A NORM INEQUALITY FOR A "FINITE-SECTION" WIENER-HOPF EQUATION

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1. Introduction

We are concerned here with establishing a norm inequality for an equation which arises in a variety of interesting problems. This seemingly simple inequality has a surprisingly large number of applications which we have brought to the reader's attention in §3.

The result concerns the equation

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} h(\theta)f(\theta)e^{-ik\theta}\,d\theta = g_k \qquad (0 \leqq k \leqq n), \tag{1.1}$$

where $f(\theta)$ is a sufficiently nice function, and where $h(\theta)$ is a polynomial of degree n in $e^{i\theta}$. The purpose is to relate the "size" of $h(\theta)$ to the "size" of $g(\theta) = \sum_{k=0}^{n} g_k e^{ik\theta}$. In particular, we find a norm inequality

$$\| h \| \leqq M \| g \|, \tag{1.2}$$

where $\| \cdot \|$ denotes the sum of the absolute values of the coefficients in the polynomials $h(\theta)$ and $g(\theta)$, and where the constant M is independent of the particular $g(\theta)$ and $h(\theta)$ involved. Such an inequality allows one to consider the convergence of a sequence of h's in terms of the corresponding sequence of g's.

Before stating the main result, let us generalize the norm used. Let $\nu(n) \geqq 1$ be a function of the integer n such that $\nu(n) \leqq \nu(m)\nu(n-m)$ for every n, m. Denote by $\mathcal{A}_\nu$ the class of functions $F(\theta)$ integrable over $-\pi \leqq \theta \leqq \pi$ with Fourier coefficients F_k such that

$$\| F \|_\nu \equiv \sum_{-\infty}^{\infty} \nu(n) \mid F_n \mid < \infty. \tag{1.3}$$

Next, let us restrict the class of functions $f(\theta)$ considered in (1.1). Let $f(\theta)$ be integrable over $-\pi \leqq \theta \leqq \pi$ with Fourier coefficients c_k, let $D_n(f) = \det(c_{i-j})$ $(i, j = 0, 1, \cdots, n)$, and let $f(\theta)$ satisfy $\log f(\theta) \in \mathcal{A}_\nu$. In terms of the notation just introduced, equation (1.1) can be written

$$\begin{pmatrix} c_0 & c_{-1} & \cdots & c_{-n} \\ c_1 & c_0 & \cdots & c_{-n+1} \\ \vdots & \vdots & \ddots & \vdots \\ c_n & c_{n-1} & \cdots & c_0 \end{pmatrix} \begin{pmatrix} h_0 \\ h_1 \\ \vdots \\ h_n \end{pmatrix} = \begin{pmatrix} g_0 \\ g_1 \\ \vdots \\ g_n \end{pmatrix}. \tag{1.4}$$

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Our main theorem is as follows:

THEOREM 1.1. *Let* $\log f(\theta) \in \mathfrak{A}_\nu$. *Then, there exist an integer* N *and a constant* M, *both depending only on* $f(\theta)$, *such that for every pair* $g(\theta)$ *and* $h(\theta)$ *of degree* $n \geqq N$ *satisfying* (1.1)

$$\| h \|_\nu \leqq M \| g \|_\nu . \tag{1.5}$$

It is easy to construct an example to show the importance of a condition like $\log f(\theta) \in \mathfrak{A}_\nu$ for the truth of Theorem 1.1. For instance take

$$f(\theta) = 1 - e^{i\theta}, \qquad g(\theta) = 1, \quad \text{and} \quad h(\theta) = \textstyle\sum_{k=0}^{n} e^{ik\theta}.$$

Clearly, $\| h \|_\nu \geqq n + 1$, while $\| g \|_\nu = \nu(0)$. An inequality like (1.5) is impossible for all n.

Equation (1.1) can be thought of as a "finite-section" Wiener-Hopf equation. An explicit solution of (1.1) can be given for $h(\theta)$ in terms of the g_k's and determinants of c_k's by inverting (1.4). However, this explicit solution does not reveal what happens when n becomes infinite, i.e., when (1.1) approaches the usual Wiener-Hopf equation. Theorem 1.1 can be used to give information in this direction (see §3).

One consequence of Theorem 1.1 worth special mention is that $D_n(f) \neq 0$ for all $n \geqq N$. This follows according to (1.5), from the fact that the only solution of the homogeneous equation (1.4) with $g_k = 0$ for all $k \geqq 0$ is $h(\theta) \equiv 0$. Thus, the existence of $h(\theta)$ satisfying (1.1) for a given $g(\theta)$ as well as the unique relationship between these two polynomials is insured for all n sufficiently large.

The proof of Theorem 1.1 is given in §2. The reader interested primarily in the applications can pass directly to §3 after reading the first two paragraphs of §2.

We wish to express our thanks to E. Reich for helping to shorten the presentation.

2. Proof of Theorem 1.1

Except in the statement of theorems we now drop the subscript on the norm.

Let the Fourier coefficients of $\log f(\theta)$ be $\{d_k\}$, and set

$$A(\theta) = \exp\left\{\textstyle\sum_0^\infty d_k e^{ik\theta}\right\}, \qquad B(\theta) = \exp\left\{\textstyle\sum_{-\infty}^{-1} d_k e^{ik\theta}\right\}.$$

We assume without loss of generality that $f(\theta)$ is actually equal to its convergent Fourier series expansion. Therefore, $f(\theta) = A(\theta)B(\theta)$. Let $\mathfrak{A}_\nu^+$ and $\mathfrak{A}_\nu^-$ denote the subsets of functions $f(\theta)$ in $\mathfrak{A}_\nu$ with Fourier coefficients $c_k = 0$ for, respectively, $k < 0$ and $k > 0$. Clearly, $A(\theta)$ and $\hat{A}(\theta) = 1/A(\theta)$ belong to $\mathfrak{A}_\nu^+$, while $B(\theta)$ and $\hat{B}(\theta) = 1/B(\theta)$ belong to $\mathfrak{A}_\nu^-$. Finally, we introduce a notation. If

$$f(\theta) = \textstyle\sum_{-\infty}^{\infty} c_k e^{ik\theta} \in \mathfrak{A}_\nu , \tag{2.1}$$

then

$$f^+(\theta) = \sum_0^\infty c_k e^{ik\theta}, \qquad f^-(\theta) = \sum_{-\infty}^{-1} c_k e^{ik\theta}. \tag{2.2}$$

We merely remark that $f^+(\theta) \in \mathfrak{A}_\nu^+$ and $f^-(\theta) \in \mathfrak{A}_\nu^-$, and that $\| f^+ \| \leqq \| f \|$, $\| f^- \| \leqq \| f \|$.

To prove Theorem 1.1 consider (suppressing dependence on θ)

$$hf = G_1 + g + G_2, \tag{2.3}$$

where G_1 and G_2 are the terms in the product hf of the form

$$G_1 = \sum_{n+1}^\infty g_k e^{ik\theta}, \qquad G_2 = \sum_{-\infty}^{-1} g_k e^{ik\theta}.$$

Relation (2.3) can be written conveniently as

$$hA = G_1 \hat{B} + g\hat{B} + G_2 \hat{B}, \qquad hB = G_1 \hat{A} + g\hat{A} + G_2 \hat{A}. \tag{2.4}$$

We finish the proof by showing from (2.4) that

$$\| G_1 \hat{A} \| \leqq \text{Const.} \| g \|, \qquad \| G_2 \hat{B} \| \leqq \text{Const.} \| g \|,$$

where the constants are independent of g and h. That this is sufficient follows from (2.3) and

$$\begin{aligned} \| h \| &\leqq \| G_1 \hat{A}\hat{B} \| + \| g\hat{A}\hat{B} \| + \| G_2 \hat{A}\hat{B} \| \\ &\leqq \| \hat{B} \| \, \| G_1 \hat{A} \| + \| \hat{A}\hat{B} \| \, \| g \| + \| \hat{A} \| \, \| G_2 \hat{B} \|. \end{aligned}$$

Let $\hat{B}_n$ (or $\hat{A}_n$) denote the Fourier coefficient of $\hat{B}$ (or $\hat{A}$), and set

$$\hat{B}(n) = \sum_{-\infty}^{-n-1} \hat{B}_m e^{im\theta} \qquad \left(\text{or} \quad \hat{A}(n) = \sum_{n+1}^\infty \hat{A}_m e^{im\theta}\right).$$

Now, $hA \in \mathfrak{A}_\nu^+$. Thus, using the first equation in (2.4) and the notation of (2.2) we have

$$G_2 \hat{B} = -(g\hat{B})^- - (G_1 \hat{B})^- = -(g\hat{B})^- - (G_1 \hat{B}(n))^-.$$

But this means

$$\begin{aligned} \| G_2 \hat{B} \| &\leqq \| g\hat{B} \| + \| G_1 \hat{B}(n) \| \\ &\leqq \| \hat{B} \| \, \| g \| + \| \hat{B}(n) A \| \, \| G_1 \hat{A} \|. \end{aligned}$$

Now, if N is sufficiently large, then $\| \hat{B}(n) A \| \leqq \varepsilon < 1$ for all $n > N$. Thus,

$$\| G_2 \hat{B} \| \leqq \| \hat{B} \| \, \| g \| + \varepsilon \| G_1 \hat{A} \|. \tag{2.5}$$

In a similar manner using the fact that hB in (2.4) has zero Fourier coefficients for $k > n$, we have when $\| \hat{A}(n) B \| \leqq \varepsilon < 1$

$$\| G_1 \hat{A} \| \leqq \| \hat{A} \| \, \| g \| + \varepsilon \| G_2 \hat{B} \|. \tag{2.6}$$

It follows readily from (2.5) and (2.6) that $\| G_2 \hat{B} \|$ and $\| G_1 \hat{A} \|$ are both bounded by Const. $\| g \|$. If $\sigma = \max(\| \hat{A} \|, \| \hat{B} \|)$, then the proof shows that one can take $M = (3 - \varepsilon)\sigma^2/(1 - \varepsilon)$.

3. Applications

In this section we demonstrate some uses of Theorem 1.1.

1. *Approach to the Wiener-Hopf equation.* Let $g \in \mathfrak{A}_\nu^+$, and let h_n be the polynomial of degree at most n in $e^{i\theta}$ such that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} h_n(\theta)f(\theta)e^{-ik\theta}\,d\theta = g_k \qquad (0 \leqq k \leqq n), \tag{3.1}$$

where $\log f(\theta)$ is in $\mathfrak{A}_\nu$. The solution of the Wiener-Hopf equation, i.e., (3.1) when $n = \infty$, is in terms of the notation of (2.2)

$$H = \hat{A}\,(g\hat{B})^+ = \textstyle\sum_0^\infty H_m e^{im\theta}. \tag{3.2}$$

Question. How close is h_n to H?

To answer this we show that if $H(n) = \sum_0^n H_m e^{im\theta}$, then for all n sufficiently large

$$\| h_n - H(n) \| \leqq M \, \| f \| \, \| H - H(n) \|, \tag{3.3}$$

where M is the constant of Theorem 1.1. Thus the convergence of h_n to H is at the same rate as the convergence of the tail of H. To show (3.3) we apply Theorem 1.1 to

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi}(H(n)-h_n)f(\theta)\,e^{-ik\theta}\,d\theta &= \frac{1}{2\pi}\int_{-\pi}^{\pi}(H(n)-H)f(\theta)\,e^{-ik\theta}\,d\theta \\ &= g_k(n) \qquad (0 \leqq k \leqq n).\end{aligned} \tag{3.4}$$

We get immediately

$$\| h_n - H(n) \| \leqq M \, \| g(n) \| \leqq M \, \| f \| \, \| H - H(n) \|.$$

2. *On a theorem of Szegö.* One theorem of Szegö concerning the behavior of $D_n(f)$ says that

$$D_n(f)/D_{n-1}(f) \to \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \log f(\theta)\,d\theta\right\}, \tag{3.5}$$

provided $f(\theta) \geqq 0$ and $f(\theta)$, $\log f(\theta)$ are both integrable. The method of proof of this theorem used in [3, p. 44] is to relate the left-hand side of (3.5) to a minimization problem. Unfortunately, this method breaks down as soon as one leaves the case of real $f(\theta)$. In the case that $\log f(\theta)$ is in $\mathfrak{A}_\nu$, we can now prove (3.5) for possibly complex-valued functions $f(\theta)$ as well.

One starts with the relation

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} u_n(e^{i\theta})f(\theta)\,e^{-ik\theta}\,d\theta = \delta_{k0} \qquad (0 \leqq k \leqq n),$$

where $u_n(e^{i\theta})$ is a polynomial of at most degree n in $e^{i\theta}$ which is uniquely determined for all sufficiently large n. The constant term in $u_n(e^{i\theta})$ is exactly $D_{n-1}(f)/D_n(f)$. From (3.3) we see that this constant term has a limit as n becomes infinite. In fact, since $g(\theta) = 1$, this limit is exactly the constant term in $H = \hat{A}(g\hat{B})^+ = \hat{A}$. Thus

$$D_{n-1}(f)/D_n(f) \to \exp(-d_0),$$

which proves the result.

3. *On a conjecture of the author.* In a previous paper [1] we made a conjecture. Later [2], we proved the conjecture for the case of real-valued functions $f(\theta)$. Using (1.5) we can now prove it in general. For any $f(\theta)$ with $D_n(f) \neq 0$ for all $n \geqq 0$ we set $(n \geqq 1)$

$$\alpha_n = \frac{(-1)^n}{D_{n-1}(f)} \begin{vmatrix} c_1 & c_0 & \cdots & c_{-n+2} \\ c_2 & c_1 & \cdots & c_{-n+3} \\ \vdots & \vdots & \ddots & \vdots \\ c_n & c_{n-1} & \cdots & c_1 \end{vmatrix},$$

(3.6)

$$\beta_n = \frac{(-1)^n}{D_{n-1}(f)} \begin{vmatrix} c_{-1} & c_{-2} & \cdots & c_{-n} \\ c_0 & c_{-1} & \cdots & c_{-n+1} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n-2} & c_{n-3} & \cdots & c_{-1} \end{vmatrix}.$$

Finally, let $\| \alpha \|_\nu = \sum \nu(m) \mid \alpha_m \mid$, and $\| \beta \|_\nu = \sum \nu(-m) \mid \beta_m \mid$, where now we assume that $\nu(n)/n^\lambda \to 0$ as $| n | \to \infty$ for some $\lambda \geqq 0$.

THEOREM 3.1. *Let $f(\theta)$ be a bounded, measurable function on $-\pi \leqq \theta \leqq \pi$, and let $D_n(f) \neq 0$ for all $n \geqq 0$. Then $\| \log f(\theta) \|_\nu < \infty$ if, and only if, $\|\alpha \|_\nu < \infty$ and $\| \beta \|_\nu < \infty$. Moreover, in either case*

$$D_n(f)/D_{n-1}(f) \to \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \log f(\theta)\, d\theta\right\}.$$

As pointed out in [2], Theorem 3.1 has interesting interpretations in terms of expansions of functions by means of the Szegö polynomials and in terms of the asymptotic behavior of $D_n(f)$. Although we discuss these connections a little bit below, we will assume from this point on that the reader is familiar with the proofs in [2] in an attempt to avoid what we consider to be essentially repetition.

Proof of Theorem 3.1. The only gap which existed in the proof of the general conjecture in [2] was to show that $\log f(\theta)$ in $\mathfrak{A}_\nu$ implies $\| \alpha \|$ and $\| \beta \|$ finite. This could be accomplished by Theorem 6.1 in [2] if we could show that the polynomials $u_n(e^{i\theta})$ and $v_n(e^{i\theta})$ of at most degree n in $e^{i\theta}$ and $e^{-i\theta}$, respectively, satisfying

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} u_n(e^{i\theta})f(\theta)\, e^{-ik\theta}\, d\theta = \delta_{k0} \qquad (0 \leqq k \leqq n),$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} v_n(e^{i\theta})f(\theta)\, e^{ik\theta}\, d\theta = \delta_{k0} \qquad (0 \leqq k \leqq n),$$

converge in norm to nonzero functions. According to (3.3), $\| u_n - \hat{A} \| \to 0$, since $H = \hat{A}\hat{B}^{+} = \hat{A} \neq 0$. A similar argument shows that v_n also converges in norm to a nonzero function. Thus, an application of Theorem 6.1 of [2] finishes the proof.

4. *The asymptotic behavior of* $D_n(f)$. Theorem 3.1 is extremely useful in analyzing the asymptotic behavior of $D_n(f)$ because of the following identity involving the quantities α_n and β_n defined in (3.6):

$$D_n(f) = c_0^{n+1} \prod_{m=1}^{n} (1 - \alpha_m \beta_m)^{n+1-m}. \tag{3.7}$$

If $\log f(\theta)$ is integrable, let

$$G(f) = \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \log f(\theta)\, d\theta\right\},$$

and let d_k be the Fourier coefficient of $\log f(\theta)$. We use (3.7) and Theorem 3.1 to prove the following generalization of a result of Szegö [5], a result of Kac [4], and a result of the author [2].

THEOREM 3.2. *Let* $\log f(\theta)$ *be integrable and such that*

$$\sum_{-\infty}^{\infty} | k |^{1/2} | d_k | < \infty. \tag{3.8}$$

Then, there exists a finite limit

$$\lim_{n\to\infty} D_n(f)/G(f)^{n+1} = \exp\left\{\sum_{m=1}^{\infty} m d_m d_{-m}\right\}. \tag{3.9}$$

By using the Wiener-Lévy Theorem [6, p. 245], condition (3.8) can be rewritten in a variety of ways. For example, we can say that if $f(\theta)$ is a continuous function with winding number zero around the origin as θ ranges over $-\pi \leqq \theta \leqq \pi$, and if $\sum | k |^{1/2} | c_k | < \infty$, then the limit in (3.9) exists. An example has already been given in [2] to show that Theorem 3.2 is precise in the sense that no moment condition of the type $\sum | k |^{\gamma} | d_k | < \infty$ with $\gamma < \frac{1}{2}$ can universally imply (3.9). We omit the proof of Theorem 3.2 since it parallels so closely the corresponding proof in [2] of its counterpart Theorem 2.1. There is one point we should mention, however. To prove Theorem 3.2 there should be a generalization of Lemma 9.1 in [2]. Such a generalization is easy to state and is essentially proved in Szegö [5]. Actually, Szegö considers only the real case, but his steps are algebraic and can be carried through essentially without change in general.

5. *A convergence equivalence.* Theorem 3.1 also has interesting consequences for Szegö polynomials associated with a function $f(\theta)$ such that $\log f(\theta) \in \mathfrak{A}_\nu$. The Szegö polynomials, $\varphi_n(z)$ and $\psi_n(z)$, are defined by

(i) $\varphi_n(z)$ and $\psi_n(z)$ are polynomials of degree n in z and $1/z$, respectively, with equal leading coefficients,

(ii) $$\frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi_n(z)\psi_m(z)f(\theta)\, d\theta = \delta_{nm} \qquad (z = e^{i\theta}).$$

A necessary and sufficient condition for the existence of φ_n and ψ_n for all n is that $D_n(f) \neq 0$ for all n.

These polynomials can be used to expand an integrable function, say $g(\theta)$, defined on $-\pi \leqq \theta \leqq \pi$. If we set $\varphi_{-n}(z) = \psi_n(z)$, then

$$g(\theta) \sim \sum_{-\infty}^{\infty} g_k \varphi_k(e^{i\theta}). \tag{3.10}$$

Of course, $g(\theta)$ also has an expansion in ordinary Fourier series

$$g(\theta) \sim \sum_{-\infty}^{\infty} G_k e^{ik\theta}. \tag{3.11}$$

Question. Are the convergence properties of $\{G_k\}$ and $\{g_k\}$ the same?

The following theorem gives an answer to this question. Once again we assume that $\nu(n)/n^{\lambda} \to 0$ as $|n| \to \infty$.

THEOREM 3.3. *Let $D_n(f) \neq 0$ and $\log f(\theta) \in \mathfrak{A}_\nu$, and let $g(\theta)$ be an integrable function over $-\pi \leqq \theta \leqq \pi$ with expansions* (3.10) *and* (3.11) *in terms of the Szegö polynomials associated with $f(\theta)$. Then $\sum \nu(m) \mid G_m \mid < \infty$ if, and only if, $\sum \nu(m) \mid g_m \mid < \infty$.*

We omit the proof and refer the interested reader to the proof of Theorem 2.2 in [2]. A connection between Theorem 3.3 and the Wiener-Lévy Theorem [6, p. 245] is also mentioned in this reference.

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