ON WEIERSTRASS PRODUCTS OF ZERO TYPE ON THE REAL AXIS

BY

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1. Introduction

Let $\mathcal{W}$ be the class of even entire functions $W(z)$ of exponential type, with real zeros only, and such that $W(0) = 1$. It follows readily from the Hadamard factorization theorem that $\mathcal{W}$ is identical with the class of all Weierstrass products $W(z) = \prod(1 - z^2/\lambda_n^2)$ with $0 < \lambda_0 \leqq \lambda_1 \leqq \lambda_2 \leqq \cdots$ and n/λ_n bounded. For a given function $T(r) > 0$, let $\mathcal{W}_T$ be that subclass of $\mathcal{W}$ consisting of those $W \in \mathcal{W}$ for which $|W(r)| = O(1) \exp(T(r))$. If $T(r)$ does not grow too fast as $r \to \infty$ and $W \in \mathcal{W}_T$, then (see (2.4)) the sequence $\{\lambda_n\}$ must have a density D, and on each nonhorizontal ray $z = re^{i\theta}$ through the origin, $|W(z)|$ grows like $|\sin(\pi D z)|$; and if $W_1, W_2 \in \mathcal{W}_T$ and

$$W(z) = W_1(z) W_2(z)$$

is their product, then (see (2.6)) type (W) = type (W_1) + type (W_2). The weakest known hypothesis on T that guarantees these conclusions is

$$\int^{\infty} r^{-2} T(r)\, dr < \infty.$$

Our main result says that if T violates this hypothesis, then the conclusions will no longer hold.

That the types need no longer add has particular significance for generalized harmonic analysis. Since a class $\mathcal{W}_T$ corresponds to the collection of Fourier transforms of generalized distributions in a class $\mathfrak{F}_T$, multiplication in $\mathcal{W}_T$ corresponding to convolution in $\mathfrak{F}_T$, and the type of $W \in \mathcal{W}_T$ corresponding to the support of the corresponding $F \in \mathfrak{F}_T$, our main result shows, independently of the recent work of Roumieu [5], the impossibility of extending the "theorem of supports" to certain classes of generalized distributions.

This paper is essentially self-contained, but a knowledge of the general background material, as discussed, say, in Chapters I, II, and V of Boas's book [1] is probably indispensable.

2. Notation, history, and statements of results

With the Weierstrass product

$$W(z) = \textstyle\prod_{n=0}^{\infty} (1 - z^2/\lambda_n^2), \qquad 0 < \lambda_0 \leqq \lambda_1 \leqq \lambda_2 \leqq \cdots, \; n/\lambda_n \text{ bounded}, \tag{2.1}$$

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we associate the functions

$$n(t) = \sum_{\lambda_n \leqq t} 1, \qquad D(t) = n(t)/t, \qquad \bar{D}(t) = \frac{1}{t} \int_0^t D(u)\, du,$$

$$h(\theta) = \limsup_{r \to \infty} r^{-1} \log | W(re^{i\theta}) | , \qquad \chi(\theta) = \liminf r^{-1} \log | W(re^{i\theta}) |$$
$$\text{for } 0 \leqq \theta < 2\pi.$$

In addition, we use the notation

$$h = h(\pi/2) = \text{type } (W(z)),$$

$$D^{\bullet} = \limsup_{t \to \infty} D(t), \qquad D_{\bullet} = \liminf_{t \to \infty} D(t), \qquad \bar{D}^{\bullet} = \limsup \bar{D}(t).$$

We state some known results.

(2.2) $h(0) = 0$ *if and only if* $h(\theta) = \pi \bar{D}^{\bullet} \mid \sin \theta \mid$ *for all* θ [6, p. 428].

(2.3) *If* $W(z) = W_1(z) W_2(z)$, *then* (*trivially*) $h \geqq \max (h_1, h_2)$.

(2.4) *If*

$$\int^{\infty} r^{-2} \log^{+} W(r)\, dr < \infty, \tag{2.5}$$

then $D_{\bullet} = D^{\bullet}$ *and* $h(\theta) = \chi(\theta) = \pi D^{\bullet} \mid \sin \theta \mid$ *for* $\theta \neq 0, \pi$ [3, p. 769].

(2.6) COROLLARY. *If* $W(z) = W_1(z) W_2(z)$ *and* $W_1(z)$ *or* $W_2(z)$ *satisfies* (2.5), *then* $h = h_1 + h_2$.

Our main result, announced in [7], is that (2.3), (2.4), and (2.6) are essentially best possible. That the conclusion $D_{\bullet} = D^{\bullet}$ of (2.4) is no longer valid if (2.5) is weakened to the condition $h(0) = 0$, is contained in [4, Theorem V].

THEOREM. *Let* $T(r)$ *be a positive increasing function defined for* $r > r_0$ *with* $T(r)/r$ *decreasing and* $T(r)/\log r$ *increasing, and such that*

$$\int^{\infty} r^{-2} T(r)\, dr = \infty. \tag{2.7}$$

Then there exist, given any $h_1, h_2 > 0$, *Weierstrass products* (2.1), $W_1(z)$ *and* $W_2(z)$, *whose types are* h_1 *and* h_2 *respectively, satisfying*

$$| W_i(r) | = O(1) e^{T(r)}, \qquad i = 1, 2, \tag{2.8}$$

but such that if $W(z) = W_1(z) W_2(z)$ *is their product, then*

$$\text{type } (W) = \max (h_1, h_2).$$

In addition, for $i = 1, 2$, $h_i = \pi D_i^{\bullet}$, $D_{\bullet i} = 0$, *and* $\chi_i(\theta) = 0$ *for* $\theta \neq 0, \pi$.

Remarks. The conditions $T(r)/r \downarrow$ and $T(r)/\log r \uparrow$ are regularity conditions on $T(r)$ and do not affect the convergence or divergence of the integral in (2.7). It would be nice to eliminate these conditions, but we have not found a way to do this. The condition $T(r)/\log r \uparrow$ can be replaced, with certain changes in the proof, by any one of several somewhat related conditions of which three examples are

(i) $T(r)/\log(r/T(r)) \uparrow$,
(ii) $r^{1/2} \leqq T(r) \leqq r/\log r$,
(iii) the function $\tau(r)$, defined by $\tau(r) = T(r)/r$, is slowly oscillating in the sense that $\tau(ar)/\tau(r) \to 1$ as $r \to \infty$ for each positive a.

There is no difficulty in modifying the proof of the theorem to give a construction of an infinite set $W_j(z), j = 1, 2, 3, \cdots$ of products (2.1) satisfying (2.8) such that

$$\prod_{j=1}^{\infty} W_j(z) = (\sin \pi z)/\pi z = \prod_{n=1}^{\infty} (1 - z^2/n^2),$$

but such that for each $W_j(z)$ and each product $W(z)$ of a finite number of the $W_j(z)$, we have $h_1 = h_2 = \cdots = h = \pi$. To do this, one need only replace the pair of functions A_1, A_2 of Section 4 by an infinite set having similar properties, and replace the constant k there by a function $k(t)$ that decreases extremely slowly to 0 as $t \to \infty$.

The first two lemmas are interesting in themselves, and we state them here. Lemma 1 states that if $D(r)$ is slowly oscillating in the sense of (2.9), then for each $\theta \neq 0, \pi$, $|W(re^{i\theta})|$ imitates the behaviour of $D(r)$. Lemma 2 enables us to make the passage from continuous mass distributions to discrete ones. As a corollary of Lemma 1 it is easily seen that if (2.9) holds, then $h(\theta) = \pi D^{\bullet} |\sin\theta|$ for $\theta \neq 0, \pi$, and by the well-known continuity of $h(\theta)$ that $h(0) = 0$, thus giving another proof of a result of Redheffer [4, Theorem II].

LEMMA 1. *If*

$$\lim_{r\to\infty} \{D(rt) - D(r)\} = 0 \tag{2.9}$$

uniformly for t in any interval $0 < \varepsilon \leqq t \leqq 1/\varepsilon$,

then for $\theta \neq 0, \pi$

$$\log |W(re^{i\theta})| = \pi r D(r) |\sin\theta| + o(r).$$

LEMMA 2. *Suppose that $\nu(r)$ is a continuously differentiable function for* $0 \leqq r < \infty$, *that* $0 \leqq \nu'(r) \leqq q < \infty$, *and that*

$$\nu(r) \geqq n(r) > \nu(r) - K \qquad \textit{for some constant } K \textit{ and all } r. \tag{2.10}$$

Then

$$\log |W(r)| \leqq \int_0^{\infty} \log |1 - r^2/t^2| \, \nu'(t)\, dt + O(\log r) \qquad \textit{as } r \to \infty. \tag{2.11}$$

3. Proofs of Lemmas 1 and 2

Proof of Lemma 1. Write $\log W(re^{i\theta}) = \log \prod (1 - r^2 e^{2i\theta}/\lambda_n^2) = \sum \log (1 - r^2 e^{2i\theta}/\lambda_n^2) = \int_0^{\infty} \log (1 - r^2 e^{2i\theta}/t^2)\, dn(t)$. For $\theta \neq 0, \pi$ we may integrate by parts. The "integrated terms" drop out if the branch of the logarithm is conveniently chosen because n/λ_n is bounded (see (2.1)), and we get, after a multiplicative change of variables,

$$\log W(re^{i\theta}) = r \int_0^\infty \frac{2e^{2i\theta}}{e^{2i\theta} - t^2} D(rt)\, dt.$$

Hence the familiar formula

$$\log | W(re^{i\theta}) | = r \int_0^\infty P(t, \theta) D(rt)\, dt, \tag{3.1}$$

where

$$P(t, \theta) = \operatorname{Re}\left\{ \frac{2e^{2i\theta}}{e^{2i\theta} - t^2} \right\} = 2 \frac{1 - t^2 \cos 2\theta}{1 - 2t^2 \cos 2\theta + t^4}.$$

For each $\theta \neq 0, \pi$, $P(t, \theta)$ is a bounded and Lebesgue integrable function of t on $(0, \infty)$, and it is well known that $\int_0^\infty P(t, \theta)\, dt = \pi \mid \sin \theta \mid$. Thus

$$\log | W(re^{i\theta}) | - \pi r\, D(r) \mid \sin \theta \mid = r \int_0^\infty \{D(rt) - D(r)\} P(t, \theta)\, dt.$$

By breaking the range of this last integral into three parts,

$$\int_0^\infty = \int_0^\varepsilon + \int_\varepsilon^{1/\varepsilon} + \int_{1/\varepsilon}^\infty,$$

it is easy to see that $\int_0^\infty \{D(rt) - D(r)\} P(t, \theta)\, dt \to 0$ as $r \to \infty$ (but not uniformly in $\theta \neq 0, \pi$), and the lemma is proved.

Remark. The hypothesis (2.9) can be replaced by the following, apparently weaker, hypothesis:

$$\lim_{r \to \infty} \{D(rt) - D(r)\} = 0 \quad \text{for each} \quad t \in (0, \infty), \tag{2.9$'$}$$

since a frequently discovered result asserts that if (2.9′) holds for a Lebesgue measurable function $D(r)$, then (2.9) actually holds. (The history of this result is too complicated for us to unravel here, and we give only the reference [2, 1.4].)

Proof of Lemma 2. For fixed r, we write, as in the proof of Lemma 1, $\log | W(r)| = \int_0^\infty L(t)\, dn(t)$, where $L(t) = \log | 1 - r^2/t^2 |$. We point out that $L(t)$ is Lebesgue integrable on $(0, \infty)$,

$$L(0+) = +\infty, \qquad L(r-) = L(r+) = -\infty, \qquad L(\infty) = 0,$$

and that $L(t)$ is decreasing and continuous in $(0, r)$ and increasing and continuous in (r, ∞). We must compare

$$Y = \int_0^\infty L(t)\, dn(t) \quad \text{and} \quad Z = \int_0^\infty L(t)\, d\nu(t).$$

We will prove that $Y < Z + O(\log r)$ where $n(r)$ may be replaced by any increasing function $\mu(r)$ satisfying $\mu(0) = 0$ and $\nu(r) \geqq \mu(r) > \nu(r) - K$ for some constant K. We assume that $\nu'(t) \geqq p > 0$. This involves no loss of generality since if we replace $\nu(t)$ by $\nu(t) + t$, and $\mu(t)$ by $\mu(t) + t$, we change Z and Y not at all because $\int_0^\infty L(t)\,dt = 0$. We may suppose without loss of generality that $\nu(0) = 0$ since suitably redefining ν on the interval [0, 1] changes the value of the integral in the conclusion (2.11) only by $O(1)$. The additional $O(1)$ is negligible compared to $O(\log r)$, which is the discrepancy allowed in (2.11).

With each large r we associate the numbers r_1 and r_2 such that

$$\nu(r_1) = \mu(r) = \nu(r_2) - K.$$

Since $\nu'(t) \geqq p$, we will have $r - r_1 \leqq r_2 - r_1 \leqq K/p$. The following inequalities hold, as can be readily verified:

$$\int_0^r L(t)\,d\mu(t) \leqq \int_0^{r_1} L(t)\,d\nu(t), \tag{3.2}$$

$$\int_r^\infty L(t)\,d\mu(t) \leqq \int_{r_2}^\infty L(t)\,d\nu(t). \tag{3.3}$$

From these inequalities we deduce that $Y \leqq Z + X$, where

$$X = -\int_{r_1}^{r_2} \log |\, 1 - r^2/t^2 \,|\, d\nu(t),$$

and we shall prove that $X \leqq O(\log r)$. Clearly,

$$X \leqq -\int_{r_1}^{r_2} \log \left| \frac{t-r}{t} \right| d\nu(t).$$

Since $r_2 - r_1 \leqq K/p$ and $\nu'(t) \leqq q$, we have

$$X \leqq -q \int_{r_1}^{r_2} \log^- \left| \frac{t-r}{r} \right| dt \leqq q(r_2 - r_1) \log r_2 - q \int_{r_1}^{r_2} \log^- |\, t - r \,|\, dt,$$

so that $X \leqq (qK/p) \log (r + K/p) + 2q$.

4. Proof of the theorem

Let us first illustrate the method of proof with a simple example to show that one may have $h_1(0) = h_2(0) = 0$, but not $h = h_1 + h_2$. Put

$$n_1(r) = \left[\int_0^r \{1 + \sin (\log \log t)\}\, dt \right],$$

$$n_2(r) = \left[\int_0^r \{1 + \cos (\log \log t)\}\, dt \right],$$

and let $W_1(z)$ and $W_2(z)$ be the Weierstrass products (2.1) over the sets whose counting functions are $n_1(t)$ and $n_2(t)$, respectively. The slow oscillations imply (by Lemma 1 and the continuity of $h_i(\theta)$) that $h_1(0) = h_2(0) = 0$. Lemma 1 shows that $W_1(iy)$ behaves very much like

$\exp\{\pi y(1 + \sin(\log\log y))\}$, and $W_2(iy)$ like $\exp\{\pi y(1 + \cos(\log\log y))\}$ as $y \to \infty$. But since sin and cos are out of phase, we get *not*

$$h = 2\pi + 2\pi = 4\pi,$$

but $h = (2 + 2^{1/2})\pi$ instead.

Beginning now the proof of the theorem, we will suppose without loss of generality that $T(r)$ is continuous and that $\lim_{r\to\infty} T(r)/r = 0$ because a function $T(r)$ satisfying the hypotheses of the theorem certainly has a continuous minorant $T^*(r)$ satisfying the hypotheses with $\lim_{r\to\infty} T^*(r)/r = 0$. Also, (2.7) implies that $T(r)/\log r \to \infty$ since we have supposed that $T(r)/\log r \uparrow$. We will not prove the "in addition" part of the theorem since it will be amply clear from the proof that each of the functions $W_1(z)$, $W_2(z)$ will satisfy the requirements of the second part. To construct these Weierstrass products $W_1(z)$ and $W_2(z)$, we take two functions $A_1(t)$ and $A_2(t)$ satisfying the following simple conditions:

(4.1) $A_1(t)$ and $A_2(t)$ are nonnegative continuously differentiable periodic functions of period 2π for $-\infty < t < \infty$.

(4.2) $A_1(t)A_2(t) \equiv 0$, i.e., $A_1(t)$ vanishes where $A_2(t)$ does not, and vice versa.

(4.3) $\max_t A_1(t) = h_1, \quad \max_t A_2(t) = h_2$.

For example, we might choose

$$A_1(t) = h_1\{\max(\sin t, 0)\}^2 \quad \text{and} \quad A_2(t) = h_2\{\min(\sin t, 0)\}^2.$$

Now define $\nu_i(t)$ (where, as throughout this section, $i = 1, 2$) by

$$\nu_i(t) = \int_0^t A_i(l(s))\,ds,$$

where $l(s)$ is the continuous function defined by

$$(4.4)\qquad \begin{aligned} l'(H(t)) &= k\frac{\log t}{t} \quad \text{for} \quad t \geqq t_0 = \max(r_0, e), \\ l(t) &= k\frac{\log t_0}{t_0}\,t \quad \text{for} \quad 0 < t < H(t_0), \end{aligned}$$

where $H(t) = T(t)/\log t$, and the constant k will be chosen later in a way that depends only on the choice of the functions $A_1(t)$ and $A_2(t)$.

Finally, we define $W_i(z)$ by

$$\log W_i(z) = \int_0^\infty \log(1 - z^2/t^2)\,dn_i(t),$$

where $n_i(t) = [\nu_i(t)]$.

LEMMA 3. *$\lim_{r\to\infty}\{A_i(l(rt)) - A_i(l(r))\} = 0$ uniformly for t in any interval $0 < \varepsilon \leqq t \leqq 1/\varepsilon$.*

The proof follows from the estimate

$$| A_i(l(rt)) - A_i(l(r))| \leqq \| A_i' \|_\infty \{\max_{rt\leqq\xi\leqq r} l'(\xi)\} r(1 - t)$$

if $t < 1$, where $\| \quad \|_\infty$ denotes the supremum of the indicated function. There is a similar estimate if $t > 1$. But from (4.4), provided that $r \geqq H(t_0)/t$ (then $H^{-1}(rt) \geqq e$), we have $l'(\xi) = k(\log H^{-1}(\xi))/H^{-1}(\xi)$, and for such r we then have

$$rl'(\xi) = kr \frac{\log H^{-1}(\xi)}{H^{-1}(\xi)} \leqq kr \frac{\log H^{-1}(rt)}{H^{-1}(rt)}$$

$$= \frac{k}{t}(rt)\frac{\log H^{-1}(rt)}{H^{-1}(rt)} = \frac{k}{t}\frac{H(y)\log y}{y} = \frac{k}{t}\frac{T(y)}{y},$$

where $y = H^{-1}(rt)$. But $(k/t)(T(y)/y) \to 0$ uniformly for $t \geqq \varepsilon > 0$ since $T(y)/y \to 0$ as $y \to \infty$.

LEMMA 4. *$D_i(r) = A_i(l(r)) + o(1)$ as $r \to \infty$, and the hypothesis of Lemma 1 is satisfied by $D_i(r)$.*

We have to prove the first part, from which the second follows, by Lemma 3. The proof is immediate, on noticing that $D_i(r) = r^{-1}\nu_i(r) + o(1)$, so that

$$D_i(r) - A_i(l(r)) = \int_0^1 \{A_i(l(rt)) - A_i(l(r))\}\, dt + o(1),$$

and by Lemma 3 the second member is $o(1)$.

LEMMA 5. *$l(r) \to \infty$ as $r \to \infty$.*

It is precisely at this point that the condition (2.7) enters the picture. We write

$$l(H(r)) \geqq \int_{t_0}^r l'(H(s))\, dH(s).$$

By (4.4) we may write this last integral as

$$\int_{t_0}^r l'(H(s))\, dH(s) = k\int_{t_0}^r \frac{\log s}{s}\, d\left(\frac{T(s)}{s}\right)$$

$$= k\int_{t_0}^r \frac{\log s - 1}{s^2}\frac{T(s)}{\log s}\, ds + O(1)$$

on integrating by parts. Since the divergence of the last integral is an easy consequence of (2.7), we are done.

From Lemma 4 and Lemma 1, we conclude that

$$\log | W_i(re^{i\theta})| = \pi r A_i(l(r)) + o(r),$$

and therefore that for $W = W_1 W_2$

$$\log | W(re^{i\theta})| = \pi r\{A_1(l(r)) + A_2(l(r))\} + o(r).$$

Since, by Lemma 5, $l(r) \to \infty$, it is clear that

$$\text{type } (W_i) = h_i ,$$

and that because of (4.2) and (4.3)

$$\text{type } (W) = \max (h_1 , h_2).$$

It remains only to verify that the W_i satisfy (2.8), which we now do. By Lemma 2, if we show that

$$Z_i = \int_0^\infty | \log 1 - r^2/t^2 | \, d\nu_i(t) \leqq T(r) \tag{4.5}$$

for large r, we will be done except for the trivial enlargement of the $O(1)$ of (2.8) to $\exp (O(\log r))$, that is, to a term of polynomial growth. We leave it to the reader to verify that by simply dropping a finite number of terms from each of the products (2.1) for $W_i(z)$, the additional factors of polynomial growth are cancelled without affecting the other conditions.

To prove (4.5), write it as

$$Z_i = - \int_0^\infty \varphi(t/r) t\nu_i''(t) \, dt,$$

where

$$\varphi(t) = \frac{1}{t} \int_0^t \log \left| 1 - \frac{1}{u^2} \right| du = \log \left| 1 - \frac{1}{t^2} \right| + \frac{1}{t} \log \left| \frac{1+t}{1-t} \right| \geqq 0.$$

Thus

$$Z_i = \int_0^\infty - \varphi(t/r) t l'(t) A_i'(l(t)) \, dt = \int_0^H + \int_H^\infty ,$$

where $H = H(r) = T(r)/\log r$ as before. Now

$$\int_0^H - \varphi(t/r) t l'(t) A_i'(l(t)) \, dt \leqq \| A_i' \|_\infty \| t l'(t) \|_\infty \int_0^H \varphi(t/r) \, dt.$$

It is easy to verify that $\int_0^H \varphi(t/r) \, dt \leqq 3H \log (r/H) \leqq 3T(r)$ and to show that $\| t l'(t) \|_\infty \leqq kT(t_0)/t_0$, so that

$$\int_0^H \leqq kK_1 T(r),$$

where K_1 is a constant that depends only on the choice of the functions A_i .

Now for sufficiently large t the function $tl'(t)$ is decreasing, and thus, for large r, we have the estimate

$$\int_H^\infty -\varphi(t/r)tl'(t)A_i'(l(t))\,dt \leqq \|A_i'\|_\infty Hl'(H)\int_H^\infty \varphi(t/r)\,dt.$$

But $Hl'(H) = kT(r)/r$ and $\int_H^\infty \varphi(t/r)\,dt \leqq r\int_0^\infty \varphi(t)\,dt$. Hence

$$\int_H^\infty \leqq kK_2\,T(r),$$

where K_2 also depends only on the choice of the A_i.

Having chosen the A_i then, we select k so that $k(K_1 + K_2) < 1$ and conclude that $Z_i \leqq T(r)$ for all sufficiently large r, and the theorem is proved.

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