A KÜNNETH FORMULA FOR COHERENT ALGEBRAIC SHEAVES

BY

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Introduction

Let X and Y be varieties over an algebraically closed ground field K, and let $\mathfrak{F}$, $\mathfrak{G}$ denote coherent sheaves on X resp. Y. Our main result (mentioned earlier in less general form, in [3]) states that $H(X \times Y, \mathfrak{F}^* \otimes_{\mathfrak{O}^*} \mathfrak{G}^*)$ and $H(X, \mathfrak{F}) \otimes_{\mathrm{K}} H(Y, \mathfrak{G})$ are canonically isomorphic, where $\mathfrak{F}^*$, $\mathfrak{G}^*$ denote the reciprocal images of $\mathfrak{F}$, $\mathfrak{G}$ on the product, $\mathfrak{O}^*$ being the sheaf of local rings on $X \times Y$ (Section 6, Theorem 1). We first establish a local form of this result for affine varieties (Proposition 8, Section 3), then derive the global form by a homological argument of a type familiar in algebraic topology. That argument is presented in some detail, since the setting is somewhat different from the usual one. We have been informed that P. Cartier and J-P. Serre have independently obtained results similar to ours (oral communication from P. Cartier).

1. Some remarks on tensor products

We require a series of elementary results concerning tensor products and product varieties. They are presented in Propositions 1–7 below. Proofs are omitted where the statements alone suffice to make the assertions evident.

PROPOSITION 1. *Let R and S be subrings of a commutative ring Ω, and suppose that R and S contain a common subfield* K. *Then the natural ring homomorphism of the Kronecker product $R \otimes_{\mathrm{K}} S$ into Ω (defined by*

$$\sum a_i \otimes_{\mathrm{K}} b_i \to \sum a_i b_i ,$$

where $a_i \in R$ and $b_i \in S$) is an injection if and only if R and S are linearly disjoint[2] *over* K.

The following proposition is proved in [4], No. 48:

PROPOSITION 2. *Let Q be a commutative ring with unit, and let $0 \to A \to B \to C \to 0$ be an exact sequence of Q-modules. Let Q^* be a ring of quotients of Q.*[3] *Then the sequence $0 \to A \otimes_Q Q^* \to B \otimes_Q Q^* \to C \otimes_Q Q^* \to 0$ is exact.*

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[2] This means that if $b_1, \cdots, b_n$ are linearly independent elements of S over K, then they are linearly independent over R. From this it follows that if $a_1, \cdots, a_m$ are linearly independent elements of R over K, then they are linearly independent over S (cf. Weil [6], Chapter 1, Proposition 3). The proposition is easily verified by taking a K-base for R and S.

[3] I.e., with respect to some multiplicatively stable subset of Q not containing zero.

PROPOSITION 3. *Let R and S be commutative rings containing a common subfield* K. *Let $0 \to A \to B \to C \to 0$ be an exact sequence of R-modules, and let G be an S-module. Put $Q = R \otimes_K S$, $A' = A \otimes_R Q$, $B' = B \otimes_R Q$, $C' = C \otimes_R Q$, and $G' = Q \otimes_S G$. Then the sequence $0 \to A' \otimes_Q G' \to B' \otimes_Q G' \to C' \otimes_Q G' \to 0$ is exact.*

Proof. We have only to show that the first homomorphism is injective, since $\otimes$ is a right-exact functor. We have

$$A' \otimes_Q G' = (A \otimes_R Q) \otimes_Q (Q \otimes_S G) \approx A \otimes_R Q \otimes_S G$$

(using the natural identification of $Q \otimes_Q Q$ with Q), and the last expression is by definition equal to $A \otimes_R (R \otimes_K S) \otimes_S G$. From the canonical isomorphisms $A \otimes_R R \approx A$ and $S \otimes_S G \approx G$ it follows that $A' \otimes_Q G'$ is canonically isomorphic to $A \otimes_K G$. Similarly, $B' \otimes_Q G' \approx B \otimes_K G$. The assertion follows at once, since K is a field.

PROPOSITION 4. *Let the rings and modules be as in Proposition 3, and let Q^* be a ring of quotients of Q. Put $A^* = A \otimes_R Q^*$, $B^* = B \otimes_R Q^*$, $C^* = C \otimes_R Q^*$, and $G^* = Q^* \otimes_S G$. Then the sequence*

$$0 \to A^* \otimes_{Q^*} G^* \to B^* \otimes_{Q^*} G^* \to C^* \otimes_{Q^*} G^* \to 0$$

is exact.

Proof. We have $A^* = A \otimes_R Q^* \approx A \otimes_R (Q \otimes_Q Q^*) = A' \otimes_Q Q^*$, etc., A' being as in Proposition 3. Thus

$$A^* \otimes_{Q^*} G^* \approx (A' \otimes_Q Q^*) \otimes_{Q^*} (Q^* \otimes_Q G')$$
$$\approx A' \otimes_Q Q^* \otimes_Q G' \approx (A' \otimes_Q G') \otimes_Q Q^*,$$

and similarly for the other two terms of the sequence in question. It follows at once that this sequence is naturally isomorphic to the tensor product with Q^* of the sequence considered in Proposition 3. The assertion then follows from Propositions 2 and 3 above.

2. Products of affine varieties

Henceforward K will denote an algebraically closed ground field. In this section and in Section 3 below, U and V will denote affine varieties over K (cf. Serre [4], No. 30 et seq.). The projections $U \times V \to U$ and $U \times V \to V$ are regular mappings which induce injections of the rings of regular functions $\Gamma(U, \mathcal{O}_U) \to \Gamma(U \times V, \mathcal{O}_{U \times V})$ and $\Gamma(V, \mathcal{O}_V) \to \Gamma(U \times V, \mathcal{O}_{U \times V})$. We shall thus regard $\Gamma(U, \mathcal{O}_U)$, $\Gamma(V, \mathcal{O}_V)$ as subrings of $\Gamma(U \times V, \mathcal{O}_{U \times V})$. Further, at points $u \in U$, $v \in V$ and the corresponding point $(u, v) \in U \times V$ the projections induce injections of the local rings $\mathcal{O}_u \to \mathcal{O}_{(u,v)}$ resp. $\mathcal{O}_v \to \mathcal{O}_{(u,v)}$, where for simplicity of notation we write $\mathcal{O}_u$ for Serre's $\mathcal{O}_{u,U}$, etc. We shall regard $\mathcal{O}_u$ and $\mathcal{O}_v$ as subrings of $\mathcal{O}_{(u,v)}$. The facts just mentioned are immediate consequences of Nos. 30–33 of [4].

PROPOSITION 5. *The rings* $\Gamma(U, \mathcal{O}_U)$ *and* $\Gamma(V, \mathcal{O}_V)$ *are linearly disjoint subrings of* $\Gamma(U \times V, \mathcal{O}_{U\times V})$ *over* K, *and they generate* $\Gamma(U \times V, \mathcal{O}_{U\times V})$. *The local rings* $\mathcal{O}_u$, $\mathcal{O}_v$ *are linearly disjoint subrings of* $\mathcal{O}_{(u,v)}$ *over* K.

The fact that $\Gamma(U, \mathcal{O}_U)$ and $\Gamma(V, \mathcal{O}_V)$ generate the ring of regular functions on the product $U \times V$ is easily seen by first embedding U and V as closed subsets of suitable affine spaces and then applying Corollaire 3, No. 44 of [4] to the product of the affine spaces and the closed subset $U \times V$. Linear disjointness is trivial. From Proposition 1 we have the

COROLLARY. *The natural ring homomorphism*

$$\Gamma(U, \mathcal{O}_U) \otimes_K \Gamma(V, \mathcal{O}_V) \to \Gamma(U \times V, \mathcal{O}_{U\times V})$$

is an isomorphism.

PROPOSITION 6. *The local ring* $\mathcal{O}_{(u,v)}$ *at a point* (u, v) *on the product* $U \times V$ *is a ring of quotients of the subring* $Q = \mathcal{O}_u \cdot \mathcal{O}_v$ *generated by* $\mathcal{O}_u$ *and* $\mathcal{O}_v$.

In fact, if $\mathfrak{m}$ denotes the maximal prime ideal of $\mathcal{O}_{(u,v)}$, then $\mathcal{O}_{(u,v)}$ is precisely the ring of quotients of Q relative to the prime ideal $Q \cap \mathfrak{m}$.

3. A local Künneth formula

We require now the notion of the *reciprocal image* on $U \times V$ of an algebraic sheaf on U or V (relative to the projections of $U \times V$ onto U, V). For details we refer to [3], §1. To recall the definition summarily, let $\mathfrak{F}$ be an algebraic sheaf on U. Then its reciprocal image $\mathfrak{F}^*$ on $U \times V$ is the sheaf $\mathcal{O}_{U\times V} \otimes_{\mathcal{O}_U} \mathfrak{F}$. Similarly, if $\mathfrak{G}$ is an algebraic sheaf on V, then its reciprocal image on the product is $\mathfrak{G}^* = \mathcal{O}_{U\times V} \otimes_{\mathcal{O}_V} \mathfrak{G}$. In particular, both $\mathcal{O}_U^*$ and $\mathcal{O}_V^*$ can be identified with $\mathcal{O}_{U\times V}$, which we shall sometimes denote by $\mathcal{O}^*$. Our Künneth formula is concerned with the cohomology of a sheaf of the type $\mathfrak{F}^* \otimes_{\mathcal{O}^*} \mathfrak{G}^* = \mathfrak{F} \otimes_{\mathcal{O}_U} \mathcal{O}_{U\times V} \otimes_{\mathcal{O}_V} \mathfrak{G}$, where $\mathfrak{F}$ and $\mathfrak{G}$ are *coherent* sheaves on U resp. V. Asterisks will be used consistently below to denote reciprocal images.

PROPOSITION 7. *Let* $0 \to \mathfrak{A} \to \mathfrak{B} \to \mathfrak{C} \to 0$ *be an exact sequence of algebraic sheaves on* U, *and let* $\mathfrak{G}$ *be an algebraic sheaf on* V. *Then the sequence* $0 \to \mathfrak{A}^* \otimes_{\mathcal{O}^*} \mathfrak{G}^* \to \mathfrak{B}^* \otimes_{\mathcal{O}^*} \mathfrak{G}^* \to \mathfrak{C}^* \otimes_{\mathcal{O}^*} \mathfrak{G}^* \to 0$ *on* $U \times V$ *is exact.*

Proof. Consider the stalks at a point $(u, v) \in U \times V$: From Proposition 1 and Proposition 5 the natural homomorphism of the ring $\mathcal{O}_u \otimes_K \mathcal{O}_v$ onto the subring $\mathcal{O}_u \cdot \mathcal{O}_v$ of $\mathcal{O}_{(u,v)}$ is an isomorphism. Then, by Proposition 6, the local ring $\mathcal{O}_{(u,v)}$ can be identified with a ring of quotients of $\mathcal{O}_u \otimes_K \mathcal{O}_v$. The assertion follows at once from Proposition 4.

We have thus a somewhat simpler proof of the main result of §2 of [3], without the restrictions imposed there.

It is scarcely necessary to point out that Proposition 7 holds equally well for an exact sequence on V rather than on U.

PROPOSITION 8. *Let $\mathfrak{F}$ be a coherent sheaf on U, and let $\mathcal{G}$ be a coherent sheaf on V. Then the natural homomorphism*

$$\Gamma(U, \mathfrak{F}) \otimes_K \Gamma(V, \mathcal{G}) \to \Gamma(U \times V, \mathfrak{F}^* \otimes_{\mathcal{O}^*} \mathcal{G}^*) \tag{1}$$

is an isomorphism.

Proof. Let us first point out explicitly how (1) is defined. If $f \in \Gamma(U, \mathfrak{F})$ and $g \in \Gamma(V, \mathcal{G})$, then with the pair (f, g) we associate the function s^* on $U \times V$ given by

$$s^*(u, v) = f(u) \otimes 1 \otimes g(v) \in \mathfrak{F}_u \otimes_{\mathcal{O}_u} \mathcal{O}_{(u,v)} \otimes_{\mathcal{O}_v} \mathcal{G}_v ,$$

where $\mathfrak{F}_u$ is the stalk of $\mathfrak{F}$ at u, etc. From the definition of $\mathfrak{F}^*$, $\mathcal{G}^*$ and $\mathfrak{F}^* \otimes_{\mathcal{O}^*} \mathcal{G}^*$ as sheaves, it follows that the function s^* is a section of $\mathfrak{F}^* \otimes_{\mathcal{O}^*} \mathcal{G}^*$. It is very easily seen that the mapping $(f, g) \to s^*$ thus defined induces (1).

Now for the special case $\mathfrak{F} = \mathcal{O}_U$ and $\mathcal{G} = \mathcal{O}_V$ it is clear from the corollary to Proposition 5 that (1) is an isomorphism. Hence the proposition holds for *free* sheaves.

Under the hypothesis that $\mathfrak{F}$ and $\mathcal{G}$ are coherent, we can find exact sequences of sheaves

$$\begin{gathered} 0 \to \mathcal{K} \to \mathfrak{M} \to \mathfrak{F} \to 0 \\ 0 \to \mathcal{L} \to \mathfrak{N} \to \mathcal{G} \to 0 \end{gathered} \tag{2}$$

where $\mathcal{K}$ is coherent and $\mathfrak{M}$ is free on U, and where $\mathcal{L}$ is coherent and $\mathfrak{N}$ is free on V (Serre [4], No. 45, Corollaire 1 and No. 13, Théorème 1). Consider first the exact sequences $\mathfrak{M} \to \mathfrak{F} \to 0$, $\mathfrak{N} \to \mathcal{G} \to 0$ and the associated (exact) sequence $\mathfrak{M}^* \otimes_{\mathcal{O}^*} \mathfrak{N}^* \to \mathfrak{F}^* \otimes_{\mathcal{O}^*} \mathcal{G}^* \to 0$ on $U \times V$. Since U and V, hence also $U \times V$, are affine, the induced sequences of modules of sections are exact ([4], No. 45, Corollaire 2), and we obtain the diagram

$$\begin{array}{ccccc} \Gamma(\mathfrak{M}^* \otimes_{\mathcal{O}^*} \mathfrak{N}^*) & \to & \Gamma(\mathfrak{F}^* \otimes_{\mathcal{O}^*} \mathcal{G}^*) & \to & 0 \\ \uparrow{\scriptstyle\alpha} & & \uparrow{\scriptstyle\beta} & & \\ \Gamma(\mathfrak{M}) \otimes_K \Gamma(\mathfrak{N}) & \to & \Gamma(\mathfrak{F}) \otimes_K \Gamma(\mathcal{G}) & \to & 0 \end{array} \tag{3}$$

in which the rows are exact; we have written $\Gamma(\mathfrak{M}^* \otimes_{\mathcal{O}^*} \mathfrak{N}^*)$ for

$$\Gamma(U \times V, \mathfrak{M}^* \otimes_{\mathcal{O}^*} \mathfrak{N}^*)$$

and $\Gamma(\mathfrak{M})$ for $\Gamma(U, \mathfrak{M})$, etc. The homomorphisms α and β in (3) are of course instances of (1). It is clear that the diagram is commutative. Since $\mathfrak{M}$ and $\mathfrak{N}$ are free, it follows from the remarks above that α is an isomorphism. Therefore β—i.e., the homomorphism (1)—is surjective.

Consider now $0 \to \mathcal{K} \to \mathfrak{M} \to \mathfrak{F} \to 0$ and the corresponding sequence $0 \to \mathcal{K}^* \otimes_{\mathcal{O}^*} \mathcal{G}^* \to \mathfrak{M}^* \otimes_{\mathcal{O}^*} \mathcal{G}^* \to \mathfrak{F}^* \otimes_{\mathcal{O}^*} \mathcal{G}^* \to 0$ on $U \times V$. By Proposition 7 this sequence is also exact. Taking the induced sequences of modules of sections we obtain an exact, commutative diagram (again omitting the names of the spaces)

$$\begin{array}{ccccccccc} 0 \to & \Gamma(\mathcal{K}^* \otimes_{\mathcal{O}^*} \mathcal{G}^*) & \to & \Gamma(\mathcal{M}^* \otimes_{\mathcal{O}^*} \mathcal{G}^*) & \to & \Gamma(\mathcal{F}^* \otimes_{\mathcal{O}^*} \mathcal{G}^*) & \to 0 \\ & \alpha \uparrow & & \beta \uparrow & & \gamma \uparrow & \\ 0 \to & \Gamma(\mathcal{K}) \otimes_K \Gamma(\mathcal{G}) & \to & \Gamma(\mathcal{M}) \otimes_K \Gamma(\mathcal{G}) & \to & \Gamma(\mathcal{F}) \otimes_K \Gamma(\mathcal{G}) & \to 0 \end{array} \tag{4}$$

where α, β, γ are the appropriate homomorphisms (1). As we have just shown, they are all surjective. Suppose first that $\mathcal{G}$ is *free*. Then β is an isomorphism, and from exactness it follows that γ is also an isomorphism. We conclude that (1) is an isomorphism if either $\mathcal{F}$ or $\mathcal{G}$ is free. But then, since $\mathcal{M}$ is free, it follows that β in (4) is an isomorphism for any coherent $\mathcal{G}$, and again it follows that γ is an isomorphism, Q.E.D.

4. Double complexes

In order to extend the local version of the Künneth formula (Proposition 8) to products of arbitrary varieties, we shall first establish some properties of complexes associated with three open coverings $\mathfrak{U} = \{U_i\}_{i \in I}$, $\mathfrak{V} = \{V_j\}_{j \in J}$, and $\mathfrak{W} = \{W_k\}_{k \in K}$ of a topological space M, generalizing the results of Chapitre I, §4 of [4]. $\mathcal{L}$ will denote a sheaf of abelian groups on M. By $S(I)$ we denote the set of all ordered $(p+1)$-tuples $s = (i_0 \cdots i_p)$ with $i_\nu \in I$ $(p = 0, 1, 2, \cdots)$ such that $U_{i_0} \cap \cdots \cap U_{i_p}$ is not empty; s is called a *p-simplex*, and the open set $U_{i_0} \cap \cdots \cap U_{i_p}$ is its *support*, also denoted by U_s or $U_{i_0 \cdots i_p}$. Similar conventions apply to J and K.

Let us first recall briefly the definition of the double complex $C(\mathfrak{U}, \mathfrak{V}; \mathcal{L}) = \sum_{p,q} C^{p,q}(\mathfrak{U}, \mathfrak{V}; \mathcal{L})$ associated with the coverings $\mathfrak{U}$ and $\mathfrak{V}$ and the sheaf $\mathcal{L}$ (Serre [4], No. 28): The elements of $C^{p,q}(\mathfrak{U}, \mathfrak{V}; \mathcal{L})$ are functions f which assign to each p-simplex $s \in S(I)$ and each q-simplex $s' \in S(J)$, such that $U_s \cap V_{s'} \neq \emptyset$, an element $f_{s,s'}$ of $\Gamma(U_s \cap V_{s'}; \mathcal{L})$. If $s = (i_0 \cdots i_p)$ and $s' = (j_0 \cdots j_q)$, we also write $f_{s,s'} = f_{i_0 \cdots i_p,\, j_0 \cdots j_q}$. $C^{p,q}(\mathfrak{U}, \mathfrak{V}; \mathcal{L})$ has an evident structure as abelian group. Homomorphisms

$$d' : C^{p,q}(\mathfrak{U}, \mathfrak{V}; \mathcal{L}) \to C^{p+1,q}(\mathfrak{U}, \mathfrak{V}; \mathcal{L})$$

and

$$d'' : C^{p,q}(\mathfrak{U}, \mathfrak{V}; \mathcal{L}) \to C^{p,q+1}(\mathfrak{U}, \mathfrak{V}; \mathcal{L})$$

are defined by[4]

$$(d'f)_{i_0 \cdots i_{p+1},\, j_0 \cdots j_q} = \sum_{\nu=0}^{p+1} (-1)^\nu \operatorname{res} f_{i_0 \cdots \hat{i}_\nu \cdots i_{p+1},\, j_0 \cdots j_q}$$

and

$$(d''f)_{i_0 \cdots i_p,\, j_0 \cdots j_{q+1}} = \sum_{\nu=0}^{q+1} (-1)^{p+\nu} \operatorname{res} f_{i_0 \cdots i_p,\, j_0 \cdots \hat{j}_\nu \cdots j_{q+1}} .$$

We have then $d'd' = d'd'' + d''d' = d''d'' = 0$. If we define

$$C^n(\mathfrak{U}, \mathfrak{V}; \mathcal{L}) = \sum_{p+q=n} C^{p,q}(\mathfrak{U}, \mathfrak{V}; \mathcal{L})$$

[4] By "res" we always mean restriction to the open set indicated by the left member of the equation.

and $d = d' + d''$, then d is a homomorphism of $C^n(\mathfrak{U}, \mathfrak{V}; \mathcal{L})$ into $C^{n+1}(\mathfrak{U}, \mathfrak{V}; \mathcal{L})$ satisfying $dd = 0$. The cohomology groups of this single complex are denoted by $H^n(\mathfrak{U}, \mathfrak{V}; \mathcal{L})$.

LEMMA. *Suppose that M occurs among the open sets U_i, and suppose that $H^n(\mathfrak{V}, \mathcal{L}) = 0$ for $n > 0$. Then $H^n(\mathfrak{U}, \mathfrak{V}; \mathcal{L}) = 0$ for $n > 0$.* (Cf. [4], No. 29, Lemma 1.)

Proof. Let $M = U_\alpha$, where α is a fixed element of I, and let

$$f = f^0 + f^1 + \cdots + f^n$$

be a cocycle in $C^n(\mathfrak{U}, \mathfrak{V}; \mathcal{L})$, where $f^p \in C^{p,n-p}(\mathfrak{U}, \mathfrak{V}; \mathcal{L})$. If $n > 0$, define $g^p \in C^{p,n-p-1}(\mathfrak{U}, \mathfrak{V}; \mathcal{L})$ for $p = 0, 1, \cdots, n-1$ by

$$(g^p)_{i_0 \cdots i_p,\, j_0 \cdots j_{n-p-1}} = (f^{p+1})_{\alpha i_0 \cdots i_p,\, j_0 \cdots j_{n-p-1}}$$

(noting that $U_{i_0 \cdots i_p} = U_{\alpha i_0 \cdots i_p}$). Put $g = g^0 + g^1 + \cdots + g^{n-1}$; it is a cochain in $C^{n-1}(\mathfrak{U}, \mathfrak{V}; \mathcal{L})$, and we have $dg = f - \bar{f}^0$, where $\bar{f}^0 \in C^{0,n}(\mathfrak{U}, \mathfrak{V}; \mathcal{L})$ is defined by $\bar{f}^0_{i_0, j_0 \cdots j_n} = \operatorname{res} f^0_{\alpha, j_0 \cdots j_n}$. Clearly $d\bar{f}^0 = 0$, and therefore $d''\bar{f}^0 = 0$. Consequently the element $\bar{f}^0$ can be considered as a cocycle in $C^n(\mathfrak{V}, \mathcal{L})$. By assumption it is the coboundary of an element $h \in C^{n-1}(\mathfrak{V}, \mathcal{L})$. Define $\bar{h}^0 \in C^{0,n-1}(\mathfrak{U}, \mathfrak{V}; \mathcal{L})$ by $\bar{h}^0_{i_0, j_0 \cdots j_{n-1}} = \operatorname{res} h_{j_0 \cdots j_{n-1}}$. Then $d\bar{h}^0 = d''\bar{h}^0 = \bar{f}^0$, and thus $f = d(g + \bar{h}^0)$, Q.E.D.

5. Triple complexes

We must now consider the triple complex

$$C(\mathfrak{U}, \mathfrak{V}, \mathfrak{W}; \mathcal{L}) = \sum\nolimits_{p,q,r} C^{p,q,r}(\mathfrak{U}, \mathfrak{V}, \mathfrak{W}; \mathcal{L})$$

associated with the three coverings $\mathfrak{U}$, $\mathfrak{V}$, $\mathfrak{W}$, of M. It is quite analogous to the double complex just discussed, viz., the group $C^{p,q,r}(\mathfrak{U}, \mathfrak{V}, \mathfrak{W}; \mathcal{L})$, which for brevity we call $A^{p,q,r}$, consists of functions f which assign to every triple of simplices $s \in S(I)$, $s' \in S(J)$, $s'' \in S(K)$, of dimensions p, q, r respectively, an element $f_{s,s',s''}$ in $\Gamma(U_s \cap V_{s'} \cap W_{s''}; \mathcal{L})$. Three homomorphisms $d': A^{p,q,r} \to A^{p+1,q,r}$, $d'': A^{p,q,r} \to A^{p,q+1,r}$, and $d''': A^{p,q,r} \to A^{p,q,r+1}$ are defined by the formulas

$$(d'f)_{i_0 \cdots i_{p+1}, s', s''} = \sum\nolimits_{\nu=0}^{p+1} (-1)^\nu \operatorname{res} f_{i_0 \cdots \hat{i}_\nu \cdots i_{p+1}, s', s''},$$

$$(d''f)_{s, j_0 \cdots j_{q+1}, s''} = \sum\nolimits_{\nu=0}^{q+1} (-1)^{p+\nu} \operatorname{res} f_{s, j_0 \cdots \hat{j}_\nu \cdots j_{q+1}, s''},$$

$$(d'''f)_{s, s', k_0 \cdots k_{r+1}} = \sum\nolimits_{\nu=0}^{r+1} (-1)^{p+q+\nu} \operatorname{res} f_{s, s', k_0 \cdots \hat{k}_\nu \cdots k_{r+1}},$$

in which s, s', and s'' have the same significance as above. The three homomorphisms satisfy $d'd' = d''d'' = d'''d''' = 0$ and

$$d'd'' + d''d' = d'd''' + d'''d' = d''d''' + d'''d'' = 0.$$

We now regard the triple complex $A = \sum_{p,q,r} A^{p,q,r}$ as a double complex $A^* = \sum_{n,r} A^{n,r}$, where $A^{n,r} = \sum_{p+q=n} A^{p,q,r}$, with the two differentials $d_1 = d' + d''$ and $d_2 = d'''$. The bigraded cohomology groups of A^* with respect to the two differentials d_1 and d_2 will be called $H_I^{n,r}(A^*)$ resp. $H_{II}^{n,r}(A^*)$.

With the triple complex A we can also associate a single complex $\sum_n A^n$, where $A^n = \sum_{p+q+r=n} A^{p,q,r}$, the differential being $d' + d'' + d''' = d_1 + d_2$. The cohomology groups of this complex will be called $H^n(\mathfrak{U}, \mathfrak{V}, \mathfrak{W}; \mathfrak{L})$. They are manifestly the same as the cohomology groups of the single complex associated with the double complex A^* and the total differential $d_1 + d_2$.

Now let s'' be any simplex in $S(K)$. Then $\mathfrak{U}$ and $\mathfrak{V}$ induce open coverings $\mathfrak{U}_{s''}$ and $\mathfrak{V}_{s''}$ of the support $W_{s''}$, and with these coverings we can associate a double complex $C(\mathfrak{U}_{s''}, \mathfrak{V}_{s''}; \mathfrak{L})$, as in Section 4 above. If $s \in S(I)$ and $s' \in S(J)$ are any two simplices, then $\mathfrak{W}$ induces an open covering $\mathfrak{W}_{s,s'}$ of $U_s \cap V_{s'}$, and with this covering is associated the usual single complex $C(\mathfrak{W}_{s,s'}, \mathfrak{L})$. From the definitions of d_1 and d_2 above, it follows by inspection that there is an isomorphism

$$(5) \qquad H_I^{n,r}(A^*) \approx \prod_{s''} H^n(\mathfrak{U}_{s''}, \mathfrak{V}_{s''}; \mathfrak{L}),$$

the product being over all r-simplices $s'' \in S(K)$ (cf. [4], No. 28, Proposition 2). Similarly there is an isomorphism

$$(6) \qquad H_{II}^{n,r}(A^*) \approx \prod_{s,s'} H^r(\mathfrak{W}_{s,s'}, \mathfrak{L}),$$

the product extended over all pairs of simplices $s \in S(I)$ and $s' \in S(J)$ the sum of whose dimensions is equal to n.

Of particular interest in the sequel is the special case $\mathfrak{W} = \mathfrak{U} \cap \mathfrak{V}$, by which we mean that $K = I \times J$ and that if $k = (i, j)$, then $W_k = U_i \cap V_j$.

PROPOSITION 9. *Let the covering $\mathfrak{W}$ of M be a common refinement of $\mathfrak{U}$ and $\mathfrak{V}$. Then the canonical homomorphism*[5] $\iota_2 : H^n(\mathfrak{W}, \mathfrak{L}) \to H^n(\mathfrak{U}, \mathfrak{V}, \mathfrak{W}; \mathfrak{L})$ *is an isomorphism for all $n \geqq 0$. If $H^n(\mathfrak{W}_{s,s'}, \mathfrak{L}) = 0$ for all $n > 0$ and for every $s \in S(I)$, $s' \in S(J)$, then the canonical homomorphism*

$$\iota_1 : H^n(\mathfrak{U}, \mathfrak{V}; \mathfrak{L}) \to H^n(\mathfrak{U}, \mathfrak{V}, \mathfrak{W}; \mathfrak{L})$$

is an isomorphism for all $n \geqq 0$. The hypotheses are automatically fulfilled if $\mathfrak{W} = \mathfrak{U} \cap \mathfrak{V}$.

[5] ι_2 is defined as follows (see [4], Nos. 27, 28): Let A_{II}^r be the subgroup of $A^{0,r} = C^{0,0,r}(\mathfrak{U}, \mathfrak{V}, \mathfrak{W}; \mathfrak{L})$ consisting of all f such that $d_1 f = 0$, and put $A_{II} = \sum_r A_{II}^r$. This is a subcomplex of A^*, and the total differential $d_1 + d_2$ coincides on it with $d_2 = d'''$. Let $\iota_2 : C(\mathfrak{W}, \mathfrak{L}) \to A_{II}$ be the canonical isomorphism defined by $(\iota_2 f)_{i_0, j_0, k_0 \cdots k_r} = \operatorname{res} f_{k_0 \cdots k_r}$ for $f \in C^r(\mathfrak{W}, \mathfrak{L})$. This isomorphism followed by the injection $A_{II} \to A^*$ then induces $\iota_2 : H^n(\mathfrak{W}, \mathfrak{L}) \to H^n(A^*) = H^n(\mathfrak{U}, \mathfrak{V}, \mathfrak{W}; \mathfrak{L})$. In an analogous way there is defined a canonical isomorphism ι_1 of $C(\mathfrak{U}, \mathfrak{V}; \mathfrak{L})$ onto a subcomplex A_I of A^*, with a resulting homomorphism $\iota_1 : H^n(\mathfrak{U}, \mathfrak{V}; \mathfrak{L}) \to H^n(A^*)$. Similar homomorphisms will occur below for other complexes; they will be denoted by ι_2 resp. ι_1, as above, with primes when it is necessary to distinguish different occurrences.

Proof. Let $s'' \in S(K)$. If $\mathfrak{W}$ refines $\mathfrak{V}$, then the covering $\mathfrak{V}_{s''}$ of the support $W_{s''}$ induced by $\mathfrak{V}$ contains $W_{s''}$ among its open sets. Then

$$H^n(\mathfrak{V}_{s''}, \mathfrak{L}) = 0$$

for $n > 0$, by Lemma 1, No. 29, [4]. If also $\mathfrak{W}$ refines $\mathfrak{U}$, then $\mathfrak{U}_{s''}$ contains $W_{s''}$ among its open sets. Then from the lemma of Section 4 above, we have $H^n(\mathfrak{U}_{s''}, \mathfrak{V}_{s''}; \mathfrak{L}) = 0$ for $n > 0$, whence $H_I^{n,r}(A^*) = 0$ for $r \geqq 0$, $n > 0$, by (5). Consequently $\iota_2 : H^n(\mathfrak{W}, \mathfrak{L}) \to H^n(A^*)$ is an isomorphism for $n \geqq 0$, by Proposition 1, No. 27 of [4]. As we have already pointed out, $H^n(A^*) = H^n(\mathfrak{U}, \mathfrak{V}, \mathfrak{W}; \mathfrak{L})$. If now $H^r(\mathfrak{W}_{s,s'}, \mathfrak{L}) = 0$ for $r > 0$ and all s, s', then $H_{II}^{n,r}(A^*) = 0$ for $r > 0$, $n \geqq 0$, by (6). It follows, again by Proposition 1, No. 27 of [4], that $\iota_1 : H^n(\mathfrak{U}, \mathfrak{V}; \mathfrak{L}) \to H^n(A^*)$ is an isomorphism for $n \geqq 0$. Finally, if $\mathfrak{W} = \mathfrak{U} \cap \mathfrak{V}$, then the set $U_s \cap V_{s'}$ is among the open sets of the covering $\mathfrak{W}_{s,s'}$, so that $H^r(\mathfrak{W}_{s,s'}, \mathfrak{L}) = 0$ for $r > 0$, by Lemme 1, No. 29, [4], Q.E.D.

Remark 1. Proposition 9 is analogous to Propositions 4 and 5, No. 29 of [4].

Remark 2. If $\mathfrak{W}$ is a refinement of $\mathfrak{U}$ and $\mathfrak{V}$, then $\iota_2^{-1}\iota_1$ defines a canonical homomorphism $H^n(\mathfrak{U}, \mathfrak{V}; \mathfrak{L}) \to H^n(\mathfrak{W}, \mathfrak{L})$. We shall show presently how this can be obtained from a cochain mapping.

The question of stability of this homomorphism vis-à-vis refinement of coverings naturally arises. Let us then consider three new open coverings $\mathfrak{U}' = \{U'_{i'}\}_{i' \in I'}$, $\mathfrak{V}' = \{V'_{j'}\}_{j' \in J'}$, and $\mathfrak{W}' = \{W'_{k'}\}_{k' \in K'}$ of M which are refinements of $\mathfrak{U}, \mathfrak{V}, \mathfrak{W}$ respectively. Let $\sigma : I' \to I$, $\tau : J' \to J$, and $\xi : K' \to K$ be maps of the index sets such that $U'_{i'} \subset U_{\sigma i'}$, $V'_{j'} \subset V'_{\tau j'}$, $W'_{k'} \subset W_{\xi k'}$. The maps σ, τ resp. σ, τ, ξ determine cochain mappings

$$(\sigma\tau) : C^{p,q}(\mathfrak{U}, \mathfrak{V}; \mathfrak{L}) \to C^{p,q}(\mathfrak{U}', \mathfrak{V}'; \mathfrak{L})$$

resp.

$$(\sigma\tau\xi) : C^{p,q,r}(\mathfrak{U}, \mathfrak{V}, \mathfrak{W}; \mathfrak{L}) \to C^{p,q,r}(\mathfrak{U}', \mathfrak{V}', \mathfrak{W}'; \mathfrak{L})$$

defined for an element f by

$$(\sigma\tau f)_{i'_0 \cdots i'_p,\, j'_0 \cdots j'_q} = \operatorname{res} f_{\sigma i'_0 \cdots \sigma i'_p,\, \tau j'_0 \cdots \tau j'_q}$$

resp.

$$(\sigma\tau\xi f)_{i'_0 \cdots i'_p,\, j'_0 \cdots j'_q,\, k'_0 \cdots k'_r} = \operatorname{res} f_{\sigma i'_0 \cdots \sigma i'_p,\, \tau j'_0 \cdots \tau j'_q,\, \xi k'_0 \cdots \xi k'_r}.$$

These homomorphisms clearly commute with the differentials and therefore induce homomorphisms

$$\alpha : H^n;(\mathfrak{U}, \mathfrak{V}\ \mathfrak{L}) \to H^n(\mathfrak{U}', \mathfrak{V}'; \mathfrak{L})$$

and

$$\beta : H^n(\mathfrak{U}, \mathfrak{V}, \mathfrak{W}; \mathfrak{L}) \to H^n(\mathfrak{U}', \mathfrak{V}', \mathfrak{W}'; \mathfrak{L}).$$

In like manner the map $\xi : K' \to K$ induces a homomorphism

$$\gamma : H^n(\mathfrak{W}, \mathfrak{L}) \to H^n(\mathfrak{W}', \mathfrak{L})$$

which is well known to be independent of the choice of ξ ([4], No. 21, Proposition 3). It is easily verified in a similar manner that α and β are independent of the particular choice of σ, τ resp. σ, τ, ξ. For example, α can be decomposed in an obvious way into $H^n(\mathfrak{U}, \mathfrak{V}; \mathfrak{L}) \to H^n(\mathfrak{U}', \mathfrak{V}; \mathfrak{L}) \to H^n(\mathfrak{U}', \mathfrak{V}'; \mathfrak{L})$, the first homomorphism induced by σ alone, the second by τ alone. A rather trivial modification of the proof of the proposition just cited shows that each homomorphism is independent of the choice of σ resp. τ. We omit the verification here. Similar remarks apply to the triple complexes.

From the cochain mappings just described and from the canonical injections ι_1 and ι_2, we obtain a diagram

$$\begin{array}{ccccc} C(\mathfrak{U}, \mathfrak{V}; \mathfrak{L}) & \xrightarrow{\iota_1} & C(\mathfrak{U}, \mathfrak{V}, \mathfrak{W}; \mathfrak{L}) & \xleftarrow{\iota_2} & C(\mathfrak{W}, \mathfrak{L}) \\ \downarrow{\scriptstyle (\sigma\tau)} & & \downarrow{\scriptstyle (\sigma\tau\xi)} & & \downarrow{\scriptstyle \xi} \\ C(\mathfrak{U}', \mathfrak{V}'; \mathfrak{L}) & \xrightarrow{\iota_1'} & C(\mathfrak{U}', \mathfrak{V}', \mathfrak{W}', \mathfrak{L}) & \xleftarrow{\iota_2'} & C(\mathfrak{W}', \mathfrak{L}) \end{array}$$

which is clearly commutative. The induced cohomology diagram is then also commutative, and all the homomorphisms are canonical. Thus we have

PROPOSITION 10. *Let $\mathfrak{U}$, $\mathfrak{V}$, $\mathfrak{W}$ and $\mathfrak{U}'$, $\mathfrak{V}'$, $\mathfrak{W}'$ be two sets of open coverings of M such that $\mathfrak{W}$ refines $\mathfrak{U}$ and $\mathfrak{V}$, such that $\mathfrak{W}'$ refines $\mathfrak{U}'$ and $\mathfrak{V}'$, and such that $\mathfrak{U}'$, $\mathfrak{V}'$, $\mathfrak{W}'$ are refinements of $\mathfrak{U}$, $\mathfrak{V}$, $\mathfrak{W}$, respectively. Then the diagram*

$$\begin{array}{ccc} H^n(\mathfrak{U}, \mathfrak{V}; \mathfrak{L}) & \xrightarrow{\iota_2^{-1}\iota_1} & H^n(\mathfrak{W}, \mathfrak{L}) \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \gamma} \\ H^n(\mathfrak{U}', \mathfrak{V}'; \mathfrak{L}) & \xrightarrow{\iota_2'^{-1}\iota_1'} & H^n(\mathfrak{W}', \mathfrak{L}) \end{array}$$

induced by these refinements is commutative for all $n \geqq 0$.

From this fact we can easily calculate $\iota_2^{-1}\iota_1$ explicitly. First take $\mathfrak{U} = \mathfrak{V} = \mathfrak{W}$. The hypotheses of Proposition 9 are fulfilled, and we obtain isomorphisms ι_2: $H^n(\mathfrak{W}, \mathfrak{L}) \to H^n(\mathfrak{W}, \mathfrak{W}, \mathfrak{W}; \mathfrak{L})$ and $\iota_1 : H^n(\mathfrak{W}, \mathfrak{W}; \mathfrak{L}) \to H^n(\mathfrak{W}, \mathfrak{W}, \mathfrak{W}; \mathfrak{L})$. There is also a canonical isomorphism[6] ι_2': $H^n(\mathfrak{W}, \mathfrak{L}) \to H^n(\mathfrak{W}, \mathfrak{W}; \mathfrak{L})$. We claim that $\iota_1 \iota_2' = \iota_2$. To show this, let $f \in C^n(\mathfrak{W}, \mathfrak{L})$ be a cocycle ($n > 0$). For each $q = 0, 1, \cdots, n-1$ define $g^q \in C^{0,q,n-q-1}(\mathfrak{W}, \mathfrak{W}, \mathfrak{W}; \mathfrak{L})$ by[7]

$$g^q_{k_0,\, k'_0\cdots k'_q,\, k''_q\cdots k''_{n-1}} = (-1)^q \operatorname{res} f_{k'_0\cdots k'_q k''_q\cdots k''_{n-1}}.$$

Clearly $d'g^q = 0$. One easily finds that for the $(n-1)$-cochain

$$g = g^0 + g^1 + \cdots + g^{n-1}$$

we have $(d' + d'' + d''')g = f' - f''$, where f', f'' are homogeneous of degrees $(0, 0, n)$ resp. $(0, n, 0)$, and where $f_{k'_0,k'_0,\, k''_0\cdots k''_n} = \operatorname{res} f_{k''_0\cdots k''_n}$ and

[6] The definition of ι_2' is analogous to the definition of ι_2, as explained in the preceding footnote.

[7] Here and below, k, k', and k'' denote elements of K.

$f_{k''_0,\, k'_0\cdots k'_n,\, k''_n} = \operatorname{res}\, f_{k'_0\cdots k'_n}$. I.e., $f' = \iota_2 f$ and $f'' = \iota_1 \iota'_2 f; f' \sim f''$. This proves the assertion for $n > 0$; it is trivial for $n = 0$.

We now calculate ι'_2 explicitly: Let $f = f^0 + f^1 + \cdots + f^n$ be a cocycle of $C^n(\mathfrak{W}, \mathfrak{W}; \mathfrak{L})$, where f^p is in $C^{p,n-p}(\mathfrak{W}, \mathfrak{W}; \mathfrak{L})$ and $n > 0$. For each $p = 0, 1, \cdots, n-1$ define $g^p \in C^{p,n-p-1}(\mathfrak{W}, \mathfrak{W}; \mathfrak{L})$ by the formula

$$g^p_{k_0\cdots k_p,\, k'_p\cdots k'_{n-1}} = (-1)^p \sum_{r=p}^{n-1} f^{r+1}_{k_0\cdots k_p k'_p\cdots k'_r,\, k'_r\cdots k'_{n-1}} .$$

From a straightforward calculation it is easily seen that for the $(n-1)$-cochain $g = g^0 + g^1 + \cdots + g^{n-1}$ we have $dg = (d' + d'')g = \bar{f} - f$, where $\bar{f} \in C^{0,n}(\mathfrak{W}, \mathfrak{W}; \mathfrak{L})$ and

$$\bar{f}_{k_0,\, k'_0\cdots k'_n} = \sum_{p=0}^{n} \operatorname{res} f^p_{k'_0\cdots k'_p,\, k'_p\cdots k'_n} . \tag{7}$$

Since $d'\bar{f} = 0$, this n-cocycle lies in the image of

$$\iota'_2 : C^n(\mathfrak{W}, \mathfrak{L}) \to C^{0,n}(\mathfrak{W}, \mathfrak{W}; \mathfrak{L}).$$

The map $f \to \bar{f}$ clearly induces the canonical isomorphism

$$\iota'^{-1}_2 : H^n(\mathfrak{W}, \mathfrak{W}; \mathfrak{L}) \to H^n(\mathfrak{W}, \mathfrak{L})$$

for $n > 0$. If f is a cocycle of degree zero, then we must have $d'f = d''f = 0$, and in this case we simply take $\bar{f} = f$. Thus we conclude that *for $n \geqq 0$ the canonical isomorphism $H^n(\mathfrak{W}, \mathfrak{W}; \mathfrak{L}) \to H^n(\mathfrak{W}, \mathfrak{L})$ is induced by the cochain mapping $\chi : C^{p,n-p}(\mathfrak{W}, \mathfrak{W}; \mathfrak{L}) \to C^n(\mathfrak{W}, \mathfrak{L})$ defined by*

$$(\chi f^p)_{k_0\cdots k_n} = f^p_{k_0\cdots k_p,\, k_p\cdots k_n} . \tag{7'}$$

Now assume again that $\mathfrak{W}$ is a refinement of $\mathfrak{U}$ and $\mathfrak{V}$, and choose maps $\sigma : K \to I$, $\tau : K \to J$ such that $W_k \subset U_{\sigma k}$ and $W_k \subset V_{\tau k}$. Then from what we have established above and from Proposition 10 we obtain a commutative diagram

$$\begin{tikzcd}
H^n(\mathfrak{U}, \mathfrak{V}; \mathfrak{L}) \arrow[dd, "\alpha"'] \arrow[dr, "\iota_2^{-1}\iota_1"] & \\
& H^n(\mathfrak{W}, \mathfrak{L}) \\
H^n(\mathfrak{W}, \mathfrak{W}; \mathfrak{L}) \arrow[ur, "\iota'^{-1}_2"'] &
\end{tikzcd}$$

from which we can express $\iota_2^{-1}\iota_1$ in terms of the cochain mappings $(\sigma\tau)$ and χ. We state the result in

PROPOSITION 11. *Let $\mathfrak{W}$ be a refinement of $\mathfrak{U}$ and $\mathfrak{V}$. Then the canonical homomorphism $\iota_2^{-1}\iota_1 : H^n(\mathfrak{U}, \mathfrak{V}; \mathfrak{L}) \to H^n(\mathfrak{W}, \mathfrak{L})$ is induced by the cochain mapping φ: $C^{p,q}(\mathfrak{U}, \mathfrak{V}; \mathfrak{L}) \to C^{p+q}(\mathfrak{W}, \mathfrak{L})$ defined by*

$$(\varphi f)_{k_0\cdots k_{p+q}} = \operatorname{res} f_{\sigma k_0\cdots \sigma k_p,\, \tau k_p\cdots \tau k_{p+q}} , \tag{8}$$

where $\sigma:K \to I$, $\tau:K \to J$ *are any maps such that* $W_k \subset U_{\sigma k}$ *and* $W_k \subset V_{\tau k}$. *In particular, if* $\mathfrak{W} = \mathfrak{U} \cap \mathfrak{V}$, *then* (8) *can be put in the form*

$$(\varphi f)_{(i_0 j_0)\cdots(i_{p+q} j_{p+q})} = \operatorname{res} f_{i_0\cdots i_p,\, j_p\cdots j_{p+q}}, \tag{8'}$$

where $(i_0 j_0) \cdots (i_{p+q} j_{p+q})$ *is a* $(p+q)$*-simplex of* $K = I \times J$.

6. A Künneth formula for coherent algebraic sheaves

Let X and Y be algebraic varieties (in the sense of Serre [4]) over an algebraically closed ground field K; let $\mathfrak{U} = \{U_i\}_{i\in I}$ resp. $\mathfrak{V} = \{V_j\}_{j\in J}$ be finite coverings of X resp. Y by open affine subsets. Put $U_i^* = U_i \times Y$, $V_j^* = X \times V_j$, and $W_{ij}^* = U_i^* \cap V_j^* = U_i \times V_j$. We obtain then three open coverings $\mathfrak{U}^* = \{U_i^*\}$, $\mathfrak{V}^* = \{V_j^*\}$, and $\mathfrak{W}^* = \{W_{ij}^*\}$ of the product variety $X \times Y$. By Proposition 9 there is a canonical isomorphism

$$H^n(\mathfrak{U}^*, \mathfrak{V}^*; \mathcal{L}^*) \approx H^n(\mathfrak{W}^*, \mathcal{L}^*) \tag{9}$$

for any sheaf $\mathcal{L}^*$ on $X \times Y$ ($n \geqq 0$).

Now let $\mathfrak{F}$ be a coherent sheaf on X, let $\mathcal{G}$ be a coherent sheaf on Y, and denote their reciprocal images on $X \times Y$ by $\mathfrak{F}^*$, $\mathcal{G}^*$. Further, let $\mathcal{O}^*$ denote the sheaf of local rings on $X \times Y$. We now apply (9) to the sheaf $\mathcal{L}^* = \mathfrak{F}^* \otimes_{\mathcal{O}^*} \mathcal{G}^*$. Consider the double cochain complex

$$C(\mathfrak{U}^*, \mathfrak{V}^*; \mathfrak{F}^* \otimes_{\mathcal{O}^*} \mathcal{G}^*) = \textstyle\sum_{p,q} C^{p,q}(\mathfrak{U}^*, \mathfrak{V}^*; \mathfrak{F}^* \otimes_{\mathcal{O}^*} \mathcal{G}^*):$$

The natural homomorphism

$$\psi:\Gamma(U_{i_0\cdots i_p}, \mathfrak{F}) \otimes_{\mathrm{K}} \Gamma(V_{j_0\cdots j_q}, \mathcal{G}) \to \Gamma(U_{i_0\cdots i_p} \times V_{j_0\cdots j_q}; \mathfrak{F}^* \otimes_{\mathcal{O}^*} \mathcal{G}^*)$$

(cf. Proposition 8) induces a homomorphism

$$\psi^{p,q}:C^p(\mathfrak{U}, \mathfrak{F}) \otimes_{\mathrm{K}} C^q(\mathfrak{V}, \mathcal{G}) \to C^{p,q}(\mathfrak{U}^*, \mathfrak{V}^*; \mathfrak{F}^* \otimes_{\mathcal{O}^*} \mathcal{G}^*)$$

defined by

$$(\psi^{p,q}(f \otimes_{\mathrm{K}} g))_{i_0\cdots i_p,\, j_0\cdots j_q} = \psi(f_{i_0\cdots i_p} \otimes_{\mathrm{K}} g_{j_0\cdots j_q})$$

for $f \in C^p(\mathfrak{U}, \mathfrak{F})$ and $g \in C^q(\mathfrak{V}, \mathcal{G})$. By Proposition 8, $\psi^{p,q}$ is an isomorphism. It follows readily that the $\psi^{p,q}$ define a natural isomorphism of the two complexes $C(\mathfrak{U}, \mathfrak{F}) \otimes_{\mathrm{K}} C(\mathfrak{V}, \mathcal{G})$ and $C(\mathfrak{U}^*, \mathfrak{V}^*; \mathfrak{F}^* \otimes_{\mathcal{O}^*} \mathcal{G}^*)$. From (9) and the Künneth formula (Cartan-Eilenberg [2], Chapter VI, Theorem 3.1), we have a natural isomorphism $H(\mathfrak{U}, \mathfrak{F}) \otimes_{\mathrm{K}} H(\mathfrak{V}, \mathcal{G}) \approx H(\mathfrak{W}^*, \mathfrak{F}^* \otimes_{\mathcal{O}^*} \mathcal{G}^*)$. From the canonical isomorphisms $H(\mathfrak{U}, \mathfrak{F}) \approx H(X, \mathfrak{F})$, $H(\mathfrak{V}, \mathcal{G}) \approx H(Y, \mathcal{G})$, and $H(\mathfrak{W}^*, \mathfrak{F}^* \otimes_{\mathcal{O}^*} \mathcal{G}^*) \approx H(X \times Y, \mathfrak{F}^* \otimes_{\mathcal{O}^*} \mathcal{G}^*)$ ([4], No. 47, Théorème 4) we obtain finally an isomorphism

$$H(X, \mathfrak{F}) \otimes_{\mathrm{K}} H(Y, \mathcal{G}) \approx H(X \times Y, \mathfrak{F}^* \otimes_{\mathcal{O}^*} \mathcal{G}^*).$$

From Proposition 10 it is easily seen that this isomorphism is independent of the choice of coverings $\mathfrak{U}$ and $\mathfrak{V}$. Therefore we have

THEOREM 1. *Let $\mathfrak{F}$ denote a coherent sheaf on a variety X, and let $\mathfrak{G}$ denote a coherent sheaf on a variety Y. Then the projections $X \times Y \to X$ resp. Y induce a canonical isomorphism*

$$H(X, \mathfrak{F}) \otimes_K H(Y, \mathfrak{G}) \approx H(X \times Y, \mathfrak{F}^* \otimes_{\mathfrak{O}^*} \mathfrak{G}^*), \tag{10}$$

where $\mathfrak{F}^*$, $\mathfrak{G}^*$ are the reciprocal images of $\mathfrak{F}$, $\mathfrak{G}$ on $X \times Y$, and $\mathfrak{O}^*$ is the sheaf of local rings on $X \times Y$. In particular we have

$$H(X, \mathfrak{O}_X) \otimes_K H(Y, \mathfrak{O}_Y) \approx H(X \times Y, \mathfrak{O}^*). \tag{11}$$

If one of the varieties, say X, is affine, then (10) *reduces to*

$$\Gamma(X, \mathfrak{F}) \otimes_K H^n(Y, \mathfrak{G}) \approx H^n(X \times Y, \mathfrak{F}^* \otimes_{\mathfrak{O}^*} \mathfrak{G}^*) \qquad (n \geqq 0). \tag{12}$$

Remark. In virtue of results of Serre ([5], No. 12, Théorèmes 1, 2, and 3) the isomorphism (10) remains valid if X and Y are projective varieties over the field of complex numbers and if $\mathfrak{F}$ and $\mathfrak{G}$ are coherent *analytic* sheaves, $\mathfrak{O}^*$ being interpreted as the sheaf of germs of holomorphic functions on $X \times Y$. A similar remark holds for (11). However, (12) is not valid for analytic sheaves.

Let $\mathfrak{U}$ and $\mathfrak{V}$ be the affine coverings used above, let $f \in C^p(\mathfrak{U}, \mathfrak{F})$, and let $g \in C^q(\mathfrak{V}, \mathfrak{G})$. Then, from the definitions, the isomorphism

$$C(\mathfrak{U}, \mathfrak{F}) \otimes_K C(\mathfrak{V}, \mathfrak{G}) \to C(\mathfrak{U}^*, \mathfrak{V}^*; \mathfrak{F}^* \otimes_{\mathfrak{O}^*} \mathfrak{G}^*)$$

maps $f \otimes_K g$ into the (p, q)-cochain whose value for simplices $(i_0 \cdots i_p)$, $(j_0 \cdots j_q)$ is equal to

$$f_{i_0 \cdots i_p} \otimes 1 \otimes g_{j_0 \cdots j_q} \in \Gamma(U_{i_0 \cdots i_q} \times V_{j_0 \cdots j_q}, \mathfrak{F}^* \otimes_{\mathfrak{O}^*} \mathfrak{G}^*).$$

From Proposition 11 we have at once

PROPOSITION 12. *The canonical isomorphism*

$$H(\mathfrak{U}, \mathfrak{F}) \otimes_K H(\mathfrak{V}, \mathfrak{G}) \approx H(\mathfrak{W}^*, \mathfrak{F}^* \otimes_{\mathfrak{O}^*} \mathfrak{G}^*),$$

and hence also the isomorphism (10), *are induced by the cochain mapping*

$$\varphi: C(\mathfrak{U}, \mathfrak{F}) \otimes_K C(\mathfrak{V}, \mathfrak{G}) \to C(\mathfrak{W}^*, \mathfrak{F}^* \otimes_{\mathfrak{O}^*} \mathfrak{G}^*)$$

defined for an element $f \otimes_K g$ with $f \in C^p(\mathfrak{U}, \mathfrak{F})$ and $g \in C^{n-p}(\mathfrak{V}, \mathfrak{G})$ by

$$(\varphi(f \otimes_K g))_{(i_0 j_0) \cdots (i_n j_n)} = \operatorname{res}\,(f_{i_0 \cdots i_p} \otimes 1 \otimes g_{j_p \cdots j_n}), \tag{13}$$

where $(i_0 j_0) \cdots (i_n j_n)$ denotes an n-simplex of $I \times J$.

7. An application

Let U be an affine variety, and let $\mathbf{P}$ be a projective space of positive dimension. Let $\mathfrak{O}_{\mathbf{P}}(h)$ be Serre's sheaf ([4], No. 54) on $\mathbf{P}$, and let $\mathfrak{O}^*(h)$ be its reciprocal image on $U \times \mathbf{P}$. Then the Künneth formula (12) gives us at once $H^n(U \times \mathbf{P}, \mathcal{O}^*(h)) \approx \Gamma(U, \mathcal{O}_U) \otimes_{\mathrm{K}} H^n(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(h))$ for all $n \geqq 0$. In particular, $H^n(U \times \mathbf{P}, \mathcal{O}^*(h)) = 0$ for $n > 0$ and $h \geqq -1$ ([4], No. 65, Proposition 8). This is the main result (Proposition 2) of §4 of [3].

8. The cohomology ring

Let X be any variety, and let Δ be the diagonal in $X \times X$. Then there is a canonical homomorphism $\mathcal{O}^* \to \mathcal{O}_\Delta$ ($\mathcal{O}^* = \mathcal{O}_{X \times X}$, $\mathcal{O}_\Delta$ being extended by zero outside of Δ). Because of the isomorphism $X \approx \Delta$, we obtain a natural mapping $H(X \times X, \mathcal{O}^*) \to H(X, \mathcal{O}_X)$. From the Künneth formula (11) there results a canonical homomorphism

$$H(X, \mathcal{O}_X) \otimes_{\mathrm{K}} H(X, \mathcal{O}_X) \to H(X, \mathcal{O}_X) \tag{14}$$

(of degree zero). This homomorphism endows $H(X, \mathcal{O}_X)$ with a structure of graded, associative K-algebra, i.e., the *cohomology ring* of X.

We can express the multiplication in $H(X, \mathcal{O}_X)$ by a formula analogous to the ordinary cup-product (cf. [1], Exposé 4.8). The result is of course a special case of (13). Take two finite coverings $\mathfrak{U}$ and $\mathfrak{V}$ of X by open affine subsets. We use the notation of Section 6 with $X = Y$. The covering $\mathfrak{W}^*$ is then a covering of $X \times X$, and the covering induced by $\mathfrak{W}^*$ on Δ ($= X$) is simply the intersection $\mathfrak{U} \cap \mathfrak{V}$. Consider the diagram below:

$$H(\mathfrak{U}, \mathcal{O}_X) \otimes_{\mathrm{K}} H(\mathfrak{V}, \mathcal{O}_X) \xrightarrow{\alpha} H(\mathfrak{W}^*, \mathcal{O}_X^* \otimes_{\mathcal{O}^*} \mathcal{O}_X^*) \xrightarrow{\beta} H(\mathfrak{W}^*, \mathcal{O}^*)$$
$$\xrightarrow{\gamma} H(\mathfrak{W}^*, \mathcal{O}_\Delta) \xrightarrow{\delta} H(\mathfrak{U} \cap \mathfrak{V}, \mathcal{O}_X).$$

The maps are as follows: α is the natural isomorphism described in Section 6; β is the map induced by identification of $\mathcal{O}_X^* \otimes_{\mathcal{O}^*} \mathcal{O}_X^*$ with $\mathcal{O}^*$; γ is induced by $\mathcal{O}^* \to \mathcal{O}_\Delta$; δ is induced by the identification of X with Δ and the resulting identification of $\operatorname{res}_\Delta \mathfrak{W}^*$ with $\mathfrak{U} \cap \mathfrak{V}$. The composite homomorphism is of course (14). Now take $\mathfrak{U} = \mathfrak{V}$. The index map $I \to I \times I$ defined by $i \to (i, i)$ induces the canonical isomorphism $H(\mathfrak{U} \cap \mathfrak{U}, \mathcal{O}_X) \to H(\mathfrak{U}, \mathcal{O}_X)$. From this, from the sequence above, and from (13), it is then easily seen that *the homomorphism* (14) *is induced by the cochain mapping*

$$C^p(\mathfrak{U}, \mathcal{O}_X) \otimes_{\mathrm{K}} C^{n-p}(\mathfrak{U}, \mathcal{O}_X) \to C^n(\mathfrak{U}, \mathcal{O}_X)$$

defined by $f \otimes_{\mathrm{K}} g \to f \smile g$, *where*

$$(f \smile g)_{i_0 \cdots i_n} = \operatorname{res} f_{i_0 \cdots i_p} \cdot g_{i_p \cdots i_n}\,. \tag{15}$$

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