Extremal transition and quantum cohomology: Examples of toric degeneration

Hiroshi Iritani and Jifu Xiao

Abstract When a singular projective variety X_{sing} admits a projective crepant resolution X_{res} and a smoothing X_{sm}, we say that X_{res} and X_{sm} are related by *extremal transition*. In this article, we study a relationship between the quantum cohomology of X_{res} and X_{sm} in some examples. For 3-dimensional conifold transition, a result of Li and Ruan implies that the quantum cohomology of a smoothing X_{sm} is isomorphic to a certain subquotient of the quantum cohomology of a resolution X_{res} with the quantum variables of exceptional curves specialized to one. We observe that similar phenomena happen for toric degenerations of $\mathrm{Fl}(1,2,3)$, $\mathrm{Gr}(2,4)$, and $\mathrm{Gr}(2,5)$ by explicit computations.

1. Introduction

Let X_{sing} be a Gorenstein normal projective variety. Suppose that X_{sing} admits a projective crepant resolution $\pi\colon X_{\mathrm{res}} \to X_{\mathrm{sing}}$ and a smoothing X_{sm} which is projective. The passage from X_{res} to X_{sm} is called the *extremal transition* (see [14]). When X_{sing} is a threefold having only ordinary double points as singularities, this is known as *conifold transition*, which has been studied by many people, for example, as a means of constructing new Calabi–Yau threefolds or finding mirrors.

This article is an attempt to understand the change of quantum cohomology under extremal transition and relate it with the following diagram:

$$X_{\mathrm{res}} \xrightarrow{\ \pi\ } X_{\mathrm{sing}} \xleftarrow{\ r\ } X_{\mathrm{sm}}, \tag{1}$$

where π is a resolution of singularities and r is a (continuous) retraction. Recall that the (small) quantum product $\star$ of a smooth projective variety X defines a commutative ring structure on $QH^*(X) = H^*(X) \otimes \mathbb{C}[\![q_1, \dots, q_r]\!]$, where the q_i's are the Novikov (quantum) variables associated to a basis of curve classes on X, and $r = \dim H^2(X)$. This defines the quantum connection (or Dubrovin connection)

$$\nabla_{q_i \frac{\partial}{\partial q_i}} = q_i \frac{\partial}{\partial q_i} + \frac{1}{z}(\phi_i \star), \quad 1 \le i \le r,$$

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with a parameter $z \in \mathbb{C}^\times$, on the trivial bundle over the q-space with fiber the cohomology group $H^*(X)$. This is flat for all values of z. Here $\phi_1, \dots, \phi_r$ is a basis of $H^2(X)$ dual to the variables $q_1, \dots, q_r$.

In the case of threefold conifold transition, Li and Ruan [12] studied the change of Gromov–Witten invariants and functoriality of quantum cohomology. In terms of the quantum connection, their result can be restated as follows.

THEOREM 1.1 (SEE THEOREM 3.5 AND COROLLARY 3.6)
Let $X_{\mathrm{res}} \to X_{\mathrm{sing}} \leftarrow X_{\mathrm{sm}}$ be a threefold conifold transition. Let $E_1, \dots, E_k$ be exceptional curves of X_{res}.

(a) *The quantum connection of X_{res} is of the form*

$$\nabla^{\mathrm{res}} = \nabla' + \sum_{i=1}^{k} N_i \frac{dq^{E_i}}{1 - q^{E_i}},$$

where ∇' is a connection which is regular along $\Delta_{\mathrm{exc}} = \{q^{E_1} = q^{E_2} = \cdots = q^{E_k} = 1\}$ and $N_i \in \mathrm{End}(H^*(X_{\mathrm{res}}))$ is a nilpotent endomorphism.

(b) *The residue endomorphisms N_i along $q^{E_i} = 1$ define the following filtration $0 \subset W \subset V \subset H^*(X_{\mathrm{res}})$:*

$$V := \bigcap_{i=1}^{k} \mathrm{Ker}(N_i), \qquad W := V \cap V^\perp = \bigcap_{i=1}^{k} \mathrm{Ker}(N_i) \cap \sum_{i=1}^{k} \mathrm{Im}(N_i). \tag{2}$$

This filtration arises from the diagram (1) as $V = \mathrm{Im}\,\pi^*$ and $W = \pi^*(\mathrm{Ker}\, r^*)$.

(c) *The connection $\nabla'|_{\Delta_{\mathrm{exc}}}$ induces a flat connection on the vector bundle $(V/W) \times \Delta_{\mathrm{exc}} \to \Delta_{\mathrm{exc}}$ which is isomorphic to the small quantum connection of X_{sm} under the isomorphism $r^* \circ (\pi^*)^{-1} \colon V/W \cong H^*(X_{\mathrm{sm}})$.*

In particular, the small quantum cohomology $QH^*(X_{\mathrm{sm}})$ of X_{sm} is isomorphic to the subquotient $(V/W, \star|_{q_{\mathrm{exc}}=1})$ of the quantum cohomology of X_{res} along the locus where all the exceptional quantum variables $q_{\mathrm{exc}} = (q^{E_1}, \dots, q^{E_k})$ equal one.

The idea that $QH^*(X_{\mathrm{sm}})$ could be described as a subquotient of $QH^*(X_{\mathrm{res}})$ with respect to a certain filtration given by monodromy arose out of a discussion of the first author with Tom Coates and Alessio Corti around 2010. We also want to draw attention to a recent paper of Lee, Lin, and Wang [11], where they studied the behavior of $(A + B)$-theory under conifold transition of Calabi–Yau threefolds.

In this article, we study analogous phenomena for higher-dimensional extremal transitions. As studied in [9] and [1], a partial flag variety admits a flat degeneration to a singular Gorenstein toric variety X_{sing}, which in turn admits a toric crepant resolution X_{res}. We study extremal transitions of $\mathrm{Fl}(1,2,3)$, $\mathrm{Gr}(2,4)$, and $\mathrm{Gr}(2,5)$ by explicit computations. A toric degeneration of $\mathrm{Fl}(1,2,3)$ and its resolution is a special case of the threefold conifold transition, and we confirm the above result. In the remaining two cases, we find analogous results together with some new phenomena, as follows.

• For $\mathrm{Gr}(2,4)$, the map $r^*\colon H^*(X_{\mathrm{sing}}) \to H^*(X_{\mathrm{sm}})$ is not surjective and the subquotient $(V/W, \star|_{q_{\mathrm{exc}}=1})$ of $H^*(X_{\mathrm{res}})$ is identified with a *proper* subring $\operatorname{Im} r^* \subsetneqq QH^*(\mathrm{Gr}(2,4))$, where V, W are defined by the residue endomorphism N as in (2). If we consider the weight filtration $\{W_\bullet\}$ associated to N, then we can extend the inclusion $(V/W, \star|_{q_{\mathrm{exc}}=1}) \hookrightarrow QH^*(\mathrm{Gr}(2,4))$ to an isomorphism $W_0/W_{-1} \cong QH^*(\mathrm{Gr}(2,4))$. The isomorphism $W_0/W_{-1} \cong QH^*(\mathrm{Gr}(2,4))$, however, involves an imaginary number.

• For $\mathrm{Gr}(2,5)$, the subquotient $(V/W, \star|_{q_{\mathrm{exc}}=1})$ is isomorphic to $QH^*(\mathrm{Gr}(2,5))$, where V, W are defined by the residue endomorphisms N_2, N_3 as in (2). In this case, $W \subset \operatorname{Im}\pi^* \subsetneqq V$ and the isomorphism $V/W \cong H^*(\mathrm{Gr}(2,5))$ coincides with $r^* \circ (\pi^*)^{-1}$ only on the subspace $\operatorname{Im}\pi^*/W$. Also, the quotient W_0/W_{-1} associated to the weight filtration $\{W_\bullet\}$ of $aN_2 + bN_3$ $(a \neq 0,\ b \neq 0)$ has dimension bigger than $\dim H^*(\mathrm{Gr}(2,5))$.

See Theorems 4.1, 5.2, 5.4, 6.1, and 6.3 for more details. Note also that $\mathrm{Fl}(1,2,3)$ and $\mathrm{Gr}(2,4)$ are hypersurfaces in toric varieties, whereas $\mathrm{Gr}(2,5)$ is not.

This article is structured as follows. In Section 2, we introduce notation on Gromov–Witten invariants and quantum cohomology. In Section 3, we study conifold transition in dimension 3 using a result of Li and Ruan [12]. In Sections 4–6, we study extremal transitions of $\mathrm{Fl}(1,2,3)$, $\mathrm{Gr}(2,4)$, and $\mathrm{Gr}(2,5)$. In Section 7, we formulate a conjecture for the change of quantum cohomology under extremal transitions of partial flag varieties.

2. Preliminaries

In this section we fix notation for Gromov–Witten invariants and quantum cohomology. For details on Gromov–Witten theory, we refer the reader to [5] and references therein. In this article we only consider cohomology classes of even degree and denote by $H^*(X)$ the even part $H^{\mathrm{ev}}(X,\mathbb{C})$ of the cohomology group with complex coefficients.

2.1. Gromov–Witten invariants

Let X be a smooth projective variety. For a second homology class $\beta \in H_2(X,\mathbb{Z})$ and nonnegative integers g,n, we denote by $\overline{M}_{g,n}(X,\beta)$ the moduli space of stable maps of degree β and genus g with n marked points. This has a virtual fundamental class $[\overline{M}_{g,n}(X,\beta)]_{\mathrm{vir}} \in H_{2D}(\overline{M}_{g,n}(X,\beta))$ of dimension $D = (1-g)(\dim X - 3) + n + \int_\beta c_1(X)$. Let $\mathrm{ev}_i\colon \overline{M}_{g,n}(X,\beta) \to X$ be the evaluation map at the ith marked point. Gromov–Witten invariants are defined by

$$\langle \gamma_1, \dots, \gamma_n \rangle^X_{g,n,\beta} = \int_{[\overline{M}_{g,n}(X,\beta)]_{\mathrm{vir}}} \mathrm{ev}_1^*(\gamma_1) \cup \cdots \cup \mathrm{ev}_n^*(\gamma_n),$$

where $\gamma_1, \dots, \gamma_n \in H^*(X)$. In this article, we are mainly interested in three-point genus 0 Gromov–Witten invariants and the associated small quantum cohomology.

2.2. Quantum cohomology

We choose a basis $\{\phi_0, \phi_1, \dots, \phi_N\}$ of $H^*(X)$ such that:

(1) ϕ_0 is the identity element of $H^*(X)$;
(2) $\phi_1, \dots, \phi_r$ form a nef integral basis for $H^2(X,\mathbb{Z})/\text{torsion}$, where r is the rank of $H^2(X,\mathbb{Z})$;
(3) ϕ_i is homogeneous.

Let $(\alpha,\beta) = \int_X \alpha \cup \beta$ denote the Poincaré pairing. Let $\{\phi^0, \dots, \phi^N\}$ denote the basis dual to $\{\phi_0, \dots, \phi_N\}$ with respect to the Poincaré pairing: $(\phi_i, \phi^j) = \delta_i^j$. Note that condition (2) above is equivalent to the condition that the cone spanned by the dual basis $\{\phi^1, \dots, \phi^r\}$ in $H^{2\dim X - 2}(X,\mathbb{R}) \cong H_2(X,\mathbb{R})$ contains the cone $\overline{\mathrm{NE}}(X)$ of effective curves (the Mori cone).

Let $q_1, \dots, q_r$ be the Novikov variables which are dual to the basis $\{\phi_1, \dots, \phi_r\}$ of $H^2(X)$. For $\beta \in H_2(X)$, we write

$$q^\beta = q_1^{\phi_1 \cdot \beta} q_2^{\phi_2 \cdot \beta} \cdots q_r^{\phi_r \cdot \beta}.$$

Note that if β is an effective class, the right-hand side only contains nonnegative powers of $q_1, \dots, q_r$. We define the Novikov ring to be $\Lambda := \mathbb{C}[\![q_1, \dots, q_r]\!]$. The small quantum product $\star$ on $H^*(X) \otimes \Lambda$ is defined by

$$(u \star v, w) = \sum_{\beta \in \mathrm{Eff}(X)} \langle u, v, w \rangle_{0,3,\beta}^X q^\beta.$$

The product $\star$ defines an associative and commutative ring structure on $H^*(X) \otimes \Lambda$. Moreover, this is graded with respect to the grading $\deg q_i = 2\rho_i$ and the usual grading on $H^*(X)$, where $c_1(X) = \sum_{i=1}^r \rho_i \phi_i$. This is called the *small quantum cohomology* and is denoted by $QH^*(X)$. The structure constants of small quantum cohomology are not known to be convergent in general (as power series in $q_1, \dots, q_r$); however, they are convergent for all the examples in this article.

2.3. Quantum connection

The quantum cohomology associates a pencil of flat connection, called the *quantum connection* or *Dubrovin connection*. This is a flat connection ∇ on the trivial $H^*(X)$-bundle over $\mathbb{C}^r$ with logarithmic singularities along the normal crossing divisor $q_1 q_2 \cdots q_r = 0$, given by

$$\nabla_{q_i \frac{\partial}{\partial q_i}} = q_i \frac{\partial}{\partial q_i} + \frac{1}{z}(\phi_i \star).$$

Here $z \in \mathbb{C}^\times$ is a parameter of the pencil. The flatness follows from the associativity of the quantum product. When we identify $v = \sum_{i=1}^r v^i \phi_i \in H^2(X)$ with the logarithmic vector field $\partial_v = \sum_{i=1}^r v^i q_i \frac{\partial}{\partial q_i}$ on $\mathbb{C}^r$, we can write the quantum connection in the following way:

$$\nabla_v = \partial_v + \frac{1}{z}(v \star).$$

In this article, we relate the quantum connections of a smoothing and a resolution.

3. Conifold transition and quantum cohomology

In this section we describe the change of quantum cohomology under conifold transition in dimension 3, using a result of Li and Ruan [12]. Our main result in this section is stated in Theorem 3.5. We observe that the quantum cohomology of a smoothing arises as a limit of the quantum cohomology of a resolution when the quantum variables associated to exceptional curves go to one.

3.1. Geometry of conifold transition

The conifold transition in dimension 3 is a surgery which replaces a $(-1,-1)$-rational curve with a real 3-sphere. In this section we describe topological properties of the conifold transition. See, for example, [14] and [16] for more background material.

Let X_{sing} be a 3-dimensional projective variety whose only singularities are ordinary double points $p_1, \dots, p_k$. Recall that an ordinary double point (or A_1-singularity) is a singularity whose neighborhood is analytically isomorphic to a neighborhood of the origin in $\{xy = zw\} \subset \mathbb{C}^4$. Let X_{res} be a small resolution of X_{sing}, and suppose that X_{sing} admits a smoothing X_{sm}. The passage from X_{res} to X_{sm} is called the *conifold transition*. Since we are interested in Gromov–Witten theory, we assume that both X_{sing} and X_{sm} are projective. The inverse image E_i of p_i in the small resolution X_{res} is a rational curve whose normal bundle is $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. The vanishing cycle $S_i \subset X_{\text{sm}}$ associated to p_i is a real 3-sphere. In topological terms, the conifold transition replaces a neighborhood $S^2 \times D^4$ of E_i with a neighborhood $D^3 \times S^3 \cong T^* S_i$ of S_i. There are two natural maps:

- a morphism $\pi : X_{\text{res}} \longrightarrow X_{\text{sing}}$ contracting the rational curves $E_1, \dots, E_k$;
- a continuous map $r : X_{\text{sm}} \longrightarrow X_{\text{sing}}$ contracting the real 3-spheres $S_1, \dots, S_k$.

They give the following correspondence between the cohomology groups of the resolution and the smoothing:

$$\begin{tikzcd}
H^*(X_{\text{res}}) & & H^*(X_{\text{sm}}) \\
& H^*(X_{\text{sing}}) \arrow[ul, "\pi^*"] \arrow[ur, "r^*"'] &
\end{tikzcd}$$

Set $E = E_1 \cup E_2 \cup \cdots \cup E_k \subset X_{\text{res}}$ and $S = S_1 \cup S_2 \cup \cdots \cup S_k \subset X_{\text{sm}}$. The relative cohomology exact sequence gives the following exact sequences:

$$\begin{array}{ccccccc}
0 \longrightarrow & H^2(X_{\text{res}}, E) & \longrightarrow & H^2(X_{\text{res}}) & \longrightarrow & H^2(E), & \\
0 \longrightarrow & H^4(X_{\text{res}}, E) & \longrightarrow & H^4(X_{\text{res}}) & \longrightarrow & 0, & \\
& 0 & \longrightarrow & H^2(X_{\text{sm}}, S) & \longrightarrow & H^2(X_{\text{sm}}) & \longrightarrow 0, \\
& H^3(S) & \longrightarrow & H^4(X_{\text{sm}}, S) & \longrightarrow & H^4(X_{\text{sm}}) & \longrightarrow 0.
\end{array}$$

Set $P=\{p_1,\dots,p_k\}\subset X_{\mathrm{sing}}$. Then we have $H^*(X_{\mathrm{res}},E)\cong H^*(X_{\mathrm{sing}},P)\cong H^*(X_{\mathrm{sm}},S)$ and $H^i(X_{\mathrm{sing}},P)\cong H^i(X_{\mathrm{sing}})$ for $i\geq 2$. Therefore, we obtain

$$\begin{aligned} &0\to H^2(X_{\mathrm{sing}})\xrightarrow{\pi^*}H^2(X_{\mathrm{res}})\to\mathbb{C}^k, \qquad \pi^*\colon H^4(X_{\mathrm{sing}})\cong H^4(X_{\mathrm{res}}),\\ &\mathbb{C}^k\to H^4(X_{\mathrm{sing}})\xrightarrow{r^*}H^4(X_{\mathrm{sm}})\to 0, \qquad r^*\colon H^2(X_{\mathrm{sing}})\cong H^2(X_{\mathrm{sm}}).\end{aligned}\tag{3}$$

Combining these sequences, we obtain

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^2(X_{\mathrm{sm}}) & \longrightarrow & H^2(X_{\mathrm{res}}) & \longrightarrow & \mathbb{C}^k,\\ \mathbb{C}^k & \longrightarrow & H^4(X_{\mathrm{res}}) & \longrightarrow & H^4(X_{\mathrm{sm}}) & \longrightarrow & 0.\end{array}\tag{4}$$

Note that the map $H^2(X_{\mathrm{res}})\to\mathbb{C}^k$ in the first sequence sends a class $\alpha\in H^2(X_{\mathrm{res}})$ to the vector $([E_1]\cdot\alpha,\dots,[E_k]\cdot\alpha)\in\mathbb{C}^k$.

LEMMA 3.1

The two sequences in (4) are dual to each other with respect to the Poincaré pairing. In other words, the image of the standard basis vector $e_i\in\mathbb{C}^k$ in $H^4(X_{\mathrm{res}})$ under the map $\mathbb{C}^k\to H^4(X_{\mathrm{res}})$ in the second sequence of (4) is the Poincaré dual of E_i.

Proof

The dual of the map $\mathbb{C}^k=H^3(S)\to H^4(X_{\mathrm{sm}},S)\cong H^4(X_{\mathrm{sing}})\cong H^4(X_{\mathrm{res}})$ is identified with the following boundary map:

$$H_4(X_{\mathrm{res}})\cong H_4(X_{\mathrm{sing}})\cong H_4(X_{\mathrm{sm}},S)\xrightarrow{\partial}H_3(S)=\mathbb{C}^k.\tag{5}$$

It suffices to show that this is given by the intersection numbers with the exceptional curves $E_1,\dots,E_k$. Take a real 4-cycle $D\subset X_{\mathrm{res}}$, and suppose that D intersects every E_i transversely. Under the conifold transition, each intersection point of D and E_i is replaced with the 3-sphere S_i. Therefore, the image of $[D]$ under (5) is given by $(E_i\cdot D)_{i=1}^k$. The lemma follows. $\square$

Note that the map $\pi^*\colon H^*(X_{\mathrm{sing}})\to H^*(X_{\mathrm{res}})$ is injective and $r^*\colon H^*(X_{\mathrm{sing}})\to H^*(X_{\mathrm{sm}})$ is surjective. The exact sequences (3) and (4) and the above lemma imply the following description of $H^*(X_{\mathrm{sm}})$ as a subquotient of $H^*(X_{\mathrm{res}})$.

PROPOSITION 3.2

Consider the filtration $0\subset W\subset V\subset H^*(X_{\mathrm{res}})$ defined by

$$W:=\sum_{i=1}^k\mathbb{C}[E_i]\subset H^4(X_{\mathrm{res}}),\qquad V:=\operatorname{Im}\pi^*\cong H^*(X_{\mathrm{sing}}).$$

Then we have $V/W\cong H^*(X_{\mathrm{sm}})$. More precisely, the following holds.

(1) *W is the annihilator of V with respect to the Poincaré pairing, that is, $W=V^\perp$. In particular, V/W has a nondegenerate pairing.*

(2) *The map r^* induces an isomorphism $V/W\cong H^*(X_{\mathrm{sm}})$ which preserves the pairing and the cup product.*

REMARK 3.3
It is a subtle problem if X_{sing} admits a smoothing or if the small resolution X_{res} is projective. In the Calabi–Yau case, there is a criterion due to Friedman [6], Kawamata [10], and Tian [17] about the smoothability of X_{sing}: X_{sing} is smoothable if and only if there exist nonzero rational numbers $\alpha_1, \dots, \alpha_k \in \mathbb{Q}^\times$ such that $\sum_{i=1}^k \alpha_i [E_i] = 0$ in $H_2(X_{\mathrm{res}})$. In the Fano case, X_{sing} is always smoothable (see [6], [15]).

3.2. A theorem of Li and Ruan

We write $\langle \cdot \rangle^{\mathrm{res}}_{g,n,d}$ for Gromov–Witten invariants for X_{res} and $\langle \cdot \rangle^{\mathrm{sm}}_{g,n,\beta}$ for Gromov–Witten invariants for X_{sm}. Li and Ruan [12] showed the following theorem.

THEOREM 3.4 ([12, THEOREM B])
Let $v_1, \dots, v_n$ be elements of $H^*(X_{\mathrm{sing}})$, and let $0 \neq \beta \in H_2(X_{\mathrm{sm}}, \mathbb{Z})$ be a nonzero degree. We have

$$\sum_{d:\pi_*(d)=\beta} \big\langle \pi^*(v_1), \dots, \pi^*(v_n) \big\rangle^{\mathrm{res}}_{g,n,d} = \big\langle r^*(v_1), \dots, r^*(v_n) \big\rangle^{\mathrm{sm}}_{g,n,\beta}.$$

The sum on the left-hand side is finite, that is, $\langle \pi^*(v_1), \dots, \pi^*(v_n) \rangle_{g,n,d}$ with $\pi_*(d) = \beta$ with a fixed β vanishes except for finitely many d.

3.3. Transition of quantum cohomology

We choose a suitable basis of $H_2(X_{\mathrm{res}}, \mathbb{Z})$. Let L be an ample line bundle over X_{sing}. Then the line bundle $\pi^* L$ is nef on X_{res}, and for any curve $C \subset X_{\mathrm{res}}$, we have $L \cdot C = 0$ if and only if C is one of the exceptional curves $E_1, \dots, E_k$. Therefore, the face $F := \{d \in \overline{\mathrm{NE}}(X_{\mathrm{res}}) : d \cdot \pi^* L = 0\}$ of the Mori cone $\overline{\mathrm{NE}}(X_{\mathrm{res}})$ is spanned by the classes of $E_1, \dots, E_k$. We choose an integral basis $d_1, \dots, d_r$ of $H_2(X_{\mathrm{res}}, \mathbb{Z})/\text{torsion}$ such that

- $d_1, \dots, d_e$ span a cone containing the face F, where† $e = \dim F$,
- $d_1, \dots, d_r$ span a cone containing $\overline{\mathrm{NE}}(X_{\mathrm{res}})$.

Let $q_1, \dots, q_r$ be the Novikov variables corresponding to the basis $d_1, \dots, d_r$. For any $d = \sum_{i=1}^r n_i d_i \in H_2(X_{\mathrm{res}}, \mathbb{Z})/\text{torsion}$, we write $q^d = q_1^{n_1} q_2^{n_2} \cdots q_r^{n_r}$. By the exact sequence (4), we have

$$\bigoplus_{i=1}^k \mathbb{C}[E_i] \longrightarrow H_2(X_{\mathrm{res}}) \xrightarrow{\pi_*} H_2(X_{\mathrm{sing}}) \cong H_2(X_{\mathrm{sm}}) \longrightarrow 0.$$

Therefore, $\pi_*(d_{e+1}), \dots, \pi_*(d_r)$ form a basis of $H_2(X_{\mathrm{sing}}) \cong H_2(X_{\mathrm{sm}})$, and $q_{e+1}, \dots, q_r$ can be identified with Novikov parameters for X_{sm}. Note that, by Li–Ruan's theorem and by the surjectivity of r^*, Gromov–Witten invariants of X_{sm} of degree $\beta \in H_2(X_{\mathrm{sm}}, \mathbb{Z})$ can be nonzero only when β is a linear combination of

† We have $e \leq k$. It is possible that $[E_1], \dots, [E_k]$ are linearly dependent.

$\pi_*(d_{e+1}), \dots, \pi_*(d_r)$ with *nonnegative* coefficients. Therefore, the quantum product of X_{sm} is defined over the ring $\mathbb{C}[\![q_{e+1}, \dots, q_r]\!]$.

Before stating the result, we explain the meaning of analytic continuation. We will consider analytic continuation of the quantum product of X_{res} across the locus $\Delta_{\mathrm{exc}} := \{q_1 = q_2 = \cdots = q_e = 1\} \subset \mathbb{C}^r$ where all the quantum variables associated to exceptional classes equal one. The map $\pi_*\colon H_2(X_{\mathrm{res}}) \to H_2(X_{\mathrm{sing}}) \cong H_2(X_{\mathrm{sm}})$ induces a ring homomorphism

$$\lim_{q_{\mathrm{exc}} \to 1} := \lim_{(q_1,\dots,q_e)\to(1,\dots,1)} \colon \mathbb{C}[q_1, \dots, q_r] \longrightarrow \mathbb{C}[q_{e+1}, \dots, q_r],$$

where q_{exc} stands for quantum variables associated to exceptional curves. However, this does not extend to a homomorphism between the Novikov rings $\mathbb{C}[\![q_1, \dots, q_r]\!]$ and $\mathbb{C}[\![q_{e+1}, \dots, q_r]\!]$. Instead, we have a map

$$\lim_{q_{\mathrm{exc}} \to 1} \colon \mathbb{C}[q_1, \dots, q_e][\![q_{e+1}, \dots, q_r]\!] \to \mathbb{C}[\![q_{e+1}, \dots, q_r]\!].$$

Thus, if $v \star w$ is defined over the ring $\mathbb{C}[q_1, \dots, q_e][\![q_{e+1}, \dots, q_r]\!]$, then we have a well-defined limit $\lim_{q_{\mathrm{exc}} \to 1} v \star w$.

THEOREM 3.5

The quantum cohomology of X_{sm} is a subquotient of the quantum cohomology of X_{res} restricted to the locus $\Delta_{\mathrm{exc}} := \{q_1 = q_2 = \cdots = q_e = 1\}$ where the Novikov variables of exceptional curves equal one. More precisely, we have the following.

(1) *The small quantum product of $v \in H^*(X_{\mathrm{res}})$ is of the form*

$$(v\star) = \sum_{i=1}^{k} \bigl(v \cdot [E_i]\bigr) \frac{q^{E_i}}{1 - q^{E_i}} N_i + R(v),$$

where $R(v) \in \mathrm{End}(H^*(X_{\mathrm{res}})) \otimes \mathbb{C}[q_1, \dots, q_e][\![q_{e+1}, \dots, q_r]\!]$ is regular along Δ_{exc}, $R(v)|_{q_{e+1}=\cdots=q_r=0}$ is the cup product by v, and $N_i \in \mathrm{End}(H^*(X_{\mathrm{res}}))$ is a nilpotent endomorphism defined by $N_i(w) = (w \cdot [E_i])[E_i]$.

(2) *The endomorphisms N_i define the filtration $0 \subset W \subset V \subset H^*(X_{\mathrm{res}})$ by*

$$V := \bigcap_{i=1}^{k} \mathrm{Ker}(N_i), \qquad W := V \cap \sum_{i=1}^{k} \mathrm{Im}(N_i).$$

This filtration coincides with the one in Proposition 3.2, that is, $W = \sum_{i=1}^{k} \mathbb{C}[E_i]$ and $V = \mathrm{Im}(\pi^*) \cong H^*(X_{\mathrm{sing}})$.

(3) *For $v, w \in V$, the limit $\lim_{q_{\mathrm{exc}} \to 1} v \star w$ exists and lies in $V \otimes \mathbb{C}[\![q_{e+1}, \dots, q_r]\!]$. Moreover, the map $r^*\colon V \to H^*(X_{\mathrm{sm}})$ satisfies*

$$r^*\Bigl(\lim_{q_{\mathrm{exc}} \to 1} v \star w\Bigr) = r^*(v) \star r^*(w).$$

Therefore, the isomorphism $V/W \cong H^*(X_{\mathrm{sm}})$ in Proposition 3.2 intertwines the quantum product of X_{res} restricted to Δ_{exc} with the quantum product of X_{sm}.

In terms of the quantum connection, we can rephrase the above result as follows.

COROLLARY 3.6

The small quantum connection ∇^{res} of X_{res} is of the form

$$\nabla^{\mathrm{res}} = \nabla' + \frac{1}{z}\sum_{i=1}^{k} N_i \frac{dq^{E_i}}{1-q^{E_i}},$$

where ∇' is a connection regular along $\Delta_{\mathrm{exc}} = \{q_1 = \cdots = q_e = 1\}$. The restriction of ∇' to Δ_{exc} induces a flat connection on the vector bundle $(V/W)\times\Delta_{\mathrm{exc}} \to \Delta_{\mathrm{exc}}$ which is isomorphic to the small quantum connection ∇^{sm} of X_{sm} under the natural isomorphism $r^*\colon V/W \cong H^*(X_{\mathrm{sm}})$.

REMARK 3.7

The filtration $0 \subset W \subset V \subset H^*(X_{\mathrm{res}})$ is the weight filtration associated to the nilpotent endomorphism $\sum_{i=1}^{k} a_i N_i$ (see, e.g., [3, Section A.2]) for a generic choice of $a_1, \ldots, a_k$. As we shall see later in Section 6 for $\mathrm{Gr}(2,5)$, however, the quantum cohomology of a smoothing does not necessarily appear as a subquotient associated with the weight filtration.

REMARK 3.8

The monodromy of the quantum connection ∇^{res} around the divisor $\{q^{E_i}=1\}$ is conjugate to $\exp(2\pi\sqrt{-1}N_i/z)$ and is unipotent.

REMARK 3.9

The residue of $(v\star)$ along the divisor $q^{E_i}=1$ is also computed by Lee, Lin, and Wang [11, Lemma 3.12].

3.4. Proof of Theorem 3.5

We set $V = \mathrm{Im}\,\pi^*$ and $W = \sum_{i=1}^{k}\mathbb{C}[E_i]$ as in Proposition 3.2. Since $V \cong H^*(X_{\mathrm{sing}})$, we may regard r^* as a map from V to $H^*(X_{\mathrm{sm}})$. Theorem 3.5(2) follows from Theorem 3.5(1) and the exact sequences (3) and (4). Thus, it suffices to prove Theorems 3.5(1) and 3.5(3). Theorem 3.5(1) follows from the following lemma.

LEMMA 3.10

Fix $\beta \in H_2(X_{\mathrm{sm}},\mathbb{Z}) \cong H_2(X_{\mathrm{sing}},\mathbb{Z})$, and take $v_1, v_2, v_3 \in H^*(X_{\mathrm{res}})$. Consider the sum

$$\sum_{d:\pi_*(d)=\beta} \langle v_1, v_2, v_3\rangle_{0,3,d}^{\mathrm{res}} q^d. \tag{6}$$

1. *If $\beta \neq 0$, then the sum is finite.*
2. *If $\beta = 0$, the sum equals*

$$\int_{X_{\mathrm{res}}} v_1 \cup v_2 \cup v_3 + \sum_{i=1}^{k} \bigl(v_1\cdot[E_i]\bigr)\bigl(v_2\cdot[E_i]\bigr)\bigl(v_3\cdot[E_i]\bigr)\frac{q^{E_i}}{1-q^{E_i}}.$$

Proof
We may assume that v_1, v_2, v_3 are homogeneous. Suppose that $\beta \neq 0$. If $v_1, v_2, v_3 \in V = \operatorname{Im} \pi^*$, then the finiteness of the sum (6) follows from Theorem 3.4. If $v_1 \notin V$, then $v_1 \in H^2(X_{\mathrm{res}})$ by homogeneity. Thus, we can use the divisor equation to factor out v_1:

$$\text{equation (6)} = \sum_{d : \pi_*(d) = \beta} (v_1 \cdot d) \langle v_2, v_3 \rangle^{\mathrm{res}}_{0,2,d} q^d.$$

If in addition $v_2, v_3 \in V$, then Theorem 3.4 again shows the finiteness of the sum. The finiteness in the other cases can be similarly shown using the divisor equation and Theorem 3.4.

Next suppose that $\beta = 0$. The $d = 0$ term in (6) gives $\int_{X_{\mathrm{res}}} v_1 \cup v_2 \cup v_3$. The only curves in X_{res} contributing to the sum (6) are multiples of the exceptional curve E_i. By the dimension axiom, we have $\deg v_1 + \deg v_2 + \deg v_3 = 6$. If one of $\deg v_1$, $\deg v_2$, $\deg v_3$ is zero, then the invariant $\langle v_1, v_2, v_3 \rangle_{0,3,d}$ is zero for $d \neq 0$. Therefore, we only need to consider the case where $v_1, v_2, v_3 \in H^2(X_{\mathrm{res}})$. Since the moduli space $M_{0,0}(X_{\mathrm{res}}, d)$ with $\pi_*(d) = 0$ consists of multiple covers of some E_i, we have

$$\begin{aligned}
\sum_{d \neq 0 : \pi_*(d) = 0} \langle v_1, v_2, v_3 \rangle^{\mathrm{res}}_{0,3,d} &= \sum_{d \neq 0 : \pi_*(d) = 0} (v_1 \cdot d)(v_2 \cdot d)(v_3 \cdot d) \langle \cdot \rangle^{\mathrm{res}}_{0,0,d} q^d \\
&= \sum_{i=1}^{k} \sum_{n=1}^{\infty} (v_1 \cdot nE_i)(v_2 \cdot nE_i)(v_3 \cdot nE_i) \frac{1}{n^3} q^{nE_i} \\
&= \sum_{i=1}^{k} (v_1 \cdot E_i)(v_2 \cdot E_i)(v_3 \cdot E_i) \frac{q^{E_i}}{1 - q^{E_i}}
\end{aligned}$$

by the multiple cover formula (see [13, Theorem 0.5.1]) for a $(-1,-1)$-curve (each multiple cover of degree n contributes $1/n^3$). The lemma is proved. □

Finally, we prove Theorem 3.5(3). Suppose that $v, w \in V$. The existence of the limit $\lim_{q_{\mathrm{exc}} \to 1} v \star w$ follows from Lemma 3.10. We claim that

$$\lim_{q_{\mathrm{exc}} \to 1} (u, v \star w) = \bigl(r^*(u), r^*(v) \star r^*(w)\bigr) \tag{7}$$

for all $u \in V$. The left-hand side equals

$$\sum_{d : \pi_*(d) = 0} \langle u, v, w \rangle^{\mathrm{res}}_{0,3,d} + \sum_{\beta \neq 0} \sum_{d : \pi_*(d) = \beta} \langle u, v, w \rangle^{\mathrm{res}}_{0,3,d} q^\beta. \tag{8}$$

By Lemma 3.10, the first term equals

$$\int_{X_{\mathrm{res}}} u \cup v \cup w = \int_{X_{\mathrm{sm}}} r^* u \cup r^* v \cup r^* w$$

since $u \cdot [E_i] = v \cdot [E_i] = w \cdot [E_i] = 0$. We also used the fact that r^* preserves the pairing (see Proposition 3.2). By Theorem 3.4, the second term of (8) equals

$$\sum_{\beta \neq 0} \bigl\langle r^*(u), r^*(v), r^*(w) \bigr\rangle^{\mathrm{sm}}_{0,3,\beta} q^\beta.$$

The claim follows. Setting $u = [E_i]$ in (7) and using the fact that $r^*([E_i]) = 0$ from Lemma 3.1, we obtain $([E_i], \lim_{q_{\mathrm{exc}}\to 1} v \star w) = 0$. This means that $\lim_{q_{\mathrm{exc}}\to 1} v \star w$ lies in V. Using again the fact that r^* preserves the pairing, we obtain from (7) that

$$\big(r^*(u), r^*\big(\lim_{q_{\mathrm{exc}}\to 1} v \star w\big)\big) = \big(r^*(u), r^*(v) \star r^*(w)\big).$$

Since r^* is surjective, this shows that $r^*(\lim_{q_{\mathrm{exc}}\to 1} v \star w) = r^*(v) \star r^*(w)$. Theorem 3.5(3) is proved.

4. Example: $\mathrm{Fl}(1,2,3)$

In this section we study a conifold transition of $\mathrm{Fl}(1,2,3)$, the space of full flags in $\mathbb{C}^3$, confirming the result in the previous section. Consider a toric degeneration of $\mathrm{Fl}(1,2,3)$ given by a family $\{F_t\}_{t\in\mathbb{C}}$ of $(1,1)$-hypersurfaces in $\mathbb{P}^2 \times \mathbb{P}^2$:

$$F_t = \big\{\big([z_1, z_2, z_3], [Z_1, Z_2, Z_3]\big) \in \mathbb{P}^2 \times \mathbb{P}^2 \;\big|\; t z_1 Z_1 + z_2 Z_2 + z_3 Z_3 = 0\big\}.$$

Then $F_t \cong \mathrm{Fl}(1,2,3)$ for $t \neq 0$ and the central fiber $X_{\mathrm{sing}} := F_0$ is a singular toric variety with an ordinary double point. This admits a small toric crepant resolution $X_{\mathrm{res}} \to X_{\mathrm{sing}}$. We study a relationship between the quantum cohomology of $\mathrm{Fl}(1,2,3)$ and X_{res}.

4.1. Quantum cohomology of $\mathrm{Fl}(1,2,3)$

The quantum cohomology ring of a flag variety is well known (see, e.g., [8], [4]). Let L_1, L_2, L_3 be the line bundles on $\mathrm{Fl}(1,2,3)$ whose fibers at a flag $0 \subset L \subset V \subset \mathbb{C}^3$ are given by L, V/L, and $\mathbb{C}^3/V$, respectively. The cohomology ring of $\mathrm{Fl}(1,2,3)$ is generated by the Chern classes $c_i := -c_1(L_i)$, $i = 1,2,3$, and

$$H^*\big(\mathrm{Fl}(1,2,3)\big) \cong \mathbb{C}[c_1, c_2, c_3]/\langle \sigma_1, \sigma_2, \sigma_3\rangle,$$

where σ_i is the ith elementary symmetric polynomial of c_1, c_2, c_3. A basis of $H^2(\mathrm{Fl}(1,2,3))$ is given by

$$p_1 := c_1 = -c_1(L_1), \qquad p_2 := c_1 + c_2 = c_1(L_3).$$

These classes span the nef cone of $\mathrm{Fl}(1,2,3)$ and satisfy the relations $p_1^2 + p_2^2 - p_1 p_2 = 0$, $p_2^3 = p_1^3 = 0$, $p_1^2 p_2 = p_1 p_2^2$. The dual basis in $H_2(\mathrm{Fl}(1,2,3))$ is

$$\beta_1 = \mathrm{PD}(p_2^2), \qquad \beta_2 = \mathrm{PD}(p_1^2).$$

These classes span the Mori cone: they are represented by fibers of the natural maps $\mathrm{Fl}(1,2,3) \to \mathrm{Fl}(2,3) \cong (\mathbb{P}^2)^\star$ and $\mathrm{Fl}(1,2,3) \to \mathrm{Fl}(1,3) \cong \mathbb{P}^2$, respectively. For an effective class $d = n_1\beta_1 + n_2\beta_2 \in H_2(\mathrm{Fl}(1,2,3))$, we write $q^d = q_1^{n_1} q_2^{n_2}$ with $q_i = q^{\beta_i}$. Since $c_1(\mathrm{Fl}(1,2,3)) = 2p_1 + 2p_2$, we have $\deg q_1 = \deg q_2 = 4$. Consider the basis of $H^*(\mathrm{Fl}(1,2,3))$ given by

$$\big\{1, p_1, p_2, p_1^2 = \mathrm{PD}(\beta_2), p_2^2 = \mathrm{PD}(\beta_1), p_1^2 p_2 = \mathrm{PD}\big([\mathrm{pt}]\big)\big\}.$$

In this basis, the quantum multiplication by p_1 and p_2 is given by the following matrices:

$$p_1\star = \begin{pmatrix} 0 & q_1 & 0 & 0 & 0 & q_1q_2 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q_1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & q_1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \qquad p_2\star = \begin{pmatrix} 0 & 0 & q_2 & 0 & 0 & q_1q_2 \\ 0 & 0 & 0 & 0 & q_2 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & q_2 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

4.2. Quantum cohomology of X_{res}

The singular fiber $X_{\mathrm{sing}} = F_0$ is a toric variety, and the corresponding fan is given by the following data. One-dimensional cones are spanned by

$$r_1 = (0,0,1), \qquad r_2 = (1,1,-1), \qquad r_3 = (0,1,0),$$
$$r_4 = (0,-1,0), \qquad r_5 = (1,0,0), \qquad r_6 = (-1,0,0),$$

and the full-dimensional cones are given by

$$\langle r_1, r_3, r_6\rangle, \qquad \langle r_1, r_4, r_5\rangle, \qquad \langle r_1, r_4, r_6\rangle, \qquad \langle r_1, r_2, r_3, r_5\rangle,$$
$$\langle r_2, r_4, r_5\rangle, \qquad \langle r_2, r_4, r_6\rangle, \qquad \langle r_2, r_3, r_6\rangle.$$

A small resolution X_{res} of X_{sing} is given by dividing the cone $\langle r_1, r_2, r_3, r_5\rangle$ into the two simplicial cones $\langle r_1, r_3, r_5\rangle$ and $\langle r_2, r_3, r_5\rangle$. Let $R_1, \dots, R_6 \in H^2(X_{\mathrm{res}})$ be the classes of the prime toric divisors corresponding to the 1-dimensional cones $\langle r_1\rangle, \dots, \langle r_6\rangle$. The cohomology ring of X_{res} is generated by $R_1, \dots, R_6$ with the relations $R_1 = R_2$, $R_2 + R_3 = R_4$, $R_2 + R_5 = R_6$, $R_1R_2 = R_3R_4 = R_5R_6 = 0$. We choose a basis $\{\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3\}$ of $H^2(X_{\mathrm{res}})$ as

$$\mathfrak{p}_1 := R_4, \qquad \mathfrak{p}_2 := R_6, \qquad \mathfrak{p}_3 := R_2.$$

They span the nef cone of X_{res} and satisfy the relations $\mathfrak{p}_1(\mathfrak{p}_1 - \mathfrak{p}_3) = \mathfrak{p}_2(\mathfrak{p}_2 - \mathfrak{p}_3) = \mathfrak{p}_3^2 = 0$. The dual basis in $H_2(X_{\mathrm{res}})$ is given by

$$\beta_1 := \mathrm{PD}(\mathfrak{p}_2\mathfrak{p}_3), \qquad \beta_2 := \mathrm{PD}(\mathfrak{p}_1\mathfrak{p}_3),$$
$$\beta_3 := \mathrm{PD}(R_3R_5) = \mathrm{PD}(\mathfrak{p}_1\mathfrak{p}_2 - \mathfrak{p}_1\mathfrak{p}_3 - \mathfrak{p}_2\mathfrak{p}_3).$$

They span the Mori cone of X_{res}. The class β_3 is represented by the exceptional curve in X_{res}.

We can compute the quantum product of X_{res} by using Givental's mirror theorem [7, Theorem 0.1]. The computation will be illustrated in the Appendix for the example in Section 6. For $d = n_1\beta_1 + n_2\beta_2 + n_3\beta_3 \in H_2(X_{\mathrm{res}})$, we write $q^d = q_1^{n_1}q_2^{n_2}q_3^{n_3}$, setting $q_i = q^{\beta_i}$. Since $c_1(X_{\mathrm{res}}) = 2\mathfrak{p}_1 + 2\mathfrak{p}_2$, we have $\deg q_1 = \deg q_2 = 4$ and $\deg q_3 = 0$. Consider the following basis of $H^*(X_{\mathrm{res}})$:

$$\{1, \mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3, \mathfrak{p}_1\mathfrak{p}_2 - \mathfrak{p}_1\mathfrak{p}_3 - \mathfrak{p}_2\mathfrak{p}_3, \mathfrak{p}_1\mathfrak{p}_3, \mathfrak{p}_2\mathfrak{p}_3, \mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3\}.$$

In this basis, the quantum product by $\mathfrak{p}_1$, $\mathfrak{p}_2$, $\mathfrak{p}_3$ is represented by the following matrices:

$$\mathfrak{p}_1\star = \begin{pmatrix} 0 & q_1 & 0 & 0 & 0 & 0 & 0 & q_1q_2q_3 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & q_1(1-q_3) & q_1q_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & -q_1(1-q_3) & q_1(1-q_3) & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & q_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

$$\mathfrak{p}_2\star = \begin{pmatrix} 0 & 0 & q_2 & 0 & 0 & 0 & 0 & q_1q_2q_3 \\ 0 & 0 & 0 & 0 & q_2(1-q_3) & 0 & q_2q_3 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -q_2(1-q_3) & 0 & q_2(1-q_3) & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & q_2 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix},$$

$$\mathfrak{p}_3\star = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & q_1q_2q_3 \\ 0 & 0 & 0 & 0 & -q_2q_3 & 0 & q_2q_3 & 0 \\ 0 & 0 & 0 & 0 & -q_1q_3 & q_1q_3 & 0 & 0 \\ 1 & 0 & 0 & 0 & q_3(q_1+q_2) & -q_1q_3 & -q_2q_3 & 0 \\ 0 & 0 & 0 & \frac{q_3}{1-q_3} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

4.3. Comparison of quantum cohomology

We write X_{sm} for $\mathrm{Fl}(1,2,3)$. Recall from Section 3.1 that we have natural maps

$$X_{\mathrm{res}} \xrightarrow{\pi} X_{\mathrm{sing}} \xleftarrow{r} X_{\mathrm{sm}}.$$

The map $\pi^*\colon H^*(X_{\mathrm{sing}}) \to H^*(X_{\mathrm{res}})$ is injective and has the image

$$\pi^*\big(H^*(X_{\mathrm{sing}})\big) = \langle 1, \mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_1\mathfrak{p}_2 - \mathfrak{p}_1\mathfrak{p}_3 - \mathfrak{p}_2\mathfrak{p}_3, \mathfrak{p}_1\mathfrak{p}_3, \mathfrak{p}_2\mathfrak{p}_3, \mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_2\rangle.$$

The map $r^*\colon H^*(X_{\mathrm{sing}}) \to H^*(X_{\mathrm{sm}})$ is surjective with kernel

$$\pi^*\big(\mathrm{Ker}(r^*)\big) = \langle \mathfrak{p}_1\mathfrak{p}_2 - \mathfrak{p}_1\mathfrak{p}_3 - \mathfrak{p}_2\mathfrak{p}_3\rangle = \big\langle \mathrm{PD}(\beta_3)\big\rangle.$$

On the second homology groups, the maps π, r induce a map†

$$(r_*)^{-1}\pi_*\colon H_2(X_{\mathrm{res}}) \to H_2(X_{\mathrm{sm}}), \qquad \beta_1 \mapsto \beta_1, \qquad \beta_2 \mapsto \beta_2, \qquad \beta_3 \mapsto 0.$$

This gives rise to the map $\lim_{q_3\to 1}\colon \mathbb{C}[q_1,q_2,q_3] \to \mathbb{C}[q_1,q_2]$ between Novikov rings. The residue of the quantum multiplication by $\mathfrak{p}_3$ on $H^*(X_{\mathrm{res}})$ along $q_3 = 1$ is

† Note that r_* on H_2 is an isomorphism.

$$N=\operatorname{Res}_{q_3=1}(\mathfrak{p}_3\star)=\begin{pmatrix}0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&-1&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{pmatrix}.$$

It is nilpotent and induces the weight filtration on $H^*(X_{\mathrm{res}})$:

$$0\subset \operatorname{Im} N\subset \operatorname{Ker} N\subset H^*(X_{\mathrm{res}}). \tag{9}$$

The computation in Sections 4.1 and 4.2 shows the following proposition, which confirms the general argument in Section 3.

THEOREM 4.1

The weight filtration (9) defined by the nilpotent operator $N=\operatorname{Res}_{q_3=1}(\mathfrak{p}_3\star)$ coincides with the filtration

$$0\subset \pi^*(\operatorname{Ker} r^*)\subset \operatorname{Im}\pi^*\subset H^*(X_{\mathrm{res}}).$$

The quantum multiplication by $\mathfrak{p}_1$, $\mathfrak{p}_2$ on $H^*(X_{\mathrm{res}})$ is regular at $q_3=1$, and the operators induced by $\lim_{q_3\to 1}\mathfrak{p}_1\star$, $\lim_{q_3\to 1}\mathfrak{p}_2\star$ on

$$\operatorname{Ker} N/\operatorname{Im} N\cong H^*\big(\mathrm{Fl}(1,2,3)\big)$$

coincide with the quantum multiplication by p_1, p_2 on $H^*(\mathrm{Fl}(1,2,3))$. Here note that $\mathfrak{p}_i\in \operatorname{Im}\pi^*$ and $p_i=r^*(\pi^*)^{-1}\mathfrak{p}_i$ for $i=1,2$.

5. Example: $\mathrm{Gr}(2,4)$

In this section we study an extremal transition of $\mathrm{Gr}(2,4)$, the space of complex two-planes in $\mathbb{C}^4$. By the Plücker embedding, $\mathrm{Gr}(2,4)$ can be realized as a quadric in $\mathbb{P}^5=\mathbb{P}(\bigwedge^2\mathbb{C}^4)$. Consider a toric degeneration of $\mathrm{Gr}(2,4)$ given by a family $\{F_t\}_{t\in\mathbb{C}}$ of quadric hyperplanes in $\mathbb{P}^5$:

$$F_t=\big\{[Z_{12},Z_{13},Z_{14},Z_{23},Z_{24},Z_{34}]\in\mathbb{P}^5\ \big|\ Z_{12}Z_{34}-Z_{13}Z_{24}+tZ_{14}Z_{23}=0\big\}.$$

Then $F_t\cong \mathrm{Gr}(2,4)$ for $t\neq 0$ and the central fiber $X_{\mathrm{sing}}:=F_0$ is a singular toric variety with a transversal A_1-singularity along $(Z_{12}=Z_{34}=Z_{13}=Z_{24}=0)\cong\mathbb{P}^1$. This singular variety admits a small toric crepant resolution $X_{\mathrm{res}}\to X_{\mathrm{sing}}$. We study a relationship between the quantum cohomology of $\mathrm{Gr}(2,4)$ and X_{res}.

5.1. Quantum cohomology of $\mathrm{Gr}(2,4)$

Let $T^\star$ be the dual tautological bundle of $\mathrm{Gr}(2,4)$. The cohomology ring of $\mathrm{Gr}(2,4)$ is generated by the Chern classes $c_1(T^\star)$ and $c_2(T^\star)$. Fix a complete flag $0\subset E_1\subset E_2\subset E_3\subset E_4=\mathbb{C}^4$ in $\mathbb{C}^4$. Consider the following cycles:

$$
\begin{aligned}
D &= \{V \in \mathrm{Gr}(2,4) : \dim(V \cap E_2) = 1\},\\
\Delta &= \{V \in \mathrm{Gr}(2,4) : V \subset E_3\},\\
C &= \{V \in \mathrm{Gr}(2,4) : E_1 \subset V \subset E_3\}.
\end{aligned}
$$

Their Poincaré duals are denoted, respectively, by d, δ, c. We know that $d = c_1(T^\star)$, $\delta = c_2(T^\star)$, and $c = d\delta = d^3/2$. The cohomology ring is given by

$$H^*\big(\mathrm{Gr}(2,4)\big) \cong \mathbb{C}[d,\delta]/\langle d^3 - 2d\delta, d^2\delta - \delta^2\rangle.$$

We choose an additive basis of $H^*(\mathrm{Gr}(2,4))$ as follows:

$$1, \qquad d, \qquad d^2, \qquad d^2 - 2\delta, \qquad d^3, \qquad d^4.$$

Let q be the Novikov variable dual to $d \in H^2(\mathrm{Gr}(2,4))$. We have $\deg q = 8$. We use the quantum Schubert calculus (see [2], [4]) to compute the quantum product of d. Under the above basis, the quantum product matrix of d is

$$
d\star = \begin{pmatrix}
0 & 0 & 0 & 0 & 2q & 0\\
1 & 0 & 0 & 0 & 0 & 2q\\
0 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix},
$$

and the quantum product matrix of δ is

$$
\delta\star = \begin{pmatrix}
0 & 0 & q & q & 0 & 0\\
0 & 0 & 0 & 0 & 2q & 0\\
\frac{1}{2} & 1 & 0 & 0 & 0 & 2q\\
-\frac{1}{2} & 0 & 0 & 0 & 0 & 0\\
0 & \frac{1}{2} & 0 & 0 & 0 & 0\\
0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0
\end{pmatrix}.
$$

5.2. Quantum cohomology of X_{res}

The fan for the singular toric variety X_{sing} is as follows. It is a 4-dimensional fan whose 1-dimensional cones are spanned by

$$
\begin{aligned}
&r_1 = (1,0,0,0), \qquad r_2 = (-1,0,1,0), \qquad r_3 = (0,0,-1,1),\\
&r_4 = (-1,1,0,0), \qquad r_5 = (0,-1,0,1), \qquad r_6 = (0,0,0,-1).
\end{aligned}
$$

This is a complete fan whose top-dimensional cones are

$$
\begin{aligned}
&\langle r_1, r_3, r_5, r_6\rangle, \qquad \langle r_1, r_2, r_4, r_6\rangle, \qquad \langle r_2, r_3, r_4, r_5, r_6\rangle,\\
&\langle r_1, r_2, r_5, r_6\rangle, \qquad \langle r_1, r_3, r_4, r_6\rangle, \qquad \langle r_1, r_2, r_3, r_4, r_5\rangle.
\end{aligned}
$$

Note that there are two nonsimplicial 4-dimensional cones. We divide these cones as follows:

- divide $\langle r_2, r_3, r_4, r_5, r_6\rangle$ into $\langle r_2, r_3, r_4, r_6\rangle$ and $\langle r_2, r_3, r_5, r_6\rangle$,
- divide $\langle r_1, r_2, r_3, r_4, r_5\rangle$ into $\langle r_1, r_2, r_3, r_4\rangle$ and $\langle r_1, r_2, r_3, r_5\rangle$.

Then we get a smooth fan. This fan corresponds to a smooth toric variety which we denote by X_{res}. Let R_i denote the class of the toric divisor corresponding to the ray $\langle r_i\rangle$. There are linear relations $R_1 = R_2 + R_4$, $R_4 = R_5$, $R_2 = R_3$, $R_3 + R_5 = R_6$. The cohomology ring of X_{res} is given by

$$H^*(X_{\mathrm{sm}}) = \mathbb{C}[R_1, R_4]/\langle R_4^2, R_1^4 - 2R_1^3 R_4\rangle.$$

We choose a basis $\{m_1, m_2\}$ of $H^2(X_{\mathrm{res}})$ as $m_1 = R_1$, $m_2 = R_4$. They span the nef cone of X_{res}. The dual basis in $H_2(X_{\mathrm{res}})$ is given by $\beta_1 = \mathrm{PD}(R_1R_2R_4)$ and $\beta_2 = \mathrm{PD}(R_1R_2R_3)$. They span the Mori cone of X_{res}. The class β_2 is represented by an exceptional curve.

We compute the quantum product of X_{res} using Givental's mirror theorem [7, Theorem 0.1] (see the Appendix for the method). For $d = n_1\beta_1 + n_2\beta_2 \in H_2(X_{\mathrm{res}})$, we write $q^d = q_1^{n_1} q_2^{n_2}$, where $q_i = q^{\beta_i}$. We have $\deg q_1 = 8$ and $\deg q_2 = 0$. We choose the following basis for the cohomology ring of X_{res}:

$$\left\{1, m_1, m_1 - 2m_2, m_1^2, m_1^2 - 2m_1m_2, m_1^3, m_1^3 - 2m_1^2 m_2, m_1^4 = 2[\mathrm{pt}]\right\}.$$

Under this basis, the quantum product matrices of the divisors m_1 and m_2 are as follows:

$$m_1\star = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & q_1(1+q_2) & q_1(1-q_2) & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & q_1(1+q_2) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -q_1(1-q_2) \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix},$$

$$m_2\star = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & q_1(1-q_2) & 2q_1q_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & q_1(1-q_2) \\ 1 & 0 & 2 & 0 & 0 & 0 & 0 & -q_1(1+q_2) \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & \frac{2(1+q_2)}{1-q_2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \frac{2(1+q_2)}{1-q_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

5.3. Comparison of quantum cohomology

The residue of the quantum product matrix of m_2 at $q_2 = 1$ is

$$N = \mathrm{Res}_{q_2=1}(m_2\star) = \begin{pmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&4&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&4&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix}.$$

The residue N defines the filtration $0 \subset W \subset V \subset H^*(X_{\mathrm{res}})$ as

$$\begin{aligned} V &:= \operatorname{Ker} N = \mathrm{Span}\{1, m_1, m_1^2, m_1^3, m_1^4, m_1^3 - 2m_1^2 m_2\},\\ W &:= \operatorname{Ker} N \cap \operatorname{Im} N = \mathbb{C}(m_1^3 - 2m_1^2 m_2). \end{aligned} \tag{10}$$

This filtration arises from the correspondence $X_{\mathrm{res}} \to X_{\mathrm{sing}} \leftarrow X_{\mathrm{sm}} := \mathrm{Gr}(2,4)$ as follows.

PROPOSITION 5.1

Let $\pi\colon X_{\mathrm{res}} \to X_{\mathrm{sing}}$ and $r\colon X_{\mathrm{sm}} = \mathrm{Gr}(2,4) \to X_{\mathrm{sing}}$ be natural maps associated to the resolution and the smoothing.

(1) *The singular cohomology group of X_{sing} is given by the table*

degree p	0	1	2	3	4	5	6	7	8
$H^p(X_{\mathrm{sing}})$	$\mathbb{C}$	0	$\mathbb{C}$	0	$\mathbb{C}$	0	$\mathbb{C}^2$	0	$\mathbb{C}$

(2) *The map $\pi^*\colon H^*(X_{\mathrm{sing}}) \to H^*(X_{\mathrm{res}})$ is injective and $\operatorname{Im}\pi^* = V$.*

(3) *The map $r^*\colon H^*(X_{\mathrm{sing}}) \to H^*(X_{\mathrm{sm}})$ is neither surjective nor injective; we have $\pi^*(\operatorname{Ker} r^*) = W$ and $\operatorname{Im} r^* = \mathrm{Span}\{1, d, d^2, d^3, d^4\}$.*

(4) *The map $r^* \circ (\pi^*)^{-1}\colon V \to H^*(X_{\mathrm{sm}})$ sends m_1^i to d^i for $0 \le i \le 4$ and $m_1^3 - 2m_1^2 m_2$ to zero.*

Proof

Note that the nonsingular locus Y of X_{sing} is isomorphic to the total space of $\mathcal{O}(1,1)^{\oplus 2}$ over $\mathbb{P}^1 \times \mathbb{P}^1$. We consider the Mayer–Vietoris exact sequence associated to Y and a neighborhood ν of the singular locus $\mathbb{P}^1$. The intersection $\nu \cap Y$ is homotopic to the 3-sphere bundle associated to $\mathcal{O}(1,1)^{\oplus 2} \to \mathbb{P}^1 \times \mathbb{P}^1$ and the cohomology of $\nu \cap Y$ can be easily computed by the Gysin sequence. We have

$$H^*(N \cap Y) = \mathbb{C}, 0, \mathbb{C}^2, 0, 0, \mathbb{C}^2, 0, \mathbb{C} \quad \text{for } * = 0,1,2,3,4,5,6,7.$$

Then the Mayer–Vietoris sequence gives the result for $H^*(X_{\mathrm{sing}})$. To prove the statement about π^*, we consider the hypercohomology spectral sequence for

$\mathbb{H}^*(X_{\mathrm{sing}}, \mathbb{R}\pi_*\underline{\mathbb{C}}) = H^*(X_{\mathrm{res}})$. Since we have

$$R^j\pi_*\underline{\mathbb{C}} = \begin{cases} \underline{\mathbb{C}} & j=0, \\ \iota_*\underline{\mathbb{C}}_{\mathbb{P}^1} & j=2, \\ 0 & \text{otherwise}, \end{cases}$$

where $\iota\colon \mathbb{P}^1 \to X_{\mathrm{sing}}$ is the inclusion of the singular locus, the spectral sequence degenerates at the E_2-term $H^j(R^i\pi_*\underline{\mathbb{C}})$; this shows that π^* is injective. Since the image of π^* contains the pullback m_1 of the ample class $\alpha := c_1(\mathcal{O}(1))$ on X_{sing}, it follows that $\operatorname{Im}\pi^* = V$. On the other hand, r^* also sends the ample class α to $d = c_1(\mathcal{O}(1)) \in H^2(X_{\mathrm{sm}})$ and it follows that $\operatorname{Im} r^* = \operatorname{Span}\{1, d, d^2, d^3, d^4\}$. Let $x \in H^6(X_{\mathrm{sing}})$ be a generator of the kernel of r^*. Then we have $\alpha \cup x = 0$ in $H^8(X_{\mathrm{sing}})$ (as otherwise we have $0 \neq r^*(\alpha \cup x) = r^*(\alpha) \cup r^*(x) = 0$). Therefore, $0 = \pi^*(\alpha \cup x) = m_1 \cup \pi^*(x)$. This shows that $\pi^*(x)$ is a multiple of $m_1^3 - 2m_1^2 m_2$. $\square$

The computation in Sections 5.1–5.2 implies the following theorem.

THEOREM 5.2

The filtration $0 \subset W \subset V \subset H^*(X_{\mathrm{res}})$ *that (10) defined by the residue* $N = \operatorname{Res}_{q_2=1}(m_2\star)$ *along* $q_2 = 1$ *matches with the filtration*

$$0 \subset \pi^*(\operatorname{Ker} r^*) \subset \operatorname{Im}\pi^* \subset H^*(X_{\mathrm{res}}).$$

The quantum products of elements in $\operatorname{Im}\pi^*$ *are regular at* $q_2 = 1$*, and the map*

$$r^* \circ (\pi^*)^{-1}\colon\ \operatorname{Im}\pi^* \to H^*\bigl(\mathrm{Gr}(2,4)\bigr)$$

intertwines the quantum product $\star|_{q_2=1}$ *on* $\operatorname{Im}\pi^* = V$ *with the quantum product on* $H^*(\mathrm{Gr}(2,4))$ *under the identification* $q_1 = q$ *of the Novikov variables. This map also preserves the Poincaré pairing.*

REMARK 5.3

Since N is self-adjoining with respect to the Poincaré pairing, we have $(\operatorname{Ker} N)^\perp = \operatorname{Im} N$. Thus, the Poincaré pairing induces a nondegenerate pairing on $V/W = \operatorname{Ker} N/(\operatorname{Ker} N \cap (\operatorname{Ker} N)^\perp)$.

In the above theorem, we identified the subquotient $(V/W, \star|_{q_2=1})$ of $H^*(X_{\mathrm{res}})$ with a *subring* of the quantum cohomology of $\mathrm{Gr}(2,4)$. We can extend this isomorphism to the whole of $H^*(\mathrm{Gr}(2,4))$ as follows. The weight filtration $W_{-2} \subset W_{-1} \subset W_0 \subset W_1 \subset W_2 = H^{\mathrm{even}}(X_{\mathrm{res}})$ associated to the nilpotent endomorphism N (see, e.g., [3, Section A.2]) is given as

$$W_{-2} = W_{-1} = \operatorname{Span}\{m_1^3 - 2m_1^2 m_2\},$$

$$W_0 = W_1 = \operatorname{Span}\{m_1^3 - 2m_1^2 m_2, m_1^2 - 2m_1 m_2, 1, m_1, m_1^2, m_1^3, m_1^4\}.$$

This is illustrated by the following table:

$$
\begin{array}{cccccccc}
 & & & & & m_1^3-2m_1^2m_2 & W_{-2}=W_{-1} & \\
\hline
1 & m_1 & m_1^2 & m_1^3 & m_1^4 & m_1^2-2m_1m_2 & & W_0=W_1 \\
\hline
 & & & & & m_1-2m_2 & & W_2
\end{array}
$$

Therefore, V/W can be regarded as a subspace of W_0/W_{-1}. We define a linear isomorphism $\theta\colon W_0/W_{-1} \cong H^*(\mathrm{Gr}(2,4))$ by

$$\theta(m_1^i) = d^i \quad \text{for } 0 \le i \le 4,$$

$$\theta(m_1^2 - 2m_1m_2) = \sqrt{-1}(d^2 - 2\delta).$$

This gives an extension of the map $r^* \circ (\pi^*)^{-1}\colon V/W \to H^*(\mathrm{Gr}(2,4))$. We have the following.

THEOREM 5.4

The quantum products of elements in W_0 are regular at $q_2 = 1$ and belong to W_0. The quantum product $\star|_{q_2=1}$ on W_0 descends to W_0/W_{-1} and θ induces an isomorphism of rings:

$$\theta\colon (W_0/W_{-1}, \star|_{q_2=1}) \cong \big(H^*\big(\mathrm{Gr}(2,4)\big), \star\big),$$

under the identification $q_1 = q$. Moreover, θ preserves the Poincaré pairing.

REMARK 5.5

It is curious that we have imaginary numbers in the isomorphism θ. The assignment $\theta\colon m_1^2 - 2m_1m_2 \mapsto \sqrt{-1}(d^2 - 2\delta)$ is uniquely determined up to sign if we require that θ coincides with $r^* \circ (\pi^*)^{-1}$ on V/W and intertwines the quantum products.

6. Example: $\mathrm{Gr}(2,5)$

In this section we study an extremal transition of the 6-dimensional Fano variety $\mathrm{Gr}(2,5)$, the space of complex two-planes in $\mathbb{C}^5$. Unlike the previous two examples in Sections 4 and 5, the image of the Plücker embedding of $\mathrm{Gr}(2,5)$ is not a hypersurface nor a complete intersection. We use the toric degeneration of $\mathrm{Gr}(2,5)$ and its crepant resolution studied by Gonciulea and Lakshmibai [9] and Batyrev, Ciocan-Fontanine, Kim, and van Straten [1].

According to [9] and [1], the Grassmannian $\mathrm{Gr}(2,5)$ admits a flat degeneration to the Gorenstein toric variety X_{sing} defined by the following 6-dimensional fan. The primitive generators of the 1-dimensional cones are

$$
\begin{aligned}
&r_1 = (1,0,0,0,0,0), && r_2 = (-1,1,0,0,0,0), && r_3 = (-1,0,1,0,0,0),\\
&r_4 = (0,-1,0,1,0,0), && r_5 = (0,0,-1,1,0,0), && r_6 = (0,0,-1,0,1,0),\\
&r_7 = (0,0,0,-1,0,1), && r_8 = (0,0,0,0,-1,1), && r_9 = (0,0,0,0,0,-1).
\end{aligned}
$$

The top-dimensional cones are

$$\langle r_2, r_3, r_4, r_5, r_6, r_7, r_8, r_9\rangle, \qquad \langle r_1, r_2, r_3, r_4, r_5, r_6, r_7, r_8\rangle,$$

$\langle r_1, r_4, r_5, r_6, r_7, r_8, r_9\rangle$, $\langle r_1, r_2, r_5, r_6, r_7, r_8, r_9\rangle$, $\langle r_1, r_2, r_3, r_4, r_5, r_8, r_9\rangle$,

$\langle r_1, r_2, r_3, r_4, r_5, r_6, r_9\rangle$, $\langle r_1, r_3, r_4, r_7, r_8, r_9\rangle$, $\langle r_1, r_3, r_4, r_6, r_7, r_9\rangle$,

$\langle r_1, r_2, r_3, r_7, r_8, r_9\rangle$, $\langle r_1, r_2, r_3, r_6, r_7, r_9\rangle$.

To obtain a crepant small resolution of $X_{\rm sing}$, we divide nonsimplicial cones as follows:

- divide $\langle r_2, r_3, r_4, r_5, r_6, r_7, r_8, r_9\rangle$ into $\langle r_2, r_3, r_5, r_6, r_7, r_9\rangle$, $\langle r_2, r_3, r_5, r_7, r_8, r_9\rangle$, $\langle r_3, r_4, r_5, r_6, r_7, r_9\rangle$, $\langle r_3, r_4, r_5, r_7, r_8, r_9\rangle$;
- divide $\langle r_1, r_2, r_3, r_4, r_5, r_6, r_7, r_8\rangle$ into $\langle r_1, r_2, r_3, r_5, r_6, r_7\rangle$, $\langle r_1, r_2, r_3, r_5, r_7, r_8\rangle$, $\langle r_1, r_3, r_4, r_5, r_6, r_7\rangle$, $\langle r_1, r_3, r_4, r_5, r_7, r_8\rangle$;
- divide $\langle r_1, r_4, r_5, r_6, r_7, r_8, r_9\rangle$ into $\langle r_1, r_4, r_5, r_6, r_7, r_9\rangle$, $\langle r_1, r_4, r_5, r_7, r_8, r_9\rangle$;
- divide $\langle r_1, r_2, r_5, r_6, r_7, r_8, r_9\rangle$ into $\langle r_1, r_2, r_5, r_6, r_7, r_9\rangle$, $\langle r_1, r_2, r_5, r_7, r_8, r_9\rangle$;
- divide $\langle r_1, r_2, r_3, r_4, r_5, r_8, r_9\rangle$ into $\langle r_1, r_2, r_3, r_5, r_8, r_9\rangle$, $\langle r_1, r_3, r_4, r_5, r_8, r_9\rangle$;
- divide $\langle r_1, r_2, r_3, r_4, r_5, r_6, r_9\rangle$ into $\langle r_1, r_2, r_3, r_5, r_6, r_9\rangle$, $\langle r_1, r_3, r_4, r_5, r_6, r_9\rangle$.

These subdivisions define a smooth toric variety $X_{\rm res}$. In this section we study a relationship between the quantum cohomology of $\mathrm{Gr}(2,5)$ and $X_{\rm res}$.

6.1. Quantum cohomology of $\mathrm{Gr}(2,5)$

We refer the reader to [2] and [4] for the quantum cohomology of $\mathrm{Gr}(2,5)$. It is well known that the Poincaré duals of the Schubert cycles form an additive basis of the cohomology ring of $\mathrm{Gr}(2,5)$. Fix a full flag $0 \subset F_1 \subset F_2 \subset \cdots \subset F_5 = \mathbb{C}^5$. The Schubert cycle $\Omega_{(a_1,a_2)} \subset \mathrm{Gr}(2,5)$, indexed by a pair (a_1, a_2) of integers satisfying $3 \ge a_1 \ge a_2 \ge 0$, is given by

$$\Omega_{(a_1,a_2)} = \left\{ V \subset \mathbb{C}^5 : \dim V = 2, \dim(V \cap F_{4-a_1}) \ge 1, V \subset F_{5-a_2} \right\}. \tag{11}$$

We denote by $\omega_{(a_1,a_2)} \in H^{2(a_1+a_2)}(\mathrm{Gr}(2,5))$ the Poincaré dual of the Schubert cycle $\Omega_{(a_1,a_2)}$. The dual basis of $\{\omega_{(a_1,a_2)}\}$ is given by $\{\omega_{(3-a_2,3-a_1)}\}$. We choose the following additive basis of $H^*(\mathrm{Gr}(2,5))$:

$$\left\{\omega_{(0,0)}, \omega_{(1,0)}, \omega_{(1,1)}, \omega_{(2,0)}, \omega_{(2,1)}, \omega_{(3,0)}, \omega_{(3,1)}, \omega_{(2,2)}, \omega_{(3,2)}, \omega_{(3,3)}\right\}.$$

Let q be the Novikov variable dual to the ample class $\omega_{(1,0)} \in H^2(\mathrm{Gr}(2,5))$. We have $\deg q = 10$. The class $\omega_{(1,0)}$ generates the small quantum cohomology ring of $\mathrm{Gr}(2,5)$, and its quantum product is given by the following matrix:

$$\omega_{(1,0)} \star = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & q & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

6.2. Quantum cohomology of X_{res}

Let R_i denote the class of the toric divisor corresponding to the ray $\mathbb{R}_{\geq 0} r_i$. We choose a basis $\{m_1, m_2, m_3\}$ of $H^2(X_{\mathrm{res}})$ as $m_1 = R_1$, $m_2 = R_2$, $m_3 = R_6$. Then we have

$$R_1 = m_1, \qquad R_2 = m_2, \qquad R_3 = m_1 - m_2,$$
$$R_4 = m_2, \qquad R_5 = m_1 - m_2 - m_3, \qquad R_6 = m_3,$$
$$R_7 = m_1 - m_3, \qquad R_8 = m_3, \qquad R_9 = m_1.$$

The cohomology ring of X_{res} is given by

$$H^\star(X_{\mathrm{sm}}) = \mathbb{C}[m_1, m_2, m_3]/\big\langle m_2^2, m_3^2, m_1^2(m_1 - m_2)(m_1 - m_2 - m_3)(m_1 - m_3)\big\rangle.$$

The classes m_1, m_2, m_3 span the nef cone of X_{res}. Let $\{\beta_1, \beta_2, \beta_3\} \subset H_2(X_{\mathrm{res}})$ be the dual basis of $\{m_1, m_2, m_3\}$; they span the Mori cone of X_{res}. For $d = n_1\beta_1 + n_2\beta_2 + n_3\beta_3 \in H_2(X_{\mathrm{res}})$, we write $q^d = q_1^{n_1} q_2^{n_2} q_3^{n_3}$, where $q_i = q^{\beta_i}$. We have $\deg q_1 = 10$, $\deg q_2 = \deg q_3 = 0$. We choose the following basis for $H^*(X_{\mathrm{res}})$:

$$\left\{ \begin{matrix} 1, m_1, m_2, m_3, m_1^2, m_1 m_2, m_1 m_3, m_2 m_3, m_1^3, m_1^2 m_2, m_1^2 m_3, \\ m_1 m_2 m_3, m_1^4, m_1^3 m_2, m_1^3 m_3, m_1^2 m_2 m_3, m_1^5, m_1^4 m_2, m_1^4 m_3, m_1^6 \end{matrix} \right\}. \tag{12}$$

We use Givental's mirror theorem [7, Theorem 0.1] to calculate the quantum product (see the Appendix for the details).

The quantum products of m_1 with cohomology classes in the chosen basis (12) are as follows:

$$\begin{aligned}
m_1 \star m_1^4 &= m_1^5 + q_1(1 + q_2 + q_3),\\
m_1 \star m_1^3 m_2 &= m_1^4 m_2 + q_1 q_2,\\
m_1 \star m_1^3 m_3 &= m_1^4 m_3 + q_1 q_3,\\
m_1 \star m_1^5 &= m_1^6 + (2m_2 + 2m_3)q_1 + (m_3 + 2m_1 - 2m_2)q_1 q_2\\
&\quad + (m_2 + 2m_1 - 2m_3)q_1 q_3 + (m_1 - m_2 - m_3)q_1 q_2 q_3,\\
m_1 \star m_1^4 m_2 &= m_1^5 m_2 + m_2 q_1 + (m_1 - m_2 + m_3)q_1 q_2 + m_2 q_1 q_3\\
&\quad + (m_1 - m_2 - m_3)q_1 q_2 q_3,\\
m_1 \star m_1^4 m_3 &= m_1^5 m_3 + m_3 q_1 + (m_1 - m_3 + m_2)q_1 q_3 + m_3 q_1 q_2\\
&\quad + (m_1 - m_2 - m_3)q_1 q_2 q_3,\\
m_1 \star m_1^6 &= 5m_2 m_3 q_1 + (5m_1 m_3 - 5m_2 m_3)q_1 q_2 + (5m_1 m_2 - 5m_2 m_3)q_1 q_3,
\end{aligned}$$

and all the other quantum products coincide with the cup products.

The quantum products of m_2 with cohomology classes in the chosen basis (12) are as follows:

$$\begin{aligned}
m_2 \star m_2 &= (m_1 - m_2)(m_1 - m_2 - m_3)\frac{q_2}{1 - q_2},\\
m_2 \star m_1 m_2 &= m_1(m_1 - m_2)(m_1 - m_2 - m_3)\frac{q_2}{1 - q_2},
\end{aligned}$$

$$\begin{aligned}
m_2 \star m_2 m_3 &= m_3(m_1 - m_2)(m_1 - m_2 - m_3)\frac{q_2}{1-q_2} \\
&\quad - (m_1 - m_2)(m_1 - m_3)(m_1 - m_2 - m_3)\frac{q_2 q_3}{(1-q_2)(1-q_2-q_3)}, \\
m_2 \star m_1^2 m_2 &= m_1^2(m_1 - m_2)(m_1 - m_2 - m_3)\frac{q_2}{1-q_2}, \\
m_2 \star m_1 m_2 m_3 &= m_1 m_3(m_1 - m_2)(m_1 - m_2 - m_3)\frac{q_2}{1-q_2} \\
&\quad - m_1(m_1 - m_2)(m_1 - m_3)(m_1 - m_2 - m_3) \\
&\quad \times \frac{q_2 q_3}{(1-q_2)(1-q_2-q_3)}, \\
m_2 \star m_1^4 &= m_1^4 m_2 + q_1 q_2, \\
m_2 \star m_1^3 m_2 &= m_1^3(m_1 - m_2)(m_1 - m_2 - m_3)\frac{q_2}{1-q_2} + q_1 q_2, \\
m_2 \star m_1^2 m_2 m_3 &= m_1^2 m_3(m_1 - m_2)(m_1 - m_2 - m_3)\frac{q_2}{1-q_2}, \\
m_2 \star m_1^5 &= m_1^5 m_2 + (m_3 + 2m_1 - 2m_2)q_1 q_2 + (m_1 - m_2 - m_3)q_1 q_2 q_3, \\
m_2 \star m_1^4 m_2 &= (m_1 - m_2 + m_3)q_1 q_2 + (m_1 - m_2 - m_3)q_1 q_2 q_3, \\
m_2 \star m_1^4 m_3 &= m_1^4 m_2 m_3 + (m_1 - m_2 - m_3)q_1 q_2 q_3 + m_3 q_1 q_2, \\
m_2 \star m_1^6 &= (5m_1 m_3 - 5m_2 m_3)q_1 q_2.
\end{aligned}$$

All the other quantum products with m_2 are the same as the cup products.

The quantum products of m_3 with cohomology classes in the chosen basis (12) are as follows:

$$\begin{aligned}
m_3 \star m_3 &= (m_1 - m_3)(m_1 - m_2 - m_3)\frac{q_3}{1-q_3}, \\
m_3 \star m_1 m_3 &= m_1(m_1 - m_3)(m_1 - m_2 - m_3)\frac{q_3}{1-q_3}, \\
m_3 \star m_2 m_3 &= m_2(m_1 - m_3)(m_1 - m_2 - m_3)\frac{q_3}{1-q_3} \\
&\quad - (m_1 - m_2)(m_1 - m_3)(m_1 - m_2 - m_3)\frac{q_2 q_3}{(1-q_3)(1-q_2-q_3)}, \\
m_3 \star m_1^2 m_3 &= m_1^2(m_1 - m_3)(m_1 - m_2 - m_3)\frac{q_3}{1-q_3}, \\
m_3 \star m_1 m_2 m_3 &= m_1 m_2(m_1 - m_3)(m_1 - m_2 - m_3)\frac{q_3}{1-q_3} \\
&\quad - m_1(m_1 - m_2)(m_1 - m_3)(m_1 - m_2 - m_3) \\
&\quad \times \frac{q_2 q_3}{(1-q_3)(1-q_2-q_3)}, \\
m_3 \star m_1^4 &= m_1^4 m_3 + q_1 q_3,
\end{aligned}$$

$$m_3 \star m_1^3 m_3 = m_1^3(m_1 - m_3)(m_1 - m_2 - m_3)\frac{q_3}{1-q_3} + q_1 q_3,$$
$$m_3 \star m_1^2 m_2 m_3 = m_1^2 m_2 (m_1 - m_3)(m_1 - m_2 - m_3)\frac{q_3}{1-q_3},$$
$$m_3 \star m_1^5 = m_1^5 m_3 + (m_2 + 2m_1 - 2m_3)q_1 q_3 + (m_1 - m_2 - m_3)q_1 q_2 q_3,$$
$$m_3 \star m_1^4 m_2 = m_1^4 m_2 m_3 + m_2 q_1 q_3 + (m_1 - m_2 - m_3)q_1 q_2 q_3,$$
$$m_3 \star m_1^4 m_3 = (m_1 - m_3 + m_2)q_1 q_3 + (m_1 - m_2 - m_3)q_1 q_2 q_3,$$
$$m_3 \star m_1^6 = (5m_1 m_2 - 5m_2 m_3)q_1 q_3.$$

All the other quantum products with m_3 are the same as the cup products.

6.3. Comparison of quantum cohomology

The quantum product of m_2 has simple poles along $q_2 = 1$ and $q_2 + q_3 = 1$; the quantum product of m_3 has simple poles along $q_3 = 1$ and $q_2 + q_3 = 1$. We define

$$N_2 := \mathrm{Res}_{q_2=1}(m_2\star)\frac{dq_2}{q_2}\Big|_{(q_2,q_3)=(1,1)},$$
$$N_3 := \mathrm{Res}_{q_3=1}(m_3\star)\frac{dq_3}{q_3}\Big|_{(q_2,q_3)=(1,1)}.$$

These are nilpotent endomorphisms. Thus, the monodromy of the quantum connection around the normal crossing divisors $(q_2 = 1)$, $(q_3 = 1)$ is unipotent. As before, the endomorphisms N_2, N_3 define the filtration $0 \subset W \subset V \subset H^*(X_{\mathrm{res}})$ by

$$V := \mathrm{Ker}(N_2) \cap \mathrm{Ker}(N_3), \qquad W := V \cap \big(\mathrm{Im}(N_2) + \mathrm{Im}(N_3)\big). \tag{13}$$

We have $\dim V = 12$ and $\dim W = 2$. The basis of V is given by

$$1, m_1, m_1^2, m_1^3, m_1^4, m_1^5, m_1^6, \alpha, m_1\alpha, m_1^2\alpha, m_1^4 m_2, m_1^4 m_3,$$

where $\alpha := m_1 m_2 + m_1 m_3 - m_2 m_3$, and the basis of W is given by

$$m_1^4 m_2 - m_1^4 m_3, \qquad 2m_1^5 - 5m_1^4 m_2.$$

Define a linear map $\theta\colon V \to H^*(X_{\mathrm{sm}})$ as follows:

$$\begin{aligned} \theta(m_1^i) &= (\omega_{(1,0)})^i, \quad 0 \le i \le 6,\\ \theta(m_1^i\alpha) &= (\omega_{(1,0)})^i\omega_{(2,0)}, \quad 0 \le i \le 2,\\ \theta(m_1^4 m_2) &= 2\omega_{(3,2)},\\ \theta(m_1^4 m_3) &= 2\omega_{(3,2)}. \end{aligned} \tag{14}$$

We have $\mathrm{Ker}\,\theta = W$, and the map θ induces an isomorphism

$$\theta\colon V/W \cong H^*\big(\mathrm{Gr}(2,5)\big).$$

Note that the quantum product of m_1 is regular along $q_2 = q_3 = 1$. Since $(m_1\star)$ commutes with $(m_2\star)$ and $(m_3\star)$, it follows that $(m_1\star)|_{q_2=q_3=1}$ commutes with

N_2 and N_3; thus, $(m_1\star)|_{q_2=q_3=1}$ descends to the quotient space V/W and defines a ring structure on V/W. The following result follows by a direct computation.

THEOREM 6.1

The quantum product on $H^*(X_{\mathrm{res}})$ at $q_2=q_3=1$ descends to a well-defined product structure on V/W. The linear isomorphism $\theta\colon V/W \cong H^*(\mathrm{Gr}(2,5))$ intertwines the quantum product $\star|_{q_2=q_3=1}$ on V/W with the quantum product on $H^*(\mathrm{Gr}(2,5))$. Moreover, θ preserves the Poincaré pairing.

REMARK 6.2

When $a\neq 0$ and $b\neq 0$, the nilpotent operator $N=aN_2+bN_3$ defines a weight filtration $\{W_\bullet\}$ independent of (a,b). The Jordan normal form of N consists of 10 Jordan blocks of size 1 (one-by-one zero matrices) and 2 Jordan blocks of size 5. Therefore, W_0/W_{-1} gives a 12-dimensional space which is bigger than $H^*(X_{\mathrm{sm}})$. The above quotient V/W corresponds to Jordan blocks of size 1.

6.4. Topology of the extremal transition of $\mathrm{Gr}(2,5)$

We study a relationship between the map θ in Theorem 6.1 and the maps on cohomology induced by the natural maps $X_{\mathrm{res}}\to X_{\mathrm{sing}}\leftarrow X_{\mathrm{sm}}=\mathrm{Gr}(2,5)$. In this section, we prove the following.

THEOREM 6.3

Let $\pi\colon X_{\mathrm{res}}\to X_{\mathrm{sing}}$ and $r\colon X_{\mathrm{sm}}=\mathrm{Gr}(2,5)\to X_{\mathrm{sing}}$ denote the natural maps associated to the extremal transition of $\mathrm{Gr}(2,5)$. Let V, W be as given in (13). We have the following commutative diagram:

$$\begin{tikzcd}
H^*(X_{\mathrm{res}})\supset V \arrow[rr, "\theta"] & & H^*\big(\mathrm{Gr}(2,5)\big)\cong V/W \\
& H^*(X_{\mathrm{sing}}) \arrow[ul, "\pi^*"] \arrow[ur, "r^*"'] &
\end{tikzcd}$$

so that $\theta\circ\pi^*=r^*$, where θ is given in (14) and

(1) $\pi^*\colon H^*(X_{\mathrm{sing}})\to H^*(X_{\mathrm{res}})$ *is injective and the image is contained in* V*;*

(2) $r^*\colon H^*(X_{\mathrm{sing}})\to H^*(X_{\mathrm{sm}})=H^*(\mathrm{Gr}(2,5))$ *is neither injective nor surjective;*

(3) $W\subset \mathrm{Im}\,\pi^*\subsetneqq V$ *and* $\pi^*(\mathrm{Ker}\,r^*)=W$.

Let us describe a degeneration of $\mathrm{Gr}(2,5)$ to X_{sing}. By the Plücker embedding, $\mathrm{Gr}(2,5)$ can be realized as the codimension 3 subvariety $X_t\subset\mathbb{P}^9$ (with $t\neq 0$) cut out by the following five equations:

$$tZ_{12}Z_{34}-Z_{13}Z_{24}+Z_{14}Z_{23}=0,$$

$$tZ_{12}Z_{35}-Z_{13}Z_{25}+Z_{15}Z_{23}=0,$$

$$tZ_{12}Z_{45} - Z_{14}Z_{25} + Z_{15}Z_{24} = 0,$$
$$tZ_{13}Z_{45} - Z_{14}Z_{35} + Z_{15}Z_{34} = 0,$$
$$tZ_{23}Z_{45} - Z_{24}Z_{35} + Z_{25}Z_{34} = 0,$$

where $(Z_{12}, Z_{13}, Z_{14}, Z_{15}, Z_{23}, Z_{24}, Z_{25}, Z_{34}, Z_{35}, Z_{45})$ are homogeneous coordinates of $\mathbb{P}^9$. The central fiber X_0 gives the singular toric variety X_{sing}. Let $z_1, z_2, \dots, z_9$ denote the homogeneous coordinates of the toric variety X_{sing} corresponding to the toric divisors $R_1, \dots, R_9$. Let $L = \mathcal{O}(R_1)$ be the line bundle on X_{sing} corresponding to the Cartier toric divisor R_1. This line bundle L defines an embedding of X_{sing} into $\mathbb{P}^9$ via the following basis of $H^0(X_{\text{sing}}, L)$:

$$Z_{12} = z_1, \qquad Z_{13} = z_6 z_7, \qquad Z_{14} = z_4 z_5 z_6, \qquad Z_{15} = z_2 z_5 z_6,$$
$$Z_{23} = z_7 z_8, \qquad Z_{24} = z_4 z_5 z_8, \qquad Z_{25} = z_2 z_5 z_8, \qquad Z_{34} = z_3 z_4,$$
$$Z_{35} = z_2 z_3, \qquad Z_{45} = z_9.$$

The image of this embedding coincides with X_0.

We start with the computation of $H^*(X_{\text{sing}})$. For a subset $\{i_1, \dots, i_k\} \subset \{1, 2, \dots, 9\}$, we write

$$V(i_1, \dots, i_k) \subset X_{\text{sing}} \qquad \text{or} \qquad \widetilde{V}(i_1, \dots, i_k) \subset X_{\text{res}}$$

for the closed toric subvarieties associated with the cone $\langle r_{i_1}, r_{i_2}, \dots, r_{i_k} \rangle$. Let $E \subset X_{\text{res}}$ denote the exceptional set of the resolution $\pi \colon X_{\text{res}} \to X_{\text{sing}}$, and let $S \subset X_{\text{sing}}$ denote the singular locus. We have

$$S = S_1 \cup S_2, \qquad E = E_1 \cup E_2$$

with $S_1 = V(2,3,4,5)$, $S_2 = V(5,6,7,8)$, $E_1 = \widetilde{V}(3,5)$, $E_2 = \widetilde{V}(5,7)$, and

$$S_1 \cong S_2 \cong \mathbb{P}^3, \qquad S_1 \cap S_2 \cong \mathbb{P}^1,$$
$$E_1 \cong E_2 \cong \mathbb{P}^1 \times \mathrm{Bl}_{\mathbb{P}^1}(\mathbb{P}^3), \qquad E_1 \cap E_2 \cong \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1.$$

Here $\mathrm{Bl}_{\mathbb{P}^1}(\mathbb{P}^3)$ denotes the blowup of $\mathbb{P}^3$ along a line $\mathbb{P}^1$. The toric variety X_{sing} has transversely conifold $\{xy = zw\}$ singularities along the smooth locus of S. Since odd cohomology groups of $E_i, S_i, E_1 \cap E_2, S_1 \cap S_2$ vanish, the Mayer–Vietoris exact sequences give

$$0 \longrightarrow H^*(S) \longrightarrow H^*(S_1) \oplus H^*(S_2) \longrightarrow H^*(S_1 \cap S_2) \longrightarrow 0,$$
$$0 \longrightarrow H^*(E) \longrightarrow H^*(E_1) \oplus H^*(E_2) \longrightarrow H^*(E_1 \cap E_2) \longrightarrow 0,$$

and thus

$$H^*(S) = \mathbb{C}, 0, \mathbb{C}, 0, \mathbb{C}^2, 0, \mathbb{C}^2 \quad \text{for } * = 0,1,2,3,4,5,6,$$
$$H^*(E) = \mathbb{C}, 0, \mathbb{C}^3, 0, \mathbb{C}^5, 0, \mathbb{C}^5, 0, \mathbb{C}^2 \quad \text{for } * = 0,1,2,3,4,5,6,7,8.$$

LEMMA 6.4

The relative cohomology group of the pair (X_{res}, E) is given by the following table.

degree p	0	1	2	3	4	5	6	7	8	9	10	11	12
$H^p(X_{\mathrm{res}},E)$	0	0	0	0	0	$\mathbb{C}$	0	$\mathbb{C}$	$\mathbb{C}^2$	0	$\mathbb{C}^3$	0	$\mathbb{C}$

The relative cohomology $H^*(X_{\mathrm{sing}},S)\cong H^*(X_{\mathrm{res}},E)$ is given by the same table.

Proof
This follows from the relative cohomology exact sequence associated with the pair (X_{res},E). Since the odd cohomology groups of X_{res} and E vanish, we have the exact sequence

$$0\longrightarrow H^{2k}(X_{\mathrm{res}},E)\longrightarrow H^{2k}(X_{\mathrm{res}})\longrightarrow H^{2k}(E)\longrightarrow H^{2k+1}(X_{\mathrm{res}},E)\longrightarrow 0$$

for each integer k. It suffices to study the restriction map $H^{2k}(X_{\mathrm{res}})\to H^{2k}(E)$. Since the spaces X_{res}, E are toric, this can be done by standard methods. We find that

- $H^0(X_{\mathrm{res}})\to H^0(E)$, $H^2(X_{\mathrm{res}})\to H^2(E)$ are isomorphisms;
- $H^4(X_{\mathrm{res}})\to H^4(E)$, $H^6(X_{\mathrm{res}})\to H^6(E)$ are injective;
- $H^8(X_{\mathrm{res}})\to H^8(E)$ is surjective.

The conclusion follows. □

LEMMA 6.5
The cohomology group of X_{sing} is given by the following table.

degree p	0	1	2	3	4	5	6	7	8	9	10	11	12
$H^p(X_{\mathrm{sing}})$	$\mathbb{C}$	0	$\mathbb{C}$	0	$\mathbb{C}$	0	$\mathbb{C}$	0	$\mathbb{C}^2$	0	$\mathbb{C}^3$	0	$\mathbb{C}$

Moreover, the map $\pi^*\colon H^*(X_{\mathrm{sing}})\to H^*(X_{\mathrm{res}})$ is injective and $\operatorname{Im}\pi^*$ has the following basis:

$$1, m_1, m_1^2, m_1^3, m_1^4, m_1^2\alpha, m_1^5, m_1^4 m_2, m_1^4 m_3, m_1^6,$$

with $\alpha = m_1m_2+m_1m_3-m_2m_3$. In particular, we have $\operatorname{Im}\pi^*\subsetneq V$.

Proof
The relative cohomology exact sequence for the pair (X_{sing},S) and the previous Lemma 6.4 give $H^i(X_{\mathrm{sing}})\cong H^i(S)$ for $i=0,1,2,3$, the exact sequences

$$0\longrightarrow H^p(X_{\mathrm{sing}})\longrightarrow H^p(S)\longrightarrow H^{p+1}(X_{\mathrm{sing}},S)\longrightarrow H^{p+1}(X_{\mathrm{sing}})\longrightarrow 0$$

for $p=4,6$, and $H^k(X_{\mathrm{sing}},S)\cong H^k(X_{\mathrm{sing}})$ for $k=8,9,10,11,12$. To determine $H^p(X_{\mathrm{sing}})$ for $p=4,5,6,7$, we use the naturality of the long exact sequence. We have the commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & H^p(X_{\mathrm{sing}}) \arrow[r] \arrow[d] & H^p(S) \arrow[r] \arrow[d] & H^{p+1}(X_{\mathrm{sing}},S) \arrow[r] \arrow[d, equal] & H^{p+1}(X_{\mathrm{sing}}) \arrow[r] & 0 \\
0 \arrow[r] & H^p(X_{\mathrm{res}}) \arrow[r] & H^p(E) \arrow[r] & H^{p+1}(X_{\mathrm{res}},E) \arrow[rr] & & 0
\end{tikzcd}$$

for $p=4,6$, where the rows are exact and the columns are induced by $\pi\colon X_{\mathrm{res}}\to X_{\mathrm{sing}}$. For both $p=4$ and $p=6$, we can show that the images of the maps $H^p(X_{\mathrm{res}})\to H^p(E)$ and $H^p(S)\to H^p(E)$ together span $H^p(E)$, and thus, $H^p(S)\to H^{p+1}(X_{\mathrm{sing}},S)$ is surjective. The first statement follows.

To show the second statement, we note that the toric divisor R_1 is Cartier and ample on X_{sing}. Therefore, the class $m_1=R_1$ on X_{res} lies in the image of $\pi^*\colon H^2(X_{\mathrm{sing}})\to H^2(X_{\mathrm{res}})$. It follows that m_1^i is a generator of $\pi^*(H^{2i}(X_{\mathrm{sing}}))\cong\mathbb{C}$ for $i=1,2,3,6$. The image of the map $\pi^*\colon H^8(X_{\mathrm{sing}})\to H^8(X_{\mathrm{res}})$ can be computed via the commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & H^8(X_{\mathrm{sing}},S) \arrow[r] \arrow[d, equal] & H^8(X_{\mathrm{sing}}) \arrow[r] \arrow[d, "\pi^*"] & 0 & \\
0 \arrow[r] & H^8(X_{\mathrm{res}},E) \arrow[r] & H^8(X_{\mathrm{res}}) \arrow[r] & H^8(E) \arrow[r] & 0
\end{tikzcd}$$

Therefore, $\pi^*(H^8(X_{\mathrm{sing}}))$ equals the kernel of the restriction $H^8(X_{\mathrm{res}})\to H^8(E)$, and we can show that it is spanned by m_1^4 and $m_1^2\alpha$. By a similar argument, we find that $\pi^*\colon H^{10}(X_{\mathrm{sing}})\cong H^{10}(X_{\mathrm{res}})$. The conclusion follows. $\square$

Finally, we compute the map $r^*\colon H^*(X_{\mathrm{sing}})\to H^*(X_{\mathrm{sm}})$.

LEMMA 6.6

The map $r^*\circ(\pi^*)^{-1}\colon \operatorname{Im}\pi^*\to H^*(X_{\mathrm{sm}})$ *sends the basis of* $\operatorname{Im}\pi^*$ *given in Lemma 6.5 as follows:*

$$\begin{aligned}
m_1^i &\longmapsto \omega_{(1,0)}^i, \quad 0\le i\le 6,\\
m_1^2\alpha &\longmapsto \omega_{(1,0)}^2\omega_{(2,0)},\\
m_1^4 m_2 &\longmapsto 2\omega_{(3,2)},\\
m_1^4 m_3 &\longmapsto 2\omega_{(3,2)}.
\end{aligned}$$

Proof

Abusing notation we write m_1 for the class of the Cartier toric divisor R_1 on X_{sing}, so that $\pi^*(m_1)=m_1$. Note that $m_1\in H^2(X_{\mathrm{sing}})$ or $\omega_{(1,0)}\in H^2(X_{\mathrm{sm}})$ is the restriction of the ample class $\mathcal{O}(1)$ on $\mathbb{P}^9$ to X_0 or to X_t (with $t\neq 0$), respectively. Therefore, r^* sends m_1 to $\omega_{(1,0)}$. The images of $m_1^4m_2, m_1^4m_3\in\pi^*(H^{10}(X_{\mathrm{sing}}))$ under $r^*\circ(\pi^*)^{-1}$ can be easily computed from the commutative diagram:

$$\begin{tikzcd}
H^{10}(X_{\mathrm{sing}}) \arrow[r, "\cup m_1"] \arrow[d, "r^*"] & H^{12}(X_{\mathrm{sing}}) \arrow[d, "r^*", "\cong"'] \\
H^{10}(X_{\mathrm{sm}}) \arrow[r, "\cup\omega_{(1,0)}", "\cong"'] & H^{12}(X_{\mathrm{sm}})
\end{tikzcd}$$

It remains to compute the image of $m_1^2\alpha \in \pi^*(H^8(X_{\mathrm{sing}}))$. By the commutative diagram

$$\begin{tikzcd} H^8(X_{\mathrm{sing}}) \arrow[r, "\cup m_1^2"] \arrow[d, "r^*"] & H^{12}(X_{\mathrm{sing}}) \arrow[d, "r^*", "\cong"'] \\ H^8(X_{\mathrm{sm}}) \arrow[r, "\cup \omega_{(1,0)}^2"] & H^{12}(X_{\mathrm{sm}}) \end{tikzcd}$$

it follows that the kernel of $m_1^2\colon H^8(X_{\mathrm{sing}}) \to H^{12}(X_{\mathrm{sing}})$ should be sent to the kernel of $\omega_{(1,0)}^2\colon H^8(X_{\mathrm{sm}}) \to H^{12}(X_{\mathrm{sm}})$ under r^*. Therefore, we have

$$r^* \circ (\pi^*)^{-1}(3m_1^2 - 5m_1^2\alpha) = a(\omega_{(3,1)} - \omega_{(2,2)})$$

for some $a \in \mathbb{C}$. This implies $r^* \circ (\pi^*)^{-1}(m_1^2\alpha) = \frac{9-a}{5}\omega_{(3,1)} + \frac{6+a}{5}\omega_{(2,2)}$. To determine a, we use the fact that $r_*[\Omega_{(1,1)}] = [V(2,9)]$, which is proved in Lemma 6.7 below. Since the map $\pi\colon \widetilde{V}(2,9) \to V(2,9)$ is birational, we have $\pi_*[\widetilde{V}(2,9)] = [V(2,9)]$. Thus,

$$\begin{aligned} m_1^2\alpha \cdot \big[\widetilde{V}(2,9)\big] &= (\pi^*)^{-1}(m_1^2\alpha) \cdot \big[V(2,9)\big] \\ &= \big(r^* \circ (\pi^*)^{-1}(m_1^2\alpha)\big) \cdot [\Omega_{(1,1)}] = \frac{6+a}{5}. \end{aligned}$$

On the other hand, $m_1^2\alpha \cdot [\widetilde{V}(2,9)] = m_1^2\alpha R_2 R_9 \cdot [X_{\mathrm{res}}] = 1$. Therefore, $a = -1$, and the conclusion follows. □

LEMMA 6.7

Consider the map $r_*\colon H_8(X_{\mathrm{sm}}) = H_8(\mathrm{Gr}(2,5)) \to H_8(X_{\mathrm{sing}})$ *between homology groups. We have* $r_*[\mathrm{Gr}(2,4)] = [V(2,9)]$, *where* $\mathrm{Gr}(2,4)$ *is identified with the Schubert cycle* $\Omega_{(1,1)}$ *in* $\mathrm{Gr}(2,5)$ *(see (11)).*

Proof

We consider the linear subspace

$$\mathbb{P}^5 = \{Z_{15} = Z_{25} = Z_{35} = Z_{45} = 0\} \subset \mathbb{P}^9$$

and restrict the family X_t to $\mathbb{P}^5$. Note that $X_t \cap \mathbb{P}^5$ is defined by the equation $tZ_{12}Z_{34} - Z_{13}Z_{24} + Z_{14}Z_{23} = 0$ in $\mathbb{P}^5$. For $t \neq 0$, $X_t \cap \mathbb{P}^5$ is identified with the image of $\Omega_{(1,1)} \cong \mathrm{Gr}(2,4)$ under the Plücker embedding. On the other hand, $X_0 \cap \mathbb{P}^5$ is identified with the toric subvariety $V(2,9)$ of $X_0 = X_{\mathrm{sing}}$. Since the family $t \mapsto X_t \cap \mathbb{P}^5$ gives a flat degeneration of $\mathrm{Gr}(2,4)$ to $V(2,9)$, the conclusion follows. □

Theorem 6.3 follows easily from the computations in Lemmas 6.4, 6.5, and 6.6.

7. Conjecture for partial flag varieties

In this section we formulate a conjecture which describes the change of quantum cohomology under the extremal transition (see [1]) of partial flag varieties. For a sequence of integers $0 < n_1 < n_2 < \cdots < n_l < n$, we consider the partial flag variety

$$\mathrm{Fl}(n_1, n_2, \ldots, n_l, n) = \{V_1 \subset V_2 \subset \cdots \subset V_l \subset \mathbb{C}^n : \dim V_i = n_i\}.$$

This space admits a flat degeneration to a Gorenstein toric Fano variety X_{sing}, and X_{sing} has a small crepant resolution X_{res}.

We recall the toric varieties X_{sing}, X_{res} from [1]. Let D, S be the following subsets of $\mathbb{Z}^2$:

$$D = \bigcup_{p=1}^{l} \big\{(i,j) \in \mathbb{Z}^2 : 0 \le i \le n - n_p - 1, 0 \le j \le n_p - 1\big\},$$

$$S = \big\{(n - n_1, 0), (n - n_2, n_1), \ldots, (n - n_l, n_{l-1}), (0, n_l)\big\}.$$

Elements of D are called *dots*, and elements of S are called *stars*. Elements of $D \cup S$ form vertices of the *ladder diagram* (see [1, Section 2]), which is an oriented graph. The set E of oriented edges of the ladder diagram consists of pairs $e = (t(e), h(e))$ with $t(e), h(e) \in D \cup S$ such that $h(e) - t(e) = (1,0)$ or $h(e) - t(e) = (0,-1)$, where $t(e)$ is the tail and $h(e)$ is the head. Consider the vector space $\mathbb{R}^D$ with the standard basis $\{\mathbf{e}_v : v \in D\}$. We set $\mathbf{e}_s = 0$ for $s \in S$. The fan Σ_{sing} of the toric variety X_{sing} is defined on $\mathbb{R}^D$; 1-dimensional cones of the fan are parameterized by E and their primitive generators are given by

$$r_e := \mathbf{e}_{h(e)} - \mathbf{e}_{t(e)}$$

for $e \in E$. The convex hull $\Delta \subset \mathbb{R}^D$ of the vectors r_e, $e \in E$, is a reflexive polytope (see [1]), and the fan Σ_{sing} is defined to be the set of cones over faces of Δ. The fan Σ_{res} of X_{res} is given by a simplicial subdivision of Σ_{sing}. For $1 \le i \le l$, a *roof* $\mathcal{R}_i$ is a collection of edges connecting the $(i+1)$th star $(n - n_{i+1}, n_i) \in S$ and the ith star $(n - n_i, n_{i-1}) \in S$ along the "boundary" of the ladder diagram (where we set $n_0 = 0$, $n_{l+1} = n$). More precisely,

$$\begin{aligned}
\mathcal{R}_i = &\big\{\big((n - n_{i+1}, n_i), (n - n_{i+1}, n_i - 1)\big)\big\} \\
&\cup \big\{\big((p, n_i - 1), (p+1, n_i - 1)\big) : n - n_{i+1} \le p \le n - n_i - 2\big\} \\
&\cup \big\{\big((n - n_i - 1, q), (n - n_i - 1, q - 1)\big) : n_{i-1} + 1 \le q \le n_i - 1\big\} \\
&\cup \big\{\big((n - n_i - 1, n_{i-1}), (n - n_i, n_{i-1})\big)\big\}.
\end{aligned}$$

A *box* of the ladder diagram is a subset of four vertices of the form

$$b = \big\{(i,j), (i+1,j), (i,j+1), (i+1,j+1)\big\} \subset D \cup S.$$

The corner $\mathcal{C}_b$ of b is the subset $\{((i,j+1),(i,j)), ((i,j),(i+1,j))\}$ of edges adjacent to the lower left vertex (i,j) of the box b. We write $\mathcal{C}_b^-$ for the upper right corner $\{((i,j+1),(i+1,j+1)), ((i+1,j+1),(i+1,j))\}$. Let Box denote the set of boxes of the ladder diagram. The fan Σ_{res} of X_{res} is a simplicial subdivision of Σ_{sing} such that $\mathcal{R}_1, \ldots, \mathcal{R}_l$ and $\mathcal{C}_b$ with $b \in \mathrm{Box}$ are primitive collections. Here we mean by a primitive collection a minimal subset P of E such that the cone spanned by $\{r_e : e \in P\}$ does not belong to the fan Σ_{res}. The corresponding toric variety X_{res} gives a small crepant resolution of X_{sing} (see [1, Section 3]). We write $\pi\colon X_{\mathrm{res}} \to X_{\mathrm{sing}}$ for the natural map.

The Mori cone of $X_{\mathrm{sm}} = \mathrm{Fl}(n_1, \dots, n_l, n)$ is a simplicial cone generated by Δ_i, where Δ_i is the class of a curve in the fiber of the natural map $\mathrm{Fl}(n_1, \dots, n_l, n) \to \mathrm{Fl}(n_1, \dots, \widehat{n_i}, \dots, n_l, n)$. We write $\overline{q}_i$ for the Novikov variable of X_{sm} corresponding to Δ_i for $1 \le i \le l$. The Mori cone of X_{res} is also a simplicial cone generated by the curve classes C_i with $1 \le i \le l$ and C_b with $b \in \mathrm{Box}$ (see [1, Section 3]), where C_i is defined by the "roof relation" $\sum_{e \in \mathcal{R}_i} r_e = 0$ and C_b is defined by the "box relation" $\sum_{e \in \mathcal{C}_b} r_e - \sum_{e \in \mathcal{C}_b^-} r_e = 0$. We write q_i, q_b for the Novikov variables corresponding to C_i, C_b, respectively. The morphism $\pi\colon X_{\mathrm{res}} \to X_{\mathrm{sing}}$ contracts the extremal rays $\mathbb{R}_{\ge 0} C_b$ with $b \in \mathrm{Box}$. We write ϕ_i with $1 \le i \le l$ and ϕ_b with $b \in \mathrm{Box}$ for the basis of $H^2(X_{\mathrm{res}})$ dual to C_i, C_b. We also write $\overline{\phi}_i$ with $1 \le i \le l$ for the basis of $H^2(X_{\mathrm{sm}})$ dual to Δ_i.

CONJECTURE 7.1

Let $X_{\mathrm{sm}} = \mathrm{Fl}(n_1, \dots, n_l, n)$, X_{sing}, X_{res} be as above.

(1) *The structure constants of the small quantum product of X_{res} are polynomials in $q_1, \dots, q_l$ with coefficients in rational functions of q_b, $b \in \mathrm{Box}$.*

(2) *The small quantum connection of X_{res} has logarithmic singularities along the normal crossing divisor $\prod_{b \in \mathrm{Box}} (q_b - 1) = 0$, and the residue endomorphisms along $q_b = 1$ (with $b \in \mathrm{Box}$) are nilpotent. More precisely, $(\phi_i \star)$ with $1 \le i \le l$ is regular along $\Delta_{\mathrm{exc}} := \{q_b = 1 \ (\forall b \in \mathrm{Box})\}$, $(\phi_b \star)$ with $b \in \mathrm{Box}$ has simple poles along $\{q_b = 1\}$ but no poles along $\{q_{b'} = 1\}$ for $b' \ne b$, and*

$$N_b := \mathrm{Res}_{q_b = 1} (\phi_b \star) \frac{dq_b}{q_b} \Big|_{\Delta_{\mathrm{exc}}}$$

is a nilpotent endomorphism which does not depend on $q_1, \dots, q_l$.

(3) *Define a filtration $0 \subset W \subset V \subset H^*(X_{\mathrm{res}})$ by $V = \bigcap_{b \in \mathrm{Box}} \mathrm{Ker}\, N_b$ and $W = V \cap \sum_{b \in \mathrm{Box}} \mathrm{Im}\, N_b$. Along the locus Δ_{exc}, the small quantum connection of X_{res} induces a residual flat connection on the bundle $(V/W) \times \Delta_{\mathrm{exc}} \to \Delta_{\mathrm{exc}}$. We have a linear map $\theta\colon V/W \to H^*(X_{\mathrm{sm}})$ which intertwines the residual flat connection with the small quantum connection of X_{sm} under the identification $q_i = \overline{q}_i$ of Novikov variables. More precisely, θ intertwines the action of $(\phi_j \star)|_{\Delta_{\mathrm{exc}}}$ on V/W with the action of $(\overline{\phi}_j \star)|_{\overline{q}_1 = q_1, \dots, \overline{q}_l = q_l}$ on $H^*(X_{\mathrm{sm}})$ for $1 \le j \le l$. Moreover, θ preserves the Poincaré pairing.*

(4) *Let $\pi\colon X_{\mathrm{res}} \to X_{\mathrm{sing}}$ denote the resolution, and let $r\colon X_{\mathrm{sm}} \to X_{\mathrm{sing}}$ denote the retraction. We have $\mathrm{Im}\, \pi^* \subset V$ and the following commutative diagram:*

$$\begin{tikzcd} H^*(X_{\mathrm{res}}) \supset V \arrow[r] & V/W \arrow[r, "\theta"] & H^*(X_{\mathrm{sm}}) \\ & H^*(X_{\mathrm{sing}}) \arrow[ul, "\pi^*"] \arrow[ur, "r^*"'] & \end{tikzcd}$$

REMARK 7.2
This conjecture is closely related to [1, Conjecture 4.1.2].

Appendix. Computing quantum cohomology of a toric variety

We explain how to compute the small quantum cohomology of a weak Fano toric manifold using Givental's mirror theorem [7, Theorem 0.1].

Let X_{res} be the toric variety in Section 6, which is a crepant resolution of a toric degeneration of $\mathrm{Gr}(2,5)$. The I-function of X_{res} is a cohomology-valued hypergeometric function given by

$$I(q,z) = e^{m \log q/z} \sum_{\beta \in H_2(X_{\mathrm{res}},\mathbb{Z})} q^\beta \prod_{i=1}^{9} \frac{\prod_{c=-\infty}^{0}(R_i + cz)}{\prod_{c=-\infty}^{R_i\cdot\beta}(R_i + cz)},$$

where we set $m\log q := \sum_{i=1}^3 m_i \log q_i$. In the case at hand, the mirror map is trivial and the mirror theorem of Givental [7, Theorem 0.1] says that $I(q,z)$ equals the J-function

$$J(q,z) = e^{m\log q/z}\Big(1 + \sum_{i=0}^{N}\sum_{\beta\neq 0}\langle \frac{\phi_i}{z(z-\psi)}\rangle_{0,1,\beta}\phi^i q^\beta\Big),$$

where $\{\phi_i\}_{i=0}^N$, $\{\phi^i\}_{i=0}^N$ are mutually dual bases of the cohomology as in Section 2. The class ψ is the first Chern class of the universal cotangent line bundle over $\overline{M}_{0,1}(X_{\mathrm{res}},\beta)$. More generally, the I-function and the J-function match under a change of coordinates (mirror map).

The method to determine the quantum product is as follows. We first find differential operators $\mathcal{D}_i(z\partial_1, z\partial_2, z\partial_3, z, q_1, q_2, q_3)$ which are polynomials in $z\partial_i := zq_i\frac{\partial}{\partial q_i}$ and z such that we have the asymptotics

$$\mathcal{D}_i I(q,z) = e^{m\log q/z}\big(\phi_i + O(z^{-1})\big), \quad 0 \le i \le N = 19.$$

Then the quantum product by m_j, $j=1,2,3$, is determined by the asymptotics

$$z\partial_j\big(\mathcal{D}_i I(q,z)\big) = e^{m\log q/z}\big(m_j \star \phi_i + O(z^{-1})\big).$$

In our case, for the choice of a basis in (12), we can take $\mathcal{D}_i$ as follows:

$$\mathcal{D}_0 = 1, \quad \mathcal{D}_1 = z\partial_1, \quad \mathcal{D}_2 = z\partial_2, \quad \mathcal{D}_3 = z\partial_3, \quad \mathcal{D}_4 = (z\partial_1)^2,$$

$$\mathcal{D}_5 = z\partial_1 z\partial_2, \quad \mathcal{D}_6 = z\partial_1 z\partial_3, \quad \mathcal{D}_7 = z\partial_2 z\partial_3, \quad \mathcal{D}_8 = (z\partial_1)^3,$$

$$\mathcal{D}_9 = (z\partial_1)^2 z\partial_2, \quad \mathcal{D}_{10} = (z\partial_1)^2 z\partial_3, \quad \mathcal{D}_{11} = z\partial_1 z\partial_2 z\partial_3, \quad \mathcal{D}_{12} = (z\partial_1)^4,$$

$$\mathcal{D}_{13} = (z\partial_1)^3 z\partial_2, \quad \mathcal{D}_{14} = (z\partial_1)^3 z\partial_3, \quad \mathcal{D}_{15} = (z\partial_1)^2 z\partial_2 z\partial_3,$$

$$\mathcal{D}_{16} = (z\partial_1)^5 - q_1(1+q_2+q_3), \quad \mathcal{D}_{17} = (z\partial_1)^4 z\partial_2 - q_1 q_2,$$

$$\mathcal{D}_{18} = (z\partial_1)^4 z\partial_3 - q_1 q_3,$$

$$\begin{aligned}\mathcal{D}_{19} = {} & (z\partial_1)^6 - zq_1(1+q_2+q_3) - q_1(1+3q_2+3q_3+q_2q_3)z\partial_1 \\ & - q_1(2+q_3)(1-q_2)z\partial_2 - q_1(2+q_2)(1-q_3)z\partial_3.\end{aligned}$$

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Iritani: Department of Mathematics, Graduate School of Science, Kyoto University, Kitashirakawa-Oiwake-cho, Sakyo-ku, Kyoto, Japan; iritani@math.kyoto-u.ac.jp

Xiao: Department of Mathematics, The Chinese University of Hong Kong, Shatin, New Territories, Hong Kong; jfxiao@math.cuhk.edu.hk