ILLINOIS JOURNAL OF MATHEMATICS
Volume 39, Number 2, Summer 1995

A NOTE ON CONFORMAL VECTOR FIELDS AND POSITIVE CURVATURE

DAGANG YANG[1]

Introduction

All known examples of compact Riemannian manifolds with positive sectional curvature carries a positively curved metric with a continuous Lie group as its group of isometries and thus carries a nontrivial vector field of infinitesimal isometries, i.e., a Killing vector field. However, due to a theorem of M. Berger [1], such a Killing vector field must be singular at least at one point if the manifold is even dimensional. This is related to a well-known conjecture that for an even dimensional closed positively curved Riemannian manifold, its Euler characteristic is positive (cf. [4]). It is easy to see that Berger's theorem remains true for conformal vector fields (see also [3]). On the other hand, the Euler characteristic of a closed odd dimensional manifold is always zero. There are many simple examples of odd dimensional closed positively curved Riemannian manifolds which carry nonsingular Killing vector fields. The simplest example is perhaps the round 3-dimensional sphere S^3, which admits 3 pointwise linearly independent Killing vector fields while any two of them do not commute. This is obvious if one considers S^3 from the Lie group theoretic point of view.

The aim of this note is to give a generalization of M. Berger's theorem to odd dimensional manifolds.

THEOREM. *On a closed odd dimensional Riemannian manifold of positive sectional curvature, each pair of commutative conformal vector fields are dependent at least at one point.*

Remark 1. Analogous to the above mentioned conjecture, one might expect the following: On a closed positively curved odd dimensional Riemannian manifold, each pair of commutative vector fields are dependent at least at one point.

Received December 1, 1992.
1991 Mathematics Subject Classification. 53C.
[1]Partially supported by a grant from the National Science Foundation.

As an application, we have:

COROLLARY. *On a compact Lie group G, except that the universal covering of G is the 3-sphere S^3, every smooth Riemannian metric conformal to a left invariant one has nonpositive sectional curvature somewhere.*

Remark 2. It is well-known that every compact Lie group carries left invariant metrics of nonnegative sectional curvature. However, these nonnegatively curved left invariant metrics cannot be conformally deformed to one with strictly positive sectional curvature.

Remark 3. This corollary is a generalization of a result by N. Wallach in [2] where he proved the corollary for left invariant metrics on G.

The results of this note were obtained while the author was at the University of Pennsylvania. The author would like to thank Professor W. Ziller for his encouragement and to Professor M. Berger for kindly referring him to the results in [3].

1. Preliminaries

Throughout this note, (M, g) will be a connected closed smooth Riemannian manifold. All global and local vector fields will be smooth ones. The capital letter X will be reserved exclusively for global conformal vector fields. Thus, whenever we have $X, X_1, X_2, \ldots,$ and etc., they will all be conformal vector fields. Two vector fields V and W are said to be commutative if their Lie bracket is a zero vector field, i.e., if

$$[V, W] = 0.$$

Recall that a nontrivial vector field X on (M, g) is said to be conformal if for all smooth vector fields V and W, we have

$$Xg(V, W) - g([X, V], W) - g(V[X, W]) = f_X g(V, W) \tag{1}$$

for some smooth function $f_X\colon M \to R$.

This function f_X can easily be determined on the open subset $\Omega = \{p \in M | X(p) \neq 0\}$. Setting $V = W = X$ in equation (1), we obtain

$$f_x = X \ln g(X, X). \tag{2}$$

Notice that Ω is dense in M. In case f_X is identically zero on M, then equation (1) is reduced to

$$Xg(V,W) - g([X,V],W) - g(V,[X,W]) = 0. \tag{3}$$

Thus X is a vector field of infinitesimal isometries, or more often called a Killing vector field.

LEMMA 1. *Let X be a conformal vector field on (M,G). For any local or global vector fields V which commutes with X, i.e., if $[X,V] = 0$, we have the following two identities*:

$$g(X,X)Xg(V,V) \equiv g(V,V)Xg(X,X) \tag{4}$$

$$g(X,X)X^2g(V,V) \equiv g(V,V)X^2g(X,X). \tag{5}$$

Proof. The identities (4) and (5) are obvious on $M \setminus \Omega$. On Ω, set $W = V$ in equation (1); we have

$$Xg(V,V) \equiv f_X g(V,V)$$

since $[X,V] = 0$. Identity (4) now follows from equation (2) and the above equation. Identity (5) is obtained by taking Lie derivative to (4) along the direction X. Q.E.D.

DEFINITION. A conformal vector field X is said to be Killing at a point $p \in M$ if $f_X(p) = 0$.

Example. Assume that $p \in M$ is a critical point of the smooth function $g(X,X)$ and $X(p) \neq 0$. Then $f_X(p) = X(p)\ln g(X,X) = 0$ and X is Killing at p.

Let ∇ be the Levi Civita connection of (M,g). Equation (1) is then equivalent to the following equation.

$$g(\nabla_V X, W) + g(V, \nabla_W X) = f_X g(V,W) \tag{6}$$

for all smooth vector fields V and W on M.

Let A_X be the tensor field of type $(1,1)$ defined by

$$A_X V = \nabla_V X \tag{7}$$

for all vectors V in the tangent space of M. It follows from equation (6) that A_X is skew-symmetric at a point $p \in M$ if and only if X is Killing at p.

Let R be the curvature tensor of type $(3, 1)$ of the Levi Civita connection ∇, so

$$R(V, W)Y = \nabla_V \nabla_W Y - \nabla_W \nabla_V Y - \nabla_{[V,W]} Y \tag{8}$$

for all vector fields V, W, and Y on M.

LEMMA 2. *Let V be a smooth local vector field on an open subset $U \subset M$ such that $[X, V] = 0$ and $g(X, V) = 0$ on U, where X is a conformal vector field on M. Assume that $g(X, X) \neq 0$ on U. Then*

$$\begin{aligned} g(V, R(V, X)X) = &-\frac{1}{2}\left\{V^2 g(X, X) + g(V, V)\frac{X^2 g(X, X)}{g(X, X)}\right\} \\ &- g(\nabla_V V, \nabla_X X) + g(\nabla_V X, \nabla_V X). \end{aligned} \tag{9}$$

Proof. Since X and V are commutative and orthogonal, $\nabla_X V = \nabla_V X$, we have

$$\begin{aligned} g(V, R(V, X)X) &= g(V, \nabla_V \nabla_X X) - g(V, \nabla_X \nabla_V X) \\ &= V_g(V, \nabla_X X) - g(\nabla_V V, \nabla_X X) \\ &\quad - Xg(V, \nabla_V X) + g(\nabla_V X, \nabla_V X), \\ g(V, \nabla_X X) = -g(\nabla_X V, X) &= -g(\nabla_V X, X) = -\tfrac{1}{2} Vg(X, X), \\ g(V, \nabla_V X) = g(V, \nabla_X V) &= \tfrac{1}{2} Xg(V, V). \end{aligned}$$

Equations (9) now follows by substituting the right hand sides of the last two equations into the first equation and then applying the second identity in Lemma 1. Q.E.D.

2. Proof of the theorem

To prove the theorem, it suffices to show that if (M, g) is an odd dimensional closed Riemannian manifold of nonnegative sectional curvature which carries two commutative pointwise linearly independent conformal vector fields, then there is a point $q \in M$ and a 2-plane σ in $T_q M$ such that the sectional curvature $K(\sigma)$ at the 2-plane σ is zero.

Thus let X_1 and X_2 be two commutative pointwise linearly independent conformal vector fields on (M, g). For each $t \in R$, set

$$X(t) = \cos tX_1 + \sin tX_2. \tag{10}$$

Thus $X(t)$ is a 1-parameter family of nonvanishing conformal vector fields on (M, g).

Consider the function

$$h\colon M \times R \to R \tag{11}$$

defined by $h(p,t) = g(X(t)(p), X(t)(p))$. h is obviously a smooth positive function periodic in the second factor of period 2π. Since M is compact, h attains its positive minimum at some point, say, $(q, t_0) \in M \times R$. Notice that by a rotation,

$$\begin{aligned}\overline{X}_1 &= \cos t_0 X_1 + \sin t_0 X_2\\ \overline{X}_2 &= -\sin t_0 X_1 + \cos t_0 X_2.\end{aligned}$$

One may, for convenience, assume that $t_0 = 0$. Thus, h attains its positive minimum value at $(q, 0) \in M \times R$. It follows that for any vector field V defined in a neighborhood of q in M, the matrix

$$\begin{bmatrix} V^2 h & V\dfrac{\partial}{\partial t}h \\ V\dfrac{\partial}{\partial t}h & \dfrac{\partial^2}{\partial t^2}h \end{bmatrix} \tag{12}$$

is nonnegatively definite at $(q,0)$, $Vh(q,0) = 0$, and $\dfrac{\partial h}{\partial t}(q,0) = 0$. This has the following implications:

$$g_{12}(q) = 0, \qquad g_{22}(q) \geq g_{11}(q) = \min h \tag{13}$$

$$A_{X_1}X_1(q) = \nabla_{X_1(q)}X_1 = 0 \tag{14}$$

$$V^2 g_{11}(q) \geq 0 \tag{15}$$

$$\left\{(g_{22} - g_{11})V^2 g_{11} - 2(Vg_{12})^2\right\}_q \geq 0 \tag{16}$$

for all vector field V defined in a neighborhood of q in M, where $g_{ij} = g(X_i, X_j)$, $i, j = 1, 2$. Moreover, X_1 is Killing at q and A_{X_1} is skew-symmetric at q.

Suppose that there is a nonzero vector $V \in T_qM$ such that $g(V, X_1(q)) = 0$, and $A_{X_1}V = 0$. One can extend V to a local vector field in a neighborhood U of q such that $g(V, X_1) = 0$ and $[X_1, V] = 0$ on U. It follows from equations (9), (14) and (15) that

$$g(V, R(V, X_1)X_1)|_q \leq 0.$$

Since (M, g) is of nonnegative sectional curvature, the sectional curvature $K(\sigma)$ of the plane σ spanned by $V(q)$ and $X_1(q)$ is

$$K(\sigma) = \frac{g(V, R(V, X_1)X_1)}{g_{11}g(V,V)}\Big|_q = 0.$$

Therefore, to complete the proof of the theorem, one need only to show:

LEMMA 3. *Assume that (M, g) is an odd dimensional closed Riemannian manifold of nonnegative sectional curvature. If h attains a positive minimum at $(q, 0)$, then the dimension of the kernel of $A_{X_1}: T_qM \to T_qM$ is at least* 3.

Proof of Lemma 3. Let $\dim M = 2n + 1 \geq 3$. Since $A_{X_1}X_1(q) = 0$ and A_{X_1} is skew-symmetric on T_qM, there is an orthonormal basis $\{e, e_1, e_2, \ldots, e_n, E_1, E_2, \ldots, E_n\}$ for T_qM and real numbers $0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ such that

$$X_1(q) = g_{11}^{1/2}(q)e \tag{17}$$

$$A_{X_1}e_i = \lambda_i E_i, \qquad A_{X_1}E_i = -\lambda_i e_i, \qquad i = 1, 2, \ldots, n. \tag{18}$$

We must show that $\lambda_1 = 0$.

Suppose on the contrary, $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. Since X_1 is conformal and $X_1(q) \neq 0$, one can easily extend $\{X_1(q), e_1, e_2, \ldots, e_n, E_1, E_2, \ldots, E_n\}$ to a local frame $Y_1, Y_2, \ldots, Y_{2n+1}$ in an open neighborhood U of q such that the following conditions are satisfied:

$$\begin{aligned} &Y_1 = X_1 \\ &Y_{2i}(q) = e_i, \qquad Y_{2i+1}(q) = E_i, \quad i = 1, 2, \ldots, n \\ &g(Y_1, Y_i) = 0, \qquad [Y_1, Y_i] = 0, \quad i = 2, 3, \ldots, 2n + 1. \end{aligned} \tag{19}$$

Since $g_{12}(q) = 0$, we have

$$X_2(q) = \sum_{i=1}^{n} (a_i e_i + b_i E_i) \tag{20}$$

for some constants a_i and b_i, $i = 1, 2, \ldots, n$. Set

$$V = \sum_{i=1}^{n} \lambda_i^{-1}(a_i Y_{2i+1} - b_i Y_{2i}). \tag{21}$$

It is clear that $V(q) \neq 0$ since $X_2(q) \neq 0$. Furthermore,

$$g(X_1, V) = 0, \quad [X_1, V] = 0 \quad \text{on } U \tag{22}$$

$$\nabla_{V(q)} X_1 = A_{X_1} V(q) = -X_2(q). \tag{23}$$

We now claim that

$$g(X_1, \nabla_{V(q)} X_2) = g(X_2, \nabla_{V(q)} X_1) = -g_{22}(q). \tag{24}$$

Indeed, since X_2 is a conformal vector field and $g(X_1, V) = 0$, it follows from equation (6) that

(25)
$$g(X_1, \nabla_{V(q)} X_2) = f_{X_2} g(X_1, V(q)) - g(V(q), \nabla_{X_1} X_2) = -g(V(q), \nabla_{X_1} X_2).$$

Since $[X_1, X_2] = 0$, we have $\nabla_{X_1} X_2 = \nabla_{X_2} X_1$, so

$$g(X_1, \nabla_{V(q)} X_2) = -g(V(q), \nabla_{X_2} X_1). \tag{26}$$

Now use the fact that X_1 is a conformal vector field and X_1 is Killing at q, i.e., $f_{X_1}(q) = 0$, equation (6) yields

$$\begin{aligned} g(V(q), \nabla_{X_2} X_1) &= f_{X_1}(q) g(X_2, V(q)) - g(X_2, \nabla_{V(q)} X_1) \\ &= -g(X_2, \nabla_{V(q)} X_1). \end{aligned} \tag{27}$$

Combine equations (23), (26), and (27), the claim (24) is proved. Thus

$$\begin{aligned} V g_{12}(q) &= V(q) g(X_1, X_2) \\ &= g(X_1, \nabla_{V(q)} X_2) + g(X_2, \nabla_{V(q)} X_1) \\ &= -2 g_{22}(q). \end{aligned} \tag{28}$$

Evaluate equation (9) at q and solve for $V^2_{g_{11}}(q)$, we obtain

(29)
$$V^2 g_{11}(q) = \{-2g(V, R(V, X_1)X_1) - g(V, V) g_{11}^{-1} X_1^2 g_{11} + 2 g_{22}\}_q.$$

It follows now from equations (13), (15), (28), and (29) that

$$\begin{aligned} &\{(g_{22} - g_{11}) V^2 g_{11} - 2(V g_{12})^2\}_q \\ &\quad = \{-(g_{22} - g_{11})(2g(V, R(V, X_1)X_1) \\ &\qquad\qquad + g(V, V) g_{11}^{-1} X_1^2 g_{11}) - 2 g_{11} g_{22} - 6 g_{22}^2\}_q \\ &\quad < 0. \end{aligned} \tag{30}$$

This contradicts with the fact that h attains its positive minimum at $(q, 0)$. Therefore, we must have $\lambda_1 = 0$. This proves that the dimension of the kernel of $A_{X_1}: T_qM \to T_qM$ is at least 3. The proofs of Lemma 3 and the theorem are now completed. Q.E.D.

3. An application

Let G be a compact connected Lie group whose universal covering is not S^3 and let g be a Riemannian metric conformal to a left invariant metric on G. Thus each right invariant vector field on G is a nonsingular conformal vector field. If G is even dimensional, then (G, g) can not have strictly positive sectional curvature since M. Berger's theorem (cf.[1]) remains true for conformal vector fields. If G is odd dimensional and $\dim G \geq 3$, we first notice that the rank of G is at least 2 except that its universal covering is S^3. Therefore, there exist at least two commutative right invariant and therefore conformal vector fields on (G, g). It follows from the theorem that (G, g) does not have strictly positive sectional curvature. Thus the corollary is proved.

References

1. M. Berger, *Trois remarques sur les varietes riemanniennes a courbure positive*, C.R. Acad. Sc. Paris Serie A. **263** (1966), 76–78.
2. N. Wallach, *Compact homogeneous Riemannian manifolds with strictly positive curvature*, Ann. Math. **96** (1972), 277–295.
3. A. Weinstein, *A fixed point theorem for positively curved manifolds*, J. Math. Mech. **18** (1968), 149–153.
4. S.T. Yau, Problem Section, #8, Seminar on differential geometry, Study 102, Princeton University Press.

Tulane University
New Orleans Louisiana