# MARKOFF PROCESSES AND POTENTIALS I

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This paper will appear in three parts. Its purpose and scope are best discussed after the state of affairs in the simplest situation has been explained.

Let $P_\tau(r, ds)$ be stationary Markoff transition measures on $\mathcal{H}$, a separable locally compact space, and suppose that the transformations $P_\tau$ of functions on $\mathcal{H}$,

$$P_\tau f.(r) \equiv \int P_\tau(r, ds)f(s),$$

leave invariant the Banach space of continuous functions vanishing at infinity and converge strongly there to the identity transformation as $\tau \to 0$. The transition measures can be realized by Markoff processes whose sample paths are continuous on the right and have limits from the left; the letter $X$ will denote such a process, $X(\tau)$ the random point of $\mathcal{H}$ obtained by fixing the time as $\tau$, and $X(\tau, \omega)$ the point of $\mathcal{H}$ obtained by fixing in addition the element $\omega$ of the basic probability space over which the process is defined.

A set $E$ in $\mathcal{H}$ is said to be nearly Borel if, for each process $X$, there are Borel sets $A$ and $B$ such that $A \subset E \subset B$ and such that, for almost all $\omega$ and for all $\tau$, the point $X(\tau, \omega)$ belongs to all three sets if it belongs to one. Such sets form a Borel field; a function measurable over the field is said to be nearly Borel measurable. The time $T$ at which a process $X$ hits a nearly Borel set $E$ is defined to be the infimum of the strictly positive $\tau$ for which $X(\tau, \omega)$ belongs to $E$, or $\infty$ if there are no such $\tau$; it is a random variable, that is to say, a measurable function on the basic probability space. If $X$ starts at a point, $T$ vanishes with probability either 0 or 1. In the latter event the point is said to be regular for $E$; for example, an interior point of $E$ is certainly regular for $E$. A point $r$ and a nearly Borel set $E$ determine a measure $H_E(r, ds)$, defined by the formula

$$H_E(r, A) \equiv \mathcal{P}\{X(T(\omega), \omega) \in A,\ T(\omega) < \infty\},$$

where $X$ is a process starting at $r$ and $\mathcal{P}\{\cdots\}$ stands for the probability of the event within the curly brackets.

A few remarks on language and notation are needed. Let $\mathcal{A}$ be the Borel field comprising the sets in $\mathcal{H}$ that are measurable for the completion of every measure defined on the topological Borel field of $\mathcal{H}$. A function on $\mathcal{H}$ is understood to be measurable over $\mathcal{A}$; a positive function may take on the values 0 and $\infty$; a measure on $\mathcal{H}$ is understood to be defined on $\mathcal{A}$ and to be the sum of countably many bounded positive measures. A kernel, say

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$K(r, ds)$, is a family of measures on $\mathcal{K}$, one for each point, which is subject to slight conditions of regularity. It defines a transformation $f \rightarrow Kf$ of positive functions,

$$Kf.(r) \equiv \int K(r, ds)f(s),$$

and a transformation $\mu \rightarrow \mu K$ of measures,

$$\mu K.(A) \equiv \int \mu(dr)K(r, A).$$

Both $P_\tau(r, ds)$ and $H_E(r, ds)$ are examples of kernels.

The kernel for potentials is taken to be

$$U(r, A) \equiv \int_0^\infty P_\tau(r, A)\, d\tau,$$

and $Uf$ or $\mu U$ is said to be the potential of the function $f$ or the measure $\mu$. We shall assume $U(r, F)$ to be bounded in $r$ whenever $F$ is a compact set, although some of the assertions, notably (ii), are true without this hypothesis. The properties of potentials are very easy to establish. It is nearly trivial, for example, that an inequality $a + Uf \geqq Ug$ must hold everywhere if it holds on the set where $g$ does not vanish and if $f$, $g$, and the constant $a$ are all positive.

A positive function $\phi$ is said to be excessive if $P_\tau \phi$ increases to $\phi$ as $\tau$ decreases to 0. Positive constants and potentials of positive functions are excessive. We shall list some of the properties of an excessive function $\phi$, denoting by $E$ and $X$ an arbitrary Borel set and process. The relation to more familiar statements from the theory of the Newtonian potential will be explained after (vi).

(i) $\phi$ is the limit of an increasing sequence of potentials of positive functions.

(ii) $\phi$ is nearly Borel measurable. For almost all $\omega$, the function $\phi(X(\tau, \omega))$ of $\tau$ is continuous on the right (in the topology of the extended reals), has limits from the left, and is finite for $\tau$ greater than $\sigma$ if $\phi(X(\sigma, \omega))$ is finite. The family of random variables $\phi(X(\tau))$ is a separable lower semimartingale, provided the expectation of $\phi(X(0))$ is finite.

(iii) The function $H_E \phi$ is excessive. If $\phi(r)$ is finite, then $H_E \phi.(r)$ is alternating of order infinity, in the sense of Choquet [3], when considered as a function of $E$.

(iv) The function $H_E \phi$ coincides, except perhaps at the points of $E$ not regular for $E$, with the infimum of the excessive functions that majorize $\phi$ on $E$.

The next assertion requires another definition and a special hypothesis. A closed set $F$ is said to be a determining set for an excessive function $\phi$ if

$H_G \phi$ coincides with $\phi$ whenever $G$ is a neighborhood of $F$. The hypothesis is that $H_G H_F f = H_F f$ for every positive function $f$, whenever $F$ is compact and $G$ a neighborhood of $F$.

(v) Let $\phi$ be everywhere finite. Then the infimum of the excessive functions that majorize $\phi$ on some (variable) neighborhood of $E$ coincides, except perhaps at the points of $E$ not regular for $E$, with the supremum of the excessive functions that nowhere exceed $\phi$ and have a compact subset of $E$ for a determining set.

The function $\phi$ has been assumed finite in order to keep the statement simple. The supremum mentioned is an excessive function; it is precisely $H_E \phi$ if $\phi$ is bounded and continuous.

A measure $\zeta$ is said to be excessive if it is finite on compact sets and if $\zeta$ majorizes $\zeta P_\tau$ for all $\tau$. A potential $\mu U$ is excessive if $\mu H_F$ is a bounded measure whenever $F$ is a compact set. The operation of passing from $\mu U$ to $(\mu H_E)U$ can be extended to a transformation $L_E$ of excessive measures. The analogues of (i), (iii), (v) hold for excessive measures, and there is also a representation theorem:

(vi) An excessive measure $\zeta$ can be written in just one way as $\mu U + \xi$, with $\xi$ an excessive measure such that $\xi = L_G \xi$ whenever the complement of $G$ is compact; $\zeta$ reduces to $\mu U$ if and only if $L_G \zeta.(F) \to 0$, for every compact set $F$, as $G$ runs through a decreasing sequence of sets whose complements are compact and exhaust $\mathcal{H}$.

In order to clarify the meaning of these statements, especially the last two, let us momentarily take $\mathcal{H}$ to be Euclidean space and the transition measures to be those of Brownian motion. We may write, after choosing a suitable scale for time,

$$U(r, ds) = \frac{ds}{\rho(r, s)},$$

with $ds$ the element of Lebesgue measure and $\rho(r, s)$ the distance from $r$ to $s$. Since $Uf$ is just the Newtonian potential of the measure $f\,ds$, the class of excessive functions coincides, according to (i), with the class of positive superharmonic functions augmented by the function which is identically infinite. The phrase *regular point*, as defined above, turns out to have the same meaning as in the theory of the Newtonian potential; $H_E(r, A)$ is the harmonic measure of $A$ relative to $r$ and $E$; and a positive superharmonic function has the compact set $F$ for a determining set if and only if it is the Newtonian potential of a measure on $F$. It follows that (iv) and (v) reduce to well known statements concerning superharmonic functions. Doob [6] has shown that a strong version of (ii) holds in the present situation, and Choquet has proved (iii). It is easily seen that there is a one-to-one correspondence $\phi \leftrightarrow \phi\, ds$ between positive superharmonic functions and excessive measures, under which $H_E\,\phi$ is associated with $L_E(\phi\,ds)$. On using this correspondence to state (vi) in terms of superharmonic functions, one obtains the criterion that a superharmonic function be a Newtonian potential and the representation of a positive superharmonic function as the sum of a harmonic function and a Newtonian potential.

Other choices of the transition measures lead to statements concerning superparabolic functions or the potentials of M. Riesz. It is pointless to give details, and we resume the description of the more general situation.

A system of terminal times is the assignment of a positive random variable $R$ to each process $X$, certain conditions of compatibility being satisfied. Let us suppose, to fix matters, that $R$ is the time $X$ hits a given Borel set $B$. The transition measures relative to the system are

$$K_\tau(r, A) \equiv \mathcal{P}\{X(\tau) \in A, R > \tau\},$$

with $X$ a process starting at $r$, and the kernel for potentials is

$$V(r, A) \equiv \int_0^\infty K_\tau(r, A)\,d\tau. \tag{1}$$

All that has been said remains true, except for changes in wording, when $P_\tau(r, ds)$ and $U(r, ds)$ are replaced by these kernels, the space $\mathfrak{K}$ by the set of points not regular for $B$, and the inequality $T < \infty$ by the inequality $T < R$ in the definition of $H_E(r, ds)$. As one example of the changes to be made, the continuity on the right asserted in (ii) holds only for $\tau$ less than $R(\omega)$, the existence of limits from the left only for $\tau$ not exceeding $R(\omega)$. If $\mathfrak{K}$ is Euclidean space, the transition measures those of Brownian motion, $B$ a closed set,—then the density function of $V(r, ds)$ relative to Lebesgue measure is precisely the Green's function of the complement of $B$, and the assertions concern functions defined and superharmonic on this complement.

The preceding statements are proved in the first two parts of the paper, the simplest terminal times being treated in this part, more general ones in the next. The third part deals with the consequences of hypotheses stronger than the ones mentioned above; a description of the results is to be found in [11].

The relation of the paper to "potential theory" is discussed in §15. We consider there a separable locally compact space $\mathfrak{K}$, the Banach space $\mathcal{C}$ of continuous functions on $\mathfrak{K}$ vanishing at infinity, the set $\mathcal{B}$ of continuous functions with compact supports, and suppose given a linear transformation $S$ from $\mathcal{B}$ to $\mathcal{C}$ having the two following properties:

(*a*) Let $a$ be a positive constant, $f$ and $g$ positive functions in $\mathcal{B}$. Then the inequality $a + Sf \geqq Sg$ holds everywhere if it holds on the support of $g$.

(*b*) The range of $S$ is dense in $\mathcal{C}$.

Such a transformation determines what may be called a theory of potentials of functions in which a principle of the maximum is valid; the dual notion is a theory of potentials of measures in which a principle of projection is valid. Matters being so, take $\mathcal{H}$ to be $\mathcal{K}$ with a single point $b$ adjoined. It is possible to construct stationary Markoff transition measures on $\mathcal{H}$, satisfying the hypothesis at the beginning of the paper, in such a manner that $Sf$ is precisely the potential $Vf$, the kernel $V(r, ds)$ being defined by (1) with $B$ reduced to the point $b$. The reader acquainted with potential theory may prefer to begin with §15, supplemented by the note [4] of Choquet and Deny.

Markoff processes, it is now clear, form a natural basis for developing the part of potential theory that can be derived from a principle of the maximum. This part includes, as we shall see in the third installment, the notion of capacity and a number of theorems usually associated with the notion of energy. The introduction of energy itself apparently requires more structure than we shall impose; the definition by an analytical expression is obvious, provided a certain symmetry holds, but the probabilistic interpretation is not clear. Problems dominated by the notion of energy are therefore not discussed.

The omission of boundary value problems is a more serious limitation in scope. Statement (ii), phrased for Brownian motion and the relative theory, is the starting point of Doob's treatment of superharmonic and superparabolic functions near the boundary of the domain of definition. The treatment carries over in large part to the situations we consider, and Doob's papers form a helpful complement to the present one.

The examples found in this paper are all trivial; they are introduced only to illustrate exceptional behavior. Brownian motion in space or in space-time is thoroughly treated in Doob's papers, and the reader should have no difficulty in writing down the details for the processes corresponding to the iesz potentials.

REGULAR PROCESSES

## 1. Hypotheses

Let $P_\tau(r, ds)$ be stationary Markoff transition measures on $\mathcal{H}$, a separable locally compact Hausdorff space. That is to say, $P_\tau(r, A)$ is defined for $\tau$ strictly positive, $r$ a point of $\mathcal{H}$ and $A$ a Borel set in $\mathcal{H}$; it is a probability measure when considered as a function of $A$ with $\tau$ and $r$ fixed; it is Borel measurable in the pair $(\tau, r)$ for $A$ fixed; and the relation

$$(1.1) \qquad \int_{\mathcal{H}} P_\sigma(r, ds) P_\tau(s, A) = P_{\sigma+\tau}(r, A)$$

holds identically.

Throughout the paper it is assumed that the transition probabilities can be realized by well behaved processes. The remainder of the section will be spent explaining this assertion.

We suppose a basic probability field $(\Omega, \mathfrak{F}, \mathcal{P})$, with a completed measure, to be chosen once for all. It is sometimes necessary, however, to consider a derived probability field $(\Omega', \mathfrak{F}', \mathcal{P}')$, with $\Omega'$ a set in $\mathfrak{F}$ having strictly positive probability, $\mathfrak{F}'$ the trace of $\mathfrak{F}$ on $\Omega'$ (the field of subsets of $\Omega'$ which belong to $\mathfrak{F}$), and $\mathcal{P}'\{A\}$ the conditional probability $\mathcal{P}\{A\}/\mathcal{P}\{\Omega'\}$. A random point defined over the smaller field will be spoken of as a random point over $\Omega'$. The symbol $d\omega$ always stands for the element of measure $\mathcal{P}\{d\omega\}$.

A random point of $\mathcal{K}$ is a function from $\Omega$ to $\mathcal{K}$ which is measurable relative to $\mathfrak{F}$ and the topological Borel field of $\mathcal{K}$. The field generated by a set of random points is the least completed subfield of $\mathfrak{F}$ over which each of the random points is measurable.

The first assumption on the transition probabilities is this: *For each probability measure on the topological Borel field of $\mathcal{K}$ there is a Markoff process on $\mathcal{K}$, with that measure as initial distribution and the $P_\tau(r, ds)$ as transition measures, whose sample functions are continuous on the right and have limits from the left.*

The name *process* is reserved for a Markoff process having these properties. A process is a family $(X(\tau))_{\tau \geqq 0}$ of random points of $\mathcal{K}$; it is usually denoted by the single letter $X$, or by $X_r$ if the initial point $X(0)$ is identically the point $r$. The sample path corresponding to $\omega$ is sometimes denoted by $X(\omega)$, for there is little danger of confusion with the random point $X(\tau)$ of $\mathcal{K}$. The initial distribution and the transition measures by no means determine a process, but they do determine the distribution of every function of sample paths with which we shall be concerned.

The other assumptions are expressed in terms of stopping times. Let $X$ be a process; $T$ a positive random variable, taking on infinite values perhaps; and $\mathcal{E}$ a subfield of $\mathfrak{F}$ which is independent of $X$. Then $T$ is said to be a stopping time for $X$ (with $\mathcal{E}$ as auxiliary field) if, for every $\sigma$, the subset $\{T < \sigma\}$ of $\Omega$ belongs to the least completed subfield of $\mathfrak{F}$ which includes both $\mathcal{E}$ and the field generated by the random points $X(\tau)$ for $\tau$ less than $\sigma$. This is the same as saying, for every strictly positive $\alpha$ the set $\{T \leqq \sigma\}$ belongs to the least completed subfield of $\mathfrak{F}$ which includes both $\mathcal{E}$ and the field generated by the $X(\tau)$ for $\tau$ less than $\sigma + \alpha$. The field $\mathcal{E}$ is somewhat arbitrary, for it may be replaced by any larger field which is also independent of $X$. A positive random variable independent of $X$—in particular, a constant—is a stopping time. A less trivial example is the infimum of the $\tau$ for which $X(\tau)$ belongs to a certain subset of $\mathcal{K}$; it will be discussed in the next section.

Suppose that $T$ is a stopping time for the process $X$, that $\mathcal{E}$ is the auxiliary field, and that $\Omega'$, the set on which $T$ is finite, has strictly positive probability. For every strictly positive $\alpha$ define the family $(Y_\alpha(\tau))_{\tau \geqq 0}$ of random points of $\mathcal{K}$ by the formula

$$Y_\alpha(\tau, \omega) = \begin{cases} X(\tau, \omega) & \textit{for } \tau < T(\omega) + \alpha, \\ X(T(\omega) + \alpha, \omega) & \textit{for } \tau \geqq T(\omega) + \alpha, \end{cases}$$

and let $\mathcal{A}_\alpha$ be the least completed subfield of $\mathfrak{F}$ which includes both $\mathcal{E}$ and the field generated by the $Y_\alpha(\tau)$. Finally, suppose $W$ to be a random point of some space and measurable over $\mathcal{A}_\alpha$ for every strictly positive $\alpha$. Speaking loosely, the dependence of $W$ on the process $X$ extends only up to and infinitesimally past the random time $T$.

The second assumption on the transition probabilities is that processes have the extended Markoff property: *Under the conditions of the preceding paragraph, the family of random points $Z(\tau)$, defined over $(\Omega', \mathfrak{F}', \mathcal{P}')$ by*

$$Z(\tau, \omega) = X(\tau + T(\omega), \omega), \qquad \tau \geqq 0, \omega \in \Omega',$$

*is a Markoff process with the $P_\tau(r, ds)$ as transition probabilities; moreover, this Markoff process and the restriction of $W$ to $\Omega'$ are conditionally independent relative to the random point $Z(0)$.*

It follows that $Z$ is a process in the strict sense explained above, for the sample paths are continuous on the right and have limits from the left.

The extended Markoff property implies a zero–one law for a process $X_r$ starting at a point $r$. Let $A$ be a subset of $\Omega$ which, for every strictly positive $\alpha$, belongs to the field generated by the $X_r(\tau)$ for $\tau$ less than $\alpha$. On taking the stopping time $T$ to vanish identically and the auxiliary field $\mathcal{E}$ to be trivial, we see that $A$ is independent of the field generated by all the $X_r(\tau)$. Since $A$ belongs to that field,

$$\mathcal{P}\{A \cap A\} = \mathcal{P}\{A\}\mathcal{P}\{A\}.$$

It is now clear that the probability of $A$ must be either 0 or 1.

The third assumption on the transition probabilities is this: *If $(T_n)$ is an increasing sequence of stopping times for the process $X$, the auxiliary field being the same for each $T_n$, then*

$$\lim X(T_n(\omega), \omega) = X(\lim T_n(\omega), \omega)$$

*for almost all $\omega$ for which the $T_n(\omega)$ are bounded.*

In particular, for each $\sigma$ the point $X(\tau, \omega)$ tends to $X(\sigma, \omega)$ with probability 1 as $\tau$ increases to $\sigma$ through a sequence of values. Since the sample paths have limits from the left, the convergence to $X(\sigma, \omega)$ holds with probability 1 as $\tau$ increases to $\sigma$ without restriction. It follows that a process has no fixed discontinuities.

The three statements in italics, referred to as hypothesis $(A)$, are assumed throughout the paper. They are fully discussed in [1, 12], which together give various sufficient conditions in terms of the transition probabilities. It is enough to suppose, for example, that the transformations $P_\tau$,

$$P_\tau f.(r) \equiv \int_{\mathcal{K}} P_\tau(r, ds) f(s),$$

leave invariant the Banach space $\mathcal{C}(\mathcal{K})$ of functions which are continuous on

$\mathcal{K}$ and vanish at infinity, and that $P_\tau$ converges strongly on $\mathcal{C}(\mathcal{K})$ to the identity transformation as $\tau \to 0$. Detailed references for the steps of the proof will be given in §15. At the moment, only observe that most familiar transition measures satisfy the hypothesis.

The following examples not only indicate the behavior that is ruled out but also show that, even when $(A)$ is satisfied, the topology of $\mathcal{K}$ may not be the one best suited to discussing processes. Let $\mathcal{K}$ be the interval $0 \leqq r < \infty$ and let $S$ be a random variable which is either identically null, identically infinite, or exponentially distributed. A particle starting from the origin remains there until time $S$ and then moves to the right with uniform velocity; a particle starting from any other point immediately begins the uniform motion to the right. The sample paths are obviously continuous. Now consider a process starting at the origin and take $S$ to be the stopping time. The second statement in italics is false if $S$ has an exponential distribution, true if $S$ is identically null or infinite. Also, the transformations $P_\tau$ send continuous functions into continuous functions only if $S$ is identically null. Matters can be set right by slightly modifying $\mathcal{K}$. If $S$ is exponentially distributed, one should split the origin into two points $a$ and $b$, with $a$ isolated and $b$ taking the place of the origin. A particle starting from $b$ immediately begins the uniform motion to the right; one starting from $a$ stays there until it occupies the point $b$ at the moment $S$ and begins the uniform motion to the right. Both hypothesis $(A)$ and the proper behavior of the transformations $P_\tau$ are now secured; it has been necessary to strengthen the topology, and then to complete the new space (by adding the point $b$) so that the sample paths can be taken continuous on the right. The same modification preserves $(A)$, if $S$ is identically infinite, and again secures the proper behavior of the transformations $P_\tau$.

There are some analytical consequences of $(A)$ that will be used without particular mention. First, $P_\tau f$ approaches $f$ boundedly as $\tau \to 0$, provided the function $f$ is bounded and continuous. This assertion follows at once from the continuity of sample paths on the right.

Second, the usual problems of measurability are trivial. Let $\mathcal{B}$ be the topological Borel field of $\mathcal{K}$. Given a bounded measure $\nu$ on $\mathcal{B}$, let $\mathcal{B}_\nu$ be the completion of $\mathcal{B}$ with respect to $\nu$; and let $\mathcal{A}$ be the intersection of all the fields $\mathcal{B}_\nu$. Consider also the product set $I \times \Omega$, where $I$ is the interval $0 \leqq \tau < \infty$; let $\mu$ be the completed product of $\mathcal{P}$ and Lebesgue measure; and let $\mathcal{D}$ be the field of definition of $\mu$. The right continuity of the sample paths implies that a process $X$, considered as the function $(\tau, \omega) \to X(\tau, \omega)$ from $I \times \Omega$ to $\mathcal{K}$, is measurable relative to the fields $\mathcal{D}$ and $\mathcal{B}$. Let $\nu$ be the measure on $\mathcal{B}$ induced by $\mu$ and this function; then $X$ is measurable relative to $\mathcal{D}$ and $\mathcal{B}_\nu$, so also relative to $\mathcal{D}$ and $\mathcal{A}$. Thus, if $f$ is a real function, defined on $\mathcal{K}$ and measurable over $\mathcal{A}$, then $f(X(\tau, \omega))$ is a function on $I \times \Omega$ which is measurable over $\mathcal{D}$. The same proof holds if Lebesgue measure on $I$ is replaced by a bounded measure defined on the Borel sets of $I$.

From now on a real function on $\mathcal{H}$ is understood to be one measurable over $\mathcal{A}$, and a measure on $\mathcal{H}$ is understood to be a positive measure which is defined on $\mathcal{A}$ and which is the countable sum of bounded measures. A measure is determined by its restriction to the field $\mathcal{B}$. The transition measures, for example, are now supposed extended to $\mathcal{A}$; it is easily verified, by an argument like the one of the next four paragraphs but much simpler, that $P_\tau(r, A)$ is measurable over $\mathcal{A}$ when considered as a function of $r$ with $\tau$ fixed and $A$ a set in $\mathcal{A}$. A set in $\mathcal{H}$ is understood to belong to $\mathcal{A}$, except during the remainder of this section.

Another kind of measurability must be proved in order to use the extended Markoff property freely. Let $\mathfrak{X}$ be the sample space for processes—that is to say, the set of all functions $x$ from $I$ to $\mathcal{H}$ which are continuous on the right and have limits from the left, $I$ being again the interval $0 \leqq \tau < \infty$. Let $\mathcal{C}_0$ be the ring of sets, closed under complements and finite unions, which is generated by the sets defined by a single relation $x(\tau) \in A$, with $A$ in $\mathcal{A}$; and take $\mathcal{C}$ to be the least Borel field including $\mathcal{C}_0$. Given a probability measure $\nu$ on $\mathcal{H}$, one can construct a measure on $\mathcal{C}_0$ by means of the customary formula using $\nu$ as the initial measure and the $P_\tau(r, ds)$ as the transition measures; hypothesis $(A)$ implies that this measure can be extended to the field $\mathcal{C}$. Denote the completion of the measure by $m_\nu$ and its field of definition by $\mathcal{C}_\nu$, or by $m_r$ and $\mathcal{C}_r$ if $\nu$ is the unit mass placed at the point $r$.

We shall now prove the following assertions concerning a set $C$ in $\mathcal{C}_\nu$. For all $r$ outside a set which belongs to the field $\mathcal{B}_\nu$ defined before and which has $\nu$ measure null, the set $C$ belongs to $\mathcal{C}_r$; we may therefore think of $m_r(C)$ as a function of $r$ defined up to a null set relative to the completed measure $\nu$. The function $m_r(C)$, after being defined arbitrarily for the exceptional values of $r$, is measurable over $\mathcal{B}_\nu$, and $m_\nu(C)$ is given by the formula

$$m_\nu(C) = \int_{\mathcal{H}} m_r(C)\nu(dr). \tag{1.2}$$

The assertions are trivial for a set in $\mathcal{C}_0$. It is clear also that the assertions hold for the limit of a monotone sequence of sets if they hold for each set in the sequence. They are therefore true for a set in $\mathcal{C}$, because this field is the least monotone class of sets including $\mathcal{C}_0$. Now, if $C$ is only known to be in $\mathcal{C}_\nu$, choose $A$ and $B$ in $\mathcal{C}$ so that

$$A \subset C \subset B, \qquad m_\nu(A) = m_\nu(C) = m_\nu(B).$$

The validity of (1.2) for $A$ and $B$ implies at once all the assertions concerning $C$.

This result, or rather an obvious generalization, will be used frequently in the following way. Let $f$ be a positive real function defined on $\mathfrak{X}$ and measurable over every field $\mathcal{C}_\nu$. Define the function $g$ on $\mathcal{H}$ to be

$$g(r) \equiv \int_{\mathfrak{X}} f(x)m_r(dx).$$

Then one has the formula

$$\int_{\mathfrak{X}} f(x) m_\nu(dx) = \int_{\mathcal{K}} g(r)\nu(dr)$$

for all measures $\nu$. It follows in particular that $g$ is measurable over the field $\mathcal{A}$ defined above. The last two equations are usually written, in another notation, as

$$g(r) \equiv \int_\Omega f(X_r(\omega))\, d\omega,$$

$$\int_\Omega f(X(\omega))\, d\omega = \int_{\mathcal{K}} g(r)\nu(dr),$$

where $X_r$ is a process starting from $r$ and $X$ is a process with initial distribution $\nu$. The hypotheses on $f$ imply that the distribution of the random variable $f(X(\omega))$ is determined by the initial distribution and the transition measures of the process; the formula may not hold if one only assumes that $f(X(\omega))$ is measurable over $\mathfrak{F}$, for then $f$ need not be measurable over $\mathcal{C}_\nu$.

A more difficult problem of measurability is treated in the next section.

## 2. Hitting a set

The purpose of this section is to establish some properties of the time a process first hits a given set in $\mathcal{K}$. In the next few paragraphs $X$ is a fixed process and $J$ a fixed compact interval $[\alpha, \beta]$. If $E$ denotes a subset of $\mathcal{K}$ then $\tilde{E}$ denotes the set of $\omega$ such that $X(\tau, \omega)$ belongs to $E$ for some $\tau$ in $J$. We shall prove that $\tilde{E}$ belongs to $\mathfrak{F}$ when $E$ is open or compact and that $\mathcal{P}\{\tilde{E}\}$, considered as a function of such simple sets, is a capacity to which Choquet's extension theorem applies.

First of all, $\tilde{E}$ increases with $E$ and the operation $E \to \tilde{E}$ preserves unions. Accordingly, if $E$ is the union of an increasing sequence of sets $E_n$ and if each $\tilde{E}_n$ belongs to $\mathfrak{F}$, then $\tilde{E}$ also belongs to $\mathfrak{F}$ and the probability of $\tilde{E}_n$ increases to that of $\tilde{E}$.

Suppose that $\tilde{E}, \tilde{E}_1, \cdots, \tilde{E}_n$ all belong to $\mathfrak{F}$, and define

$$A(i \cdots l) \equiv E \cup E_i \cup \cdots \cup E_l, \qquad 1 \leqq i < \cdots < l \leqq n.$$

Then

$$-\mathcal{P}\{\tilde{E}\} - \sum (-1)^k \sum \mathcal{P}\{\tilde{A}(i_1 \cdots i_k)\}$$

is defined and lies between 0 and 1, for it is the probability that the point $X(\tau)$ meets each of the sets $E_i$, but not the set $E$, as $\tau$ ranges over $J$. In the language of [3], the function $\mathcal{P}\{\tilde{E}\}$ of $E$ is alternating of order infinity on the class where it is defined.

If the point $X(\tau, \omega)$ belongs to a certain open set $G$, then so does

$$X(\tau + \sigma, \omega)$$

for all sufficiently small positive $\sigma$, because the sample paths are continuous on the right. Hence $\tilde{G}$ is the union of the subsets $\{X(\tau) \in G\}$ of $\Omega$, with $\tau$ ranging over $\beta$ and the rationals in $J$. It follows that $\tilde{G}$ belongs to $\mathfrak{F}$, for each set $\{X(\tau) \in G\}$ does so.

Define $T(\omega)$ to be the infimum of the $\tau$ in $J$ for which $X(\tau, \omega)$ belongs to the open set $G$, or $\beta + 1$ if there are no such $\tau$. The argument of the preceding paragraph, slightly modified, shows that $T$ is measurable over $\mathfrak{F}$ and that it is indeed a stopping time for $X$, the auxiliary field being trivial. Also, the point $X(T(\omega), \omega)$ belongs to the closure of $G$ unless $T(\omega)$ has the value $\beta + 1$.

Now consider a compact set $F$. Choose a decreasing sequence of open neighborhoods $G_n$ of $F$, whose closures are compact and shrink to $F$, and define $T_n$ as in the preceding paragraph but in terms of $G_n$. The $T_n$ increase with $n$ to a limit $T$; the set $\Omega'$ on which $T$ does not exceed $\beta$ belongs to $\mathfrak{F}$ and obviously includes $\tilde{F}$. On the other hand, $\tilde{F}$ includes $\Omega'$ up to a set of probability null, because the point $X(T(\omega), \omega)$, being almost certainly the limit of $X(T_n(\omega), \omega)$, belongs to $F$ for almost all $\omega$ in $\Omega'$. It is now clear that $\tilde{F}$ belongs to $\mathfrak{F}$ and that the probability of $\tilde{G}_n$ decreases to that of $\tilde{F}$.

The argument also proves that $\mathcal{P}\{\tilde{E}\}$, considered as a function of compact sets, is continuous on the right. That is to say, for every compact $F$ and every strictly positive $\delta$, there is an open neighborhood $G$ of $F$ such that

$$\mathcal{P}\{\tilde{F}\} \leqq \mathcal{P}\{\tilde{E}\} \leqq \mathcal{P}\{\tilde{F}\} + \delta$$

for every compact set $E$ lying between $F$ and $G$. Indeed, $G$ may be taken to be one of the $G_n$ of the preceding paragraph.

If $G$ is open, the probability of $\tilde{G}$ is the supremum of the probability of $\tilde{F}$ as $F$ ranges over the compact subsets of $G$. For, an open set in $\mathcal{H}$ is the union of an increasing sequence of compact sets.

Consider $\mathcal{P}\{\tilde{E}\}$ as a function of $E$, defined for sets which are either compact or open. The facts already proved show that this function is a capacity to which the extension theorem in §30 of [3] may be applied. Precisely, every analytic set $E$ has a compact subset $F$ and an open neighborhood $G$ for which the probabilities of $\tilde{F}$ and $\tilde{G}$ differ by an arbitrarily small amount, so that $\tilde{E}$ itself belongs to $\mathfrak{F}$. Hence $\mathcal{P}\{\tilde{E}\}$ is defined at least on the class of analytic subsets of $\mathcal{H}$, and it is alternating of order infinity and continuous on the right there. It is also clear, now and later on, that the probabilities we are speaking of depend only on the initial distribution and the transition probabilities of the process $X$.

For the moment let $E'$ denote the set of $\omega$ for which either $X(\tau, \omega)$ or $X(\tau-, \omega)$, the limit from the left, belongs to $E$ for some $\tau$ in $J$. Clearly $E'$ includes $\tilde{E}$; at times we shall need the fact that both sets have the same probability. This is certainly true if $E$ is open, for then $E'$ exceeds $\tilde{E}$ by at most the set on which $X(\alpha-, \omega)$ differs from $X(\alpha, \omega)$, where $\alpha$ is the left endpoint of $J$. The assertion for an analytic set $E$ is proved by observing that $G'$ includes $E'$ whenever $G$ is a neighborhood of $E$ and that the probability of $G'$ can be made close to that of $\tilde{E}$.

Let $J$ now be an interval open at one end or both, and define $\tilde{E}$ and $E'$ as before. The results above imply that both sets belong to $\mathfrak{F}$ and have the same probability, provided of course $E$ is an analytic set. It is also easy to see that there is an increasing sequence of compact subsets $F_n$ of $E$ such that the probability of $\tilde{F}_n$ increases to that of $\tilde{E}$. The sequence may depend upon the initial distribution of the process $X$ unless more hypotheses are made; for example, let all processes be constant in time and let $E$ be a set which is not the countable union of compact sets. A decreasing sequence of open neighborhoods of $E$ with similar properties does not generally exist, as one can see by considering uniform motion on a line; this fact explains the restrictions made in the next paragraph.

Let $J$ be the interval $0 < \tau \leqq \beta$ and let $E$ be an analytic set to which the initial distribution of $X$ attributes no mass; then there is a decreasing sequence of open neighborhoods $G_n$ of $E$ such that the probability of $\tilde{G}_n$ decreases to that of $\tilde{E}$. First suppose $E$ to have no point in common with the closed support $A$ of the initial distribution. According to Choquet's extension theorem, there is a decreasing sequence of open neighborhoods $G_n$ of $E$ such that the probability of $X(\tau)$ belonging to $G_n$ for some $\tau$ in the closed interval $0 \leqq \tau \leqq \beta$ decreases to the corresponding probability for $E$. The desired sequence of neighborhoods is obtained by taking the intersections of the $G_n$ with the complement of $A$, for $X(0)$ can belong to one of the new neighborhoods only with probability null. In dealing with the general situation, consider an increasing sequence of closed sets $A_k$, in the complement of $E$, whose union bears all the mass of the initial distribution, and let $\Omega_k$ be the set of $\omega$ for which $X(0, \omega)$ belongs to $A_k$. By what has already been proved, for each $k$ there is a sequence $G_{k,1}, G_{k,2}, \cdots$ which serves for the restriction of the process $X$ to $\Omega_k$; one may also assume that $G_{k,l}$ decreases as either $k$ or $l$ increases. Matters being so, the sequence $G_{1,1}, G_{2,2}, \cdots$ meets the requirements.

The notation introduced so far will not be used permanently. However, $F$ usually stands for a closed set and $G$ for an open set. We shall now interpret the preceding results in terms of the time a process first hits a set.

Given a process $X$ and an analytic set $E$, define $T$, *the time $X$ hits $E$*, by taking $T(\omega)$ to be the infimum of the strictly positive $\tau$ for which $X(\tau, \omega)$ belongs to $E$, or $\infty$ if there are no such $\tau$. Clearly $X(T(\omega), \omega)$ belongs to the closure of $E$ if $T(\omega)$ is finite. The results above show that $T$ is measurable over $\mathfrak{F}$ and that it is indeed a stopping time for $X$, the auxiliary field being trivial. Note that $T$ coincides, except on a set of probability null, with the infimum of the strictly positive $\tau$ for which either $X(\tau)$ or $X(\tau-)$ belongs to $E$.

In the next two propositions, which merely restate what has already been proved, the symbols $X$, $T$, $E$ have the meaning just given them.

PROPOSITION 2.1. *There is an increasing sequence of compact subsets $F_n$ of $E$ such that the time $X$ hits $F_n$ decreases to $T$ with probability* 1.

PROPOSITION 2.2. *If the initial distribution of $X$ attributes no mass to $E$, then there is a decreasing sequence of open neighborhoods $G_n$ of $E$ such that the time $X$ hits $G_n$ increases to $T$ with probability* 1.

The sequences may depend upon both the set $E$ and the process $X$.

Let $X_r$ be a process starting from a point $r$ and $T_r$ the time it hits the analytic set $E$. The zero-one law implies that $T_r$ vanishes with probability either 0 or 1. The point $r$ is said to be regular for $E$ if the probability is 1; this definition does not depend upon the choice of the process starting from $r$. An interior point of $E$ is certainly regular for $E$ but not regular for the complement of $E$. If the transition probabilities define uniform motion on a line and if $E$ is a bounded interval, open or closed but not reduced to a point, then just one endpoint of $E$ is regular for $E$. A sufficient condition for $r$ to be regular for a set $E$ is that

$$\limsup_{\tau \to 0} P_\tau(r, E) > 0.$$

This criterion is useful in discussing simple examples; it follows at once from the zero-one law. Regular points in Brownian motion are treated thoroughly in [6, 7].

The set $E$ is said to be negligible if $T_r$ is infinite with probability 1 for every $r$. This amounts to saying, no process hits $E$ in finite time with strictly positive probability.

PROPOSITION 2.3. *Let $T$ be the time the process $X$ hits the analytic set $E$ and let $\Omega'$ be the set on which $T$ is finite. For almost all $\omega$ in $\Omega'$ the point $X\big(T(\omega), \omega\big)$ either belongs to $E$ or is regular for $E$.*

Denote by $A$ the set of points which neither belong to $E$ nor are regular for $E$. The last paragraphs of §1 and the definition of regular imply that the time a process hits $E$ is strictly positive with probability 1, if the initial distribution of the process is concentrated on $A$. This being so, let $\Omega''$ be the subset of $\Omega'$ on which $X\big(T(\omega), \omega\big)$ belongs to $A$, and suppose that the probability of $\Omega''$ is strictly positive. Take $Z$ to be the process

$$Z(\tau, \omega) \equiv X\big(\tau + T(\omega), \omega\big), \qquad \tau \geqq 0, \omega \in \Omega'',$$

defined over $\Omega''$. For every $\omega$ in $\Omega''$ the point $X(\tau, \omega)$ lies outside $E$ for

$$0 < \tau \leqq T(\omega);$$

and by the first remark, for almost all $\omega$ in $\Omega''$ there is a strictly positive number $\alpha(\omega)$ such that the point $Z(\tau, \omega)$ lies outside $E$ for $0 < \tau < \alpha(\omega)$. These assertions are in contradiction with the definition of $T$, so that the probability of $\Omega''$ must vanish.

PROPOSITION 2.4. *Let $X$ be a process and $E$ an analytic set for which no point is regular. Then, for almost all $\omega$, the point $X(\tau, \omega)$ belongs to $E$ for at most countably many values of $\tau$.*

The proof is omitted since it is the same as the latter half of the proof of Theorem 7.2 of [7].

We shall now sketch how one can prove the measurability of certain stopping times used later. Let $f$ be a real Borel measurable function on $\mathfrak{X}$, which we assume finite for ease of exposition, and let $X$ be a process. Define $T(\omega)$ to be the infimum of the $\tau$ for which

$$\sup_{0 \leqq \sigma \leqq \tau} | f(X(0, \omega)) - f(X(\sigma, \omega)) | \geqq 1,$$

or $\infty$ if there are no such $\tau$. If $X$ is a process starting from the point $r$, then $T$ is just the time it hits the set where $f$ differs from $f(r)$ by at least 1. If $X$ has an arbitrary initial distribution, the application of the preceding results is not so obvious. The set on which $T$ exceeds a given $\tau$ is precisely the set where the inequality above fails to hold. Let $B_{n,k}$ be the set of $\omega$ defined by the inequalities

$$\frac{k}{n} \leqq f(X(0, \omega)) \leqq \frac{k+1}{n},$$

$$\inf_{0 \leqq \sigma \leqq \tau} f(X(\sigma, \omega)) > \frac{k+1}{n} - 1,$$

$$\sup_{0 \leqq \sigma \leqq \tau} f(X(\sigma, \omega)) < \frac{k}{n} + 1.$$

The $B_{n,k}$ are measurable, since each inequality defines a measurable set; the last one, for example defines the complement of the set of $\omega$ such that $X(\sigma, \omega)$ belongs, for some $\sigma$ in $[0, \tau]$, to the subset of $\mathfrak{X}$ where $f$ is at least $k/n + 1$. Now, the set on which $T$ exceeds $\tau$ is just the union of all the $B_{n,k}$.

The notion of Borel measurability must be extended slightly, in order to avoid hampering restrictions in later discussions. A function $g$ is said to be nearly Borel measurable if, for every process $X$, there are Borel measurable functions $f$ and $h$ such that the inequalities $f \leqq g \leqq h$ hold everywhere and such that for almost all $\omega$ the equations

$$f(X(\tau, \omega)) = g(X(\tau, \omega)) = h(X(\tau, \omega))$$

hold for all $\tau$. A set is nearly Borel if its characteristic function is nearly Borel measurable. The nearly Borel sets form a Borel field, and a function is nearly Borel measurable if and only if it is measurable over this field.

A set $B$ is said to be nearly analytic if, for every process $X$, there are analytic sets $A$ and $C$ such that $A \subset B \subset C$ and such that for almost all $\omega$ the three assertions

$$X(\tau, \omega) \in A, \qquad X(\tau, \omega) \in B, \qquad X(\tau, \omega) \in C,$$

have the same truth value for every $\tau$. The set $B$ is obviously nearly Borel if $A$ and $C$ can always be chosen as Borel sets. Observe that a nearly analytic set is measurable over the field $\mathfrak{A}$.

The results of this section have been phrased for Borel measurable functions or analytic sets, but we shall use them for nearly Borel measurable functions or nearly analytic sets, as they carry over trivially. The need of the extension will appear in §5.

## 3. Notation and conventions

The conventions regarding measures and real functions on $\mathcal{H}$ have been explained at the end of §1, where $\mathcal{A}$ was defined to be a certain extension of the topological Borel field of $\mathcal{H}$. The functions considered are positive, except in a few sections, and they are usually permitted to take on infinite values.

A kernel, say $H(r, ds)$, is a function of a point in $\mathcal{H}$ and a set in $\mathcal{A}$ which is a measure, when considered as a function of the set, and a positive function measurable over $\mathcal{A}$, when considered as a function of the point. The kernel defines a transformation $f \to Hf$ of positive functions by the formula

$$Hf.(r) \equiv \int_{\mathcal{H}} H(r, ds)f(s). \tag{3.1}$$

It also defines a transformation $\mu \to \mu H$ of measures,

$$\mu H.(A) \equiv \int_{\mathcal{H}} \mu(dr)H(r, A), \tag{3.2}$$

under certain conditions of finiteness, which are verified in all instances arising in this paper. Either transformation determines the kernel. The kernel or one of the transformations will sometimes be denoted simply by $H$. If $K(r, ds)$ is a second kernel, then $HK$, $HKf$, $\mu HK$ have the obvious meanings. Equation (1.1) could be written $P_\sigma P_\tau = P_{\sigma+\tau}$, for example.

The notation of the preceding sections for random elements will be used with a good deal of freedom, in an effort to reduce the complexity of expressions occurring frequently and so avoid a bewildering number of abbreviations. The principal abuses are the suppression of the argument $\omega$; the denoting a function and one of its values by the same symbol, the two notions being distinguished by a qualifying phrase; and a suggestive, rather than correct, notation for composite functions. The following examples should make matters clear. Let $X$ be a process, $T$ a positive random variable, $\Omega'$ the set on which $T$ is finite. Then $X(T)$ stands for the function

$$\omega \to X(T(\omega), \omega)$$

defined over $\Omega'$. The *random point* $X(\tau + T)$ of $\mathcal{H}$ similarly means the function $\omega \to X(\tau + T(\omega), \omega)$ over $\Omega'$, with $\tau$ understood to be fixed; whereas the *process* $X(\tau + T)$ means the process $Z$ defined over $\Omega'$ by the formula

$$Z(\tau, \omega) = X(\tau + T(\omega), \omega), \qquad \tau \geqq 0, \omega \in \Omega', \tag{3.3}$$

with $\tau$ now variable. An integral

$$\int_\Omega d\omega \int_0^{S(\omega)} f(X(\tau, \omega))d\tau$$

is usually written as

$$\int_{\Omega} d\omega \int_0^S f(X(\tau))\,d\tau. \tag{3.4}$$

From now on most quantities defined will depend upon a positive, usually strictly positive, parameter $\lambda$. It is seldom necessary to vary the parameter, so that the symbol $\lambda$ is ordinarily omitted from the notation; it appears as a superscript when it must be indicated.

Let $S^1$ be a strictly positive random variable which is independent of all processes considered and which has the density function $e^{-\sigma}$ for $\sigma$ positive. The *terminal time* $S$ is the random variable $S^1/\lambda$; it has the density function $\lambda e^{-\lambda\sigma}$ for positive $\sigma$ if $\lambda$ is strictly positive, and it is identically infinite if $\lambda$ vanishes. The variation of many quantities with the parameter is obvious, because the terminal time increases as the parameter decreases. The reason for speaking of $S$ as the terminal time will become clear in the next few sections.

It is worth spending some time on the following situation, which will arise frequently. $T$ is a stopping time for the process $X$; the terminal time $S$ is independent of the pair $(X, T)$; the set $\Omega'$ where $T$ is less than $S$ has strictly positive probability. Matters being so, let $S'$ be the restriction of $S - T$ to $\Omega'$, let $Z$ be the process $X(\tau + T)$ defined over $\Omega'$ by (3.3), and let $H(ds)$ be the measure

$$H(A) \equiv \mathcal{P}\{T < S, X(T) \in A\}, \qquad A \subset \mathcal{K}. \tag{3.5}$$

An easy calculation shows that the pair $(Z, S')$ behaves over the field $\Omega'$ precisely as a pair $(Y, S)$ behaves over the original field $\Omega$, where $Y$ is a process with the initial distribution $H(ds)/\mathcal{P}\{\Omega'\}$. Furthermore, $S'$ is independent of the restrictions of $T$ and $X$ to $\Omega'$. If $S$ is identically infinite, the assertions are of course included in the statement of the extended Markoff property in §1.

These properties are often used to transform integrals like (3.4). Define the function $\phi$ by

$$\phi(r) = \int_{\Omega} d\omega \int_0^S f(X_r(\tau))\,d\tau, \qquad r \in \mathcal{K},$$

where $X_r$ is a process starting from the point $r$. Then, by the remarks above and the last part of §1, the integral (3.4) can be written

$$\begin{aligned} \int_{\Omega'} d\omega \int_T^S f(X(\tau))\,d\tau + \cdots &= \int_{\Omega'} d\omega \int_0^{S'} f(Z(\tau))\,d\tau + \cdots \\ &= \int_{\mathcal{K}} H(ds)\phi(s) + \cdots, \end{aligned} \tag{3.6}$$

the dots standing for two other integrals which must be discussed in another way.

If $T$ is the time $X$ hits a nearly analytic $E$, then the measure $H(ds)$ is called the distribution of (first) hits of $E$ (before time $S$) by the process $X$, the words in parentheses usually being omitted.

The following notation is frequently used. $E$ is a nearly analytic set, $X_r$ a process starting from a point, $T_r$ the time $X_r$ hits $E$, and $\Omega_r$ the set on which $T_r$ is less than the terminal time $S$. Both the point $r$ and the set $E$ are variable, subject to the restrictions of the moment. $H_E(r, ds)$ is the distribution of hits of $E$ starting from $r$, that is to say, the measure

$$H_E(r, A) \equiv \mathcal{P}\{T_r < S, X_r(T_r) \in A\}, \qquad A \subset \mathcal{K}. \tag{3.7}$$

It is clearly a kernel in the sense explained before, so that $H_E f$ and $\mu H_E$ have the meanings given in (3.1) and (3.2).

The transition probabilities relative to the terminal time $S$ are denoted by $H_\tau(r, ds)$,

$$\begin{aligned} H_\tau(r, A) &\equiv \mathcal{P}\{\tau < S, X_r(\tau) \in A\} \\ &\equiv e^{-\lambda\tau} P_\tau(r, A). \end{aligned} \tag{3.8}$$

The equation $H_\sigma H_\tau = H_{\sigma+\tau}$ follows at once from either form of the definition.

Observe that the equality $S = T$ can hold only with probability null, if $T$ is independent of the terminal time. This remark is needed in several passages to the limit.

It will be noted that many propositions are stated only for $\lambda$ strictly positive. The extension to vanishing parameter may take on another form and usually requires an additional hypothesis; the discussion is deferred until §§12–14.

In §10 the simple terminal time $S$, which serves only to render certain integrals convergent, will be replaced by a system of terminal times which depend upon the processes. Probabilistic arguments are only slightly modified by the extension, whereas alternative analytical arguments often become more complicated. This fact explains why a proof is sometimes conducted using the terminal time $S$ rather than the explicit convergence factor $e^{-\lambda\tau}$.

## Excessive functions

### 4. Potentials of functions

The relative transition probabilities $H_\tau(r, ds)$ defined in (3.8) give rise to the dual notions, potential of a function and potential of a measure. The kernel for either one is $U(r, ds)$,

$$U(r, A) \equiv \int_0^\infty H_\tau(r, A) d\tau, \qquad A \subset \mathcal{K}.$$

For each $r$ the kernel is a measure of total mass $1/\lambda$, and $U(r, A)$ is Borel measurable in $r$ if the set $A$ is Borel measurable.

The potential of the positive function $f$ is the function $Uf$. For example,

$1/\lambda$ is the potential of 1, so that a positive constant is a potential if $\lambda$ is strictly positive. Fubini's theorem implies at once that

$$Uf.(r) = \int_{\Omega} d\omega \int_0^S f(X_r(\tau))\,d\tau, \tag{4.1}$$

where $X_r$ is a process starting from $r$. This interpretation is the source of many proofs. It is clear that the potential of a positive function increases as $\lambda$ decreases.

PROPOSITION 4.1. *A bounded continuous function $f$ is determined by its potential, provided $\lambda$ is strictly positive.*

For, $H_\sigma f$ tends boundedly to $f$ as $\sigma \to 0$, according to a remark toward the end of §1, so that

$$\frac{1}{\tau}(Uf - H_\tau\, Uf) \equiv \frac{1}{\tau}\int_0^\tau H_\sigma f\, d\sigma$$

approaches $f$ as $\tau \to 0$. The proposition is without much interest.

PROPOSITION 4.2. *If $f$ is positive and $E$ is nearly analytic, then*

$$H_E\, Uf \leqq Uf, \tag{4.2}$$

*and equality holds if $f$ vanishes outside $E$.*

Express the right member of (4.1) as the sum of three integrals,

$$\int_{\Omega_r}\int_{T_r}^{S} \cdots + \int_{\Omega_r}\int_0^{T_r} \cdots + \int_{\Omega-\Omega_r}\int_0^{S} \cdots$$

where $T_r$ is the time $X_r$ hits $E$ and $\Omega_r$ is the set on which it is less than $S$. The last two integrals are positive, and they vanish if $f$ is null outside $E$. The first integral is just $H_E\, Uf.(r)$ according to (3.6). The expression makes it clear that $H_E\, Uf$ increases with $E$.

PROPOSITION 4.3. *Let $f$ be positive and vanish outside the nearly analytic set $E$. Then $Uf$ is determined by its restriction to $E$. The inequality $Uf \leqq Ug$ holds everywhere if it holds on $E$ and $g$ is positive.*

The proposition follows at once from the preceding one if $E$ is closed, for then each measure $H_E(r, dt)$ is concentrated on $E$. If $E$ is not closed, consider the function $h$ which coincides with $f$ on a closed subset $F$ of $E$ and vanishes elsewhere. For each $r$ there is an increasing sequence of closed subsets of $E$ with the property that $Uh.(r)$ increases to $Uf.(r)$ as $F$ runs through the sequence. It follows that $Uf.(r)$ is the supremum of $H_F\, Uf.(r)$ as $F$ ranges over the closed subsets of $E$, and the proof is complete.

One sees as a corollary that $Uf$ has the same supremum on $E$ as on $\mathcal{K}$. This is an immediate consequence if $\lambda$ is strictly positive, for then 1 is a potential. The result carries over to vanishing $\lambda$, since $U^\lambda f$ increases to $U^0 f$ as $\lambda \to 0$.

Potentials of positive functions are instances of excessive functions, which are studied in the next two sections. In the remainder of this section we exhibit a method of defining potentials to be used later on.

Let $h$ be a positive function, $Z$ a positive random variable which has the density function $e^{-\sigma}$ for $\sigma$ positive and which is independent of the terminal time $S$ as well as of all processes considered. To each process $X$ assign a stopping time $R$ by taking $R(\omega)$ to be the infimum of the $\tau$ for which

$$\int_0^\tau h(X(\sigma,\omega))d\sigma \geqq Z(\omega), \tag{4.3}$$

or $\infty$ if there are no such $\tau$. Observe that $R$ decreases as $h$ increases. The system of times assigned in this way to the various processes is said to be determined by $h$, the variable $Z$ being only auxiliary. It is an instance of the systems of terminal times discussed in §10; the simple terminal time $S$, or rather an equivalent one, is obtained when $h$ is the constant $\lambda$. The reader may find §10 helpful in reading the next few pages, but no part of that section is needed in the proofs.

Let $\phi$ be a positive function on $\mathfrak{H}$, and define the function $\psi$ by the formula

$$\psi(r) = \int_{\Omega^*} \phi(X(R))d\omega, \tag{4.4}$$

where $X$ is a process starting at the point $r$, the time $R$ is the one just defined, and $\Omega^*$ is the set where $R$ is less than $S$. Note that $\psi(r)$ becomes the probability that $R$ is less than $S$, if $\phi$ is taken to be the constant 1.

PROPOSITION 4.4. *Let $\lambda$ be strictly positive, let $\phi$ be bounded, and let $h$ have the property that, for every process $X$ and every finite $\tau$, the integral*

$$\int_0^\tau h(X(\sigma))d\sigma$$

*is finite with probability* 1. *The function $\psi$ defined by* (4.4) *is then the potential of the function $(\phi - \psi)h$. If $\phi$ is the potential of a positive function, then $\psi$ increases with $h$ and the difference $\phi - \psi$ is positive.*

In the proof we shall use the abbreviations

$$a(\tau) = \phi(X(\tau)), \qquad b(\tau) = h(X(\tau)),$$
$$c(\sigma, \tau) = \exp\left\{-\int_\sigma^\tau h(X(\alpha))d\alpha\right\},$$

with $X$ a process starting at the point $r$, and the relation

$$1 - c(0, \tau) = \int_0^\tau b(\sigma)c(\sigma, \tau)d\sigma,$$

which is verified by integration by parts.

Since the conditional probability that $R$ exceeds $\tau$, given the path $X(\omega)$, is

$$\exp\left\{-\int_0^\tau h(X(\alpha, \omega))d\alpha\right\},$$

we may write

$$\begin{aligned} \psi(r) &= -\int_\Omega d\omega \int_0^S a(\tau)d_\tau\, c(0, \tau) \\ &= \int_\Omega d\omega \int_0^S a(\tau)b(\tau)c(0, \tau)d\tau \qquad (4.5) \\ &= \int_\Omega d\omega \int_0^S a(\tau)b(\tau)d\tau - \int_\Omega d\omega \int_0^S a(\tau)b(\tau)[1 - c(0, \tau)]d\tau. \end{aligned}$$

The first integral in the last member is the potential of $\phi h$. The second integral is

$$\begin{aligned} &\int_\Omega d\omega \int_0^S a(\tau)b(\tau)d\tau \int_0^\tau b(\sigma)c(\sigma, \tau)d\sigma \\ &\qquad = \int_\Omega d\omega \int_0^S b(\sigma)d\sigma \int_\sigma^S a(\tau)b(\tau)c(\sigma, \tau)d\tau \qquad (4.6) \\ &\qquad = \int_\Omega d\omega \int_0^S b(\sigma)d\sigma \int_\sigma^S a(\tau)d_\tau\, c(\sigma, \tau) \\ &\qquad = \int_\Omega d\omega \int_0^S b(\sigma)\, \psi(X(\sigma))d\sigma, \end{aligned}$$

that is to say, the potential of $h\psi$. The last equality in (4.6) is justified by the simple Markoff property of processes, the independence of $S$ and $X$, and the exponential distribution of $S$. The reader will observe, on reading §10, that the last step is also valid if $S$ is replaced by a system of terminal times.

The full force of our working hypotheses was not used in the proof; the proposition holds, in fact, for transition probabilities that can be realized by measurable processes. Also, the terminal time may be taken identically infinite if the integrals all converge.

If $\phi$ is the potential of a positive function, say $f$, then

$$\psi(r) = \int_{\Omega^*} d\omega \int_R^S f(X(\tau))d\tau, \qquad (4.7)$$

where $X$ is a process starting from $r$ and $\Omega^*$ is the set on which $R$, the time assigned to $X$, is less than $S$. This representation shows that $\psi$ nowhere exceeds $\phi$ and that it increases with $h$, for $R$ decreases as $h$ increases.

The following remarks will be needed when the proposition is used.

Let $A$ be the set where $h$ is strictly positive. On comparing the expression of $\psi(r)$ in (4.4) with the third member of (4.5), which shows that $\phi$ may vary arbitrarily outside $A$ without changing $\psi$, one sees that the point $X(R(\omega), \omega)$

belongs to $A$ for almost all $\omega$ such that $R(\omega)$ is less than $S(\omega)$, so also for almost all $\omega$ such that $R(\omega)$ is finite, because $R$ and $S$ are independent. It is easily shown that the point $X(R(\omega), \omega)$ is regular for $A$ for almost all such $\omega$, assuming the notion to make sense for $A$, and the assertion remains valid even if $h$ does not satisfy the hypotheses of the proposition.

Suppose now that $E$ is a nearly analytic set having the property that the sample paths of a process starting at a point of $E$ remain in $E$ for an initial interval of time with probability 1. Let $X$ be an arbitrary process, $T$ the time it hits $E$, $\alpha$ a strictly positive number, and $B(\omega)$ the set of $\sigma$ in the interval $(T(\omega), T(\omega) + \alpha)$ for which $X(\sigma, \omega)$ belongs to $E$. For almost all $\omega$ such that $T(\omega)$ is finite, the point $X(T(\omega), \omega)$ either belongs to $E$ or is regular for $E$ and the Lebesgue measure of $B(\omega)$ is strictly positive, according to the hypotheses on $E$. Let $h$ run through an increasing sequence of functions which vanish outside $E$ and tend to infinity at every point of $E$; the preceding remark implies that, for almost all $\omega$ such that $T(\omega)$ is finite, the time $R(\omega)$ assigned by $h$ to a sample path of the process $X$ decreases to a limit which is less than $T(\omega) + \alpha$. Since $\alpha$ is arbitrary and $R$ is never less than $T$, the time $R$ decreases to $T$ with probability 1. Equation (4.7) now shows that $\psi$ increases to $H_E\,\phi$, if $\phi$ is the potential of a positive function.

Finally, suppose that $h$ is bounded above by $\alpha$ and that $\phi$ is the potential of a function $f$ which is bounded below by the positive constant $\beta$. In the notation used in (4.7)

$$\phi(r) - \psi(r) = \int_{\Omega^*} d\omega \int_0^R f(X(\tau))d\tau \geqq \beta \int_{\Omega^*} R\, d\omega \geqq \frac{\beta}{\lambda + \alpha}.$$

Specialize $h$ to $\alpha\chi$, with $\chi$ the characteristic function of a set $A$; we see that

$$\liminf_{\alpha \to \infty} \alpha(\phi - \psi) \geqq \beta$$

everywhere on $A$. We have observed before that $\psi$ increases with $\alpha$; when $\phi$ is taken to be the constant 1, the limit of $\psi(r)$ as $\alpha$ becomes large is just the probability that the numbers $\tau$ in the interval $(0, S(\omega))$ for which $X(\tau, \omega)$ belongs to $A$ form a set with strictly positive Lebesgue measure.

## 5. Excessive functions

A positive function $\phi$ on $\mathcal{K}$ is excessive (relative to the terminal time $S$) if it is measurable over the field $\mathcal{A}$ and if $H_\tau\,\phi$ increases everywhere to $\phi$ as $\tau$ decreases to 0. If the parameter vanishes and the transition probabilities define Brownian motion in three dimensions, the class of excessive functions is just the class of positive superharmonic functions augmented by the function which is identically infinite; this statement is an easy consequence of [6] or of Proposition 5.3.

A function is excessive for one value of $\lambda$ if and only if it is excessive for every greater value. If $\phi$ is excessive, then $H_\tau\,\phi$ decreases as $\tau$ increases and as $\lambda$ increases.

A positive constant is excessive. So also is the potential of a positive function $f$, for

$$H_\tau\, Uf \equiv \int_\tau^\infty H_\sigma f\, d\sigma \to \int_0^\infty H_\sigma f\, d\sigma \equiv Uf.$$

The limit of an increasing sequence of excessive functions is itself excessive.

The most important excessive function we shall study is the probability of hitting a nearly analytic set $E$. As usual, let $X_r$ be a process starting from the point $r$ and let $T_r$ be the time it hits $E$. Then $\Phi_E(r)$ is defined to be the probability that $X_r(\sigma)$ belongs to $E$ for some $\sigma$ in the open interval $(0, S)$ or, equivalently, the probability that $T_r$ is less than $S$. By the last few paragraphs of §1, this function is measurable over $\mathfrak{A}$, and $H_\tau \Phi_E.(r)$ is the probability of the joint event that $S$ exceeds $\tau$ and $X(\sigma)$ belongs to $E$ for some $\sigma$ in the open interval $(\tau, S)$. So $\Phi_E$ is indeed excessive. If $\lambda$ is strictly positive, a point $r$ is regular for $E$ if and only if $\Phi_E(r)$ is 1.

Another example of an excessive function is $H_E\, Uf$, with $f$ positive. Given $r$ and $\tau$, let $T$ be the infimum of the $\sigma$ greater than $\tau$ for which $X_r(\sigma)$ belongs to $E$, and let $\Omega'$ be the set where $T$ is less than $S$. Clearly, $T$ decreases to $T_r$ as $\tau \to 0$, so that

$$H_\tau H_E\, Uf.(r) \equiv \int_{\Omega'} d\omega \int_T^S f(X_r(\sigma))\, d\sigma$$

increases to

$$H_E\, Uf.(r) \equiv \int_{\Omega_r} d\omega \int_{T_r}^S f(X_r(\sigma))\, d\sigma,$$

with $\Omega_r$ the set where $T_r$ is less than $S$. The calculation proves again that $\Phi_E(r)$ is excessive, at least when $\lambda$ is strictly positive and constants are potentials, for $H_E(r, \mathfrak{K})$ is another way of writing $\Phi_E(r)$.

There are two results of importance in this section, Proposition 5.3 and Theorem 5.6. The one concerns approximation of an excessive function by potentials, the other the composition of an excessive function with a process.

Proposition 5.1. *An excessive function $\phi$ is the limit of an increasing sequence of bounded excessive functions.*

Let $\psi$ be the minimum of $\phi$ and the positive constant $\alpha$. This function will later be shown to be excessive, but at the moment only the inequality $H_\tau \psi \leqq \psi$ is obvious. The inequality implies that $H_\tau \psi$ increases as $\tau$ decreases. Let $\bar\psi$, the adjusted minimum, be the limit as $\tau \to 0$. It is excessive, for $H_\tau \bar\psi$ is the limit of $H_\sigma \psi$ as $\sigma$ decreases to $\tau$, and it increases with $\alpha$. As $\alpha$ tends to infinity through a sequence of values

$$\lim \bar\psi \geqq \lim H_\tau \psi = H_\tau \phi, \qquad \tau > 0,$$

by monotone convergence. Since $\tau$ is arbitrary the first limit must be $\phi$ itself.

PROPOSITION 5.2. *Let $\phi$ be excessive and $r$ a point such that $H_\tau \phi.(r)$ is finite for all $\tau$ and tends to 0 as $\tau \to \infty$. Then, at the point $r$, the potential of*

$$(\phi - H_\tau \phi)/\tau$$

*increases to $\phi(r)$ as $\tau$ decreases to 0.*

For, the value of the potential at $r$ is

$$\lim_{\alpha \to \infty} \frac{1}{\tau} \int_0^\alpha [H_\sigma \phi.(r) - H_\sigma H_\tau \phi.(r)] d\sigma$$

$$= \frac{1}{\tau} \int_0^\tau H_\sigma \phi.(r) d\sigma - \lim_{\alpha \to \infty} \frac{1}{\tau} \int_\alpha^{\alpha+\tau} H_\sigma \phi.(r) d\sigma$$

$$= \frac{1}{\tau} \int_0^\tau H_\sigma \phi.(r) d\sigma,$$

which obviously increases to $\phi(r)$.

In particular, $\phi$ is the limit of an increasing sequence of potentials of positive functions if all points satisfy the hypothesis of the proposition. This is true, for example, if $\lambda$ is strictly positive and $\phi$ is bounded. The next proposition follows from this remark and Proposition 5.1; the succeeding one, from the calculation above and dominated convergence.

PROPOSITION 5.3. *If $\lambda$ is strictly positive, then every excessive function is the limit of an increasing sequence of bounded potentials of positive functions.*

PROPOSITION 5.4. *Let $\lambda$ be strictly positive and let $\phi$ be a bounded excessive function such that $(\phi - H_\tau \phi)/\tau$ converges boundedly to a function $f$ as $\tau \to 0$. Then $\phi$ is the potential of $f$.*

Let $E$ be a nearly analytic set and $\phi$ an excessive function. Then $H_E \phi$ is also excessive and it is majorized by $\phi$, according to Propositions 4.2, 5.3, and the remarks at the beginning of this section.

PROPOSITION 5.5. *If $\phi$ is excessive and if $r$ is regular for the nearly analytic set $E$, then*

$$\inf_E \phi \leqq \phi(r) \leqq \sup_E \phi. \tag{5.1}$$

If $\alpha$ is the infimum of $\phi$ on $E$ and $F$ a compact subset of $E$, then

$$\phi(r) \geqq H_F \phi.(r) \geqq \alpha H_F(r, \mathcal{H}),$$

because the distribution of hits of $F$ is concentrated on $F$. The first inequality in (5.1) is proved by letting $F$ run through the sequence of Proposition 2.1 and observing that $H_F(r, \mathcal{H})$ tends to 1 because $r$ is regular for $E$.

In proving the second inequality we assume $\lambda$ to be strictly positive, for $\phi$ remains excessive as $\lambda$ increases. Also, by Proposition 5.3, it is enough to prove the inequality for $\phi$ the bounded potential of a positive function $f$.

Let $T$ be the time a process $X_r$ starting from $r$ hits the compact subset $F$ of $E$, and let $\Omega'$ be the set on which $T$ is less than $S$. Then $Uf.(r)$ can be written

$$\int_{\Omega'} d\omega \int_T^S f(X_r(\tau))d\tau + \int_{\Omega'} d\omega \int_0^T \cdots d\tau + \int_{\Omega-\Omega'} d\omega \int_0^S \cdots d\tau.$$

The first integral is $H_F\, Uf.(r)$, which does not exceed the supremum of $Uf$ on $E$. As for the other two integrals, they approach 0 as $F$ runs through the sequence of Proposition 2.1, because $T$ tends to 0 with probability 1, the probability of $\Omega - \Omega'$ tends to 0, and the finiteness of the integral in (4.1) allows one to use dominated convergence.

THEOREM 5.6. *An excessive function $\phi$ is nearly Borel measurable. For every process $X$ and for almost all $\omega$, the function $\phi(X(\tau, \omega))$ of $\tau$ is continuous on the right, and it is finite for all $\tau$ greater than $\sigma$ if $\phi(X(\sigma, \omega))$ is finite.*

The continuity is defined using the topology of the extended reals, since an excessive function may take on infinite values.

The second sentence of the theorem will be proved first, assuming $\phi$ to be nearly Borel measurable, and the result will be used in proving the first sentence.

If $X$ is a process starting at a point $r$, then $\phi(X(\tau))$ approaches $\phi(r)$ with probability 1 as $\tau \to 0$. Indeed, let $A$ be the set where $\phi$ exceeds $\phi(r) + \alpha$ and $B$ the set where $\phi$ is less than $\phi(r) - \alpha$, with $\alpha$ strictly positive. (Take $B$ to be the set where $\phi$ is less than $1/\alpha$, if $\phi(r)$ is infinite.) The point $r$ is regular for neither $A$ nor $B$, according to Proposition 5.5, so that the time $T$ at which $X$ hits $A \cup B$ is strictly positive with probability 1. Now, $\phi(X(\tau))$ lies between $\phi(r) + \alpha$ and $\phi(r) - \alpha$, or $1/\alpha$ if $\phi(r)$ is infinite, for all $\tau$ less than $T$.

Let $X$ be any process. The proof at the end of §2 shows that the stopping time $T$, defined by taking $T(\omega)$ to be the infimum of the $\tau$ for which

$$\sup_{0 \leqq \sigma \leqq \tau} | \phi(X(\sigma, \omega)) - \phi(X(0, \omega)) | \geqq \alpha,$$

is measurable over $\mathfrak{F}$, and the result just proved, together with the argument at the end of §1, shows that $T$ is strictly positive with probability 1. The continuity of $\phi(X(\tau))$ on the right, with probability 1, is now established by repeating the latter half of the proof of Theorem 7.2 of [7]. Alternatively, one can give an indirect proof using a suitable stopping time; this method, however, requires another proof of measurability.

Let $E$ be the set on which $\phi$ is infinite. By Proposition 5.5, every point regular for $E$ belongs to $E$; so Proposition 2.3 implies that each measure $H_E(r, ds)$ is concentrated on $E$. Consequently, the measure $H_E(r, ds)$ vanishes if $\phi$ is finite at the point $r$, for $H_E\, \phi$ nowhere exceeds $\phi$. Suppose $r$ to be such a point and $T$ to be the time a process starting at $r$ hits $E$. Clearly, $T$ cannot be less than $S$ with strictly positive probability; it is therefore infinite with probability 1, because $T$ and $S$ are independent. It follows quickly from this fact, by an argument using conditional probabilities, that $\phi(X(\tau))$ is finite for all $\tau$, with probability 1, if $\phi(X(0))$ is finite with probability 1.

Let $X$ be any process and $F$ a closed set on which $\phi$ is finite. By the preceding paragraph and the extended Markoff property, the function $\phi(X(\tau))$ is, with probability 1, finite for all $\tau$ greater than the time $X$ hits $F$; and, by Proposition 2.1, the set $F$ may be replaced in this assertion by the set where $\phi$ is finite. The proof of the second sentence of the theorem is now complete, for an excessive function known to be nearly Borel measurable.

In proving that an excessive function is nearly Borel measurable, we assume, as we may, that $\lambda$ is strictly positive. Consider first a bounded potential $Ug$, with $g$ a positive function, and let $X$ be a process having initial distribution $\mu$. Choose Borel measurable functions $f$ and $h$ so that

$$f \leqq g \leqq h, \qquad \int_{\mathcal{K}} \mu U.(ds) f(s) = \int_{\mathcal{K}} \mu U.(ds) h(s).$$

These relations imply that, for every $\tau$, the equation

$$Uf.(X(\tau, \omega)) = Uh.(X(\tau, \omega)) \tag{5.2}$$

holds with probability 1, because the first member cannot exceed the second and the two have the same expectation, the measure $\mu H_\tau U$ being majorized by $\mu U$. By Fubini's theorem, for every $\omega$ in a certain set of probability 1 equation (5.2) holds for all $\tau$ outside a set of Lebesgue measure null, and so for a set dense in the positive reals. Now, the properties of the transition probabilities mentioned in the first paragraph of §1 ensure that $Uf$ and $Uh$ are Borel measurable, so that the two members of (5.2) are continuous on the right in $\tau$ with probability 1, by what has been proved before. Matters being so, for almost all $\omega$ equation (5.2) is true for all $\tau$ and the two members have the common value $Ug.(X(\tau, \omega))$. Hence $Ug$ is nearly Borel measurable.

The extension of this result to arbitrary excessive functions is an immediate consequence of Proposition 5.3 and the fact that the set on which a sequence of Borel measurable functions converges is a Borel set. The proof of the theorem is now complete.

We shall prove one more proposition before discussing the theorem.

PROPOSITION 5.7. *The minimum of two excessive functions is itself excessive.*

Let $\phi$ and $\psi$ be excessive, let $\phi(r)$ be the value of the minimum at $r$, and let $X$ be a process starting at $r$. As $\tau \to 0$,

$$\begin{aligned}\lim \int_{\mathcal{K}} H_\tau(r, ds) \min \{\phi(s), \psi(s)\} &= \lim \int_\Omega e^{-\lambda\tau} \min \{\phi(X(\tau)), \psi(X(\tau))\}\, d\omega \\ &\geqq \int_\Omega \lim \min \{\phi(X(\tau)), \psi(X(\tau))\}\, d\omega \\ &= \phi(r),\end{aligned}$$

by Fatou's lemma and the preceding theorem. On the other hand, the limit cannot exceed $\phi(r)$.

It is worth spending a few minutes treating certain problems of measurability related to Theorem 5.6, if only to show that the notion of nearly Borel measurable function is not a thin generality; afterwards we shall discuss the theorem itself.

First of all, an excessive function need not be Borel measurable. A trivial example is given by taking $\mathcal{H}$ to be the reals and all processes to be constant in time; a function is excessive if it is positive and measurable over the field $\mathcal{A}$. Indeed, there are good reasons for relaxing the measurability, and declaring every positive function to be excessive. Simple translation in the plane is practically the same example. A less trivial one is Brownian motion in the plane, the motion being stopped the moment a particle hits a given curve; here an excessive function is Borel measurable on the complement of the curve, but perhaps not so on the curve.

Every excessive function is Borel measurable if the integral

$$\int_0^\infty \gamma(\tau)\ P_\tau(r, A)d\tau$$

is Borel measurable in $r$ whenever $A$ is a set in $\mathcal{A}$ and $\gamma$ is Borel measurable on the positive reals. For, if $\phi$ is excessive, then

$$\phi = \lim_{T\to 0} \frac{1}{T} \int_0^T e^{-\lambda\sigma}\ P_\sigma\ \phi\ d\sigma,$$

and the integral is Borel measurable. The condition on the transition probabilities is verified if all the measures $U(r, ds)$, for one value of $\lambda$, are absolutely continuous with respect to some one measure. Similar conditions on the transition measures $P_\tau(r, ds)$ are more restrictive, ruling out uniform motion on a line, for example. The condition at the beginning of the paragraph is itself undesirable, because it may be lost in forming Cartesian products. By the way, the reason some propositions are stated for nearly analytic sets rather than for nearly Borel sets is that in some applications it is necessary to pass from a nearly Borel set in a product space to its projection on one of the factor spaces.

Let us turn to the implications of the theorem. Doob proved the theorem for certain processes, Brownian motion in space or in space-time being among them, and he used it in studying boundary value problems. It will suffice here to describe the consequences we shall use.

Let $X$ be a process, $\mathcal{F}_\sigma$ the field generated by the random points $X(\tau)$ for $\tau$ not exceeding $\sigma$, and $\phi$ an excessive function. It is clear from the definitions that the family of random variables $e^{-\lambda\tau}\phi(X(\tau))$ is a lower semimartingale relative to the fields $\mathcal{F}_\sigma$, provided the expectation of $\phi(X(0))$ is finite. This semimartingale is separable, nearly all the sample functions being continuous on the right according to the theorem; so the results of martingale theory may be used. In particular, with probability 1 finite limits from the left exist at all values of $\tau$ as well as a finite limit as $\tau$ becomes infinite. The assertion remains true, except for the finiteness, even if the expectation of the initial variable is infinite and the family is not properly a semimartingale. To see this, first observe that the proof of the existence of finite limits is immediate, using conditional probabilities, when $\phi(X(0))$ is finite with probability 1. In the general case, let $E$ be the set where $\phi$ is finite, let $T'$ be the time $X$ hits a compact subset $F$ of $E$, and let $\Omega'$ be the set where $T'$ is finite. The existence of finite limits from the left is proved for almost all $\omega$ in $\Omega'$ and all $\tau$ greater than $T'(\omega)$ by the extended Markoff property and the preceding remark. On letting $F$ run through the sequence of Proposition 2.1, we see that $T'$ may be replaced by $T$, the time $X$ hits $E$. This settles the existence of finite limits for values of $\tau$ greater than $T$. At other values, and at infinity if $T$ is infinite, the limit from the left is trivially infinite. It is the existence of limits from the left, rather than continuity on the right, that is used in most proofs.

A set in $\mathcal{K}$ is said to be approximately null if it is a null set for each measure $U(r, ds)$, and an assertion concerning a variable point of $\mathcal{K}$ is said to hold approximately everywhere if it is true except when the point belongs to a certain approximately null set. These definitions do not depend upon the value of $\lambda$.

PROPOSITION 5.8. *The points belonging to a nearly analytic set $E$ but not regular for $E$ form an approximately null set.*

Let the parameter $\lambda$ be given a strictly positive value. The set of points not regular for $E$ is defined by the inequality $\Phi_E < 1$, where $\Phi_E$ is the excessive function defined at the beginning of the section, so that the exceptional set mentioned in the proposition is nearly analytic, being the intersection of two nearly analytic sets. The proof will be completed by showing that some point of a nearly analytic set $A$ is regular for $A$ unless the set is approximately null. Suppose $r$ chosen so that $U(r, A)$ is strictly positive; then $U(r, F)$ is strictly positive for some compact subset $F$ of $A$; by Proposition 2.4, some point is regular for $F$, hence for $A$; and this point must belong to $F$, because $F$ is compact.

It is sometimes useful to know that the set on which an excessive function $\phi$ has infinite values is negligible, in the sense of §2, if and only if it is approximately null. Only one implication need be proved, because a negligible set is clearly approximately null by Fubini's theorem. Suppose, then, that a process $X$ starting from the point $r$ hits the set $E$ where $\phi$ is infinite with strictly positive probability, and choose $\sigma$ and $\alpha$, both strictly positive, so that with probability $\alpha$ a sample path of the process meets $E$ at some time after $\sigma$. Define a new process $Z$ by

$$Z(\tau, \omega) = X(\tau + R(\omega), \omega),$$

where $R$ is a random variable independent of $X$ and distributed uniformly over the interval $(0, \sigma)$. The initial distribution of $Z$ is

$$\mu(A) \equiv \frac{1}{\sigma}\int_0^\sigma P_\tau(r, A)d\tau, \qquad A \subset \mathfrak{K},$$

which is absolutely continuous with respect to $U(r, ds)$, and the sample paths of $Z$ hit $E$ with probability at least $\alpha$. According to Theorem 5.6, this can happen only if $Z(0)$ belongs to $E$ with the same probability, so that $U(r, E)$ is strictly positive.

The proper restriction to place on an excessive function $\phi$ is that it be infinite only on a negligible set. Then, for every process $X$, almost all the random functions $\phi(X(\tau))$ are finite for all strictly positive $\tau$.

Observe that an excessive function $\phi$ is determined once it is known approximately everywhere, for

$$\phi = \lim_{\tau \to 0} \frac{1}{\tau}\int_0^\tau H_\sigma \phi \, d\sigma.$$

Similarly, an inequality between two excessive functions holds everywhere if it holds approximately everywhere.

## 6. Characterization of $H_E \phi$

In this section $\phi$ denotes an excessive function and $E$ a nearly analytic set; only the supplementary restrictions on $\phi$ or $E$ will be mentioned.

Let $\psi$ be an excessive function for which the inequality $\psi \geqq \phi$ holds on $E$. According to Theorem 5.6 and the definition of regular points, the inequality is also true at all points regular for $E$, so that $H_E \psi$, and hence $\psi$ itself, majorizes $H_E \phi$ everywhere. The next proposition now follows from the remark that $H_E \phi$ coincides with $\phi$ at the points regular for $E$.

PROPOSITION 6.1. *If all points of $E$ are regular for $E$, then $H_E \phi$ is the least excessive function which majorizes $\phi$ on $E$.*

In a sense the proposition cannot be bettered: If $r$ belongs to $E$ but is not regular for $E$, take $\phi$ as the constant 1; then $H_E \phi.(r)$ is less than 1 for $\lambda$ strictly positive. It is therefore necessary to consider similar propositions permitting an exceptional set. The next two theorems prepare the way.

THEOREM 6.2. *Let $X$ be a process and $(T_n)$ an increasing sequence of stopping times for $X$ whose limit is $T$, the time $X$ hits $E$, with probability 1. Then $\Phi_E(X(\tau, \omega)) \to 1$ as $\tau$ increases to $T(\omega)$, for almost all $\omega$ such that $T(\omega)$ is finite and greater than $T_n(\omega)$ for all $n$.*

The function $\Phi_E$ is, of course, the probability of hitting $E$ before the time $S$. We shall assume $\lambda$ to be strictly positive, for $\Phi_E$ decreases as $\lambda$ increases; we shall also assume $X$, $\mathcal{E}_n$, $S$ to be independent for each $n$, where $\mathcal{E}_n$ is the auxiliary field used in defining $T_n$. The terminal time is then finite, and the sets

$$(6.1) \qquad \begin{array}{ll} \Omega' : T < S,\ T_n < T \text{ for all } n, & \Omega'_n : T_n < S,\ T_n < T, \\ \Omega'' : T < S,\ T_n = T \text{ for some } n, & \Omega''_n : T_n < S,\ T_n = T, \end{array}$$

satisfy the relations

$$(6.2)\qquad \begin{aligned} &\Omega'_n \supset \Omega'_{n+1} \supset \Omega', &\quad \mathcal{P}\{\Omega'_n\} \to \mathcal{P}\{\Omega'\},\\ &\Omega''_n \supset \Omega''_{n+1} \supset \Omega'', &\quad \mathcal{P}\{\Omega''_n\} \to \mathcal{P}\{\Omega''\}.\end{aligned}$$

By the extended Markoff property, with stopping time $T_n$,

$$(6.3)\qquad \mathcal{P}\{T < S\} = \int_{\Omega'_n} \Phi_E(X(T_n))d\omega + \mathcal{P}\{\Omega''_n\}.$$

Now, $\Phi_E(X(\tau, \omega))$ has a limit $\psi(\omega)$, for almost all $\omega$, as $\tau$ increases to $T(\omega)$; and the preceding equation becomes

$$\mathcal{P}\{T < S\} = \int_{\Omega'} \psi(\omega)d\omega + \mathcal{P}\{\Omega''\},$$

as $n \to \infty$. Since $\psi$ is bounded by 1 and since the left member is the probability of $\Omega' \cup \Omega''$, the function $\psi$ must have the value 1 almost everywhere on $\Omega'$. The proposition is now proved by letting $\lambda$ tend to 0, or by observing that $S$ may be chosen independent of all the other quantities mentioned.

THEOREM 6.3. *Let $X$ be a process whose initial distribution attributes no mass to the set of points belonging to $E$ but not regular for $E$. Then there is a decreasing sequence of nearly analytic sets $E_n$ having the following properties: Almost all sample paths of a process starting at a point of $E_n$ remain in $E_n$ for an initial interval of time. Each point of $E$ is regular for $E \cup E_n$. The time $T_n$ at which $X$ hits $E \cup E_n$ increases with probability 1 to $T$, the time $X$ hits $E$. For almost all $\omega$ such that $T(\omega)$ is finite, $T_n(\omega)$ coincides with $T(\omega)$ for sufficiently large $n$.*

During the proof $\lambda$ is to be given some strictly positive value. Assume first that $\Phi_E$ is less than $\alpha$ on $E$, with $\alpha$ less than 1; then no point of $E$ is regular for $E$, so that the initial distribution of $X$ attributes no mass to $E$. Let the $G_n$ be the open sets mentioned in Proposition 2.2, and take $E_n$ to be the intersection of $G_n$ with the part of $\mathcal{K}$ where $\Phi_E$ is less than $\alpha$. The $E_n$ have the first three properties mentioned in the proposition, by Theorem 5.6 and the fact that $E_n$ includes $E$. In verifying the fourth property, let $\Omega'$ be the set defined by the conditions

$$T(\omega) < \infty, \qquad T_n(\omega) < T(\omega) \textit{ for all } n.$$

According to the preceding theorem, $\Phi_E(X(T_n))$ approaches 1 almost everywhere on $\Omega'$. On the other hand, $X(T_n(\omega), \omega)$ is regular for $E_n$ for almost all $\omega$ in $\Omega'$, and the value of $\Phi_E$ at such a point does not exceed $\alpha$. Consequently, $\Omega'$ is a null set,—that is to say, the $E_n$ have the fourth property.

Given an arbitrary nearly analytic set $E$, take $A_k$ to be the subset of $E$ where $\Phi_E$ is less than $1 - 1/k$ and let $R_k$ be the time $X$ hits $A_k$. These sets clearly satisfy the additional condition imposed at the beginning of the proof, so that one can find nearly analytic sets $A_k^n$ having the first property mentioned in the proposition and such that

$$A_k^n \supset A_k^{n+1} \supset A_k\,, \qquad \mathcal{P}\{R_k^n \neq R_k\,, R_k^n < n\} < 2^{-n-k},$$

where $R_k^n$ is the time $X$ hits $A_k^n$ , Let us set

$$E_n = \bigcup_k A_k^n\,.$$

The $E_n$ decrease with $n$ and have the first property. They also have the second property, for a point of $A_k$ is regular for $A_k^n$ and a point of $E$ not belonging to some $A_k$ is regular for $E$ itself. The remaining two properties follow from the relation

$$T_n = \min\{T, \inf_k R_k^n\},$$

which implies that, after a set of probability less than $2^{-n}$ is excepted, one of the two assertions $T_n(\omega) = T(\omega)$ or $T_n(\omega) > n$ must be true.

The next theorem ranks with Theorem 5.6 in importance.

THEOREM 6.4. *If $\lambda$ is strictly positive, then $H_E\,\phi$ coincides, except perhaps at the points belonging to E but not regular for E, with the infimum of the excessive functions which majorize $\phi$ on E.*

By the remarks at the beginning of the section, it suffices to prove that the infimum does not exceed $H_E\,\phi$ at any point outside $E$.

First, assume that the restriction of $\phi$ to $E$ is bounded, say by $\alpha$, take the sets $E_n$ to be those of the preceding theorem, and let $A_n$ be the intersection of $E \cup E_n$ with the set where $\phi$ is less than $\alpha + 1$. According to Theorem 5.6, the $A_n$ have all the properties of the sets $E \cup E_n$ , and in addition $\phi$ does not exceed $\alpha + 1$ at any point belonging to $A_n$ or regular for $A_n$ . Now specialize the process of the preceding theorem to one starting at a point $r$ outside $E$, let $T$ or $T_n$ be the time it hits $E$ or $A_n$ , and let $\Omega'$, $\Omega_n$ , $\Omega_n'$ be the sets

$$\Omega' : T < S, \qquad \Omega_n : T_n < S, \qquad \Omega_n' : T_n = T < S.$$

Then $\Omega'$ includes $\Omega_n'$ , and

$$\mathcal{P}\{\Omega_n - \Omega_n'\} \to 0, \qquad \mathcal{P}\{\Omega_n'\} \to P\{\Omega'\},$$

because the terminal time is finite. Since every point of $E$ is regular for $A_n$ , the excessive function $H_{A_n}\,\phi$ coincides with $\phi$ on $E$; moreover, its value at $r$ is

$$H_{A_n}\,\phi.(r) \equiv \int_{\Omega_n} \phi(X(T_n))d\omega,$$

the integrand being bounded by $\alpha + 1$ almost everywhere on $\Omega_n$ because the point $X(T_n(\omega), \omega)$ is regular for $A_n$ for almost all $\omega$ in $\Omega_n$ . As $n$ increases the integral decreases to

$$H_E\,\phi.(r) \equiv \int_{\Omega'} \phi(X(T))d\omega,$$

according to the relations above and the fact that $T_n(\omega)$ ultimately coincides with $T(\omega)$, for almost all $\omega$ in $\Omega'$. The proof under the additional hypothesis is now complete. We give one application before going on.

PROPOSITION 6.5. *Let $\lambda$ be strictly positive, let $r$ lie outside $E$, and let $\Phi_E(r)$ vanish. Then there is an excessive function $\psi$ which is infinite at all points of $E$ and less than* 1 *at $r$.*

Take the function $\phi$ of the theorem to be the constant 1. Then $H_E\,\phi$ reduces to $\Phi_E$ and vanishes at $r$. So, for each natural number $n$, there is an excessive function with values at least 1 on $E$ and less than $2^{-n}$ at $r$, and $\psi$ may be taken to be their sum.

We continue the proof of the theorem, assuming now that $\phi$ is finite at every point of $E$. Let $\varepsilon$ be strictly positive, $r$ outside $E$, and $H_E\,\phi.(r)$ finite. If $E_k$ is the part of $E$ where $\phi$ is less than $k$, one can choose a nearly analytic set $B_k$ including $E_k$ so that

$$H_{B_k}\,\phi.(r) \leqq H_{E_k}\,\phi.(r) + \varepsilon 2^{-k}$$

and so that every point of $E_k$ is regular for $B_k$. According to inequality (11.10), which can be proved immediately after Theorem 5.3 and is placed in a later section only because of the context,

$$H_B\,\phi.(r) - H_E\,\phi.(r) \leqq \sum [H_{B_k}\,\phi.(r) - H_{E_k}\,\phi.(r)] \leqq \varepsilon,$$

with $B$ the union of the $B_k$. The excessive function $H_B\,\phi$ coincides with $\phi$ on $E$, for every point of that set is regular for $B$, and the proof has again been completed under a restrictive condition.

Finally, consider an arbitrary excessive function $\phi$, let $r$ be a point outside $E$, and let $F$ be the part of $E$ where $\phi$ is infinite. If $\Phi_F(r)$ is strictly positive, then $H_F\,\phi.(r)$ is infinite, so that $H_E\,\phi.(r)$ is also infinite and there is nothing to prove. If $\Phi_F(r)$ vanishes, let $\psi$ be an excessive function which is infinite on $F$ and small at $r$, and let $\psi'$ be an excessive function which majorizes $\phi$ on $E - F$ and is not much greater than $H_E\,\phi.(r)$ at $r$; the existence of such functions has already been proved. It is now clear, on considering the properties of $\psi + \psi'$, that the proof of the theorem is complete.

The theorem evidently characterizes $H_E\,\phi$, for $\lambda$ strictly positive, because the exceptional set is approximately null. In particular, the probability $\Phi_E$ of hitting $E$ before the terminal time is the greatest excessive function which is majorized by every excessive function exceeding 1 on $E$, provided $\lambda$ is strictly positive.

The theorem must be restated if $\lambda$ vanishes, as one sees on considering Brownian motion in the plane, with $E$ a point and $\phi$ a constant. The simplest version will be discussed in §13.

A set $E$ is said to be nearly open if it is nearly analytic and if, for every process starting at a point of $E$, almost all sample paths of the process remain in $E$ for an initial interval of time. There is another characterization of

$H_E\,\phi$ for such sets. By Proposition 4.2, $H_E\,\phi$ majorizes the potential $Uf$ if $f$ is a positive function vanishing outside $E$ and $Uf$ nowhere exceeds $\phi$. The next theorem asserts that $H_E\,\phi$ can be approximated by these potentials, provided $E$ is nearly open.

THEOREM 6.6. *If $\lambda$ is strictly positive and $E$ nearly open, there is a sequence of positive functions, each one vanishing outside $E$, whose potentials increase to $H_E\,\phi$ everywhere. The functions may be taken to have compact supports included in $E$, if $E$ is the countable union of closed sets.*

Suppose first that $\phi$ is bounded, and let $\psi$ be the potential mentioned in Proposition 4.4. One verifies, using Theorem 5.3, that $\psi$ increases with the function $h$ and that the difference $\phi - \psi$ is positive, for these assertions are true when $\phi$ is the potential of a positive function. Now let $h$ run through a sequence of functions which increase to infinity at every point of $E$ and which vanish outside $E$—or outside some variable compact subset of $E$, if $E$ is a countable union of closed sets. The time assigned to a process by $h$ decreases with probability 1 to the time the process hits $E$. These facts, together with the definition of $\psi$ and the continuity of $\phi(X(\tau))$ on the right, imply that $\psi$ increases to $H_E\,\phi$ everywhere.

If $\phi$ is unbounded, let $\psi_{n,h}$ be the potential defined as $\psi$ above, but with $\phi$ replaced by the minimum of $\phi$ and the positive integer $n$. This potential increases with $n$ and with $h$, so that the theorem is proved by letting $n$ tend to infinity as $h$ runs through the sequence described before.

We shall now discuss determining sets for an excessive function $\phi$. Let $\mathcal{K}_c$ be the one point compactification of $\mathcal{K}$, or $\mathcal{K}$ itself if the space is already compact. When we speak of $H_G$ and so on, $G$ being a set in $\mathcal{K}_c$, we mean the kernel or other quantity defined by the part of $G$ in $\mathcal{K}$. A closed set $F$ in $\mathcal{K}_c$ is said to be a determining set for $\phi$ if $H_G\,\phi$ coincides with $\phi$ whenever $G$ is a neighborhood of $F$. The notion is very coarse; a finer one requires ramification of the point at infinity.

Let $\phi$ be bounded, $\lambda$ strictly positive, $F$ a determining set for $\phi$, and $G_n$ a decreasing sequence of compact neighborhoods of $F$ in $\mathcal{K}_c$ which shrink to $F$. If $X$ is a process starting at a point $r$ outside $F$, the time $T_n$ at which it hits $G_n$ increases with probability 1 to $T$, the time it hits $F$; so the probability of $\Omega_n$, the set where $T_n$ is less than $S$, decreases to the probability of $\Omega'$, the set where $T$ is less than $S$. This behavior and the equation

$$\phi(r) \equiv H_{G_n}\,\phi.(r) \equiv \int_{\Omega_n} \phi(X(T_n))d\omega, \tag{6.4}$$

together imply that

$$\phi(r) = \int_{\Omega^1} \lim_{\tau \nearrow T} \phi(X(\tau))d\omega + \int_{\Omega^2} \phi(X(T))d\omega. \tag{6.5}$$

The set $\Omega^2$ is the part of $\Omega'$ on which $T_n(\omega)$ coincides with $T(\omega)$ for some $n$, and $\Omega^1$ is the remaining part of $\Omega'$; the obvious integrand for the first integral is the limit of $\phi(X(T_n))$, but it may be replaced by the one used because limits from the left exist. The equation shows that $\phi$ is determined by its values and limiting values at the points of a determining set. A similar result, $\Omega'$ being defined by a weak inequality, holds when $\lambda$ vanishes. A more detailed study is to be found in Doob's papers.

PROPOSITION 6.7. *If $\lambda$ is strictly positive and if $G$ is a neighborhood of a determining set for $\phi$, there is a sequence of positive functions, each one vanishing outside $G$, whose potentials increase to $\phi$.*

PROPOSITION 6.8. *An excessive function majorizes $\phi$ everywhere if it does so on a neighborhood of a determining set for $\phi$.*

The first proposition follows from Theorem 6.6, the second from the remark after Proposition 5.4.

PROPOSITION 6.9. *Let $E$ be open in $\mathcal{K}_c$. Then $H_E\,\phi$ is the supremum of the excessive functions which are majorized by $\phi$ and have a determining set included in $E$.*

According to the preceding proposition, $H_E\,\phi$ is an upper bound for the excessive functions mentioned. The proof is completed, when $\lambda$ is strictly positive, by Theorem 6.6 and the remark that the support of a function is a determining set for the potential of the function; the extension to vanishing $\lambda$ is straightforward.

PROPOSITION 6.10. *If $\lambda$ is strictly positive and $\phi$ bounded, then $\phi$ has a least determining set.*

Let $F$ be the intersection of all determining sets for $\phi$. Every neighborhood of $F$ includes the intersection of a finite number of determining sets, for such sets are compact. In proving $F$ to be a determining set, it therefore suffices to establish that the intersection of two determining sets $F_1$ and $F_2$ is itself a determining set. Given a neighborhood $G$ of the intersection, choose a neighborhood $G_i$ of $F_i$ so that $G_1 - G$ and $G_2 - G$ have disjoint closures, and take $E_i$ to be the union of $G_i$ with $G$. Let $X$ be a process starting from a point $r$, let $T$ be the time it hits $G$, and let $\Omega'$ be the set where $T$ is less than $S$. We shall approximate $T$ by an increasing sequence of stopping times $T_n$ defined recursively: $T_1$ is the time $X$ hits $E_1$, $T_{2k}$ is the infimum of the $\tau$ greater than $T_{2k-1}$ for which $X(\tau)$ belongs to $E_2$, and $T_{2k+1}$ is the infimum of the $\tau$ greater than $T_{2k}$ for which $X(\tau)$ belongs to $E_1$; the value of one of these times is to be infinite if there are no such $\tau$. The equality of $T_n(\omega)$ and $T_{n+1}(\omega)$ is equivalent to that of $T_n(\omega)$ and $T(\omega)$, because of the way $G_1$ and $G_2$ were chosen. Also, $T_n(\omega) \to \infty$ if all the $T_n(\omega)$ are distinct; else, some sample path would fail to have a limit from the left at a certain finite time. The probability of $\Omega_n$, the set where $T_n$ is less than $S$, therefore decreases to the probability of

$\Omega'$; and $T_n(\omega)$ coincides with $T(\omega)$, for almost all $\omega$ in $\Omega'$ and for sufficiently large $n$. Since $\phi$ is bounded,

$$\psi_n(r) \equiv \int_{\Omega_n} \phi(X(T_n))d\omega \rightarrow \int_{\Omega'} \phi(X(T))d\omega \equiv H_G\,\phi.(r).$$

On the other hand, all the $\psi_n$ coincide with $\phi$, for

$$\psi_{2k} = H_{E_2}\,\psi_{2k-1}\,, \qquad \psi_{2k+1} = H_{E_1}\,\psi_{2k}\,,$$

by the extended Markoff property. The proof is now complete.

The hypothesis that $\phi$ be bounded cannot be entirely avoided, as shown by the following example: $\mathcal{H}$ is the extended reals with the origin deleted. A particle starting from $+\infty$ or $-\infty$ remains fixed; one finding itself at a finite point $r$ at time $\tau$ moves with velocity $+1$ or $-1$, according as $r$ is positive or negative, and has a probability $d\tau$ of jumping to the point $-r$ during the interval $(\tau, \tau + d\tau)$. The function $\phi$,

$$\phi(\pm\infty) = 0, \qquad \phi(r) = e^{|r|} \; if\; r \neq \pm\infty,$$

is excessive if $\lambda$ has the value 1. But, for strictly positive $a$, the two intervals

$$-\infty \leqq r \leqq -a, \qquad a \leqq r \leqq +\infty,$$

are disjoint and each is a determining set for $\phi$.

Even simpler examples show that boundedness of $\phi$ does not suffice when $\lambda$ vanishes. However, the proposition holds for all $\lambda$ if, instead of boundedness, it is assumed that $\phi$ is majorized by a potential which is finite approximately everywhere. The proof is practically the same.

The least determining set for the potential of a positive bounded continuous function is the support of the function, provided $\lambda$ is positive. The least determining sets for more general excessive functions may behave unexpectedly. For example, the least determining set for $\Phi_E$ may have no point in common with $E$, even when $E$ is compact. Consider the following situation: $\mathcal{H}$ is the interval $b \leqq r < \infty$ with an isolated point $a$ adjoined. A particle starting at a point of the interval travels with uniform velocity to the right; one starting at $a$ remains there a period of time distributed exponentially, then jumps to $b$ and begins the uniform motion. The least determining set for $\Phi_{\{b\}}$ reduces to $a$. If the interval is replaced by the real line, the rest of the description being unchanged, then $\{a, b\}$ is the least determining set. If now the uniform motion on the line is replaced by Brownian motion, $\{b\}$ is the least determining set.

It can be proved, under additional hypotheses on the transition measures, that for every point of $\mathcal{H}$ there is essentially one excessive function whose least determining set reduces to the point, and that the infimum of the excessive functions which majorize a given excessive function $\phi$ on some neighborhood of a fixed closed set $F$ coincides approximately everywhere with the supremum of the excessive functions which are majorized everywhere by $\phi$ and have $F$

for a determining set. It is desirable to find the least hypotheses which imply these statements.

We shall set the problem before the reader by developing the consequences of the following hypothesis:

(*B*) *If $G$ is a neighborhood of the compact set $F$, then $H_G H_F$ coincides with $H_F$ for all $\lambda$.*

This is equivalent to saying, $F$ is a determining set for every excessive function of the form $H_F \phi$. In deriving a less obvious restatement we shall suppose $\lambda$ to be strictly positive. Consider a process $X$ starting at a point $r$ outside $G$, let $T$ be the time it hits $G$, and partition the set where $T$ is less than $S$ into $\Omega'$ and $\Omega''$, the second set being defined by the condition that $X(T)$ belong to $F$. The extended Markoff property on the one hand, and hypothesis ($B$) on the other, give the two equations

$$\Phi_F(r) = \int_{\Omega'} \Phi_F(X(T))d\omega + \mathcal{P}\{\Omega''\}$$

$$\Phi_F(r) = \int_{\Omega'} \Phi_F(X(T))d\omega + \int_{\Omega''} \Phi_F(X(T))d\omega,$$

so that the integrand of the last integral has the value 1 almost everywhere on $\Omega''$. Since $\lambda$ is strictly positive and the terminal time is independent of the process, the point $X(T(\omega), \omega)$ must be regular for $F$ for almost all $\omega$ in $\Omega''$; equivalently, the measure $H_G(r, ds)$ must attribute no mass to the points which belong to $F$ but are not regular for $F$. In the last statement the compact set $F$ may be replaced by an arbitrary nearly analytic subset $E$ of $G$. One sees this by choosing, for each $r$, a sequence of compact subsets $F_n$ of $E$ so that $H_G(r, F_n)$ increases to $H_G(r, E)$, and noting that a point regular for $F_n$ is also regular for $E$. Thus ($B$) is equivalent to the stronger statement obtained by allowing $F$ to be a nearly analytic set.

Given an excessive function $\phi$ and a nearly analytic set $E$, define a function $\phi_E$ in the following manner. Let $X$ be a process starting at $r$, let $T$ be the time it hits $E$, and partition the set where $T$ is less than $S$ into $\Omega'$ and $\Omega''$, the first being the set where $T$ is strictly positive, $X(T)$ belongs to $E$, and $X(\tau)$ is continuous on the left at $T$. Set

$$\phi_E(r) \equiv \int_{\Omega'} \lim \phi(X(\tau))\, d\omega + \int_{\Omega''} \phi(X(T))\, d\omega, \tag{6.6}$$

the limit being taken as $\tau$ increases to $T$. Assuming $\phi$ to be bounded, one first shows that $\phi_E$ is excessive; the verification is straightforward at a point not regular for $E$, and quite simple at a regular point if Theorem 5.6 is used. Standard martingale theorems, together with the discussion in §5, prove $\phi_E$ to be an increasing function of $E$. These results are extended to unbounded $\phi$ by a passage to the limit using Theorem 5.3. Finally, once the point $r$ is fixed, $\phi_F(r)$ increases to $\phi_E(r)$ as $F$ runs through a certain sequence of compact subsets of $E$. To see this, let the sets $F_n$ be the ones mentioned in Proposition 2.1 and choose an increasing sequence of compact subsets $F'_n$ of $E$ so that, for almost all $\omega$ in $\Omega''$, the point $X(T(\omega), \omega)$ belongs to some $F'_n$; the required sets may be taken to be $F_n \cup F'_n$, as one sees immediately from Theorem 5.6 and the definitions.

The function $\phi_E$ can be defined another way when $\phi$ is bounded and $\lambda$ strictly positive; it then coincides, except perhaps at the points belonging to $E$ but not regular for $E$, with the infimum of the excessive functions which majorize $\phi$ on some neighborhood of $E$. The assertion is evident at a point regular for $E$. Moreover, $\phi_E$ nowhere exceeds one of the functions mentioned, because it is majorized by $H_G\phi$ whenever $G$ is in a neighborhood of $E$. We shall prove a strengthened version of Proposition 2.2 before completing the proof.

Let $X$ be a process, $T$ the time it hits $E$, and $\Omega^*$ the set where $T$ is finite and both $X(\tau)$ and $X(\tau-)$ belong to the complement of $E$ for $0 \leqq \tau \leqq T$. The results of §2 imply that $\Omega^*$ is changed only by a null set if the condition on $X(\tau-)$ is omitted; this remark will be needed at the end of the paragraph. For every $\omega$ define a compact subset $C(\omega)$ of $\mathcal{K}$ in the following way: $C(\omega)$ is empty for $\omega$ not in $\Omega^*$; for $\omega$ in $\Omega^*$ it comprises exactly the points $X(\tau)$ and $X(\tau-)$ as $\tau$ ranges over the closed interval $[0, T]$. If $B$ is a subset of $\mathcal{K}$, denote by $\tilde{B}$ the set of $\omega$ for which $B \cap C(\omega)$ is not empty; the set $\tilde{B}$ certainly belongs to $\mathfrak{F}$ whenever $B$ is nearly analytic. The function $\mathcal{P}\{\tilde{F}\}$, where $F$ denotes a variable compact set, is alternating of order infinity because of the probability interpretation. It is also continuous on the right,—that is to say, for every compact $F$ and every strictly positive $\alpha$, there is an open neighborhood $G$ of $F$ such that $\mathcal{P}\{\tilde{F'}\}$ is less than $\mathcal{P}\{\tilde{F}\} + \alpha$ for every compact subset $F'$ of $G$. The proof is merely the observation that $\mathcal{P}\{\tilde{G}\}$ decreases to $\mathcal{P}\{\tilde{F}\}$ as the open set $G$ shrinks to $F$, because each set $C(\omega)$ is compact. It follows from Choquet's extension theorem that the function $\mathcal{P}\{\tilde{F}\}$ can be extended to analytic sets, remaining continuous on the right, and it is easily verified that the extended function has precisely the value $\mathcal{P}\{\tilde{B}\}$ at the analytic set $B$. Now choose an analytic set $E'$ including $E$ so that, for almost all $\omega$ and for all $\tau$, the point $X(\tau, \omega)$ belongs to $E'$ if and only if it belongs to $E$. According to the remark at the beginning of the paragraph, $\tilde{E}'$ and $\tilde{E}$ differ at most by a null set, and of course the probability of $\tilde{E}$ vanishes by definition. Consequently, by the right continuity, there is a decreasing sequence of open neighborhoods $B_n$ of $E'$ such that the probability of $\tilde{B}_n$ tends to 0.

Specialize $X$ to the process occurring in (6.6), let $r$ be a point outside $E$, let the sets $G_n$ be the sets mentioned in Proposition 2.2, and take $G'_n$ to be $G_n \cap B_n$. The $G'_n$ form a decreasing sequence of open neighborhoods of $E$, and the time $T_n$ at which $X$ hits $G'_n$ increases to $T$ with probability 1. The choice of the $B_n$ implies that, for almost all $\omega$ for which $T(\omega)$ is finite and $X(T(\omega), \omega)$ lies outside $E$, the time $T(\omega)$ coincides with $T_n(\omega)$ when $n$ is sufficiently large. Consider an $\omega$ such that $T(\omega)$ is finite, $X(T(\omega), \omega)$ belongs to

$E$, and $T(\omega)$ coincides with some $T_n(\omega)$; clearly the point $X(\tau, \omega)$ is a discontinuous function of $\tau$ at $T(\omega)$, for it lies outside $G_n$ for $\tau$ less than $T_n(\tau)$. Finally, for almost all $\omega$ such that $T(\omega)$ is finite and greater than all $T_n(\omega)$, the point $X(\tau, \omega)$ is continuous in $\tau$ at $T(\omega)$ by the last part of hypothesis $(A)$. Matters being so, let $\Omega_n$ be the set where $T_n$ is less than $S$ and let $\Omega'$ and $\Omega''$ have the same meaning as in (6.6). The argument leading to (6.5) gives the relation

$$\begin{aligned} \lim H_{G'_n} \phi.(r) &= \lim \int_{\Omega_n} \phi(X(T_n))\, d\omega \\ &= \int_{\Omega'} \lim_{\tau \nearrow T} \phi(X(\tau))\, d\omega + \int_{\Omega''} \phi(X(T))\, d\omega \qquad (6.7) \\ &= \phi_E(r), \end{aligned}$$

under the assumptions that $\phi$ is bounded and $\lambda$ strictly positive. The verification that $\phi_E$ coincides with a certain infimum, except on an approximately null set, is now complete.

The next theorem should be compared with Theorem 8.4 of [7]. Doob's language is a little different from ours, and he treats the relative theory with vanishing parameter, but the two theorems are essentially the same.

THEOREM 6.11. *If $(B)$ is true and $\lambda$ strictly positive, $\phi_E$ is the supremum of the bounded excessive functions which nowhere exceed $\phi$ and have a compact subset of $E$ for a determining set.*

If the theorem is true for bounded $\phi$ it is also true for unbounded $\phi$; one has only to consider the minimum of $\phi$ and a constant which is allowed to grow large. We shall therefore assume $\phi$ to be bounded in the proof.

Let $\psi$ be an excessive function majorized by $\phi$ and having the compact subset $F$ of $E$ for a determining set. Then

$$\psi = \psi_F \leqq \phi_F \leqq \phi_E ,$$

the steps being justified by equation (6.7) with $\psi$ and $F$ replacing $\phi$ and $E$, the majorization of $\psi$ by $\phi$, and the fact that $\phi_E$ increases with $E$. Observe that $(B)$ has not been used so far.

It remains to prove that $\phi_E$ can be approximated at every point by the functions mentioned in the theorem. First assume $E$ to be compact, let $G$ be any neighborhood of $E$, and choose a decreasing sequence of compact neighborhoods $F_n$ of $E$ which shrink to $E$. Clearly, $H_{F_n} \phi.(r)$ tends to $\phi_E(r)$ if $r$ is regular for $E$, since all terms have the value $\phi(r)$. The convergence holds also at a point $r$ outside $E$, since the $F_n$ behave just like the sets $G'_n$ in (6.7) when $E$ is compact. And, for such points $r$, the measure $H_G(r, ds)$ attributes no mass to the remaining set of points where the convergence may not take place. Accordingly

$$H_G \phi_E = H_G(\lim H_{F_n} \phi) = \lim(H_G H_{F_n} \phi) = \lim H_{F_n} \phi = \phi_E ,$$

except perhaps at the points belonging to $E$ but not regular for $E$; hypothesis $(B)$ justifies the next to last step, since $F_n$ is finally included in $G$. On the other hand, the extreme members clearly have the same value at a point of $E$. Thus $\phi_E$ itself has $E$ for a determining set. We have proved a little more than was stated in the theorem, assuming $E$ to be compact.

If $E$ is not compact, consider $\phi_F$ as $F$ varies over the compact subsets of $E$. It is one of the functions mentioned in the theorem, by what has just been proved; and for each fixed $r$, according to a remark following the definition, $\phi_E(r)$ is the limit of $\phi_F(r)$ as $F$ runs through an appropriate sequence. The proof of the theorem is now complete.

The function $\phi_E$ is at least as great as $H_E\,\phi$. The two coincide if $E$ is open, because then $\Omega'$ is empty and the first integral in (6.6) vanishes; so the theorem implies Proposition 6.9, which was proved without using $(B)$. It is also clear that $\phi_E$ and $H_E\,\phi$ are the same if $\phi$ is a bounded continuous function or if it is the potential of a positive function and finite except on a negligible set. The statement is true for much wider classes of functions when the transition probabilities are sufficiently regular; when they give the classical or the Riesz potentials, for example, $\phi$ may be the potential of any measure which attributes no mass to a set of capacity null.

The part of the theorem proved without using $(B)$ is sometimes useful. By way of illustration, suppose that there is a bounded excessive function which does not vanish identically and which has a compact subset of $E$ for a determining set; then $\Phi_E$ can not vanish identically, so that $E$ is not negligible.

There are other versions of the theorem; the one stated was chosen because it is closest to Doob's theorem. Hypothesis $(B)$ can be dispensed with by changing the definition of determining set, requiring only that equation (6.5) be true; the theorem is then nearly obvious. A more significant change is to define $\phi_E$ as the excessive function that coincides, except on an approximately null set, with the infimum of the excessive functions which majorize $\phi$ on some neighborhood of $E$; the word *bounded* should then be deleted from the statement of the theorem.

I have not found simple and general conditions on the transition measures to ensure the truth of hypothesis $(B)$. It is implied by the rather elaborate hypothesis in §2 of [11], but that is a good deal too restrictive.

## Excessive measures

### 7. Potentials of measures

Recall that a measure on $\mathcal{K}$ is always a countable sum of positive bounded measures. We shall say that a measure $\mu$ majorizes a measure $\nu$ on the set $A$ if the inequality $\mu(B) \geqq \nu(B)$ holds for all subsets of $A$, and that a sequence of measures $\mu_n$ increases if $\mu_{n+1}$ majorizes $\mu_n$ everywhere. A measure $\mu$ which is finite on compact sets is determined by the values of $\int f\,d\mu$ as $f$ ranges over the positive continuous functions with compact supports. A sequence of measures $\mu_n$ is said to converge weakly to the measure $\mu$ if $\int f\, d\mu_n$ converges to $\int f\, d\mu$ for all such functions $f$. A nucleus for a measure is a set bearing all the mass of the measure.

The potential of the measure $\mu$ is the measure $\mu U$. For example, the measure $U(r, ds)$ is the potential of the unit mass placed at the point $r$. The potential of $\mu$ has total mass $\mu(\mathcal{H})/\lambda$, so that it is bounded if $\mu$ is bounded and $\lambda$ is strictly positive.

The notation

$$\langle \mu, f \rangle \equiv \int_{\mathcal{H}} \mu U.(ds) f(s) \equiv \int_{\mathcal{H}} \mu(dr)\, Uf.(r) \tag{7.1}$$

will be used occasionally in this part, more frequently later on. Now $Uf.(r)$ can be written $\langle r, f \rangle$, with $r$ understood to be the unit mass placed at the point $r$. If $\mu$ is a probability measure, there is the interpretation

$$\langle \mu, f \rangle = \int_{\Omega} d\omega \int_0^S f(X(\tau))\, d\tau,$$

where $X$ is a process having $\mu$ as initial distribution. A similar interpretation for an arbitrary measure is obtained by considering a quantity of matter, initially distributed according to the given measure, and the paths the particles traverse.

Proposition 7.1. *If $\lambda$ is strictly positive, a bounded measure $\mu$ is determined by its potential.*

If $f$ is bounded and continuous then $H_\tau f$ converges boundedly to $f$ as $\tau \to 0$, so that

$$\begin{aligned} \int_{\mathcal{H}} \mu(dr) f(r) &= \lim_{T \to 0} \frac{1}{T} \int_{\mathcal{H}} \mu(dr) \int_0^T H_\sigma f.(r)\, d\sigma \\ &= \lim_{T \to 0} \frac{1}{T} \int_{\mathcal{H}} [\mu U.(ds) - \mu U H_T.(ds)]\, f(s). \end{aligned}$$

Now, the last expression is obviously determined by the function $f$ and the measure $\mu U$.

The projection of the measure $\mu$ on the nearly analytic set $E$ is defined to be the measure $\mu H_E$. Since

$$\int_{\mathcal{H}} \mu H_E\, U.(ds) f(s) = \int_{\mathcal{H}} \mu(dr) H_E\, Uf.(r), \tag{7.2}$$

a number of results are easily translated from potentials of functions to potentials of measures.

Proposition 7.2. *In the notation above, $\mu U$ majorizes $\mu H_E\, U$ everywhere, and the two measures have the same restriction to $E$. If $\nu$ is a measure such that $\nu U$ majorizes $\mu U$, then also $\nu H_E\, U$ majorizes $\mu H_E\, U$.*

The proof is relation (7.2) and Propositions 4.2 and 4.3.

We shall say that a set $E$ is nearly open if it is nearly analytic and if, for every process starting at a point of $E$, almost all sample paths of the process remain in $E$ for an initial interval of time.

PROPOSITION 7.3. *Let $\lambda$ be strictly positive. The potential of $\mu$ is determined by its restriction to a nearly open nucleus for $\mu$. Also, $\nu U$ majorizes $\mu U$ everywhere if it does so on a nearly open nucleus for $\mu$.*

Let $E$ be a nearly open nucleus for $\mu$ and let $f$ be a positive function. According to Theorem 6.6 there is a sequence of positive functions $f_n$, each one vanishing outside $E$, whose potentials increase to $H_E\, Uf$. It follows from (7.2) and the equation $\mu H_E = \mu$ that

$$\langle \mu, f \rangle = \lim \langle \mu, f_n \rangle,$$

so that the first assertion is proved. The second is proved similarly.

PROPOSITION 7.4. *Let $\lambda$ be strictly positive. If $E$ is nearly open, $\mu H_E\, U$ is the least potential which majorizes $\mu U$ on $E$.*

The potential of $\mu H_E$ coincides on $E$ with $\mu U$, by Proposition 7.2, so that one has only to show it to be majorized by every potential $\nu U$ which majorizes $\mu U$ on $E$. Now,

$$\begin{aligned} \langle \mu H_E , f \rangle &= \lim \langle \mu, f_n \rangle \\ &\leqq \lim \langle \nu, f_n \rangle \leqq \langle \nu, f \rangle, \end{aligned}$$

where $f$ and $f_n$ have the same meaning as in the preceding proof. The hypothesis that $\lambda$ be strictly positive is unnecessary in the last two propositions, as can be seen from Proposition 8.3.

PROPOSITION 7.5. *If $E$ is nearly open, then $\mu H_E\, U$ is the supremum of all potentials $\nu U$, with $\nu$ ranging over the measures on $E$ whose potentials are majorized by $\mu U$ everywhere.*

The potential of $\mu H_E$ is an upper bound for the $\nu U$, because $\nu = \nu H_E$ for every measure on $E$. In establishing the approximation of $\mu H_E\, U$ by the $\nu U$, we shall use the latter part of §4. Let $h$ be a bounded positive function vanishing outside $E$ and define the kernel $K(r, ds)$ by the formula

$$K(r, A) \equiv \mathcal{P}\{R_r < S, X_r(R_r) \in A\}, \qquad A \subset \mathcal{K},$$

where $X_r$ is a process starting at $r$ and $R_r$ is the time assigned to it by $h$. For every $r$, the measure $K(r, ds)$ is concentrated on $E$. Now let $h$ run through a sequence of functions which increase to infinity at all points of $E$. For every point $r$ and every positive function $f$,

$$\int_{\mathcal{K}} K(r, ds) \int_{\mathcal{K}} U(s, dt) f(t) \equiv \int_{\Omega_r'} d\omega \int_{R_r}^{S} f(X_r(\tau))\, d\tau,$$

with $\Omega_r'$ the set where $R_r$ is less than $S$, increases to

$$\int_{\mathcal{K}} H_E(r, ds) \int_{\mathcal{K}} U(s, dt) f(t) \equiv \int_{\Omega_r} d\omega \int_{T_r}^{S} f(X_r(\tau))\, d\tau,$$

where $T_r$ is the time $X_r$ hits $E$ and $\Omega_r$ is the set where it is less than $S$. So the potential of $\mu K$, which is a measure on $E$, increases to the potential of $\mu H_E$.

Observe that the measures $\mu K$ may be taken to have supports included in $E$, if $E$ is the countable union of closed sets.

Let $\mathfrak{M}$ be the class of measures whose projections on compact sets are bounded. This is the largest class of measures for which there is a satisfactory theory of potentials.

A measure $\mu$ belongs to $\mathfrak{M}$ if and only if the integral $\int \Phi_F\, d\mu$ is finite whenever $F$ is compact, because of the evident relation

$$\mu H_F.(F) = \int_{\mathcal{K}} \mu(dr) \Phi_F(r).$$

A measure $\mu$ in $\mathfrak{M}$ is clearly finite on compact sets. Its potential has the same property when $\lambda$ is strictly positive. For, if $\chi$ is the characteristic function of the compact set $F$, then

$$\begin{aligned} \langle \mu, \chi \rangle &= \int_{\mathcal{K}} \mu(dr) H_F\, U\chi.(r) \\ &= \int_{\mathcal{K}} \mu H_F.(ds)\, U\chi.(s) \leqq \frac{1}{\lambda}\, \mu H_F.(\mathcal{K}), \end{aligned}$$

because of Proposition 4.2 and the fact that $U\chi$ is bounded by $1/\lambda$. In §9, when a little more is assumed of the transition probabilities, we shall see that this property characterizes a measure in $\mathfrak{M}$.

Fix a sequence of open sets $G_n$ with the following properties: $G_0$ is empty, the closure of $G_n$ is compact and is included in $G_{n+1}$, the union of the $G_n$ is $\mathcal{K}$. Then $H_{G_{n+1}} H_{G_n}$ coincides with $H_{G_n}$, because every point of an open set is regular for the set. Now consider a measure $\mu$ in $\mathfrak{M}$ and its projections $\mu_n$ on the $G_n$. It is clear that the projections are bounded measures, and we shall prove that they determine the original measure $\mu$. Let $\mu^n$ be the restriction of $\mu$ to $G_{n+1} - G_n$ and let $\mu_k^n$ be the projection of $\mu^n$ on $G_k$. The mass of $\mu_k^n$ tends to 0 as $n$ becomes large, for $\sum_n \mu_k^n$ is the bounded measure $\mu_k$. Also, $\mu_k^n$ coincides with $\mu^n$ for $k$ greater than $n$. Finally, if $A$ is included in $G_l$ and $n$ is greater than $l$, then $\mu_k^n(A)$ decreases as $k$ increases from $l$ to $n$, and vanishes for $k$ greater than $n$. Matters being so,

$$\begin{aligned} \lim_{k \to \infty} \mu_k(A) &= \lim_{k \to \infty} \sum_n \mu_k^n(A) \\ &= \sum_n \mu^n(A) = \mu(A), \end{aligned}$$

provided $A$ is included in some $G_l$.

Proposition 7.6. *If $\lambda$ is strictly positive, a measure $\mu$ in $\mathfrak{M}$ is determined by its potential.*

By Proposition 7.1 and what has just been proved, it is enough to show that $\mu U$ determines each $\mu_n U$. Consider an open set $G$ and a positive function $f$. There is a sequence of positive functions $f_n$ whose potentials increase to $H_G Uf$; it follows that

$$\langle \mu H_G, f \rangle = \lim \langle \mu, f_n \rangle,$$

and the assertion is proved.

## 8. Excessive measures

A measure $\zeta$ on $\mathfrak{X}$ is excessive (relative to the terminal time $S$) if it is finite on compact sets and if it majorizes $\zeta H_\tau$ for every $\tau$. It follows that $\zeta H_\tau$ increases as $\tau$ decreases, and we shall prove that the limit as $\tau \to 0$ is precisely $\zeta$. If $f$ is a positive continuous function with compact support, then

$$\lim_{\tau \to 0} \int_{\mathfrak{X}} \zeta H_\tau.(dr)\, f(r) = \lim_{\tau \to 0} \int_{\mathfrak{X}} \zeta(dr) H_\tau f.(r)$$
$$\geqq \int_{\mathfrak{X}} \zeta(dr) f(r),$$

by Fatou's lemma and the fact that $H_\tau f$ approaches $f$ everywhere. On the other hand, the limit certainly cannot exceed the last integral. The assertion is now obvious.

The set of excessive measures is closed under addition, multiplication by positive constants, taking the limit of a decreasing sequence, and also taking the limit of an increasing sequence if the limit is finite on compact sets. The minimum of two excessive measures $\zeta_1$ and $\zeta_2$ is excessive,—the minimum being defined as $f(r)\zeta(dr)$, where

$$\zeta \equiv \zeta_1 + \zeta_2, \qquad f_i(r)\zeta(dr) \equiv \zeta_i(dr), \qquad f \equiv \min(f_1, f_2).$$

The minimum of a countable collection of excessive measures is excessive, for it is the limit of a decreasing sequence of excessive measures. Finally, the potential of a measure in $\mathfrak{M}$ is excessive if $\lambda$ is strictly positive.

A measure is excessive for one value of $\lambda$ if and only if it is excessive for every greater value.

Proposition 8.1. *Let $\zeta$ be an excessive measure such that $\zeta H_\tau.(F) \to 0$ as $\tau \to \infty$, whenever $F$ is compact. Then the potential of $(\zeta - \zeta H_\tau)/\tau$ increases to $\zeta$ as $\tau \to 0$.*

The proof is omitted, as it is like the proof of Proposition 5.2 and somewhat simpler. For $\lambda$ strictly positive, the hypothesis on $\zeta$ is satisfied if the measure is bounded, or if it remains excessive when $\lambda$ is slightly decreased.

Proposition 8.2. *If $\lambda$ is strictly positive, then an excessive measure $\zeta$ is the limit of an increasing sequence of bounded potentials.*

Denote by $U'$ the kernel corresponding to a value $\lambda'$ of the parameter slightly greater than $\lambda$, so that $\zeta$ is the limit of an increasing sequence of $\lambda'$-potentials $\mu_n U'$, and consider an increasing sequence of open sets $G_n$ which have compact closures and whose union is $\mathcal{K}$. Let $\nu_n$ be a bounded measure majorized by $\mu_n$ and such that

$$\nu_n U'.(G_n) > \mu_n U'.(G_n) - 1/n.$$

Then the $\lambda'$-potentials $\nu_n U'$ also converge to $\zeta$ from below, although they may not form an increasing sequence. The $\lambda$-potential $\nu_n U$ is bounded, because $\nu_n$ is bounded, and it majorizes $\nu_n U'$. The measure $\zeta_n$,

$$\zeta_n \equiv \min\left(\zeta, \sum_1^n \nu_k U\right),$$

is therefore a $\lambda$-excessive bounded measure which increases to $\zeta$, and the sequence of bounded $\lambda$-potentials

$$n[\zeta_n - \zeta_n H_{1/n}]U \equiv n \int_0^{1/n} \zeta_n H_\sigma \, d\sigma$$

also increases to $\zeta$.

The operation of passing from a potential $\mu U$ to the potential $\mu H_E U$, where $E$ is a given nearly analytic set, will now be extended to an excessive measure $\zeta$. Suppose first that $\lambda$ is strictly positive, and choose a sequence of measures $\mu_n$ whose potentials increase to $\zeta$. Then $\mu_n H_E U$ also increases with $n$, and we take $L_E \zeta$ to be the limit as $n \to \infty$. There is an alternative definition, which shows incidentally that the dependence on the particular choice of the $\mu_n$ is only apparent. Let $f$ and the $f_k$ be positive functions such that $Uf_k$ increases to $H_E Uf$. Then

$$\begin{aligned}\lim_n \langle \mu_n H_E, f\rangle &= \lim_n \lim_k \langle \mu_n, f_k\rangle \\ &= \lim_k \lim_n \langle \mu_n, f_k\rangle \\ &= \lim_k \int_{\mathcal{K}} \zeta(ds) f_k(s),\end{aligned}$$

the change in the order of the limits being permissible because $\langle \mu_n, f_k\rangle$ increases with $k$ and with $n$. The definition of $L_E \zeta$ for vanishing $\lambda$ will be given during the proof of the next proposition; for the moment suppose $\lambda$ to be strictly positive.

It should be noted that the transformation $L_E$ cannot usually be derived from a kernel. A good many of its properties follow directly from the definition. For example, $L_E \zeta$ increases with $\zeta$ and with $E$; if $\zeta$ is the limit of an increasing sequence of excessive measures $\zeta_n$, then $L_E \zeta$ is the limit of $L_E \zeta_n$; if $G$ is an open subset of $E$ then $L_G L_E$ coincides with $L_G$; and $L_E \zeta$ is an excessive measure, majorized everywhere by $\zeta$ and coinciding with $\zeta$ on the set $E$.

It is sometimes useful to know that $L_E \zeta$ is the limit of an increasing sequence of potentials $\nu_n U$, the $\nu_n$ being bounded measures whose supports are compact subsets of $E$. First of all, the measures $\mu_n$ of the definition can be chosen bounded. Next, by Proposition 2.1 and a simple selection, there is an increasing sequence of compact subsets $F_k$ of $E$ such that for each $n$ the potential $\mu_n H_{F_k} U$ increases to $\mu_n H_E U$ as $k \to \infty$. So $\nu_n$ may be taken to be the projection of $\mu_n$ on $F_n$.

An excessive measure $\zeta$ majorizes a potential $\mu U$ everywhere if it does so on a nearly open nucleus $E$ for the measure $\mu$. To see this, let $f$ and the $f_n$ be positive functions such that $Uf_n$ increases to $H_E Uf$ and each $f_n$ vanishes outside $E$. Then

$$\langle \mu, f \rangle = \langle \mu H_E, f \rangle = \lim \langle \mu, f_n \rangle$$
$$\leqq \lim \int_{\mathfrak{X}} \zeta(ds) f_n(s) = \int_{\mathfrak{X}} L_E \zeta.(ds) f(s).$$

PROPOSITION 8.3. *Let $E$ be nearly open and $\zeta$ excessive. Then $L_E \zeta$ is the least excessive measure which majorizes $\zeta$ on $E$.*

The proof will be given first for $\lambda$ strictly positive and the result will be used in defining $L_E$ for vanishing $\lambda$.

Let $\lambda$ be strictly positive. Since $L_E \zeta$ coincides with $\zeta$ on $E$, we have only to prove that an excessive measure majorizes $L_E \zeta$ everywhere if it does so on $E$; the statement follows directly from the two paragraphs preceding the proposition.

This result shows that, for a fixed nearly open set $E$ and a fixed measure $\zeta$, the measure $L_E \zeta$ decreases as the parameter $\lambda$ increases—provided, of course, one considers only the strictly positive $\lambda$ for which $\zeta$ is excessive—because the class of excessive measures increases with $\lambda$. We shall remove the restriction on $E$ as a preliminary to defining $L_E$ generally for vanishing $\lambda$.

Let $\lambda$ be strictly positive; let $\lambda'$ be a greater value of the parameter, say $\lambda + \alpha$; let $E$ be a nearly analytic set; and let $\zeta$ be a $\lambda$-excessive measure. Denote by $U'$, $H'_E$ and so on the quantities corresponding to the value $\lambda'$ of the parameter. An integration by parts shows that the kernel $U$ can be written $U' + \alpha UU'$, so that the $\lambda$-potential of a measure $\mu$ can be written as the $\lambda'$-potential of the measure $\mu + \alpha\mu U$. Consequently,

$$L'_E \zeta = \lim_n (\mu_n + \alpha \mu_n U) H'_E U,$$

where the $\mu_n$ are measures whose $\lambda$-potentials increase to $\zeta$. The majorization of $L'_E \zeta$ by $L_E \zeta$ will therefore be proved once the inequality

$$H_E U \geqq H'_E U' + \alpha U H'_E U' \tag{8.1}$$

is established. The inequality is true whenever $E$ is open, according to the preceding paragraph, and the following argument extends it to nearly analytic sets. Denote by $A(E, r, C)$ and $B(E, r, C)$ the values of the two kernels in

(8.1) at the pair $(r, C)$. These values are the same if $r$ is regular for $E$. Now consider a point $r$ outside $E$, let $X$ be a process starting from $r$, and let $T$ be the time $X$ hits $E$. Take $Z$ to be a positive random variable which is independent of $X$ and the terminal time $S'$ and which has the distribution

$$\mathcal{P}\{Z = 0\} = \frac{\lambda}{\lambda + \alpha}, \qquad \mathcal{P}\{Z > \tau\} = \frac{\alpha}{\lambda + \alpha} e^{-\lambda\tau},$$

and let $Y$ be the process $X(\tau + Z)$. Then one has the interpretation

$$\begin{aligned} B(E, r, C) &= \frac{\lambda + \alpha}{\lambda} \int_{\Omega} d\omega \int_{\mathcal{H}} H'_E(X(Z), ds)\, U'(s, C) \\ &= \frac{\lambda + \alpha}{\lambda} \int_{\Omega'} d\omega \int_{T'}^{S'} \chi(Y(\tau))\, d\tau, \end{aligned}$$

where $\chi$ is the characteristic function of $C$, $T'$ is the time $Y$ hits $E$, and $\Omega'$ is the set where $T'$ is less than $S'$. Since the points belonging to $E$ but not regular for $E$ form a null set relative to the measure $U(r, ds)$, it is only with vanishing probability that $Y(0)$ is such a point. Hence, by Proposition 2.2, there is a decreasing sequence of open neighborhoods $G_n$ of $E$ such that $T_n$ and $T'_n$, the times $X$ and $Y$ hit $G_n$, decrease to $T$ and $T'$ with probability 1. The inequality

$$A(G_n, r, C) \geqq B(G_n, r, C),$$

which is valid because $G_n$ is open, therefore becomes

$$A(E, r, C) \geqq B(E, r, C) \tag{8.2}$$

on passing to the limit. This inequality has been established approximately everywhere, and both members are $\lambda'$-excessive functions of $r$. It follows from the last paragraph of §5 that (8.2) must hold identically.

If $\zeta$ is 0-excessive, define $L_E^0 \zeta$ to be the limit of $L_E^\lambda \zeta$ as $\lambda \to 0$. The definition makes sense, and the properties of $L_E^\lambda \zeta$ are carried over by monotone convergence. In particular, the proposition is true for vanishing $\lambda$. For, if $\xi$ is 0-excessive and majorizes $\zeta$ on $E$, then it is also $\lambda$-excessive and majorizes $L_E^\lambda \zeta$ for $\lambda$ strictly positive; the majorization is obviously preserved on passing to the limit $\lambda \to 0$.

A closed subset $F$ of the compact space $\mathcal{H}_c$ defined in §6 is a determining set for the excessive measure $\zeta$ if $L_G \zeta$ coincides with $\zeta$ for every open neighborhood $G$ of $F$ in $\mathcal{H}_c$; here $L_G$ is the transformation defined by the part of $G$ in $\mathcal{H}$.

Proposition 8.4. *Let $E$ be an open set in $\mathcal{H}_c$ and $\zeta$ an excessive measure. Then $L_E \zeta$ is the supremum of the excessive measures majorized by $\zeta$ and having a determining set in $E$.*

The proof is similar to that of Proposition 6.9. We shall not state the analogues of the other propositions at the end of §6. It should be remarked, however, that the support of the measure $\mu$ is the least determining set for the potential $\mu U$, provided $\lambda$ is strictly positive and $\mu$ is a measure in $\mathfrak{M}$.

## 9. Representation as potentials

In this section we make the following assumption, in which $\mathcal{C}(\mathcal{H})$ is the Banach space of functions continuous on $\mathcal{H}$ and vanishing at infinity.

(*C*) *For $\lambda$ strictly positive, $Uf$ belongs to $\mathcal{C}(\mathcal{H})$ if $f$ does so.*

Simple examples show that some such condition is needed in proving the theorems of this section. First, consider uniform motion with velocity 1 on the interval $0 < r < \infty$, taking $\lambda$ to be 1. The measure $U(r, ds)$ vanishes in the interval $0 < s \leqq r$ and coincides with the measure of $e^{r-s}\, ds$ in the interval $r < s < \infty$; here $ds$ is the element of Lebesgue measure. The bounded excessive measure $e^{-s}\, ds$ is not a potential, but it is the limit of an increasing sequence of potentials. In this example $Uf.(r)$ is continuous but may not vanish as $r \to 0$.

Next, adjoin the point 0 to the interval, with the prescription that a particle starting from there remains fixed forever. Again $e^{-s}\, ds$ is not a potential. This time $Uf$ vanishes at infinity but may not be continuous.

PROPOSITION 9.1. *Let $\lambda$ be strictly positive and let the $\mu_n$ be measures whose masses are bounded in $n$ and whose potentials increase with $n$. Then $\mu_n$ converges weakly to a measure $\mu$, and $\mu_n U$ converges to $\mu U$.*

Some subsequence of the measures, say $\mu_n'$, converges weakly to a measure $\mu$. If $f$ belongs to $\mathcal{C}(\mathcal{H})$, then

$$\langle \mu_n', f \rangle \to \langle \mu, f \rangle,$$

because $Uf$ also belongs to $\mathcal{C}(\mathcal{H})$, so that $\mu_n' U$ increases to $\mu U$. According to Proposition 7.1, the measure $\mu$ is determined by its potential and is therefore independent of the choice of the subsequence $\mu_n'$. The convergence of the full original sequence is now proved by a familiar argument. Observe that there is no escape of mass to infinity; $\mu_n(\mathcal{H})$ tends to $\mu(\mathcal{H})$, because the mass of a measure is $\lambda$ times the mass of its potential.

THEOREM 9.2. *If $\lambda$ is strictly positive, a bounded excessive measure is a potential.*

PROPOSITION 9.3. *Let $\lambda$ be strictly positive and let the $\mu_n$ be bounded measures whose potentials decrease with $n$. Then $\mu_n$ converges weakly to a measure $\mu$, whose potential is the limit of $\mu_n U$.*

The theorem follows at once from Propositions 8.1 and 9.1. The proof of the proposition is like the preceding one.

The main result of this section is a characterization of the excessive measures which are potentials. A number of preliminary facts are needed; the parameter $\lambda$ is understood to be strictly positive throughout the discussion, this restriction being mentioned only in the theorems.

For every compact set $A$ there is a positive continuous function $g$, with compact support, whose potential exceeds 1 at every point of $A$. To see this, let $g$ run through a sequence of such functions which increase to $2\lambda$ everywhere; the corresponding potentials are continuous and increase to 2 everywhere, so that Dini's theorem justifies the assertion. If $\mu$ is a measure whose potential is finite on compact sets, then

$$\begin{aligned}\mu H_A.(\mathcal{K}) &\leqq \int_A \mu H_A.(dr)\, Ug.(r)\\ &\leqq \int_{\mathcal{K}} \mu U.(ds) g(s),\end{aligned}$$

the last member being finite. According to this implication and what was said in §7, a measure belongs to $\mathfrak{M}$ if and only if its potential is finite on compact sets.

If $\zeta$ is an excessive measure and $A$ a compact set, then $L_A\,\zeta$ is a bounded measure and therefore, by Theorem 9.2, the potential of a bounded measure. In proof, consider a sequence of measures $\mu_n$ on $A$ whose potentials increase to $L_A\,\zeta$. If $g$ is the same function as in the preceding paragraph, then

$$\mu_n(\mathcal{K}) \leqq \int_A \mu_n(dr)\, Ug.(r) \leqq \int_{\mathcal{K}} \zeta(dr) g(r),$$

the last member being finite, say with value $\alpha$, because $g$ is bounded and has compact support. The mass of each $\mu_n\, U$ is bounded by $\alpha/\lambda$, which also serves as a bound for the mass of $L_A\,\zeta$. Clearly, it would be enough to assume $A$ to be included in a compact set.

In the remainder of this section, the $G_n$ are a fixed sequence of open sets, the union of the $G_n$ being $\mathcal{K}$, the closure of $G_n$ being compact and included in $G_{n+1}$. We denote the complement of $G_n$ by $F_n$ and write

$$L_n \equiv L_{G_n}, \qquad L_n' \equiv L_{F_n}, \qquad H_n \equiv H_{G_n}, \qquad H_n' \equiv H_{F_n},$$

by way of abbreviation. The relation $L_n L_m = L_n$ holds for $m$ greater than $n$, because $G_n$ is then an open subset of $G_m$.

Let $A$ be a compact set and $\nu$ a measure in $\mathfrak{M}$. We shall need to know that $\nu H_n'\, U.(A)$ is small if $n$ is large. The statement is immediate if $\nu$ is bounded; the mass of $\nu H_n'$ then decreases to 0, because the terminal time is finite and the $F_n$ are closed sets which decrease to the empty set. If $\nu$ is unbounded, write it as $\nu_1 + \nu_2$, with $\nu_1$ bounded and $\nu_2\, U.(A)$ small; then

$$\nu_1 H_n'\, U.(A) \to 0, \qquad \nu_2 H_n'\, U.(A) \leqq \nu_2\, U.(A).$$

We shall see in a moment the significance of this property.

THEOREM 9.4. *If $\lambda$ is strictly positive, an excessive measure $\zeta$ can be written as $\xi + \mu U$, where $\mu$ is a measure in $\mathfrak{M}$ and $\xi$ an excessive measure which coincides with $L'_n \xi$ for every $n$.*

By the preceding remarks, $L_n \zeta$ is the potential of a bounded measure $\mu_n$, and $\mu_n$ coincides with the projection of $\mu_m$ on $G_n$ if $m$ exceeds $n$. Let $A$ be a compact set, say included in $G_l$; then $\mu_n(A)$ decreases as $n$ increases, at least for values of $n$ greater than $l$. Define the measure $\mu$ by setting

$$\mu(A) = \lim_n \mu_n(A)$$

for every compact set $A$. If $g$ is a bounded positive function which vanishes outside a compact set, then $\int g \, d\mu_n$ ultimately decreases as $n$ increases, and the limit is $\int g \, d\mu$. Accordingly, for every positive bounded function $f$,

$$\begin{aligned} \langle \mu, f \rangle &= \lim_n \lim_k \int_{G_n} \mu_k(dr) Uf.(r) \\ &\leqq \lim_k \langle \mu_k, f \rangle \leqq \int_{\mathcal{K}} \zeta(dr) f(r), \end{aligned}$$

the last expression being finite if $f$ vanishes outside a compact set. So $\mu$ belongs to $\mathfrak{M}$ and $\zeta$ majorizes its potential.

The positive measure $\xi$ is defined by setting

$$\xi(A) = \zeta(A) - \mu U.(A)$$

for every compact set $A$. To prove that $\xi$ is excessive, consider the measures $\mu_m U - \mu'_n U$, where $m$ is greater than $n$ and $\mu'_n$ is the restriction of $\mu$ to $G_n$. These measures are all excessive, since $\mu_m$ majorizes $\mu'_n$; they are all majorized by $\zeta$; and they increase as $m$ increases or as $n$ decreases. Now,

$$\xi = \lim_{n \to \infty} \lim_{m \to \infty} (\mu_m U - \mu'_n U),$$

and each limit is an excessive measure according to the second paragraph of §8.

The proof will be complete once the equality

$$\zeta - L'_n \zeta = \mu U - \mu H'_n U$$

is established. Since both members vanish on $F_n$, it is enough to prove that

$$L_n \zeta - L_n L'_n \zeta = \mu H_n U - \mu H'_n H_n U,$$

because the operator $L_n$ changes an excessive measure only outside $G_n$. The measure on the right attributes to a compact set $A$ the mass

$$\int_{\mathcal{K}} \mu(dr)[H_n U.(r, A) - H'_n H_n U.(r, A)].$$

Since the integrand is bounded and vanishes outside the closure of $G_n$, which is compact, this integral can be written

$$\lim_m \int_{\mathcal{H}} \mu_m(dr)[H_n U.(r, A) - H'_n H_n U.(r, A)]$$
$$= \lim_m [L_n L_m \zeta.(A) - L_n L'_n L_m \zeta.(A)]$$
$$= L_n \zeta.(A) - L_n L'_n \zeta.(A),$$

the passage to the limit being justified by the fact that $L_m \zeta$ increases to $\zeta$. The proof is now complete.

It should be noted that the decomposition is unique and independent of the particular sequence of sets $G_n$. For $\xi$ is determined by the equation

$$\xi(A) = \lim_n L'_n \zeta.(A) - \lim_n \mu H'_n U.(A) = \lim_n L'_n \zeta.(A),$$

which holds for every compact set according to a remark preceding the theorem; and given two sequences of sets $G'_n$ and $G''_n$, one can construct a third sequence which includes infinitely many of the $G'_n$ and infinitely many of the $G''_n$.

In the language of §8, the measure $\xi$ can be described as an excessive measure having the point at infinity for a determining set.

THEOREM 9.5. *If $\lambda$ is strictly positive, an excessive measure $\zeta$ is the potential of a measure in $\mathfrak{M}$ if and only if $L'_n \zeta(A) \to 0$ for every compact set $A$.*

THEOREM 9.6. *If $\lambda$ is strictly positive, an excessive measure is the potential of a measure in $\mathfrak{M}$ if it is majorized by such a potential.*

PROPOSITION 9.7. *Let $\lambda$ be strictly positive, let $\nu$ and the $\mu_n$ be measures in $\mathfrak{M}$. If*

$$\mu_n U \leqq \mu_{n+1} U \leqq \nu U$$

*for all $n$, then $\mu_n$ converges weakly to a measure $\mu$ in $\mathfrak{M}$, and $\mu_n U$ increases to $\mu U$.*

PROPOSITION 9.8. *Let $\lambda$ be strictly positive and let the $\mu_n$ be measures in $\mathfrak{M}$ whose potentials form a decreasing sequence. Then $\mu_n$ converges weakly to a measure $\mu$, and $\mu_n U$ decreases to $\mu U$.*

The proofs are omitted, as they are quite straightforward.

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