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RESTRICTION THEOREMS RELATED TO ATOMS

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Introduction

Let $\mathbb{R}^n$ be n-dimensional real Euclidean space and let S^{n-1} be the unit sphere in $\mathbb{R}^n$. Suppose that $d\sigma = d\sigma(x')$ is the element of Lebesgue measure on S^{n-1} so that the measure of S^{n-1} is 1. If $d\mu = \psi d\sigma$ is a measure with smooth density ψ, then from [9] or [10] we know that the Fourier transform of $d\mu$ satisfies $d\hat{\mu}(\xi) = O(|\xi|^{-\varepsilon})$ as $|\xi| \to \infty$, for some $\varepsilon > 0$. It turns out that if the density ψ is merely in $L^p(d\sigma)$, for some $p > 1$, then there is still an average decrease of $d\hat{\mu}$ at infinity along any ray emanating from the origin. More precisely, suppose that ψ is in $L^p(d\sigma)$, then

$$(*) \qquad R^{-1}\int_O^R |d\hat{\mu}(\rho\xi)|^2 d\rho \leq A(R|\xi|)^{-\varepsilon},$$

where $\varepsilon < (1-p^{-1})/2$, and A is a positive constant independent of $R|\xi|$ (see [10]). The estimate $(*)$ has the following application.

Let $\Omega(x)|x|^{-n}$ be a homogeneous function of degree $-n$, with $\Omega \in L^p(S^{n-1})$, for some $p > 1$, and $\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0$. Let $r \to b(r)$ be a bounded function on $(0, \infty)$. We consider the distribution $K = \text{P.V.}b(|x|)\Omega(x)|x|^{-n}$ and study the boundedness of the operator Tf which is defined by $Tf = f * K$. This operator was studied extensively and its boundedness properties were established in R. Fefferman [7], Namazi [8], Duoandikoetxea and Rubio de Francia [4] and Chen [1]. In his new significant book [9], by using $(*)$, E. M. Stein gives an alternative proof to conclude that, under the restriction $n \geq 2$, the mapping $f \to f * K$ extends to a bounded operator in $L^2(\mathbb{R}^n)$. Meanwhile, he points out that the condition $b \in L^\infty(0, \infty)$ can be replaced by a weaker condition (see pages 372–373 in [10]; also see [4]):

$$(1) \qquad R^{-1}\int_O^R |b(\rho)|^2 d\rho \leq A \text{ for all } R > 0.$$

In this paper, we shall study $d\mu = \psi d\sigma$ where the density ψ is an atom. As an application, we will prove that if $\Omega(x')$ is merely in the Hardy space $H^1(S^{n-1})$ with mean zero property and if, for some $p > 1$, the radial function $b(|x|)$ satisfies

$$(1') \qquad R^{-1}\int_O^R |b(\rho)|^p d\rho \leq A \text{ for all } R > O,$$

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then the operator $Tf = f*K$, with $K = \text{P.V.}\, b(|x|)\Omega(x')|x|^{-n}$, extends to a bounded operator in $L^2(\mathbb{R}^n)$, for $n > 1$. Clearly, our result significantly improves the above mentioned L^2 boundedness property. It also improves the result in our previous paper [5]. The proofs in this paper are modifications of those in [5].

Recall that the Poisson kernel on S^{n-1} is defined by

$$P_{ry'}(x') = (1-r^2)/|ry' - x'|^n,$$

where $0 \le r < 1$ and $x', y' \in S^{n-1}$. For any $f \in \mathcal{S}'(S^{n-1})$, we define the radial maximal function $P^+f(x')$ by

$$P^+f(x') = \sup_{0 \le r < 1} \left| \int_{S^{n-1}} f(y') P_{rx'}(y')\, d\sigma(y') \right|,$$

where $\mathcal{S}'(S^{n-1})$ is the space of Schwartz distributions on S^{n-1}.

The Hardy space $H^1(S^{n-1})$ is the linear space of distribution $f \in \mathcal{S}'(S^{n-1})$ with the finite norm $\|f\|_{H^1(S^{n-1})} = \|P^+f\|_{L^1(S^{n-1})} < \infty$. The Hardy space $H^1(S^{n-1})$ was studied in [2] (see also [3]). In particular, a well-known result is $L^1(S^{n-1}) \supseteq H^1(S^{n-1}) \supseteq L^q(S^{n-1})$ for any $q > 1$. Another important property of $H^1(S^{n-1})$ is the atomic decomposition of $H^1(S^{n-1})$, which will be reviewed in the following:

An *exceptional atom* is an L^∞ function $E(x)$ satisfying $\|E\|_\infty \le 1$.

A *regular atom* is an L^∞ function $a(x)$ that satisfies

(i) $\quad \operatorname{supp}(a) \subset \{x' \in S^{n-1}, |x' - x_0'| < \rho \text{ for some } x_0' \in S^{n-1} \text{ and } \rho > 0\}$,

(ii) $$\int_{S^{n-1}} a(\xi')\, d\sigma(\xi') = 0,$$

(iii) $$\|a\|_\infty \le \rho^{-n+1}.$$

From [3], we find that any $\Omega \in H^1(S^{n-1})$ has an atomic decomposition $\Omega(\xi') = \sum \lambda_j a_j(\xi')$, where the a_j's are either exceptional atoms or regular atoms and $\sum |\lambda_j| \le C\|\Omega\|_{H^1(S^{n-1})}$.

We have the following restriction theorem for atoms:

THEOREM 1. *Let I_k be the interval $(2^k, 2^{k+1})$. Suppose that $a(\xi')$ is an atom on S^{n-1}. Then for any $q > 1$,*

$$\sum_{k=0}^{\infty} \left(\int_{I_k} t^{-1} \left| \int_{S^{n-1}} a(\xi') e^{it\langle x', \xi' \rangle}\, d\sigma(\xi') \right|^q dt \right)^{1/q} < A, \tag{2}$$

where A is a constant independent of $x' \in S^{n-1}$ and the atom $a(x)$.

Theorem 1 has the following consequence.

THEOREM 2. *Suppose that Ω is a homogeneous function of degree zero and satisfies the mean zero property $\int_{S^{n-1}} \Omega(\xi')\,d\sigma(\xi') = 0$. If $b(x)$ satisfies (1′) for some $p > 1$ and Ω is a function in $H^1(S^{n-1})$, $n > 1$, then the operator $Tf = f * K$, with $K = \text{P.V. } b(|x|)\Omega(x)|x|^{-n}$, is bounded in $L^2(\mathbb{R}^n)$.*

As an analogue of formula $(*)$, we have the following:

THEOREM 3. *Suppose that $d\sigma = d\sigma(x')$ is the Lebesgue measure on S^{n-1}. If $d\mu = \psi d\sigma$ with a density ψ in $H^1(S^{n-1})$, then we have*

$$(**) \qquad R^{-1}\int_0^R |d\hat{\mu}(t\xi)|\,dt = o(1) \text{ as } R|\xi| \to \infty.$$

Proof of Theorem 1

We first prove Theorem 1 for a regular atom $a(\xi')$. Let

$$A_k = \left\{ \int_{I_k} t^{-1} \left| \int_{S^{n-1}} a(\xi') e^{it\langle x', \xi'\rangle}\, d\sigma(\xi') \right|^q dt \right\}^{1/q}.$$

We will prove the theorem in the two different cases $n > 2$ and $n = 2$, respectively.

Case $n > 2$. For a regular atom $a(x')$ with $\text{supp}(a) \subseteq B(x_0', \rho) \subseteq S^{n-1}$, without loss of generality, we may assume that ρ is very small. Let $\mathbf{1} = (1, 0, \dots, 0)$ be the north pole of S^{n-1}. By a rotation we can assume that $x' = \mathbf{1}$. Let $\xi' = (s, \xi_2, \dots, \xi_n)$; then

$$A_k \le A \left\{ \int_{I_k} t^{-1} |F\hat{\ }(t)|^q\, dt \right\}^{1/q},$$

where

$$F(s) = (1 - s^2)^{(n-3)/2} \chi_{(-1,1)}(s) \int_{S^{n-2}} a(s, (1 - s^2)^{1/2} y')\, d\sigma(y'),$$

$d\sigma(y')$ is the Lebesgue measure on S^{n-2} and $F\hat{\ }(t)$ is the Fourier transform of $F(s)$. Now we easily see that $\int_{\mathbb{R}} F(s)\,ds = 0$. Furthermore, we can check that, up to a constant independent of the atom $a(\xi')$, F is a regular atom on $\mathbb{R}$. Since the computation is tedious but similar to the simplest case $\text{supp}(a) \subseteq B(\mathbf{1}, \rho)$, we examine this fact for $\text{supp}(a) \subseteq B(\mathbf{1}, \rho)$ (more details can be found in [6]). For small ρ, we may assume

$$\text{supp}(a) \subseteq \{\xi' = (s, \xi_2, \dots, \xi_n) \in S^{n-1} : (s-1)^2 + \xi_2^2 + \cdots + \xi_n^2 < \rho^2\}.$$

Clearly this implies $\text{supp}(F) \subseteq (1 - \rho^2/2, 1)$ and $\|F\| \le C\rho^{-2}$. Now assume that the atom F has support $(s_0 - r, s_0 + r)$. If $2^{k+1} \le r^{-1}$, by the cancellation condition of $F(s)$, we easily see that

$$A_k \le r\left\{\int_{I_k} t^{q-1}\,dt\right\}^{1/q} = A\,2^k r.$$

Thus, we obtain

$$\sum_{2^{k+1}\le r^{-1}} A_k \le A\,r \sum_{2^{k+1}\le r^{-1}} 2^k \le A. \tag{3}$$

To estimate A_k for $2^{k+1} \ge r^{-1}$, using Hölder's inequality, we obtain

$$A_k \le \left(\int_{I_k} t^{-2}\,dt\right)^{1/2q}\left(\int_{\mathbb{R}} |F\,\hat{(t)}|^{2q}\,dt\right)^{1/2q}.$$

By the Hausdorff-Young inequality, we have

$$A_k \le A\,2^{-k/2q}\|F\|_{L^{2q/2q-1}(\mathbb{R})} \le A\,r^{-1/2q}2^{-k/2q}.$$

Therefore,

$$\sum_{2^k\ge 1/r} A_k \le A\,r^{-1/2q}\sum_{2^k\ge 1/r} 2^{-k/2q} \le A. \tag{4}$$

This proves the theorem for the case $n > 2$.

Case $n = 2$. In this case $\Sigma_1 = \mathbb{T}$, the one-dimensional torus. As before, we may assume that $\operatorname{supp}(a) \subseteq (-\rho, \rho)$. Let $x' = (\cos\alpha, \sin\alpha)$. Then for $2^k \le \rho^{-1}$,

$$A_k = \left(\int_{I_k} t^{-1}\left|\int_{-\pi}^{\pi} a(\theta)(e^{it\cos(\theta-\alpha)} - 1)\,d\theta\right|^q dt\right)^{1/q} \le A(2^k\rho).$$

Thus $\sum_{2^k\le 1/\rho} A_k \le A$.

Next we only prove the case $\cos\alpha \ne 0$ and $\sin\alpha \ne 0$, since the estimates for these two cases are easier than in the prior case. Also we assume $\sin\alpha > 0$. For $1/\rho \le 2^k \le \rho^{-2}$,

$$\begin{aligned}
A_k &= \left(\int_{I_k} t^{-1}\left|\int_{-\pi}^{\pi} a(\theta)e^{it\cos\theta\cos\alpha}e^{it\sin\theta\sin\alpha}\,d\theta\right|^q dt\right)^{1/q}\\
&= \left|\int_{2^k\cos\alpha}^{2^{k+1}\cos\alpha} t^{-1}\left|\int_{-\pi}^{\pi} a(\theta)e^{it(\cos\theta-1)}e^{it\tan\alpha\sin\theta}\,d\theta\right|^q dt\right|^{1/q}\\
&\le \left|\int_{2^k\cos\alpha}^{2^{k+1}\cos\alpha} t^{-1}\left|\int_{-\pi}^{\pi} a(\theta)e^{it\tan\alpha\sin\theta}\,d\theta\right|^q dt\right|^{1/q}\\
&\quad + A\left|\int_{2^k\cos\alpha}^{2^{k+1}\cos\alpha} t^{-1+q}\left(\int_{-\pi}^{\pi}|a(\theta)(\cos\theta-1)|\,d\theta\right)^q dt\right|^{1/q}\\
&= B_k + C_k.
\end{aligned}$$

It is easy to see that $C_k \le A\ \rho^2(\int_0^{2^{k+1}} t^{-1+q}\, dt)^{1/q} \le A\ \rho^2 2^k$.

$$\begin{aligned}
B_k &= \left| \int_{2^k\cos\alpha}^{2k+1\cos\alpha} t^{-1} \left| \int_{-\pi}^{\pi} a(\theta) e^{it\tan\alpha(\sin\theta-\theta)} e^{it\tan\alpha\theta}\, d\theta \right|^q dt \right|^{1/q} \\
&= \left(\int_{2^k\sin\alpha}^{2^{k+1}\sin\alpha} t^{-1} \left| \int_{-\pi}^{\pi} a(\theta) e^{it(\sin\theta-\theta)} e^{it\theta}\, d\theta \right|^q dt \right)^{1/q} \\
&\le \left(\int_{2^k\sin\alpha}^{2^{k+1}\sin\alpha} t^{-1} \left| \int_{-\pi}^{\pi} a(\theta) e^{it\theta}\, d\theta \right|^q dt \right)^{1/q} \\
&\quad + A \left(\int_{2^k\sin\alpha}^{2^{k+1}\sin\alpha} t^{-1+q} \left| \int_{-\pi}^{\pi} \right| a(\theta)(\sin\theta - \theta)|\, d\theta|^q\, dt \right)^{1/q} \\
&= D_k + E_k.
\end{aligned}$$

Clearly, $E_k \le A\rho^2 2^k$.

If $2^k \le (\rho\sin\alpha)^{-1}$, then D_k can be bounded by

$$\left(\int_{2^k\sin\alpha}^{2^{k+1}\sin\alpha} \left| \int_{-\pi}^{\pi} a(\theta)\{e^{it\theta} - 1\}\, d\theta \right|^q t^{-1}\, dt \right)^{1/q} \le A\rho 2^{k+1}\sin\alpha.$$

If $2^k \ge (\rho\ \sin\alpha)^{-1}$, then, by Hölder's inequality and Hausdorff-Young's inequality, we can find that

$$\begin{aligned}
D_k &= \left(\int_{2^k\sin\alpha}^{2^{k+1}\sin\alpha} |\hat{a}(t)|^q t^{-1}\, dt \right)^{1/q} \\
&\le \left(\int_{2^k\sin\alpha}^{2^{k+1}\sin\alpha} t^{-2}\, dt \right)^{1/2q} \left(\int_{\mathbb{R}} |\hat{a}(t)|^{2q}\, dt \right)^{1/2q} \\
&\le A(\sin\alpha\ 2^k)^{-1/2q} \|a\|_{2q/(2q-1)} \le A(2^k\sin\alpha)^{-1/2q}\rho^{-1/2q}.
\end{aligned}$$

This proves that

$$\begin{aligned}
\sum_{\rho^{-1}\le 2^k \le \rho^{-2}} A_k &\le \sum_{2^k\le\rho^{-2}} C_k + \sum_{\rho^{-1}\le 2^k\le\rho^{-2}} B_k \\
&\le A + \sum_{2^k\le\rho^{-2}} E_k + \sum_{\rho^{-1}\le 2^k\le\rho^{-2}} D_k \\
&\le A + A\ \rho\sin\alpha \sum_{2^k\le(\rho\sin\alpha)^{-1}} 2^k + A \sum_{(\rho\ \sin\alpha)^{-1}\le 2^k} 2^{-k/2q}(\sin\alpha\ \rho)^{-1/2q} \\
&\le A.
\end{aligned}$$

Finally, we estimate $\sum_{2^k \geq 1/\rho^2} A_k$. Since

$$
\begin{aligned}
A_k \leq & \left(\int_{I_k} t^{-1} \left| \int_0^{\pi} a(\theta + \alpha) e^{it\cos\theta} \, d\theta \right|^q dt \right)^{1/q} \\
& + \left(\int_{I_k} t^{-1} \left| \int_{-\pi}^{0} a(\theta + \alpha) e^{it\cos\theta} \, d\theta \right|^q dt \right)^{1/q} = A_{k,1} + A_{k,2},
\end{aligned}
$$

and since the estimates for $A_{k,1}$ and $A_{k,2}$ are exactly the same, without loss of generality we assume $A_k = A_{k,1}$.

Using Hölder's inequality and changing variables $u = \cos\theta$, we have

$$
\begin{aligned}
A_k & \leq 2^{-k/2q} \left(\int_{\mathbb{R}} \left| \int_0^{\rho} a(\theta + \alpha) e^{it\cos\theta} \, d\theta \right|^{2q} \right)^{1/2q} \\
& \leq A 2^{-k/2q} \left(\int_{\mathbb{R}} \left| \int_{\cos\rho}^{1} a(\alpha + \cos^{-1} u)(1 - u^2)^{-1/2} e^{itu} \, du \right|^{2q} dt \right)^{1/q}.
\end{aligned}
$$

Thus, by the Hausdorff-Young inequality again, we find that A_k is bounded by

$$
A\, 2^{-k/2q} \left(\int_{\mathbb{R}} \chi_{(\cos\rho, 1)^{(t)}} |a(\alpha + \cos^{-1} t)(1 - t^2)^{-1/2}|^{2q/(2q-1)} dt \right)^{(2q-1)/2q},
$$

where $\chi_{(\cos\rho,\,1)^{(t)}}$ is the characteristic function of the interval $(\cos\rho, 1)$. Changing variables again, we obtain that

$$
\begin{aligned}
A_k & \leq A\, 2^{-k/2q} \|a\|_{\infty} \left(\int_0^{\rho} |\sin\theta|^{-1/(2q-1)} \, d\theta \right)^{(2q-1)/2q} \\
& \leq A\, 2^{-k/2q} \rho^{-1/q}.
\end{aligned}
$$

Thus we have $\sum_{2^k \geq \rho^{-2}} A_k \leq A$, which completes the proof of Theorem 1 for the regular atom $a(\xi')$. If $a(\xi')$ is an exceptional atom, we can view it as a regular atom supported in S^{n-1} without the cancellation condition. Thus we assume the support of $a(x)$ is contained in a ball with radius $\rho = 1$. If we examine the proof for the case of regular atom $a(x)$, we find that we actually did not use the cancellation condition of $a(x)$ to prove

$$
\sum_{2^k \geq r^{-1}} A_k \leq A \text{ for } n > 2 \quad \text{and} \quad \sum_{2^k \geq \rho^{-2}} A_k \leq A \text{ when } n = 2.
$$

Now, letting $\rho = 1$ and $r = 1$ and mimicking the proof for a regular atom, one has no difficulty proving the theorem for the case of an exceptional atom. This completes the proof for Theorem 1.

Proof of Theorem 2

Let $\hat{K}$ be the Fourier transform of K. By Plancherel's theorem, we only need to prove that

$$\|\hat{K}\|_\infty \le A\|\Omega\|_{H^1(S^{n-1})}. \tag{5}$$

In fact, let $x = |x|x'$ with $x' \in S^{n-1}$. Then by Hölder's inequality one easily sees that, up to a constant, $K\hat{~}(x)$ is bounded by

$$\left(\int_0^1 |b(t|x|^{-1})|^p\,dt\right)^{1/p}\left(\int_0^1\left|\int_{S^{n-1}}\Omega(\xi')(e^{it\langle x',\xi'\rangle}-1)\,d\sigma(\xi')\right|^q t^{-q}\,dt\right)^{1/q}$$
$$+\int_1^\infty |b(t|x|^{-1})|\left|\int_{S^{n-1}}\Omega(\xi')e^{it\langle x',\xi'\rangle}\,d\sigma(\xi')\right|t^{-1}\,dt = I(x)+II(x).$$

Clearly, by (1′), one has

$$\begin{aligned} I(x) &\le A\left(|x|\int_0^{|x|^{-1}}|b(t)|^p\,dt\right)^{1/p}\left(\int_0^1 dt\right)^{1/q}\|\Omega\|_{L^1(S^{n-1})}\\ &\le A\|\Omega\|_{H^1(S^{n-1})} \quad \text{since } \|\Omega\|_{L^1(S^{n-1})}\le A\|\Omega\|_{H^1(S^{n-1})}. \end{aligned}$$

For $II(x)$, we recall that $\Omega(\xi') = \sum\lambda_j a_j(\xi')$, where the a_j's are either exceptional atoms or regular atoms and $\sum|\lambda_j| \le A\|\Omega\|_{H^1(S^{n-1})}$. Therefore it remains to prove that for any atom $a(\xi')$,

$$II_a(x) = \int_1^\infty\left|b(t/|x|)\int_{S^{n-1}}a(\xi')e^{it\langle x',\xi'\rangle}\,d\sigma(\xi')\right|t^{-1}\,dt \le A \tag{6}$$

with a constant A independent of $a(\xi')$ and $x \in \mathbb{R}^n$. In fact,

$$\begin{aligned} II_a(x) &= \sum_{k=0}^\infty\int_{2^k}^{2^{k+1}}\left|b(t/|x|)\int_{S^{n-1}}a(\xi')e^{it\langle x',\xi'\rangle}\,d\sigma(\xi')\right|t^{-1}\,dt\\ &= \sum_{k=0}^\infty L_k(x). \end{aligned}$$

Now by Hölder's inequality and (1′), we find that $L_k(x)$ is dominated by

$$\left(\int_{I_k}|b(t/|x|)|^p\,dt\right)^{1/p}\left(\int_{I_k}t^{-q}\left|\int_{S^{n-1}}a(\xi')e^{it\langle\xi',x'\rangle}\,d\sigma(\xi')\right|^q dt\right)^{1/q}$$
$$\le A\left(|x|2^{-k-1}\int_0^{2^{k+1}/|x|}|b(t)|^p\,dt\right)^{1/p}A_k \le A\,A_k,$$

where $A_k = (\int_{I_k}t^{-1}|\int_{S^{n-1}}a(\xi')e^{it\langle\xi',x'\rangle}\,d\sigma(\xi')|^q\,dt)^{1/q}$. Now Theorem 2 is easily proved by using Theorem 1.

Proof of Theorem 3

To prove the theorem, by changing variables, we only need to prove that as $R \to \infty$,

$$(7) \qquad R^{-1} \int_0^R |d\hat{\mu}(tx')|\, dt = o(1) \quad \text{uniformly for } x' \in S^{n-1}.$$

We know that $\psi \in H^1(S^{n-1})$ has an atomic decomposition $\psi = \sum_{k=1}^{\infty} c_k a_k$ with $\sum |c_k| < \infty$. So for any $\varepsilon > 0$, there exists an N such that $\sum_{k=N}^{\infty} |c_k| < \varepsilon$. For an atom $a(x)$, we let $d\mu_a = a\, d\sigma$. Then it is obvious that

$$R^{-1} \int_0^R |d\hat{\mu}_a(tx')|\, dt \le A$$

with a constant A independent of x', R and the atom $a(x)$. Therefore one easily sees that to prove (7), it suffices to prove

$$(8) \qquad \lim_{R\to\infty} R^{-1} \int_0^R |d\hat{\mu}_a(tx')|\, dt = 0.$$

By Hölder's inequality, we have

$$R^{-1} \int_0^R |d\mu_a(tx')|\, dt \le A \left(R^{-1} \int_0^R |d\hat{\mu}_a(tx')|^2\, dt \right)^{1/2}.$$

Since each atom $a(x)$ is an L^p function, (8) follows easily from (*). Theorem 3 is proved.

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