On formation of singularity of spherically symmetric nonbarotropic flows

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Abstract We study an initial boundary value problem on a ball for the heat-conductive system of compressible Navier–Stokes–Fourier equations, in particular, a criterion for the breakdown of the classical solution. For smooth initial data away from vacuum, we prove that the classical solution which is spherically symmetric loses its regularity in a finite time if and only if the density *concentrates* or *vanishes* or the velocity becomes unbounded around the center. One possible situation is that a vacuum ball appears around the center and the density may concentrate on the boundary of the vacuum ball simultaneously.

1. Introduction and main results

We are concerned with the heat-conductive system of compressible Navier–Stokes–Fourier equations, which reads as

$$\begin{cases} \rho_t + \nabla\cdot(\rho U) = 0, \\ (\rho U)_t + \nabla\cdot(\rho U\otimes U) + \nabla P = \mu\triangle U + (\mu+\lambda)\nabla(\nabla\cdot U), \\ c_V\big((\rho\theta)_t + \nabla\cdot(\rho\theta U)\big) + P\nabla\cdot U = \kappa\triangle\theta + \Psi[\nabla U], \end{cases} \tag{1.1}$$

where

$$\Psi[\nabla U] = 2\mu\big(\mathcal{D}(U)\big)^2 + \lambda(\nabla\cdot U)^2, \quad \mathcal{D}(U) = \frac{\nabla U + \nabla U^t}{2}, \tag{1.2}$$

$t\ge 0$, $x\in\Omega\subset\mathbb{R}^N$ $(N=2,3)$, and where $\rho=\rho(t,x)$, $U=U(t,x)$, and $\theta=\theta(t,x)$ are the density, fluid velocity, and temperature, respectively. Additionally, $P=P(\rho,\theta)$ is the pressure given by a state equation

$$P(\rho) = R\rho\theta. \tag{1.3}$$

The shear viscosity μ, the bulk one λ, and the heat conductivity κ are constants satisfying the physical hypothesis

$$\mu,\kappa>0, \quad \mu+\frac{N}{2}\lambda\ge 0. \tag{1.4}$$

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The domain Ω is a bounded ball with a radius b, namely,

$$\Omega = B_b = \{x \in \mathbb{R}^N; |x| \le b < \infty\}. \tag{1.5}$$

We study an initial boundary value problem for (1.1) with the initial condition

$$(\rho, U, \theta)(0, x) = (\rho_0, U_0, \theta_0)(x), \quad x \in \Omega, \tag{1.6}$$

and the boundary condition

$$U = 0, \qquad \frac{\partial \theta}{\partial n} = 0, \qquad x \in \partial\Omega, \qquad \vec{n} \text{ is an outnormal vector.} \tag{1.7}$$

We are looking for the smooth spherically symmetric solution (ρ, U) of the problem (1.1), (1.5), and (1.6), which takes the form

$$\rho(t,x) = \rho\big(t,|x|\big), \qquad U(t,x) = u\big(t,|x|\big)\frac{x}{|x|}, \qquad \theta(t,x) = \theta\big(t,|x|\big). \tag{1.8}$$

Then, for the initial data to be consistent with the form (1.7), we assume that the initial data (ρ_0, U_0) also takes the form

$$\rho_0 = \rho_0\big(|x|\big), \qquad U_0 = u_0\big(|x|\big)\frac{x}{|x|}, \qquad \theta_0 = \theta_0\big(|x|\big). \tag{1.9}$$

In this paper, we further assume that the initial density is uniformly positive, that is,

$$\rho_0 = \rho_0\big(|x|\big) \ge \underline{\rho} > 0, \quad x \in \Omega, \tag{1.10}$$

for a positive constant $\underline{\rho}$.

Here, it is noted that, since the assumption (1.7) implies that

$$U(t,x) + U(t,-x) = 0, \quad x \in \Omega, \tag{1.11}$$

we necessarily have $U(t,0) = 0$ (also $U_0(0) = 0$) as long as classical solutions are concerned.

There are many results about the existence of local and global strong solutions in time of the isentropic system of compressible Navier–Stokes equations when the initial density is uniformly positive (see [1], [13], [24], [14], [19]–[21], [25], [26] and their generalization in [15]–[17], [23] to the full system including the law of conservation of energy). On the other hand, for the initial density allowing vacuum, the local well-posedness of strong solutions of the isentropic and heat-conductive system was established by Cho and Kim [3], [4]. For strong solutions with spatial symmetries, the authors in [5] proved the global existence of radially symmetric strong solutions of the isentropic system in an annular domain, even allowing vacuum initially.

However, it still remains open whether there exist global strong solutions which are spherically symmetric in a ball. The main difficulties lie on the lack of estimates of the density and velocity near the center. In the case in which a vacuum appears, it is worth noting that Xin [27] established a blow-up result which shows that if the initial density has a compact support, then any smooth solution to the Cauchy problem of the full system of compressible Navier–Stokes equations without heat conduction blows up in a finite time (see more generalizations in [2], [28]).The same blow-up phenomenon occurs also for the isentropic system. Indeed, Zhang and Fang [29, Theorem 1.8] showed that if $(\rho, U) \in C^1([0,T]; H^k)$ $(k>3)$ is a spherically symmetric solution to the Cauchy problem with the compact supported initial density, then the upper limit of T must be finite. To deal with large discontinuous data, Hoff and Jenssen [7] established global weak solutions of the symmetric compressible heat-conductive flows. On the other hand, it is unclear whether the strong (classical) solutions lose their regularity in a finite time when the initial density is uniformly away from vacuum. Therefore, it is important to study the mechanism of the possible blowup of smooth solutions, which is a main issue in this paper.

In the spherical coordinates, the original system (1.1) under the assumption (1.8) takes the form

$$\begin{cases} \rho_t + (\rho u)_\xi = 0, \\ (\rho u)_t + (\rho u^2)_\xi + P_r = \nu u_{\xi r}, \\ c_V((\rho\theta)_t + (\rho u \theta)_\xi) + P u_\xi = \kappa \theta_{r\xi} + \nu (u_\xi)^2 - \frac{2(N-1)\mu}{r^{N-1}}(r^{N-2}u^2)_r, \end{cases} \tag{1.12}$$

where

$$\nu = 2\mu + \lambda, \qquad \frac{\partial}{\partial_\xi} = \frac{\partial}{\partial r} + \frac{N-1}{r}. \tag{1.13}$$

Without loss of generality, we assume that $c_V = 1$ and $N = 3$.

Now, we consider the following Lagrangian transformation:

$$t = t, \qquad h = \int_0^r \rho(t,s) s^2\, ds, \qquad \eta = (\rho r^2)^{-1}. \tag{1.14}$$

Then, it follows from (1.12) that

$$h_t = -\frac{u}{\eta}, \qquad r_t = u, \qquad r_h = \eta, \tag{1.15}$$

and the system (1.12) can be further reduced to

$$\begin{cases} (r^2\eta)_t = (r^2 u)_h \quad (\Longleftrightarrow \eta_t = u_h), \\ u_t = r^2(-R\frac{\theta}{r^2\eta} + \nu(\frac{u_h}{r_h} + \frac{2}{r}u)_h) \\ (u_t + Rr^2(\frac{\theta}{r^2\eta})_h = \nu r^2 (\frac{(r^2u)_h}{r^2\eta})_h), \\ \theta_t = -R\theta\frac{(r^2u)_h}{r^2\eta} + \nu r^2\eta(\frac{u_h}{r_h} + \frac{2u}{r})^2 - 4\mu(ru^2)_h + \kappa(\frac{r^2\theta_h}{r_h})_h \\ \quad = -R\theta\frac{(r^2u)_h}{r^2\eta} + \lambda r^2\eta(\frac{u_h}{r_h} + \frac{2}{r}u)^2 + 2\mu r^2\eta(\frac{u_h^2}{r_h^2} + \frac{2u^2}{r^2}) + \kappa(\frac{r^2\theta_h}{r_h})_h. \end{cases} \tag{1.16}$$

The initial boundary value problem for system (1.16) is

$$\begin{gathered} (u,\eta,\theta)(0,h) = (u_0,\eta_0,\theta_0) \quad (\eta_0 > 0, \theta_0 > 0), \\ u(t,0) = u(t,1) = 0, \qquad \theta_h(t,0) = \theta_h(t,1) = 0, \end{gathered} \tag{1.17}$$

where $t \geq 0$, $h \in [0,1]$, and

$$1 = \int_0^b \rho_0(r) r^2\, dr = \int_0^b \rho(t,r) r^2\, dr, \tag{1.18}$$

according to the conservation of mass. Note that

$$r(t,0)=0, \qquad r(t,1)=b. \tag{1.19}$$

We denote E_0 as the initial energy

$$E_0=\int_0^1\Big\{\frac{u_0^2}{2}+R(r_0^2\eta_0-\log r_0^2\eta_0-1)+(\theta_0-\log\theta_0-1)\Big\}\,dh. \tag{1.20}$$

Our main result is stated as follows.

THEOREM 1.1

Assume that the initial data (ρ_0,U_0,θ_0) satisfy (1.8), (1.9), (1.10), and

$$(\rho_0,U_0,\theta_0)\in H^3(\Omega). \tag{1.21}$$

Let (ρ,U,θ) be a classical spherically symmetric solution to the initial boundary value problem (1.1), (1.5), (1.6), and (1.7) in $[0,T]\times\Omega$, and let T^* be the upper limit of T, that is, the maximal time of existence of the classical solution. If $T^*<\infty$, then we have

$$\lim\sup_{(t,|x|)\to(T^*,0)}\Big(\rho\big(t,|x|\big)+\frac{1}{\rho}\big(t,|x|\big)+|U|\big(t,|x|\big)\Big)=\infty. \tag{1.22}$$

REMARK 1.1

The local existence of a smooth solution with initial data as in Theorem 1.1 is classical and can be found, for example, in [4] and references therein. So the maximal time T^* is well defined.

REMARK 1.2

There are several results on the blow-up criterion for strong and classical solutions to the isentropic and heat-conductive system (1.1) (see [9], [10], [12], [8], [22], [6] and references therein). Especially, the authors in [8] established the following Serrin-type blow-up criterion:

$$\lim\sup_{T\nearrow T^*}\big(\|\rho\|_{L^\infty(0,T;L^\infty)}+\|U\|_{L^r(0,T;L^s)}\big)=\infty, \tag{1.23}$$

for any $r\in[2,\infty]$ and $s\in(3,\infty]$ satisfying

$$\frac{2}{r}+\frac{3}{s}\le 1. \tag{1.24}$$

Theorem 1.1 asserts that the formation of a singularity is only due to the concentration or cavitation of the density and velocity around the center. More precisely, the density anywhere away from the center is bounded up to the maximal time. Also recall that

$$U(t,0)=0, \quad \text{for all } t<T^*, \tag{1.25}$$

as far as the classical solution is concerned. It indicates the possible loss of regularity of velocity at the center.

REMARK 1.3

Theorem 1.1 may be viewed as an extension of recent work in [11] where the authors established a blow-up criterion for barotropic spherically symmetric Navier–Stokes equations.

2. Proof of Theorem 1.1

We only prove the case when $N=3$ since the case $N=2$ is even simpler. Throughout this section, we assume that (ρ, U, θ) is a classical spherically symmetric solution with the form (1.8) to the initial boundary value problem (1.1), (1.5), (1.6), and (1.7) in $[0,T]\times\Omega$, and the maximal time T^*, the upper limit of T, is finite. We denote by C generic positive constants only depending on the initial data and the maximal time T^*.

By simple calculations, we have the following estimates.

LEMMA 2.1

We have

$$\int_0^1 \eta\,dh = \int_0^1 \eta_0\,dh, \tag{2.1}$$

$$\int_0^1 r^2\eta\,dh = \int_0^1 (r^2\eta)_0\,dh = \frac{b^3}{3}, \tag{2.2}$$

$$\int_0^1 \Big(\frac{1}{2}u^2+\theta\Big)\,dh = \int_0^1 \Big(\frac{1}{2}u_0^2+\theta_0\Big)\,dh. \tag{2.3}$$

Also, we have the following basic energy equality.

LEMMA 2.2

We have

$$\mathcal{E}(t)+\int_0^t \mathcal{V}(\tau)\,d\tau = \mathcal{E}(0), \tag{2.4}$$

where

$$\begin{aligned}
\mathcal{E}(t) &= \int_0^1 \Big\{\frac{u^2}{2}+R(r^2\eta-\log r^2\eta-1)+(\theta-\log\theta-1)\Big\}\,dh,\\
\mathcal{V}(t) &= \int_0^1 \Big\{\frac{\lambda r^2\eta}{\theta}\Big(\frac{u_h}{r_h}+\frac{2}{r}u\Big)^2+2\frac{\mu r^2\eta}{\theta}\Big(\frac{u_h^2}{r_h^2}+\frac{2u^2}{r^2}\Big)+\kappa\frac{r^2\theta_h^2}{r_h\theta^2}\Big\}\,dh\\
&= \int_0^1 \Big\{\Big(\lambda+\frac{2}{3}\mu\Big)\frac{r^2\eta}{\theta}\Big(\frac{u_h}{r_h}+\frac{2}{r}u\Big)^2+\frac{\mu r^2\eta}{\theta}\Big(\frac{u_h}{r_h}-\frac{u}{r}\Big)^2+\kappa\frac{r^2\theta_h^2}{r_h\theta^2}\Big\}\,dh,
\end{aligned} \tag{2.5}$$

or equivalently,

$$\int_\Omega \rho S\,dx+\int_0^t\int_\Omega\Big(\frac{1}{\theta}\Psi[\nabla U]+\kappa\frac{|\nabla\theta|^2}{\theta^2}\Big)\,dx\,ds = \int_\Omega \rho_0 S_0\,dx, \tag{2.6}$$

where

$$S = R\Phi(\rho^{-1}) + \Phi(\theta) + \frac{1}{2}|U|^2, \qquad \Phi(s) = s - \log s - 1. \tag{2.7}$$

In order to prove Theorem 1.1, we can show the following stronger characterization of the blow-up criterion, that is,

$$\lim \sup_{(t,h)\to(T^*-,0)} \Big(\rho(t,h) + \frac{1}{\rho}(t,h) + \|U\|_{L^2(t,T^*;L^\infty(B_h))}\Big) = \infty. \tag{2.8}$$

We argue by contradiction. For the original system (1.1), assume that there exist a small r_1, $\varepsilon > 0$ and a constant C such that

$$\rho(t,r) + \frac{1}{\rho}(t,r) + |U|(t,r) \le C, \quad \text{for } (t,r) \in (T^* - \varepsilon, T^*) \times [0, r_1]. \tag{2.9}$$

Through the Lagrangian transformation (1.14), one immediately concludes that, for system (1.16), there exists a small constant $h_1 > 0$ such that

$$\rho(t,h) + \frac{1}{\rho}(t,h) + |U|(t,h) \le C, \quad \text{for } (t,h) \in (T^* - \varepsilon, T^*) \times [0, h_1]. \tag{2.10}$$

Denote

$$h_0 = \frac{1}{2}h_1. \tag{2.11}$$

Recall blow-up criterion (1.23) by taking $r = \infty$ and $s = 2$; it amounts to proving the following proposition.

PROPOSITION 2.3

For $h_0 = \frac{1}{2}h_1$, there exists a constant C depending on h_0 such that

$$\rho(T,h) + \|U\|_{L^2(0,T;L^\infty(B_h^c))} \le C(h_0), \quad \textit{for } (T,h) \in (T^* - \varepsilon, T^*) \times [h_0, 1]. \tag{2.12}$$

To do that, we prepare the next lemma, which gives a relationship between r and y.

LEMMA 2.4

There exist a positive constant C independent of T and two strict increasing functions $\Theta_i : [0,1] \to [0,\infty)$ such that

$$r(t,h) \ge C\Theta_1(h) \tag{2.13}$$

and

$$b^3 - r(t,h)^3 \ge C\Theta_2(h), \tag{2.14}$$

for all $0 \le t < T^*$.

Proof

For $s \ge 0$, set

$$G(s) = s\log s - s + 1. \tag{2.15}$$

Obviously, G is a convex function in $(0,\infty)$. By Jensen's inequality, one has that

$$G\Big(\frac{\int_{B_r}\rho\,dx}{|B_r|}\Big)\le\frac{\int_{B_r}G(\rho)\,dx}{|B_r|}\Longleftrightarrow G\Big(\frac{C_0h}{r^3}\Big)\le\frac{C_1\int_\Omega G(\rho)\,dx}{r^3}. \tag{2.16}$$

Consequently, the uniform estimates for $r(t,h)$ follow immediately from entropy inequality (2.6) and (2.16). That is, given $h>0$ there is a strict increasing function $\Theta_1(h)$ such that

$$r(t,h)\ge\Theta_1(h),\qquad \Theta_1(0)=0,\qquad \text{and}\qquad \Theta_1(h)>0\quad \text{for } h>0. \tag{2.17}$$

By a similar step, one can obtain (2.14) by the following Jensen's inequality:

$$G\Big(\frac{\int_{B_r^c}\rho\,dx}{|B_r^c|}\Big)\le\frac{\int_{B_r^c}G(\rho)\,dx}{|B_r^c|}. \tag{2.18}$$

□

We are now in a position to establish the pointwise estimates of the density away from the center. To do that, we first write the density in the following form. One may refer to [18] for a similar representation in the one-dimensional case.

LEMMA 2.5

We have

$$r^2\eta(t,h)=\frac{1}{\mathcal{B}(t,h)\mathcal{Y}(t,h)}\Big((r^2\eta)_0(h)+\int_0^t\frac{R}{\nu}\mathcal{B}(\tau,h)\mathcal{Y}(\tau,h)\theta(\tau,h)\,d\tau\Big). \tag{2.19}$$

Here

$$\begin{aligned}\mathcal{B}(t,h)=\exp\frac{1}{\nu}\Big\{&\int_{h_0}^h\frac{u_0}{r_0^2}\,d\xi-\int_{h_0}^1\frac{u}{r^2}\,d\xi-\Big(\int_{h_0}^1 r^2\eta\,dh\Big)^{-1}\\&\times\Big(\int_{h_0}^1(r^2\eta)_0\int_{h_0}^h\frac{u_0}{r_0^2}\,d\xi\,dh-\int_{h_0}^1(r^2\eta)\int_{h_0}^h\frac{u}{r^2}\,d\xi\,dh\\&+\nu\int_{h_0}^1(r^2\eta)\,dh-\nu\int_{h_0}^1(r^2\eta)_0\,dh\Big)\Big\}\end{aligned} \tag{2.20}$$

and

$$\begin{aligned}\mathcal{Y}(t,h)=\exp\Big\{\frac{1}{\nu}\int_0^t\Big\{&-\int_{h_0}^h\frac{2u^2}{r^3}\,d\xi+\Big(\int_{h_0}^1r^2\eta\,dh\Big)^{-1}\\&\times\Big\{\int_{h_0}^1\Big(u^2+R\theta+r^2\eta\int_{h_0}^h\frac{2u^2}{r^3}\,d\xi\Big)\,dh\\&+\Big(\int_0^{h_0}(r^2u)_h\,dh\Big)\Big(\int_0^\tau\sigma(s,h_0)\,ds\Big)\Big\}\Big\}\,d\tau\Big\},\end{aligned} \tag{2.21}$$

where

$$\sigma(t,h)=-R\frac{\theta}{r^2\eta}+\nu\frac{(r^2\eta)_t}{r^2\eta}. \tag{2.22}$$

Proof
In view of (1.16), we have

$$\frac{1}{r^2}u_t = \Big(-R\frac{\theta}{r^2\eta} + \nu\frac{(r^2\eta)_t}{r^2\eta}\Big)_h \triangleq \big(\sigma(t,h)\big)_h. \tag{2.23}$$

Thus, for $h > h_0 > 0$, integrating (2.23) over (h_0, h), we deduce that

$$\Big(\int_{h_0}^h \frac{u}{r^2}\,d\xi\Big)_t + \int_{h_0}^h \frac{2u^2}{r^3}\,d\xi = \sigma(t,h) - \sigma(t,h_0). \tag{2.24}$$

Multiplying by $r^2\eta/\nu$ on both sides of (2.24) yields

$$\frac{r^2\eta}{\nu}\Big\{\Big(\int_{h_0}^h \frac{u}{r^2}\,d\xi\Big)_t + \int_{h_0}^h \frac{2u^2}{r^3}\,d\xi + \sigma(t,h_0)\Big\} = -\frac{R}{\nu}\theta + (r^2\eta)_t, \tag{2.25}$$

which is

$$(r^2\eta)_t - \frac{1}{\nu}\Big\{\Big(\int_{h_0}^h \frac{u}{r^2}\,d\xi\Big)_t + \int_{h_0}^h \frac{2u^2}{r^3}\,d\xi + \sigma(t,h_0)\Big\}(r^2\eta) = \frac{R}{\nu}\theta. \tag{2.26}$$

Denote

$$\mathcal{A}(t,h) = -\frac{1}{\nu}\Big\{\Big(\int_{h_0}^h \frac{u}{r^2}\,d\xi\Big)_t + \int_{h_0}^h \frac{2u^2}{r^3}\,d\xi + \sigma(t,h_0)\Big\}. \tag{2.27}$$

In view of (2.26) and (2.27), one has

$$(r^2\eta)(t,h) = \exp\Big(-\int_0^t \mathcal{A}\,d\tau\Big)\cdot\Big\{(r^2\eta)_0(h) + \int_0^t \exp\Big(\int_0^\tau \mathcal{A}\,ds\Big)\cdot\frac{R}{\nu}\theta\,d\tau\Big\}. \tag{2.28}$$

On the other hand, recalling (2.23) and (2.24) gives

$$\sigma(t,h_0) = -\Big(\int_{h_0}^h \frac{u}{r^2}\,d\xi\Big)_t - \int_{h_0}^h \frac{2u^2}{r^3}\,d\xi - R\frac{\theta}{r^2\eta} + \nu\frac{(r^2\eta)_t}{r^2\eta}. \tag{2.29}$$

Multiplying by $r^2\eta$ on both sides of (2.29) gives the first term as

$$\begin{aligned} -(r^2\eta)\Big(\int_{h_0}^h \frac{u}{r^2}\,d\xi\Big)_t &= -\Big\{(r^2\eta)\int_{h_0}^h \frac{u}{r^2}\,d\xi\Big\}_t + (r^2\eta)_t\int_{h_0}^h \frac{u}{r^2}\,d\xi \\ &= -\Big\{(r^2\eta)\int_{h_0}^h \frac{u}{r^2}\,d\xi\Big\}_t + \Big\{(r^2u)\int_{h_0}^h \frac{u}{r^2}\,d\xi\Big\}_h - (r^2u)\frac{u}{r^2}. \end{aligned} \tag{2.30}$$

Integrating (2.29) on $[h_0,1]\times[0,t]$ and taking into account the boundary conditions (1.17) yield

$$\begin{aligned} &\int_0^t\int_{h_0}^1 r^2\eta\sigma(\tau,h_0)\,dh\,d\tau \\ &= -\int_0^t\int_{h_0}^1\Big\{(r^2\eta)\int_{h_0}^h \frac{u}{r^2}\,d\xi\Big\}_t\,dh\,d\tau + \int_0^t\int_{h_0}^1\Big\{(r^2u)\int_{h_0}^h \frac{u}{r^2}\,d\xi\Big\}_h\,dh\,d\tau \\ &\quad - \int_0^t\int_{h_0}^1\Big\{u^2 + R\theta + (r^2\eta)\int_{h_0}^h \frac{2u^2}{r^3}\,d\xi\Big\}\,dh\,d\tau + \nu\int_0^t\int_{h_0}^1 (r^2u)_h\,dh\,d\tau. \end{aligned} \tag{2.31}$$

Therefore,

$$\int_0^t \int_{h_0}^1 r^2\eta\sigma(\tau,h_0)\,dh\,d\tau = \int_{h_0}^1 \Big((r^2\eta)_0 \cdot \int_{h_0}^h \frac{u_0}{r_0^2}\,d\xi\Big)\,dh - \int_{h_0}^1 \Big((r^2\eta)\cdot \int_{h_0}^h \frac{u}{r^2}\,d\xi\Big)\,dh$$
$$- \int_0^t \int_{h_0}^1 \Big\{u^2 + R\theta + (r^2\eta)\int_{h_0}^h \frac{2u^2}{r^3}\,d\xi\Big\}\,dh\,d\tau \tag{2.32}$$
$$+\nu \int_{h_0}^1 (r^2\eta)\,dh - \nu\int_{h_0}^1 (r^2\eta)_0\,dh.$$

In view of the boundary conditions and the first part of (1.16), the left-hand side of (2.32) can be written as

$$\int_0^t \Big(\int_{h_0}^1 r^2\eta\,dh\Big)\Big(\int_0^\tau \sigma(s,h_0)\,ds\Big)'\,d\tau$$
$$= \Big(\int_{h_0}^1 r^2\eta\,dh\Big)\Big(\int_0^t \sigma(\tau,h_0)\,d\tau\Big)$$
$$- \int_0^t \Big(\int_{h_0}^1 (r^2\eta)_t\,dh\Big)\Big(\int_0^\tau \sigma(s,h_0)\,ds\Big)\,d\tau \tag{2.33}$$
$$= \Big(\int_{h_0}^1 r^2\eta\,dh\Big)\Big(\int_0^t \sigma(\tau,h_0)\,d\tau\Big)$$
$$+ \int_0^t \Big(\int_0^{h_0} (r^2u)_h\,dh\Big)\Big(\int_0^\tau \sigma(s,h_0)\,ds\Big)\,d\tau.$$

Collecting (2.28)–(2.33), we complete the proof of Lemma 2.5. □

We immediately have the following corollary.

COROLLARY 2.6

Given $0 < h_0 < 1$, *for* $h_0 \le h \le 1$, *there exists a constant* C *depending on* h_0 *such that*

$$C^{-1}(h_0) \le \mathcal{B}(t,h), \mathcal{Y}(t,h) \le C(h_0) \tag{2.34}$$

and

$$C^{-1}(h_0)\exp\{C^{-1}(h_0)(t-\tau)\} \le \frac{\mathcal{Y}(t,h)}{\mathcal{Y}(\tau,h)}$$
$$\le C(h_0)\exp\{C(h_0)(t-\tau)\}, \quad 0 \le \tau < t. \tag{2.35}$$

Proof

First,

$$\int_{h_0}^1 r^2\eta\,dh = \int_{h_0}^1 \frac{1}{\rho}\,dh = \int_{r(h_0)}^b r^2\,dr = \frac{b^3 - r(h_0)^3}{3}. \tag{2.36}$$

In view of (2.14), one gets

$$0 < C\Theta_2(h_0) \le \int_{h_0}^1 r^2\eta\,dh \le b^3. \tag{2.37}$$

Similarly,

$$0 \le \int_{h_0}^1 \frac{u^2}{r^3}\,d\xi = \int_{r(h_0)}^b \frac{\rho u^2 r^2}{r^3}\,dr \le Cr(h_0)^{-3} \le C\Theta_1(h_0)^{-3}, \tag{2.38}$$

$$\begin{aligned}\Big|\int_{h_0}^1 \frac{u}{r^2}\,dh\Big| &\le \min_{h\ge h_0} r(t,h)^{-2}\int_{h_0}^1 |u|\,dh\\ &\le C\Theta_1(h_0)^{-2}\Big(\int_0^1 u^2\,dh\Big)^{1/2}(1-h_0)^{1/2} \le C\Theta_1(h_0)^{-2}.\end{aligned} \tag{2.39}$$

Also, one can verify that

$$\int_{h_0}^1 r^2\eta\int_{h_0}^h \frac{2u^2}{r^3}\,d\xi\,dh \tag{2.40}$$

is bounded.

Observe that

$$\begin{aligned}\int_{h_0}^1 r^2\eta\int_{h_0}^h \frac{2u^2}{r^3}\,d\xi\,dh &= \int_{h_0}^1\Big\{\Big(\frac{r^3}{3}\int_{h_0}^h \frac{2u^2}{r^3}\,d\xi\Big)_h - \frac{r^3}{3}\frac{2u^2}{r^3}\Big\}\,dh\\ &= \frac{b^3}{3}\int_{h_0}^1 \frac{2u^2}{r^3}\,d\xi - \int_{h_0}^1 \frac{2u^2}{3}\,dh\end{aligned} \tag{2.41}$$

is bounded from below and above by a constant $C(h_0)$.

To finish the proof, it suffices to bound

$$\int_0^t \sigma(\tau,h_0)\,d\tau. \tag{2.42}$$

Recalling that the right-hand side of (2.32) is bounded by some constant $C(h_0)$ and (2.33), one has

$$\begin{aligned}&\Big|\int_0^t \sigma(\tau,h_0)\,d\tau\Big|\\ &\le C(h_0) + C(h_0)\int_0^t\Big(\Big|\int_0^{h_0}(r^2u)_h\,dh\Big|\Big)\Big(\Big|\int_0^\tau \sigma(s,h_0)\,ds\Big|\Big)\,d\tau.\end{aligned} \tag{2.43}$$

On the other hand, with the help of (2.10), we obtain

$$\begin{aligned}\int_0^{h_0}(r^2u)_h\,dh &= \int_0^{h_0}\Big(r^3\frac{u}{r}\Big)_h\,dh\\ &= \int_0^{h_0} 3r\eta u\,dh + \int_0^{h_0} r^3\Big(\frac{u}{r}\Big)_h\,dh\\ &\le C\max_{0\le h\le h_0}|r^2\eta|^{1/2}\Big(\int_0^{h_0}\eta\,dh\Big)^{1/2}\Big(\int_0^{h_0}u^2\,dh\Big)^{1/2}\end{aligned}$$

$$
\begin{aligned}
(2.44)\qquad &+\Big(\int_0^{h_0}\frac{r^4}{\eta\theta}\Big(\frac{u}{r}\Big)_h^2\Big)^{1/2}\Big(\int_0^{h_0}r^2\eta\theta\,dh\Big)^{1/2}\\
&\le C(h_0)+C(h_0)\Big(\int_0^{h_0}\frac{r^4}{\eta\theta}\Big(\frac{u}{r}\Big)_h^2\Big)^{1/2}\\
&= C(h_0)+C(h_0)\Big(\int_0^{h_0}\frac{r^2\eta}{\theta}\Big(\frac{u_h}{r_h}-\frac{u}{r}\Big)^2\Big)^{1/2}\\
&\le C(h_0)+C(h_0)\mathcal{V}(t)^{1/2},
\end{aligned}
$$

where we used energy inequality, (2.1), (2.3), and (2.46) in Lemma 2.7.

Consequently, the desired bound for $\sigma(\tau,h_0)$ follows immediately from Gronwall's inequality, (2.43), and (2.44).

Thus finishes the proof of Corollary 2.6. □

Hence, substituting (2.20) and (2.21) into (2.19), we finally arrive at the following.

LEMMA 2.7

We have

$$C^{-1}(h_0)\le\rho(t,h)\le C(h_0),\quad 0<h_0\le h, \tag{2.45}$$

$$C^{-1}\le\int_0^1\theta\,dh\le C, \tag{2.46}$$

and

$$\int_0^t\max_{h\in[h_0,1]}u^2(\tau,h)\,d\tau\le C(h_0). \tag{2.47}$$

REMARK 2.1

Theorem 1.1 follows immediately from Lemma 2.7.

Proof

The right-hand side of (2.45) is a direct consequence of the fact that $\theta>0$, (2.19), and (2.34). It remains to show that the upper bound of $r^2\eta$ is $1/\rho$.

Step 1. Multiplying by $1/\theta$ on both sides of the third part of (1.16) yields

$$
\begin{aligned}
(\log\theta)_t&=-R\big(\log(r^2\eta)\big)_t+\Big(\lambda+\frac{2}{3}\mu\Big)\frac{r^2\eta}{\theta}\Big(\frac{u_h}{r_h}+\frac{2}{r}u\Big)^2+\frac{\mu r^2\eta}{\theta}\Big(\frac{u_h}{r_h}-\frac{u}{r}\Big)^2\\
&\quad+\kappa\Big(\frac{1}{\theta}\cdot\frac{r^2\theta_h}{r_h}\Big)_h+\kappa\frac{\theta_h}{\theta^2}\cdot\frac{r^2\theta_h}{r_h}.
\end{aligned}\tag{2.48}
$$

Integrating (2.48) over $[0,1]\times[0,t]$ and recalling (1.4) and (1.17) lead to

$$\Big\{\int_0^1\log\theta\,dh-\int_0^1\log\theta_0\,dh\Big\}\le -R\Big\{\int_0^1\log(r^2\eta)\,dh-\int_0^1\log(r^2\eta)_0\,dh\Big\}. \tag{2.49}$$

Applying Jensen's inequality to (2.49) and (2.2) gives

$$
\begin{aligned}
\log\int_0^1 \theta\, dh &\geq \int_0^1 \log\theta\, dh \\
&\geq \int_0^1 \log\theta_0\, dh - R\int_0^1 \log(r^2\eta)\, dh + R\int_0^1 \log(r^2\eta)_0\, dh \\
&\geq \int_0^1 \big\{\log\theta_0 + R\log(r^2\eta)_0\big\}\, dh - R\log\int_0^1 r^2\eta\, dh \\
&= C_1,
\end{aligned}
\tag{2.50}
$$

which gives the desired bound for $\int_0^1 \theta\, dh$.

Step 2. Apply the mean value theorem to the continuous function $\theta(t,h)$ to get

$$
\forall t>0, \quad \exists h(t)\in[h_0,1], \quad \text{such that} \quad \theta\big(t,h(t)\big) = \frac{\int_{h_0}^1 \theta(t,h)\, dh}{1-h_0}. \tag{2.51}
$$

Therefore, for $h \geq h_0$

$$
\begin{aligned}
\theta(t,h)^{1/2} &= \theta\big(t,h(t)\big)^{1/2} + \int_{h(t)}^h \frac{\theta_h}{2\theta(t,\xi)^{1/2}}\, d\xi \\
&\leq C(1-h_0)^{-1}\Big(\int_0^1 \theta\, dh\Big)^{1/2} + \Big(\int_{h_0}^1 \theta\, d\xi\Big)^{1/2}\Big(\int_{h_0}^1 \frac{\theta_h^2}{4\theta^2}\, d\xi\Big)^{1/2} \\
&\leq C\Big\{1+\max_{h\in[h_0,1]} r^2\eta(t,h)\int_{h_0}^1 \frac{\theta_h^2}{r^2\eta\theta^2}\, d\xi\Big\}^{1/2} \\
&\leq C(h_0)\big\{1+\max_{h\in[h_0,1]} r^2\eta(t,h)\cdot\mathcal{V}(t)\big\}^{1/2}.
\end{aligned}
\tag{2.52}
$$

Consequently,

$$
\theta(t,h) \leq C(h_0)\big\{1+\max_{h\in[h_0,1]} r^2\eta(t,h)\cdot\mathcal{V}(t)\big\}, \quad h_0\leq h\leq 1. \tag{2.53}
$$

Step 3. Observe that

$$
\begin{aligned}
r^2\eta(t,h) &= \frac{1}{\mathcal{B}(t,h)\mathcal{Y}(t,h)}\Big((r^2\eta)_0(h) + \int_0^t \frac{R}{\nu}\mathcal{B}(\tau,h)\mathcal{Y}(\tau,h)\theta(\tau,h)\, d\tau\Big) \\
&\leq \frac{1}{\mathcal{B}(t,h)\mathcal{Y}(t,h)}(r^2\eta)_0(h) \\
&\quad + C\int_0^t \frac{\mathcal{B}(\tau,h)}{\mathcal{B}(t,h)}\cdot\frac{\mathcal{Y}(\tau,h)}{\mathcal{Y}(t,h)}\big\{1+\max_{h\in[h_0,1]} r^2\eta(t,h)\cdot\mathcal{V}(t)\big\}.
\end{aligned}
\tag{2.54}
$$

Hence,

$$
\begin{aligned}
\max_{h\in[h_0,1]} r^2\eta(t,h) &\leq C + C\int_0^t \exp\big\{-\alpha(t-\tau)\big\}\big\{1+\max_{h\in[h_0,1]} r^2\eta(\tau,h)\cdot\mathcal{V}(\tau)\big\}\, d\tau \\
&\leq C + C\int_0^t \exp\big\{-\alpha(t-\tau)\big\}\max_{h\in[h_0,1]} r^2\eta(\tau,h)\cdot\mathcal{V}(\tau)\, d\tau.
\end{aligned}
\tag{2.55}
$$

Write

$$E(t) = \int_0^t \exp\{-\alpha(t-\tau)\} \max_{h\in[h_0,1]} r^2\eta(\tau,h)\cdot\mathcal{V}(\tau)\,d\tau. \tag{2.56}$$

One immediately has that

$$\begin{aligned} E_t &\le \max_{h\in[h_0,1]} r^2\eta(t,h)\cdot\mathcal{V}(t) \le (C+CE)\mathcal{V}(t) - \alpha E, \\ &E_t + \big(\alpha - C\mathcal{V}(t)\big)E \le C\mathcal{V}(t). \end{aligned} \tag{2.57}$$

Applying Gronwall's inequality to (2.57) yields

$$E \le C\exp\Big\{-\int_0^t \alpha - C\mathcal{V}(\tau)\,d\tau\Big\} \times C\int_0^t \exp\Big\{\int_0^\tau \alpha - C\mathcal{V}(s)\,ds\Big\}\mathcal{V}(\tau)\,d\tau. \tag{2.58}$$

The upper bound of $r^2\eta$ follows from (2.55) and (2.58).

Step 4. It suffices to establish a bound for the velocity. Indeed,

$$\begin{aligned} \Big(\frac{u}{r}\Big)^2(h) &\le \Big\{\int_{h_0}^h \Big|\Big(\frac{u}{r}\Big)_h\Big|\Big\}^2 \le \int_{h_0}^h \frac{r^4}{\theta r_h}\Big(\frac{u}{r}\Big)_h^2 dh \cdot \int_{h_0}^h \frac{\theta r_h}{r^4}\,dh \\ &\le C(h_0)\int_{h_0}^1 \frac{r^2\eta}{\theta}\Big(\frac{u_h}{r_h} - \frac{u}{r}\Big)^2 dh. \end{aligned} \tag{2.59}$$

That is,

$$\max_{h\in[h_0,1]} \Big(\frac{u}{r}\Big)^2(h) \le C(h_0)\mathcal{V}(t) \in L^1(0,T). \tag{2.60}$$

To conclude, (2.47) is a direct consequence of (2.60).

This finishes the proof of Lemma 2.7. □

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