 0$, then $\int \phi \restriction A = \int \phi$; this is because, in the construction above, $\langle (E_i^*, t_i)\rangle_{i \in \mathbf{N}}$ will be a generalized McShane partition of S subordinate to Δ.

See also 2E below.

1O THEOREM. *Let $(S, \mathfrak{T}, \Sigma, \mu)$ be a σ-finite outer regular quasi-Radon measure space and h: $S \to \mathbf{R}$ a function. Then h is McShane integrable iff it is integrable in the ordinary sense, and the two integrals are equal.*

Proof. (a) If h is integrable in the ordinary sense, it is Bochner integrable, and therefore McShane integrable, by 1K.

(b) If h is McShane integrable, it is measurable; this is a special case of 3C below, but to avoid suspicion of circularity I sketch a proof here. Suppose, if possible, otherwise. Then there are $\alpha < \beta$ in $\mathbf{R}$ and $E \in \Sigma$ such that $0 < \mu E < \infty$ and $\mu^* A = \mu^* B = \mu E$, where

$$A = \{t : t \in E, h(t) \leq \alpha\} \quad \text{and} \quad B = \{t : t \in E, h(t) \geq \beta\}.$$

As remarked at the end of 1N, the McShane integrals (McS) $\int h \restriction E$, (McS) $\int h \restriction A$ and (McS) $\int h \restriction B$ must all be equal. But applying 1Cc to $(h \restriction A) - \alpha$ we see that (McS) $\int h \restriction A \leq \alpha \mu_A A = \alpha \mu E$, and similarly (McS) $\int h \restriction B \geq \beta \mu E$; which is impossible, because $\alpha \mu E < \beta \mu E$.

If we now set $F = \{s : h(s) \geq 0\}$, then we have a McShane integral (McS) $\int h \restriction F$. Now if g: $F \to [0, \infty[$ is any function which is integrable in the ordinary sense, and dominated by h, we must have

$$\int_F g = (\mathrm{McS}) \int g \leq (\mathrm{McS}) \int h \restriction F;$$

because h is measurable, it follows that $\int_F h$ is defined. Similarly, $\int_{S \setminus F} h$ is defined, so that h is integrable.

1P *The Pettis integral.* Let (S, Σ, μ) be a measurable space and X a Banach space. Recall that a function ϕ: $S \to X$ is *Pettis integrable* if for every $E \in \Sigma$ there is a $w_E \in X$ such that $\int_E f(\phi(x)) \mu(dx)$ exists and is equal to $f(w_E)$ for every $f \in X^*$; in this case w_S is the *Pettis integral* of ϕ, and the map $E \mapsto w_E$: $\Sigma \to X$ is the *indefinite Pettis integral* of ϕ.

1Q THEOREM. *Let $(S, \mathfrak{T}, \Sigma, \mu)$ be a σ-finite outer regular quasi-Radon measure space and X a Banach space. If ϕ: $S \to X$ is McShane integrable, with McShane integral w, then it is Pettis integrable, with Pettis integral w.*

Proof. For every $E \in \Sigma$ we have a McShane integral w_E of $\phi \restriction E$, by 1N. If $g \in X^*$ then $g\phi \restriction E$: $E \to \mathbf{R}$ is McShane integrable, with integral $g(w_E)$, by Proposition 1C. But we have seen in 1O that this means that the ordinary integral $\int_E g\phi$ exists and is equal to $g(w_E)$. As g is arbitrary, ϕ is Pettis integrable, with indefinite Pettis integral $E \mapsto w_E$; and the Pettis integral of ϕ is $w_S = w$.

Remark. This generalises Theorem 2C of [10].

2. Further basic properties of the generalized McShane integral

I give some technical results which will enable us to move more freely in the later parts of this paper.

2A We need to recall some well-known facts concerning vector measures. Suppose that Σ is a σ-algebra of sets and X a Banach space.

(i) Let us say that a function $\nu\colon \Sigma \to X$ is 'weakly countably additive' if $f(\nu(\bigcup_{i\in\mathbf{N}} E_i)) = \sum_{i\in\mathbf{N}} f(\nu E_i)$ for every disjoint sequence $\langle E_i\rangle_{i\in\mathbf{N}}$ in Σ and every $f \in X^*$. The first fact is that in this case ν is countably additive, that is, $\sum_{i\in\mathbf{N}} \nu E_i$ is unconditionally summable to $\nu(\bigcup_{i\in\mathbf{N}} E_i)$ for the norm topology whenever $\langle E_i\rangle_{i\in\mathbf{N}}$ is a disjoint sequence of measurable sets with union E ([19], 2-6-1; [3], p. 22, Corollary 4).

In particular, an indefinite Pettis integral is always countably additive.

(ii) If now μ is a measure with domain Σ such that $\nu E = 0$ whenever $\mu E = 0$, then for every $\varepsilon > 0$ and there is a $\delta > 0$ such that $\|\nu E\| \le \varepsilon$ whenever $\mu E \le \delta$.

In particular, if ν is an indefinite Pettis integral, it is absolutely continuous in this sense with respect to the original measure.

(iii) Thirdly, suppose that $\langle \nu_n\rangle_{n\in\mathbf{N}}$ is a sequence of countably additive functions from Σ to X such that $\nu E = \lim_{n\to\infty} \nu_n E$ exists in X, for the weak topology of X, for every $E \in \Sigma$: then ν is countably additive. (Use Nikodým's theorem ([1], p. 90) to see that ν is weakly countably additive.)

2B Lemma. *Let $(S, \mathfrak{T}, \Sigma, \mu)$ be a σ-finite outer regular quasi-Radon measure space, X a Banach space and $\phi\colon S \to X$ a McShane integrable function. Then for any $\varepsilon > 0$ there is a gauge $\Delta\colon S \to \mathfrak{T}$ such that*

$$\left\| \sum_{i\le n} \mu E_i \phi(t_i) - \int_E \phi \right\| \le \varepsilon$$

whenever $\langle (E_i, t_i)\rangle_{i\le n}$ is a partial McShane partition of S subordinate to Δ, and $E = \bigcup_{i\le n} E_i$.

Proof. Recall that by 1N–1Q we can identify $E \mapsto \nu E = \int_E \phi$ as the indefinite Pettis integral of ϕ. Let $\Delta\colon S \to \mathfrak{T}$ be a gauge such that

$$\limsup_{n\to\infty} \left\| \int \phi - \sum_{i\le n} \mu E_i \phi(t_i) \right\| \le \tfrac{1}{2}\varepsilon$$

whenever $\langle (E_i, t_i)\rangle_{i\in\mathbf{N}}$ is a generalized McShane partition of S subordinate to Δ. Now let $\langle (E_i, t_i)\rangle_{i\le n}$ be a partial McShane partition of S subordinate to Δ, and $E = \bigcup_{i\le n} E_i$. Let $\langle (F_i, u_i)\rangle_{i\in\mathbf{N}}$ be a generalized McShane partition of $S\setminus E$, subordinate to Δ, such that

$$\limsup_{m\to\infty}\left\|\nu(S\setminus E) - \sum_{i\le m}\mu F_i\phi(u_i)\right\| \le \tfrac{1}{2}\varepsilon.$$

(Readers will have no difficulty in dealing separately with the case $E = S$.)

If we set

$$E_{n+1+i} = F_i,\ t_{n+1+i} = u_i$$

for $i \in \mathbf{N}$, then $\langle (E_i, t_i)\rangle_{i\in\mathbf{N}}$ is a generalized McShane partition of S subordinate to Δ. So

$$\begin{aligned}\left\|\nu E - \sum_{i\le n}\mu E_i\phi(t_i)\right\| &\\ \le \limsup_{m\to\infty}&\left(\left\|\nu S - \sum_{i\le n+m}\mu E_i\phi(t_i)\right\| + \left\|\nu(S\setminus E) - \sum_{i<m}\mu F_i\phi(u_i)\right\|\right)\\ \le \varepsilon,&\end{aligned}$$

as required.

2C Proposition. *Let $(S, \mathfrak{T}, \Sigma, \mu)$ be a σ-finite outer regular quasi-Radon measure space, X a Banach space and ϕ: $S \to X$ a McShane integrable function. Then there is a gauge Δ: $S \to \mathfrak{T}$ such that $\sum_{i\in\mathbf{N}}\mu E_i\phi(t_i)$ exists, as an unconditional sum for the norm topology of X, whenever $\langle (E_i, t_i)\rangle_{i\in\mathbf{N}}$ is a generalized McShane partition of S subordinate to Δ.*

Proof. Let w be the McShane integral of ϕ. Let $\langle S_k\rangle_{k\in\mathbf{N}}$ be an increasing sequence of open sets of finite measure with union S. For each $k \in \mathbf{N}$ let Δ_k: $S \to \mathfrak{T}$ be a gauge such that

$$\left\|\sum_{i\le n}\mu E_i\phi(t_i) - \int_E \phi\right\| \le 2^{-k}$$

whenever $\langle (E_i, t_i)\rangle_{i\le n}$ is a partial McShane partition of S subordinate to Δ_k and $E = \bigcup_{i\le n} E_i$ (see 2B). For $t \in S$ set $k(t) = \min\{k: t \in S_k, \|\phi(t)\| \le k\}$,

$$\Delta(t) = S_{k(t)} \cap \bigcap_{j\le k(t)} \Delta_j(t);$$

then Δ is a gauge on S.

Suppose that $\langle(E_i, t_i)\rangle_{i\in\mathbf{N}}$ is a generalized McShane partition of S subordinate to Δ, and $\varepsilon > 0$. Let $p \in \mathbf{N}$ be such that $2^{-p} \leq \varepsilon$. Set

$$I = \{i: i \in \mathbf{N},\, k(t_i) \leq p\},\ J = \mathbf{N}\setminus I.$$

By the remarks in 2A, $\sum_{i\in J}\int_{E_i}\phi$ exists in X and there is an $r_0 \in \mathbf{N}$ such that

$$\left\|\sum_{i\in J,\, m\leq i\leq n}\int_{E_i}\phi\right\| \leq \varepsilon$$

whenever $r_0 \leq m \leq n$. Next, $E_i \subseteq S_p$ for every $i \in I$, so $\sum_{i\in I}\mu E_i$ is finite and there is an $r_1 \geq r_0$ such that $p\sum_{i\in I,\, i\geq r_1}\mu E_i \leq \varepsilon$.

Now suppose that $r_1 \leq m \leq n$. We have

$$\left\|\sum_{i\in J,\, m\leq i\leq n}\int_{E_i}\phi\right\| \leq \varepsilon,$$

$$\left\|\sum_{i\in J,\, m\leq i\leq n}\mu E_i\phi(t_i) - \sum_{i\in J,\, m\leq i\leq n}\int_{E_i}\phi\right\| \leq 2^{-p} \leq \varepsilon$$

(because $E_i \subseteq \Delta(t_i) \subseteq \Delta_p(t_i)$ for $i \in J$),

$$\left\|\sum_{i\in I,\, m\leq i\leq n}\mu E_i\phi(t_i)\right\| \leq p\sum_{i\in I,\, m\leq i\leq n}\mu E_i \leq \varepsilon.$$

Adding,

$$\left\|\sum_{m\leq i\leq n}\mu E_i\phi(t_i)\right\| \leq 3\varepsilon.$$

As ε is arbitrary, $\langle\sum_{i\leq n}\mu E_i\phi(t_i)\rangle_{n\in\mathbf{N}}$ is a Cauchy sequence and has a limit in X.

Of course the same argument applies to any rearrangement of $\langle(E_i, t_i)\rangle_{i\in\mathbf{N}}$, so the series $\sum_{i\in\mathbf{N}}\mu E_i\phi(t_i)$ sums unconditionally.

2D Corollary. *Let $(S, \mathfrak{T}, \Sigma, \mu)$ be a σ-finite outer regular quasi-Radon measure space and X a Banach space. Then a function ϕ: $S \to X$ is McShane integrable, with McShane integral w, iff for every $\varepsilon > 0$ there is a gauge Δ: $S \to \mathfrak{T}$ such that $\sum_{i\in\mathbf{N}}\mu E_i\phi(t_i)$ exists and $\|\sum_{i\in\mathbf{N}}\mu E_i\phi(t_i) - w\| \leq \varepsilon$ whenever $\langle(E_i, t_i)\rangle_{i\in\mathbf{N}}$ is a generalized McShane partition of S subordinate to Δ.*

2E Proposition. *Let $(S, \mathfrak{T}, \Sigma, \mu)$ be a σ-finite outer regular quasi-Radon measure space and X a Banach space. Let $E \in \Sigma$ and let ϕ: $S \to X$ be a* *function which is zero on $S \setminus E$. Then ϕ is McShane integrable iff $\phi \restriction E$ is McShane integrable, and in this case the integrals are equal.*

Proof. The case $E = \emptyset$ is trivial and as usual I will ignore it. We have already seen in Theorem 1N that if ϕ is McShane integrable then $\phi \restriction E$ is McShane integrable. Now suppose that $\phi \restriction E$ is McShane integrable with integral w. Let $\varepsilon > 0$. By 2B, there is a gauge $\Delta_0 \colon E \to \mathfrak{T}$ such that

$$\left\| \sum_{i \le n} \mu H_i \phi(t_i) - \int_H \phi \right\| \le \varepsilon$$

whenever $\langle (H_i, t_i) \rangle_{i \le n}$ is a partial McShane partition of E subordinate to Δ_0 and $H = \bigcup_{i \le n} H_i$. Let ν be the indefinite integral of $\phi \restriction E$, so that ν is countably additive (1P, 2A(i)). By 2A(ii) we can find a $\delta > 0$ such that $\|\nu H\| \le \varepsilon$ whenever $H \subseteq E$ and $\mu H \le \delta$; now there is a closed set $F \subseteq E$ such that $\mu(E \setminus F) \le \delta$ (see the second sentence of 1Bc).

For each $n \in \mathbf{N}$ choose an open set $G_n \supseteq E$ such that $\mu(G_n \setminus E) \le 2^{-n}\varepsilon/(n+1)$. Now define $\Delta \colon S \to \mathfrak{T}$ by setting

$$\Delta(t) = \begin{cases} \Delta_0(t) \cap G_n & \text{if } t \in E,\ n \le \|\phi(t)\| < n+1, \\ S \setminus F & \text{if } t \in S \setminus E. \end{cases}$$

Let $\langle (E_i, t_i) \rangle_{i \in \mathbf{N}}$ be a generalized McShane partition of S subordinate to Δ. For $i \in \mathbf{N}$ set $E_i' = E_i \cap E$ if $t_i \in E$, $\emptyset$ otherwise; now set $U_n = \bigcup_{i \le n} E_i'$ for $n \in \mathbf{N}$, and $U = \bigcup_{n \in \mathbf{N}} U_n$. By the choice of Δ_0,

$$\left\| \sum_{i \le n} \mu E_i' \phi(t_i) - \nu U_n \right\| \le \varepsilon \ \forall n \in \mathbf{N}.$$

Next,

$$\begin{aligned} &\left\| \sum_{i \le n} \mu E_i \phi(t_i) - \sum_{i \le n} \mu E_i' \phi(t_i) \right\| \\ &\quad = \left\| \sum_{i \le n,\, t_i \in E} \mu(E_i \setminus E) \phi(t_i) \right\| \\ &\quad \le \sum_{k \in \mathbf{N}} \sum_{i \le n,\, t_i \in E,\, k \le \|\phi(t_i)\| < k+1} \mu(E_i \setminus E) \|\phi(t_i)\| \\ &\quad \le \sum_{k \in \mathbf{N}} (k+1) \mu(G_k \setminus E) \\ &\quad \le \sum_{k \in \mathbf{N}} 2^{-k} \varepsilon = 2\varepsilon, \end{aligned}$$

so

$$\left\| \sum_{i \le n} \mu E_i \phi(t_i) - \nu U_n \right\| \le 3\varepsilon$$

for every n. Now $\nu U = \lim_{n \to \infty} \nu U_n$, and $\mu(F \setminus U) = \mu(F \setminus \bigcup_{i \in \mathbf{N}} E_i) = 0$ because $F \cap E_i = \emptyset$ whenever $t_i \notin E$; so $\mu(E \setminus U) \le \delta$ and $\|w - \nu U\| \le \varepsilon$. Accordingly

$$\limsup_{n \to \infty} \left\| \sum_{i \le n} \mu E_i \phi(t_i) - w \right\| \le 4\varepsilon.$$

As ε is arbitrary, ϕ is McShane integrable, with integral w.

2F Corollary. *If $(S, \mathfrak{T}, \Sigma, \mu)$ is a σ-finite outer regular quasi-Radon measure space, X is a Banach space and ϕ: $S \to X$ is a McShane integrable function, then $\phi \times \chi(E)$ is McShane integrable for every $E \in \Sigma$.*

Proof. For $\phi \restriction E = (\phi \times \chi(E)) \restriction E$.

2G Corollary. *If $(S, \mathfrak{T}, \Sigma, \mu)$ is a σ-finite outer regular quasi-Radon measure space, X is a Banach space and ϕ: $S \to X$ is zero almost everywhere, then it is McShane integrable, with integral* 0.

Proof. For if $E = \{t: \phi(t) \ne 0\}$, $\int \phi \restriction E = 0$.

3. The Talagrand integral

I come now to a discussion of the relationship between the McShane integral, as I have defined it, and the Talagrand integral.

3A Definitions. Let (S, Σ, μ) be a probability space and X a Banach space, with dual X^*.

(a) A function ϕ: $S \to X$ is *Talagrand integrable*, with *Talagrand integral* w, if

$$w = \lim_{n \to \infty} \frac{1}{n} \sum_{i < n} \phi(s_i)$$

for almost all sequences $\langle s_i \rangle_{i \in \mathbf{N}} \in S^{\mathbf{N}}$, where $S^{\mathbf{N}}$ is given its product probability. (See [20], Theorem 8.)

(b) Recall that a set A of real-valued functions on S is *stable* (in Talagrand's terminology) if for every $E \in \Sigma$, with $\mu E > 0$, and all real numbers $\alpha < \beta$, there are $m, n \ge 1$ such that $\mu^*_{m+n} Z(A, E, m, n, \alpha, \beta) <$

$(\mu E)^{m+n}$, where throughout the rest of this paper I write $Z(A, E, I, J, \alpha, \beta)$ for

$$\{(t,u): t \in E^I, u \in E^J, \exists f \in A, f(t(i)) \leq \alpha\ \forall i \in I, f(u(j)) \geq \beta\ \forall j \in J\},$$

and μ^*_{m+n} is the ordinary product outer measure on $S^m \times S^n$.

(c) Now if X is a Banach space, a function ϕ: $S \to X$ is *properly measurable* if $\{h\phi$: $h \in X^*$, $\|h\| \leq 1\}$ is stable. Talagrand proved ([20], Theorem 8) that ϕ is Talagrand integrable iff it is properly measurable and the upper integral $\overline{\int}\|\phi(t)\|\mu(dt)$ is finite.

(d) We shall need to know that if $A \subseteq \mathbf{R}^S$ is stable, then $\overline{A}$, the closure of A in $\mathbf{R}^S$ for the usual product topology of $\mathbf{R}^S$, is stable; this is because

$$Z(\overline{A}, E, I, J, \alpha, \beta) \subseteq Z(A, E, I, J, \alpha', \beta') \text{ whenever } \alpha < \alpha' < \beta' < \beta.$$

The next proposition requires a lemma about gauges in quasi-Radon spaces.

3B Lemma. *Let $(S, \mathfrak{T}, \Sigma, \mu)$ be a quasi-Radon probability space and Δ: $S \to \mathfrak{T}$ a gauge. Then*

(a) $\{x$: $x \in S^{\mathbf{N}}$, $\mu(\bigcup_{i \in \mathbf{N}}\Delta(x(i))) = 1\}$ *has outer measure* 1 *in* $S^{\mathbf{N}}$;

(b) *writing μ_n for the quasi-Radon product measure on S^n, we have*

$$\lim_{n\to\infty} \overline{\int}\mu\Big(\bigcup_{i<n}\Delta(u(i))\Big)\mu_n(du) = 1.$$

Remark. The definition and properties of product quasi-Radon measures are sketched in [7], A7E and discussed in detail in [6]. For the purposes of this paper it would be enough to prove the lemma with μ_n the ordinary product measure of S^n. The crucial fact is that both product measures satisfy Fubini's theorem in the sense that if I, J are disjoint sets and $\mu_I, \mu_J, \mu_{I\cup J}$ the measures of S^I, etc., then for any $\mu_{I\cup J}$-measurable set $W \subseteq S^{I\cup J}$ we have almost every section $W_u = \{v$: $u \frown v \in W\}$ measurable, and $\int\mu_J(W_u)\mu_I(du) = \mu_{I\cup J}W$.

Proof. (a) Suppose, if possible, otherwise.

(i) Set $h(x) = \mu(\bigcup_{i\in\mathbf{N}}\Delta(x(i)))$ for each $x \in S^{\mathbf{N}}$. For any set I let μ_I be the product quasi-Radon measure on S^I.

There is supposed to be a closed set $W \subseteq S^{\mathbf{N}}$ such that $\mu_{\mathbf{N}}W > 0$ and $h(x) < 1$ for every $x \in W$. Set

$$T = \bigcup_{n\in\mathbf{N}}\big\{u: u \in S^n, \mu_{\mathbf{N}\setminus n}\{v: v \in S^{\mathbf{N}\setminus n}, uv \in W\} > 0\big\}.$$

For $u \in T$ set $g(u) = \mu(\bigcup_{i<\mathrm{dom}(u)} \Delta(u(i)))$. Note that every member of T has a proper extension which is still in T. Choose a sequence $\langle u_n \rangle_{n\in\mathbf{N}}$ in T as follows. u_0 is to be the empty sequence. Given $u_n \in T$, choose $u_{n+1} \in T$ such that u_{n+1} properly extends u_n and $g(u_{n+1}) \geq \sup\{g(u): u_n \subset u \in T\} - 2^{-n}$. Now we see that if $u_n \in S^{k(n)}$ for each n, $\langle k(n) \rangle_{n\in\mathbf{N}}$ is strictly increasing, so $x = \bigcup_{n\in\mathbf{N}} u_n \in S^{\mathbf{N}}$, also, for each $n \in \mathbf{N}$,

$$\{v: v \in S^{\mathbf{N}\setminus k(n)}, (x \restriction k(n))^\frown v \in W\} \neq \emptyset,$$

so $x \in W$ because W is closed. Consequently $h(x) < 1$.

Let $F \subseteq S \setminus \bigcup_{i\in\mathbf{N}} \Delta(x(i))$ be a non-empty self-supporting closed set, so that $\mu(F \cap G) > 0$ for every open set G meeting F. Then, in particular, $\mu(F \cap \Delta(t)) > 0$ for every $t \in F$, so there is a $\delta > 0$ such that $\mu^* D > 0$, where

$$D = \{t: t \in F, \mu(F \cap \Delta(t)) \geq \delta\}.$$

(ii) Because $\langle g(u_n) \rangle_{n\cap\mathbf{N}}$ is a bounded sequence, there is an $n \in \mathbf{N}$ such that $g(u_{n+1}) - g(u_n) + 2^{-n} < \delta$. We have

$$\mu_{\mathbf{N}\setminus k(n)}\{v: u_n^\frown v \in W\} > 0,$$

while

$$\mu^*_{\mathbf{N}\setminus k(n)}\{v: \exists i \geq k(n), v(i) \in D\} = 1,$$

so there is some $i \geq k(n)$ such that

$$\mu^*_{\mathbf{N}\setminus k(n)}\{v: u_n^\frown v \in W, v(i) \in D\} > 0.$$

Set $m = i + 1$,

$$E = \{w: w \in S^{m\setminus k(n)}, \mu_{\mathbf{N}\setminus m}\{y: u_n^\frown w^\frown y \in W\} > 0\};$$

then E is $\mu_{m\setminus k(n)}$-measurable and

$$\mu_{\mathbf{N}\setminus k(n)}\{v: u_n^\frown v \in W, v \restriction m \setminus k(n) \notin E\} = 0.$$

Consequently there is a $v \in S^{\mathbf{N}\setminus k(n)}$ such that $v \restriction m \setminus k(n) \in E$ and $v(i) \in D$. But now consider

$$u = u_n^\frown (v \restriction m \setminus k(n)).$$

We see that $u \in T$ and $u_n \subset u$, so

$$g(u) \leq g(u_{n+1}) + 2^{-n}.$$

On the other hand, $u(i) \in D$, so

$$g(u) - g(u_n) \geq \mu\Big(\Delta(u(i)) \setminus \bigcup_{j<k(n)} \Delta(u(j))\Big)$$
$$\geq \mu(\Delta(u(i)) \cap F) \geq \delta.$$

Thus

$$g(u_{n+1}) \geq g(u) - 2^{-n} \geq g(u_n) - 2^{-n} + \delta,$$

contrary to the choice of n.

This contradiction proves the first part of the lemma.

(b) The second part follows. For each $n \in \mathbf{N}$ define $h_n: S^{\mathbf{N}} \to [a,b]$ by setting

$$h_n(x) = \mu\Big(\bigcup_{i<n} \Delta(x(i))\Big) \quad \forall x \in S^{\mathbf{N}}.$$

Then $\lim_{n\to\infty} h_n(x) = h(x)$ for every x, so

$$1 = \overline{\int} h(x)\mu_{\mathbf{N}}(dx) = \lim_{n\to\infty} \overline{\int} h_n(x)\mu_{\mathbf{N}}(dx) = \lim_{n\to\infty} \overline{\int} \mu\Big(\bigcup_{i<n} \Delta(u(i))\Big)\mu_n(du),$$

as required.

3C Proposition. *Let $(S, \mathfrak{T}, \Sigma, \mu)$ be a quasi-Radon probability space and X a Banach space. Let $\phi: S \to X$ be a McShane integrable function, and write*

$$C = \{f\phi: f \in X^*, \|f\| \leq 1\}.$$

Then any countable subset of C is stable.

Proof. (a) Let A be a countable subset of C. Take $E \in \Sigma$, with $\mu E > 0$, and $\alpha < \beta$ in $\mathbf{R}$. For $m, n \geq 1$ set $H_{mn} = Z(A, E, m, n, \alpha, \beta)$; note that as A is countable, H_{mn} is μ_{m+n}-measurable. I seek an m with $\mu_{2m} H_{mm} < (\mu E)^{2m}$.

Set $\varepsilon = \frac{1}{6}(\beta - \alpha)\mu E > 0$. By Lemma 2B above, there is a gauge $\Delta: S \to \mathfrak{T}$ such that

$$\Big\| \sum_{i\leq n} \mu H_i \phi(t_i) - \int_H \phi \Big\| \leq \varepsilon$$

whenever $\langle (E_i, t_i)\rangle_{i \le n}$ is a partial McShane partition of S subordinate to Δ and $\bigcup_{i \le n} E_i = H$. The set E, with its induced topology and measure, is a quasi-Radon measure space. So we may apply 3B to E, with an appropriate normalization of its measure, to see that there is an $m \in \mathbf{N}$ such that $\mu_m^* D > 0$, where

$$D = \left\{ t\colon t \in E^m,\ \mu\left(\bigcup_{i<m} E \cap \Delta(t(i)) \right) \ge \tfrac{3}{4}\mu E \right\}.$$

Suppose, if possible, that $\mu_{2m} H_{mm} = (\mu E)^{2m}$. Then H_{mm} must meet D^2; take $t, u \in D$ such that $(t, u) \in H_{mm}$. Set

$$H = \bigcup_{i<m} \Delta(t(i)) \cap \bigcup_{i<m} \Delta(u(i));$$

then $\mu H \ge \frac{1}{2}\mu E$.

Choose disjoint covers $\langle E_i \rangle_{i<m}$, $\langle F_i \rangle_{i<m}$ of H by measurable sets such that $E_i \subseteq \Delta(t(i))$ and $F_i \subseteq \Delta(u(i))$ for each $i < m$. Then we must have

$$\left\| \sum_{i<m} \mu E_i \phi(t(i)) - \mu F_i \phi(u(i)) \right\| \le 2\varepsilon.$$

Now $(t, u) \in H_{mm}$, so there is an $f \in A$ such that $f(t(i)) \le \alpha$ and $f(u(i)) \ge \beta$ for every $i < m$. f is of the form $h\phi$ for some h of norm at most 1, so

$$\left| \sum_{i<m} \mu E_i . f(t(i)) - \mu F_i . f(u(i)) \right| \le 2\varepsilon.$$

However, $f(t(i)) \le \alpha$ for each i and $\sum_{i<m} \mu E_i = \mu H$, so

$$\sum_{i<m} \mu E_i . f(t(i)) \le \alpha \mu H;$$

similarly $\sum_{i<m} \mu F_i . f(u(i)) \ge \beta \mu H$, and we get

$$2\varepsilon \ge (\beta - \alpha)\mu H \ge (\beta - \alpha)\tfrac{1}{2}\mu E = 3\varepsilon,$$

which is absurd.

This shows that A is indeed stable.

3D Corollary. *Let $(S, \mathfrak{T}, \Sigma, \mu)$ be a quasi-Radon probability space and X a Banach space such that the unit ball of X^* is w*-separable; let $\phi\colon S \to X$ be a McShane integrable function. Then ϕ is properly measurable, and if $\overline{\int} \|\phi(s)\| \mu(ds) < \infty$, then ϕ is Talagrand integrable.*

Proof. Let B_0 be a countable w^*-dense subset of the unit ball B of X^*. Then $A = \{f\phi\colon f \in B_0\}$ is stable, by 3C. But because $f \mapsto f\phi\colon X^* \to \mathbf{R}^S$ is continuous for the w^*-topology on X^* and the pointwise topology of $\mathbf{R}^S$, $C = \{f\phi\colon f \in B\}$ is the pointwise closure of A, and is therefore stable, by 3Ad. Accordingly ϕ is properly measurable. Now the second clause follows by Talagrand's theorem.

Remark. These generalise Proposition 2L and Corollary 2M of [10]. Note that they become false if we omit the hypothesis that the unit ball of X^* is w^*-separable; see [10], 3A.

3E Corollary. *Let $(S, \mathfrak{T}, \Sigma, \mu)$ be a σ-finite outer regular quasi-Radon measure space and X a Banach space. If $\phi\colon S \to X$ is McShane integrable then its indefinite Pettis integral has totally bounded range.*

Proof. (a) Consider first the case $\mu S = 1$. By 4-1-5 of [19], it is enough to show that $C = \{f\phi\colon f \in X^*,\ \|f\| \leq 1\}$ is totally bounded for $\|\ \ \|_1$. This will be so iff every countable subset of C is totally bounded. But 3C shows that any countable subset of C is stable, and therefore totally bounded by [19], 9-5-2.

(b) It follows at once that the result is true whenever $\mu S < \infty$. For the general case, let $\varepsilon > 0$. The indefinite integral ν of ϕ is countably additive, so there must be a set $E \subseteq S$, of finite measure, such that $\|\nu H\| \leq \varepsilon$ whenever $H \subseteq S \setminus E$ is measurable. Now $\{\nu H\colon H \in \Sigma,\ H \subseteq E\}$ is totally bounded, being the range of the indefinite integral of $\phi \restriction E$, so is covered by finitely many ε-balls; and therefore the range of ν itself is covered by finitely many 2ε-balls.

4. Convergence theorems

I generalize and refine some results from [10].

4A Theorem. *Let $(S, \mathfrak{T}, \Sigma, \mu)$ be a σ-finite outer regular quasi-Radon measure space and X a Banach space. Let $\langle \phi_n \rangle_{n \in \mathbf{N}}$ be a sequence of McShane integrable functions from S to X, and suppose that $\phi(t) = \lim_{n \to \infty} \phi_n(t)$ exists in X, for the weak topology of X, for almost every $t \in S$. If moreover the limit*

$$\nu E = \lim_{n \to \infty} \int_E \phi_n$$

exists in X, for the weak topology, for every $E \in \Sigma$, then ϕ is McShane integrable and $\int \phi = \nu S$.

Proof. Fix $\varepsilon > 0$.

(a) Let $h: S \to \mathbf{R}$ be a strictly positive function such that $\int h \leq \varepsilon$. Let Γ be the set

$$\left\{(r, \alpha_0, \dots, \alpha_n): r, n \in \mathbf{N},\ \alpha_0, \dots, \alpha_n \in \mathbf{Q} \cap [0,1],\ \sum_{i \leq n} \alpha_i = 1\right\}.$$

For $\gamma = (r, \alpha_0, \dots, \alpha_n) \in \Gamma$, write

$$r_\gamma = r, \quad \phi_\gamma = \sum_{i \leq n} \alpha_i \phi_i : S \to X$$

and

$$A_\gamma = \Big\{t: t \in S,\ \sup_{i \in \mathbf{N}} \|\phi_i(t)\| \leq r,\ h(t) \geq (r+1)^{-1},$$
$$\|\phi(t) - \phi_\gamma(t)\| \leq h(t)\Big\}.$$

Note that $\mu^* A_\gamma \leq (r+1)\int h < \infty$; choose a measurable set $V_\gamma \supseteq A_\gamma$ such that $\mu V_\gamma = \mu^* A_\gamma$.

(b) If $\gamma \in \Gamma$ and $H \subseteq V_\gamma$ is a measurable set then

$$\left\| \nu H - \int_H \phi_\gamma \right\| \leq \int_H h.$$

For take any f in the unit ball of X^*. Then

$$f(\nu H) = \lim_{n \to \infty} f\left(\int_H \phi_n\right) = \lim_{n \to \infty} \int_H f\phi_n.$$

But the sequence $\langle f\phi_n \rangle_{n \in \mathbf{N}}$ of measurable functions is uniformly bounded on $A_\gamma \cap H$, which has the same outer measure as H; so in fact it is uniformly bounded almost everywhere on H, and by Lebesgue's theorem

$$\lim_{n \to \infty} \int_H f\phi_n = \int_H \left(\lim_{n \to \infty} f\phi_n\right) = \int_H f\phi.$$

Now

$$\left| f\left(\nu H - \int_H \phi_\gamma\right)\right| = \left| \int_H (f\phi - f\phi_\gamma)\right| \leq \int_H |f\phi - f\phi_\gamma| \leq \int_H h$$

because $|f\phi(t) - f\phi_\gamma(t)| \leq \|\phi(t) - \phi_\gamma(t)\| \leq h(t)$ for every $t \in A_\gamma \cap H$, and therefore for almost every $t \in H$. As f is arbitrary, $\|\nu H - \int_H \phi_\gamma\| \leq \int_H h$.

(c) Because $\phi(t)$ is in the norm-closed convex hull of $\{\phi_n(t): n \in \mathbf{N}\}$ for every t, and this is always a bounded set, $S = \bigcup_{\gamma \in \Gamma} A_\gamma$, and we can find a disjoint family $\langle A'_\gamma \rangle_{\gamma \in \Gamma}$ of sets, covering S, such that $A'_\gamma \subseteq A_\gamma$ for every γ. Let $\langle \varepsilon_\gamma \rangle_{\gamma \in \Gamma}$ be a family of strictly positive real numbers such that $\sum_{\gamma \in \Gamma}(r_\gamma + 1)\varepsilon_\gamma \le \varepsilon$.

For each γ, let $\delta_\gamma > 0$ be such that $\|\nu E\| \le \varepsilon_\gamma$ whenever $\mu E \le \delta_\gamma$ (see 2A(ii)–(iii) above); let $G_\gamma \supseteq V_\gamma$ be an open set such that $\mu(G_\gamma \setminus V_\gamma) \le \min(\varepsilon_\gamma, \delta_\gamma)$. Let $\Delta_\gamma: S \to \mathfrak{T}$ be a gauge such that

$$\left\| \int_E \phi_\gamma - \sum_{i \le n} \mu E_i \phi_\gamma(t_i) \right\| \le \varepsilon_\gamma$$

whenever $\langle (E_i, t_i) \rangle_{i \le n}$ is a partial McShane partition of S subordinate to Δ_γ and $E = \bigcup_{i \le n} E_i$; such exists by 2B (using 1Ca–b to see that ϕ_γ is McShane integrable). Applying 2B again (or 1J), this time to h, there is a gauge Δ^*: $S \to \mathfrak{T}$ such that $\sum_{i \le n} \mu E_i h(t_i) \le 2\varepsilon$ whenever $\langle (E_i, t_i) \rangle_{i \le n}$ is a partial McShane partition of S subordinate to Δ^*. Now set

$$\Delta(t) = \Delta_\gamma(t) \cap G_\gamma \cap \Delta^*(t)$$

for every $t \in A'_\gamma$, $\gamma \in \Gamma$; then $\Delta: S \to \mathfrak{T}$ is a gauge.

(d) Let $\langle (E_i, t_i) \rangle_{i \in \mathbf{N}}$ be a generalized McShane partition of S subordinate to Δ. I seek to estimate $\nu S - x_n$, where $x_n = \sum_{i \le n} \mu E_i \phi(t_i)$. Fix n for the moment.

Set $I_\gamma = \{i: i \le n,\ t_i \in A'_\gamma\}$ for each γ; of course all but finitely many of the I_γ are empty. For $i \in I_\gamma$, set $E'_i = E_i \cap V_\gamma$. We have $E_i \subseteq \Delta(t_i) \subseteq G_\gamma$, so

$$\sum_{i \in I_\gamma} \mu(E_i \setminus E'_i) \le \varepsilon_\gamma,$$

and

$$\sum_{i \in I_\gamma} \mu(E_i \setminus E'_i) \|\phi(t_i)\| \le r_\gamma \varepsilon_\gamma,$$

because $\|\phi(t)\| \le r_\gamma$ for $t \in A_\gamma$. Consequently, if we write

$$y_0 = \sum_{i \le n} \mu E'_i \phi(t_i),$$

we shall have $\|x_n - y_0\| \le \sum_{\gamma \in \Gamma} r_\gamma \varepsilon_\gamma \le \varepsilon$.

For each $i \le n$, let $\gamma(i)$ be such that $t_i \in A'_{\gamma(i)}$. Then we have

$$\|\phi(t_i) - \phi_{\gamma(i)}(t_i)\| \le h(t_i) \quad \text{for each } i.$$

So

$$\sum_{i\le n}\mu E_i'\|\phi(t_i)-\phi_{\gamma(i)}(t_i)\|\le\sum_{i\le n}\mu E_i h(t_i)\le 2\varepsilon,$$

because $E_i\subseteq\Delta^*(t_i)$ for each i. Accordingly, writing

$$y_1=\sum_{i\le n}\mu E_i'\phi_{\gamma(i)}(t_i),$$

we have $\|x_n-y_1\|\le 3\varepsilon$.

Set $H_\gamma'=\bigcup\{E_i': i\in I_\gamma\}$ for each γ. Because $E_i'\subseteq\Delta_\gamma(t_i)$ for each $i\in I_\gamma$, we have

$$\left\|\sum_{i\in I_\gamma}\mu E_i'\phi_\gamma(t_i)-\int_{H_\gamma'}\phi_\gamma\right\|\le\varepsilon_\gamma.$$

Consequently, writing

$$y_2=\sum_{\gamma\in\Gamma}\int_{H_\gamma'}\phi_\gamma,$$

we have $\|y_1-y_2\|\le\Sigma_{\gamma\in\Gamma}\varepsilon_\gamma\le\varepsilon$ and $\|x_n-y_2\|\le 4\varepsilon$.

Next, for any γ, $H_\gamma'\subseteq V_\gamma$, so we have

$$\left\|\nu H_\gamma'-\int_{H_\gamma'}\phi_\gamma\right\|\le\int_{H_\gamma'}h,$$

by (b) above. So writing $y_3=\Sigma_{\gamma\in\Gamma}\nu H_\gamma'$ we have $\|y_2-y_3\|\le\int h$ and $\|x_n-y_3\|\le 5\varepsilon$.

If we set $H_\gamma=\bigcup\{E_i: i\in I_\gamma\}$, then $\mu(H_\gamma\setminus H_\gamma')\le\delta_\gamma$, so that

$$\|\nu H_\gamma-\nu H_\gamma'\|\le\varepsilon_\gamma\quad\text{for each }\gamma.$$

Accordingly $\|x_n-y_4\|\le 6\varepsilon$, where

$$y_4=\sum_{\gamma\in\Gamma}\nu H_\gamma=\nu\Big(\bigcup_{\gamma\in\Gamma}H_\gamma\Big)=\nu\Big(\bigcup_{i\le n}E_i\Big).$$

Thus

$$\left\|\nu\Big(\bigcup_{i\le n}E_i\Big)-\sum_{i\le n}\mu E_i\phi(t_i)\right\|\le 6\varepsilon.$$

Because ν is countably additive (2A(iii)) and $\nu E = 0$ whenever $\mu E = 0$,

$$\limsup_{n\to\infty}\left\|\nu S - \sum_{i\le n}\mu E_i\phi(t_i)\right\| \le 6\varepsilon.$$

This shows that ϕ is McShane integrable, with integral νS.

Remark. This strengthens and generalizes Theorem 2I of [10].

4B Corollary. *Let $(S, \mathfrak{T}, \Sigma, \mu)$ be a σ-finite outer regular quasi-Radon measure space and X a Banach space. Let ϕ: $S \to X$ be a Pettis integrable function and $\langle E_i\rangle_{i\in\mathbf{N}}$ a cover of S by measurable sets. Suppose that $\phi \restriction E_i$ is McShane integrable for each i. Then ϕ is McShane integrable.*

Proof. Apply 4A with $\phi_n(t) = \phi(t)$ for $t \in \bigcup_{i\le n}E_i$, 0 elsewhere.

4C Corollary. *Let $(S, \mathfrak{T}, \Sigma, \mu)$ be a σ-finite outer regular quasi-Radon measure space and X a separable Banach space. Then a function ϕ: $S \to X$ is McShane integrable iff it is Pettis integrable.*

Proof. If ϕ is McShane integrable then it is Pettis integrable, by 1Q. Conversely, if it is Pettis integrable, then for each $k \in \mathbf{N}$, set

$$S_k = \{t: t \in S, \|\phi(t)\| \le k\}.$$

Because X is separable, every S_k is measurable; moreover, $\phi \restriction S_k$ is Bochner integrable, therefore McShane integrable, by 1K. Now ϕ itself is McShane integrable by 4B.

4D Corollary. *Let $(S, \mathfrak{T}, \Sigma, \mu)$ be a σ-finite outer regular quasi-Radon measure space and X a Banach space. Let ϕ: $S \to X$ be a Pettis integrable function which is measurable in the sense that $\phi^{-1}[G] \in \Sigma$ for every norm-open set $G \subseteq X$. If* **either** *$(S, \mathfrak{T}, \Sigma, \mu)$ is a Radon measure space* **or** *there is no real-valued-measurable cardinal, then ϕ is McShane integrable.*

Proof. The point is that there is a separable closed linear subspace Y of X such that $\mu(S \setminus E) = 0$, where $E = \phi^{-1}[Y]$; see [5], §2. Now $\phi \restriction E$ is McShane integrable by 4C and $\phi \restriction S \setminus E$ is McShane integrable by 2G.

Remark. For $S = [0, 1]$ this is Theorem 17 of [13].

4E Theorem. *Let $(S, \mathfrak{T}, \Sigma, \mu)$ be a σ-finite outer regular quasi-Radon measure space and X a Banach space. Let $\langle\phi_n\rangle_{n\in\mathbf{N}}$ be a sequence of McShane integrable functions from S to X such that $\phi(t) = \lim_{n\to\infty}\phi_n(t)$ exists in X, for* *the weak (resp. norm) topology of X, for almost every $t \in S$. If*

$$C = \{f\phi_n : f \in X^*, \|f\| \leq 1, n \in \mathbf{N}\}$$

is uniformly integrable, then ϕ is McShane integrable, and $\int \phi = \lim_{n \to \infty} \int \phi_n$ for the weak (resp. norm) topology of X.

Proof. (a) Let $f \in X^*$. Then $\langle f\phi_n \rangle_{n \in \mathbf{N}}$ is a uniformly integrable sequence of real-valued functions which is convergent at almost every point of S to $f\phi$. Consequently $\lim_{n \to \infty} \int_E f\phi_n$ is defined for every $E \in \Sigma$, and has modulus at most $M\|f\|$, where $M = \sup_{h \in C} \int |h| < \infty$. We therefore have an indefinite Dunford integral $\nu: \Sigma \to X^{**}$ of ϕ, and $(\nu E)(f) = \lim_{n \to \infty} \int_E f\phi_n$ for every $f \in X^*$, $E \in \Sigma$; evidently ν is additive. In fact it is countably additive. For if $\langle E_n \rangle_{n \in \mathbf{N}}$ is an increasing sequence in Σ, with union E, then $\lim_{n \to \infty} M_n = 0$, where $M_n = \sup_{h \in C} \int_{E \setminus E_n} |h|$, and $\|\nu E - \nu E_n\| \leq M_n$ for each n.

Now following the argument for Theorem 4A line by line we find that it proves that ϕ is McShane integrable as a function from S to X^{**}, with McShane integral νS. But of course νS is now approximated, in norm, by sums of the form $\sum_{i \leq n} \mu E_i \phi(t_i)$, which belong to X, and therefore $\nu S \in X$ and ϕ is McShane integrable as a function from S to X.

(b) This deals with the case in which $\langle \phi_n(t) \rangle_{n \in \mathbf{N}}$ is weakly convergent for almost all t. Observe that we must have $\int_F \phi = \nu F$, the weak limit of $\langle \int_F \phi_n \rangle_{n \in \mathbf{N}}$, for every $F \in \Sigma$ (applying the result to $\langle \phi_n \restriction F \rangle_{n \in \mathbf{N}}$).

Now suppose that in fact $\langle \phi_n(t) \rangle_{n \in \mathbf{N}}$ is norm-convergent for almost all t, and let $\varepsilon > 0$. Because C is uniformly integrable, there are a set $E \in \Sigma$, of finite measure, and a $\delta > 0$ such that $\int_F |h| \leq \varepsilon$ whenever $F \in \Sigma$ and $\mu(F \cap E) \leq \delta$; so that $\|\int_F \phi_n\| \leq \varepsilon$ and $\|\int_F \phi\| \leq \varepsilon$ whenever $n \in \mathbf{N}$, $F \in \Sigma$ and $\mu(E \cap F) \leq \delta$.

For each $n \in \mathbf{N}$, set

$$A_n = \{t : t \in E, \|\phi_m(t) - \phi(t)\| \leq \varepsilon/(1 + \mu E) \quad \text{for every } m \geq n\}.$$

Then $\mu(E \setminus \bigcup_{n \in \mathbf{N}} A_n) = 0$ so there is an $n \in \mathbf{N}$ such that $\mu^* A_n \geq \mu E - \delta$. Let $G \in \Sigma$ be such that $A_n \subseteq G \subseteq E$ and $\mu G = \mu^* A_n$. Then

$$\begin{aligned} \left\| \int_G \phi - \int_G \phi_m \right\| &= \left\| \int_{A_n} \phi - \int_{A_n} \phi_m \right\| \\ &\leq \overline{\int}_{A_n} \|\phi - \phi_m\| \\ &\leq \varepsilon \mu^* A_n / (1 + \mu E) \\ &\leq \varepsilon \end{aligned}$$

for every $m \geq n$. But also $\| \int_{S \setminus G} \phi \| \leq \varepsilon$ and $\| \int_{S \setminus G} \phi_m \| \leq \varepsilon$ for every m, because $\mu(E \setminus G) \leq \delta$, so $\| \int \phi - \int \phi_m \| \leq 3\varepsilon$ for every $m \geq n$. As ε is arbitrary, this shows that $\int \phi$ is the norm limit of $\langle \int \phi_n \rangle_{n \in \mathbf{N}}$, as claimed.

4F Corollary. *Let $(S, \mathfrak{T}, \Sigma, \mu)$ be a σ-finite outer regular quasi-Radon measure space and X a Banach space. Let $\langle \phi_n \rangle_{n \in \mathbf{N}}$ be a sequence of McShane integrable functions from S to X such that $\phi(t) = \lim_{n \to \infty} \phi_n(t)$ exists in X, for the weak (resp. norm) topology of X, for almost every $t \in S$. If*

$$\overline{\int} \sup_{n \in \mathbf{N}} \| \phi_n(t) \| \mu(dt) < \infty,$$

then ϕ is McShane integrable, and $\int \phi = \lim_{n \to \infty} \int \phi_n$ for the weak (resp. norm) topology of X.

4*G Problems.* I conclude with some questions left open by the work above.

(a) Suppose that (S, Σ, μ) is a σ-finite measure space, X is a Banach space, and ϕ: $S \to X$ is a function. Suppose that $\mathfrak{T}_1$ and $\mathfrak{T}_2$ are two topologies on S making it an outer regular quasi-Radon measure space. If ϕ is McShane integrable for $\mathfrak{T}_1$, must it be McShane integrable for $\mathfrak{T}_2$?

The point here is that a given measure space can have a wide variety of different quasi-Radon topologies on it. Consider, for instance, the case in which $S = [0, 1]$ and μ is Lebesgue measure. In this case we have the usual topology; the right-facing Sorgenfrey (or 'half-open interval') topology, generated by sets of the form $[s, t[$; the left-facing Sorgenfrey topology; and, for any strong lifting, the associated lifting topology ([14], p. 58, or [6], 3G). All of these make μ quasi-Radon. I believe that I can prove that the usual topology and the two Sorgenfrey topologies give the same McShane integrable functions; for lifting topologies there seem to be difficulties.

I should remark that if the unit ball of X^* is w^*-separable, then all quasi-Radon topologies on S give the same McShane integrable functions; see [9].

(b) Suppose that $(S, \mathfrak{T}, \Sigma, \mu)$ is a quasi-Radon probability space, X is a Banach space, and ϕ: $S \to X$ a *Pettis* integrable function. Does it follow that the indefinite integral of ϕ has totally bounded range?

This problem arises in the context of 3E. I showed there that the indefinite integral of a McShane integrable function has totally bounded range. But the question is, whether this is due to the special properties of the McShane integral, or to the special properties of the underlying measure space (S, Σ, μ). (Indefinite Pettis integrals in general do not always have totally bounded ranges; see [11], 2D, or [19], 13-3-3.)

(c) Suppose that $(S, \mathfrak{T}, \Sigma, \mu)$ is a σ-finite outer regular quasi-Radon measure space, that X is a Banach space and that ν: $\Sigma \to X$ is a function. Under what conditions will ν be the indefinite integral of a McShane integrable function from S to X? This can only be so if μ is countably additive (2A(i)) and has totally bounded range (3E). These conditions are certainly not sufficient ([10], 3C); I do not know of any useful general sufficient condition, discounting such as 'X has the Radon-Nikodým property', which makes ν the indefinite integral of a Bochner integrable function.

Acknowledgement. I should like to thank I. Dobrakov and A. Martellotti for useful conversations.

References

1. J. Diestel, *Sequences and series in Banach spaces*, Springer, New York, 1984.
2. N. Dunford and J.T. Schwartz, *Linear operators I*, Interscience, New York, 1958.
3. J. Diestel and J.J. Uhl Jr., *Vector measures* Amer. Math. Soc., Providence, R.I., 1977.
4. D.H. Fremlin, *Topological Riesz spaces and measure theory*. Cambridge University Press, Cambridge, 1974.
5. ______, *Measurable functions and almost continuous functions*, Manuscripta Math. **33** (1981), 387–405.
6. ______, *Quasi-Radon measure spaces*, Note of 2.6.82, University of Essex.
7. ______, *Consequences of Martin's Axiom*. Cambridge University Press, Cambridge, 1984.
8. ______, *On the Henstock and McShane integrals of vector-valued functions*, Illinois J. Math., to appear.
9. ______, *The McShane and Birkhoff integrals of vector-valued functions*, University of Essex Mathematics Department Research Report 92–10.
10. D.H. Fremlin and J. Mendoza, *On the integration of vector-valued functions*, Illinois J. Math. **38** (1994), 127–147.
11. D.H. Fremlin and M. Talagrand, *A decomposition theorem for additive set-functions, with applications to Pettis integrals and ergodic means*, Math. Zeitschr., **168** (1979), 117–142.
12. R.J. Gardner and W.F. Pfeffer, "*Borel measures*" in *Handbook of set-theoretic topology*, North-Holland, Amsterdam, pp. 961–1043.
13. R.A. Gordon, *The McShane integral of Banach-valued functions*, Illinois J. Math. **34** (1990), 557–567.
14. A. Ionescu Tulcea and C. Ionescu Tulcea, *Topics in the theory of lifting*. Springer, New York, 1969.
15. K. Kunen and J.E. Vaughan, *Handbook of set-theoretic topology*, North-Holland, Amsterdam, 1984.
16. E.J. McShane, *Unified integration*, Academic Press, San Diego, 1983.
17. B. Rodriguez-Salinas, Quasi-Radon measures and Radon measures of type ($\mathscr{H}$), Rend. Circ. Mat. Palermo **90** (1991), 142–152.
18. B. Rodriguez-Salinas and P. Jimenez Guerra, *Medidas de radón de tipo* ($\mathscr{H}$) *en espacios topológicos arbitrarios*, Mem. Real Acad. Ciencias Madrid, Serie de Ciencias Exactas no. 10, 1979.
19. M. Talagrand, *Pettis integral and measure theory*, Mem. Amer. Math. Soc., no. 307, 1984.
20. ______, *The Glivenko-Cantelli problem*, Ann. Probab. **15** (1987), 837–870.

University of Essex
Colchester, England