Approximation by Walsh–Marcinkiewicz means on the Hardy space $H_{2/3}$

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Abstract The main aim of this paper is to find necessary and sufficient conditions for the convergence of Walsh–Marcinkiewicz means in terms of the modulus of continuity on the Hardy space $H_{2/3}$.

1. Introduction

The convergence almost everywhere of Walsh–Fejér means $\sigma_n f$ was proved by Fine [3]. Weak-type $(1,1)$-inequality for maximal operator σ^* can be found in Zygmund [34] for the trigonometric series, in Schipp [19] for Walsh series, and in Pál and Simon [18] for bounded Vilenkin series. Moreover, Fujji [5] and Simon [21] verified that σ^* is bounded from H_1 to L_1. Weisz [31] generalized this result and proved the boundedness of σ^* from the martingale space H_p to the space L_p for $p>1/2$. Simon [22] gave a counterexample, which shows that the boundedness does not hold for $0<p<1/2$. The counterexample for $p=1/2$ is due to Goginava [7] (see also [1], [2]). In [23] the second author proved that there exists a martingale $f\in H_{1/2}$ such that the Fejér means of f are not uniformly bounded in the space $L_{1/2}$.

In [8], [24], and [25] it was proven that the maximal operator $\widetilde{\sigma}_p^*$ defined by

$$\widetilde{\sigma}_p^* := \sup_{n\in\mathbb{N}} \frac{|\sigma_n|}{(n+1)^{1/p-2}\log^{2[1/2+p]}(n+1)},$$

where $0<p\le 1/2$ and $[1/2+p]$ denotes the integer part of $1/2+p$, is bounded from the Hardy space H_p to the space L_p. It was also proven that the rate of the weights $\{(n+1)^{1/p-2}\log^{2[1/2+p]}(n+1)\}_{n=1}^{\infty}$ in the nth Fejér mean is given exactly. For Walsh–Kaczmarz system analogical theorems are proven in [12] and [26].

Móricz and Siddiqi [14] investigated the approximation properties of some special Nörlund means of Walsh–Fourier series of L_p-functions in norm. Fridly, Manchanda, and Siddiqi [4] improved and extended the results of Móricz and

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Siddiqi [14], among them in H_p-norm, where $0<p<1$. The second author [27] and [28] gave a necessary and sufficient condition for the convergence of Fejér means in terms of modulus of continuity on the Hardy space H_p $(0<p\le 1/2)$. In [6] Goginava investigated the behavior of Cesàro means of Walsh–Fourier series in detail. For the two-dimensional case, approximation properties of Nörlund and Cesàro means were considered by Nagy (see [17], [15]).

For two-dimensional trigonometric Fourier partial sums $S_{j,j}(f)$ Marcinkiewicz [13] proved that the means $\mathcal{M}_n(f)$ of a function $f\in L\log L([0,2\pi]^2)$ converges almost everywhere to f as $n\to\infty$. For two-dimensional Walsh–Fourier series Weisz [33] proved that the maximal operator $\mathcal{M}^*(f)$ is bounded from the dyadic martingale Hardy space $H_p(G^2)$ to the space $L_p(G^2)$ for $p>2/3$. In the case $p=2/3$ Goginava [7] proved that $\mathcal{M}^*$ is not bounded from the Hardy space $H_{2/3}(G^2)$ to the space $L_{2/3}(G^2)$. By interpolation it follows that $\mathcal{M}^*$ is not bounded from the Hardy space $H_p(G^2)$ to the space weak-$L_p(G^2)$ for $0<p<2/3$.

That is, the end point of the boundedness of the maximal operator $\mathcal{M}^*$ is $p=2/3$. This means that it is interesting to discuss what does happen here. Goginava [9] also proved that $\mathcal{M}^*$ is bounded from the Hardy space $H_{2/3}(G^2)$ to the space weak-$L_{2/3}(G^2)$.

The first author [16] proved that the maximal operator $\widetilde{\mathcal{M}}^*$ defined by

$$\widetilde{\mathcal{M}}^*:=\sup_{n\in\mathbb{N}}\frac{|\mathcal{M}_n|}{\log^{3/2}(n+1)}$$

is bounded from the Hardy space $H_{2/3}(G^2)$ to the space $L_{2/3}(G^2)$. As a corollary we get

$$\|\mathcal{M}_n f\|_{2/3}\le c\log^{3/2}(n+1)\|f\|_{H_{2/3}}. \tag{1}$$

In [16] the first author also proved that the sequence $\{\log^{3/2}(n+1)\}_{n=1}^{\infty}$ is important for the maximal operator $\widetilde{\mathcal{M}}^*$. That is, the order of deviant behavior of the nth Marcinkiewicz means was given exactly.

Now, we continue our investigation at the end point $p=2/3$. The main aim of this paper is to find a necessary and sufficient condition for the convergence of Walsh–Marcinkiewicz means in terms of the modulus of continuity on the Hardy space $H_{2/3}(G^2)$.

2. Definitions and notation

Now, we give a brief introduction to the theory of dyadic analysis (see [20], [30]). Let $\mathbb{N}_+$ denote the set of positive integers $\mathbb{N}:=\mathbb{N}_+\cup\{0\}$. Let $\mathbb{Z}_2$ denote the discrete cyclic group of order 2, that is, $\mathbb{Z}_2=\{0,1\}$, where the group operation is modulo 2 addition and every subset is open. The Haar measure on $\mathbb{Z}_2$ is given such that the measure of a singleton is $1/2$. Let G be the complete direct product of the countable infinite copies of the compact groups $\mathbb{Z}_2$. The elements of G are of the form $x=(x_0,x_1,\dots,x_k,\dots)$ with coordinates $x_k\in\{0,1\}$ $(k\in\mathbb{N})$. The group operation on G is the coordinatewise addition, the measure (denoted by μ) is the product measure, and the topology is the product topology. The compact

Abelian group G is called the Walsh group. A base for the neighborhoods of G can be given in the following way:

$$\begin{aligned} I_0(x) &:= G, \\ I_n(x) &:= I_n(x_0, \dots, x_{n-1}) \\ &:= \bigl\{y \in G : y = (x_0, \dots, x_{n-1}, y_n, y_{n+1}, \dots)\bigr\} \end{aligned}$$

$(x \in G, n \in \mathbb{N})$. These sets are called dyadic intervals. Let $0 = (0 : i \in \mathbb{N}) \in G$ denote the null element of G, and let $I_n := I_n(0) (n \in \mathbb{N})$. Set

$$e_n := (0, \dots, 0, 1, 0, \dots) \in G,$$

the nth coordinate of which is 1 and the rest are zeros $(n \in \mathbb{N})$.

For $k \in \mathbb{N}$ and $x \in G$ let

$$r_k(x) := (-1)^{x_k}$$

denote the kth Rademacher function. If $n \in \mathbb{N}$, then $n = \sum_{i=0}^{\infty} n_i 2^i$ can be written, where $n_i \in \{0, 1\}$ $(i \in \mathbb{N})$, that is, n is expressed in the number system of base 2. Let $|n| := \max\{j \in \mathbb{N} : n_j \neq 0\}$, that is, $2^{|n|} \leq n < 2^{|n|+1}$.

The Walsh–Paley system is defined as the sequence of Walsh–Paley functions

$$w_n(x) := \prod_{k=0}^{\infty} \bigl(r_k(x)\bigr)^{n_k} = (-1)^{\sum_{k=0}^{|n|} n_k x_k} \quad (x \in G, n \in \mathbb{N}).$$

The Dirichlet kernels are defined as

$$D_n := \sum_{k=0}^{n-1} w_k, \qquad D_0 := 0.$$

The 2^nth Dirichlet kernels have the following form (see, e.g., [20]):

$$D_{2^n}(x) = \begin{cases} 2^n & \text{if } x \in I_n, \\ 0 & \text{if } x \notin I_n. \end{cases} \tag{2}$$

The norm (or quasinorm) of the space $L_p(G)$ is defined by

$$\|f\|_p := \Bigl(\int_G \bigl|f(x)\bigr|^p \, d\mu(x)\Bigr)^{1/p} \quad (0 < p < \infty).$$

The space weak-$L_p(G)$ consists of all measurable functions f for which

$$\|f\|_{\text{weak-}L_p} := \sup_{\lambda > 0} \lambda \mu(f > \lambda)^{1/p} < +\infty.$$

The σ-algebra generated by the dyadic intervals of measure 2^{-k} will be denoted by F_k $(k \in \mathbb{N})$. Denote by $f = (f^{(n)}, n \in \mathbb{N})$ a martingale with respect to $(F_n, n \in \mathbb{N})$ (for details see, e.g., [30]). The maximal function of a martingale f is defined by

$$f^* = \sup_{n \in \mathbb{N}} |f^{(n)}|.$$

In the case $f \in L_1(G)$, the maximal function can also be given by

$$f^*(x) = \sup_{n\in\mathbb{N}} \frac{1}{\mu(I_n(x))} \left| \int_{I_n(x)} f(u)\, d\mu(u) \right|, \quad x \in G.$$

For $0 < p < \infty$ the Hardy martingale space $H_p(G)$ consists of all martingales for which

$$\|f\|_{H_p} := \|f^*\|_p < \infty.$$

If $f \in L_1(G)$, then it is easy to show that the sequence $(S_{2^n} f : n \in \mathbb{N})$ is a martingale. If $f = (f^{(0)}, f^{(1)}, \dots)$ is a martingale, then the Walsh–Fourier coefficients are defined in the following way:

$$\widehat{f}(i) = \lim_{k\to\infty} \int_G f^{(k)}(x) w_i(x)\, d\mu(x).$$

The Walsh–Fourier coefficients of $f \in L_1(G)$ are the same as the ones of the martingale $(S_{2^n} f : n \in \mathbb{N})$ obtained from f.

The partial sums of the Walsh–Fourier series are defined as

$$S_m(f; x) := \sum_{i=0}^{m-1} \widehat{f}(i) w_i(x).$$

For $n = 1, 2, \dots$ and a martingale f the nth Fejér means and Fejér kernel of the Walsh–Fourier series of the function f are given by

$$\sigma_n(f; x) = \frac{1}{n} \sum_{j=0}^{n-1} S_j(f; x), \qquad K_n(x) := \frac{1}{n} \sum_{k=0}^{n-1} D_k(x).$$

The σ-algebra generated by the dyadic two-dimensional $(I_n(x^1) \times I_n(x^2))$-square of measure $2^{-n} \times 2^{-n}$ is denoted by $F_{n,n}$ $(n \in \mathbb{N})$. Denote by $f = (f_{n,n}, n \in \mathbb{N})$ the one-parameter martingale with respect to $F_{n,n}$ $(n \in \mathbb{N})$. The definitions of the spaces $L_p(G^2)$, weak-$L_p(G^2)$, and $H_p(G^2)$ are given analogously to those in the one-dimensional case.

The Kronecker product $(w_{n,m} : n, m \in \mathbb{N})$ of two Walsh system is said to be a two-dimensional Walsh system. Thus,

$$w_{n,m}(x^1, x^2) = w_n(x^1) w_m(x^2).$$

If $f \in L_1(G^2)$, then the numbers $\widehat{f}(n,m) = \int_{G^2} f w_{n,m}\, d\mu$ $(w_{n,m} : n, m \in \mathbb{N})$ is said to be the (n,m)th Walsh–Fourier coefficient of f. We can extend this definition to the martingales in the usual way. Denote by $S_{n,m}$ the (n,m)th rectangular partial sum of the Walsh–Fourier series of a martingale f. Namely,

$$S_{n,m}(f; x^1, x^2) := \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \widehat{f}(i,j) w_{i,j}(x^1, x^2).$$

A bounded measurable function a is a p-atom if there exists a dyadic two-dimensional cube I^2 such that

$$\int_{I^2} a\, d\mu = 0, \qquad \|a\|_\infty \le \mu(I^2)^{-1/p}, \qquad \operatorname{supp} a \subset I^2.$$

The dyadic Hardy martingale spaces $H_p\ (G^2)$ for $0<p\leq 1$ have an atomic characterization. Namely the following theorem is true (see [32]).

THEOREM W (WEISZ [32, THEOREM 1, P. 359])
A martingale $f=(f_{n,n},n\in\mathbb{N})$ is in $H_p(G^2)$ $(0<p\leq 1)$ if and only if there exists a sequence $(a_k,k\in\mathbb{N})$ of p-atoms and a sequence $(\mu_k,k\in\mathbb{N})$ of real numbers such that for every $n\in\mathbb{N}$,

$$\sum_{k=0}^{\infty}\mu_k S_{2^n,2^n}a_k=f_{n,n} \tag{3}$$

and

$$\sum_{k=0}^{\infty}|\mu_k|^p<\infty.$$

Moreover, $\|f\|_{H_p}\backsim\inf\left(\sum_{k=0}^{\infty}|\mu_k|^p\right)^{1/p}$, where the infimum is taken over all decompositions of f of the form (3).

The concept of modulus of continuity in $H_p(G^2)$ $(0<p\leq 1)$ is given by

$$\omega(1/2^n,f)_{H_p}:=\|f-S_{2^n,2^n}f\|_{H_p}.$$

The nth Marcinkiewicz–Fejér mean of a martingale f is defined by

$$\mathcal{M}_n(f;x^1,x^2):=\frac{1}{n}\sum_{k=0}^{n}S_{k,k}(f;x^1,x^2).$$

The two-dimensional Dirichlet kernels and Marcinkiewicz–Fejér kernels are defined by

$$D_{k,l}(x^1,x^2)=D_k(x^1)D_l(x^2),\qquad K_n(x^1,x^2):=\frac{1}{n}\sum_{k=0}^{n}D_{k,k}(x^1,x^2).$$

Let the maximal operators $\mathcal{M}^*$ and $\mathcal{M}^{\#}$ be given by

$$\mathcal{M}^*(f)=\sup_{n\geq 1}\left|\mathcal{M}_n(f)\right|,\qquad \mathcal{M}^{\#}(f)=\sup_{n\in\mathbb{N}}\left|\mathcal{M}_{2^n}(f)\right|.$$

For the maximal operator $\mathcal{M}^{\#}$ Goginava [10] proved that the following is true.

THEOREM G (GOGINAVA [10, THEOREMS 1, 2, P. 38])
The maximal operator $\mathcal{M}^{\#}$ is bounded from the Hardy space $H_{1/2}(G^2)$ to the space weak-$L_{1/2}(G^2)$ and is not bounded from the Hardy space $H_p(G^2)$ to the space $L_p(G^2)$ for $0<p\leq 1/2$.

For the martingale

$$f=\sum_{n=0}^{\infty}(f_n-f_{n-1})$$

the conjugate transforms are defined as

$$\widetilde{f^{(t)}}=\sum_{n=0}^{\infty} r_n(t)(f_n-f_{n-1}),$$

where $t\in G$ is fixed. Note that $\widetilde{f^{(0)}}=f$. It is well known (see [30]) that

$$\|\widetilde{f^{(t)}}\|_{H_p(G^2)}=\|f\|_{H_p(G^2)}, \qquad \|f\|^p_{H_p(G^2)}\sim\int_{[0,1)}\|\widetilde{f^{(t)}}\|_p^p\,dt. \tag{4}$$

3. Formulation of main results

THEOREM 1

(a) *Let*

$$\omega\Big(\frac{1}{2^k},f\Big)_{H_{2/3}}=o\Big(\frac{1}{k^{3/2}}\Big)\quad \textit{as } k\to\infty. \tag{5}$$

Then

$$\big\|\mathcal{M}_n(f)-f\big\|_{H_{2/3}}\to 0\quad \textit{when } n\to\infty.$$

(b) *There exists a martingale $f\in H_{2/3}$, for which*

$$\omega\Big(\frac{1}{2^{2^k}},f\Big)_{H_{2/3}}=O\Big(\frac{1}{2^{3k/2}}\Big)\quad \textit{as } k\to\infty$$

and

$$\big\|\mathcal{M}_n(f)-f\big\|_{2/3}\nrightarrow 0\quad \textit{as } n\to\infty.$$

During the proof of our main theorem we will use the following lemma of Goginava [11].

LEMMA 1 (GOGINAVA [11, LEMMA 4.2, P. 1954])

Let

$$x^1\in I_{4A}(0,\dots,0,x^1_{4m}=1,0,\dots,0,x^1_{4l}=1,x^1_{4l+1},\dots,x^1_{4A-1})$$

and

$$x^2\in I_{4A}(0,\dots,0,x^2_{4l}=1,x^1_{4l+1},\dots,x^1_{4q-1},1-x^1_{4q},x^2_{4q+1},\dots,x^2_{4A-1}).$$

Then

$$n_{A-1}\big|K_{n_{A-1}}(x^1,x^2)\big|\geq 2^{4q+4l+4m-3},$$

where $n_A=2^{4A}+2^{4A-4}+\dots+2^4+2^0$.

4. Proof of the theorem

Proof of Theorem 1

During the proof we follow the method of the second author in [28] and [29], but we have to make the necessary changes. Moreover, the proof is based on the result of the first author [16] discussing the properties of the maximal operator $\widetilde{\mathcal{M}}^*$. Combining (1) and (4) we have

$$
\begin{aligned}
\|\mathcal{M}_n f\|_{H_{2/3}}^{2/3} &= \int_{[0,1)} \big\|(\widetilde{\mathcal{M}_n f})^{(t)}\big\|_{2/3}^{2/3}\, dt = \int_{[0,1)} \|\mathcal{M}_n \widetilde{f^{(t)}}\|_{2/3}^{2/3}\, dt \\
&\le c\log(n+1) \int_{[0,1)} \|\widetilde{f^{(t)}}\|_{H_{2/3}}^{2/3}\, dt \\
&= c\log(n+1) \int_{[0,1)} \|f\|_{H_{2/3}}^{2/3}\, dt \\
&= c\log(n+1)\|f\|_{H_{2/3}}^{2/3}.
\end{aligned} \tag{6}
$$

Let $2^N < n \le 2^{N+1}$. The inequality (6) implies

$$
\begin{aligned}
\|\mathcal{M}_n f - f\|_{H_{2/3}}^{2/3} &\le \|\mathcal{M}_n f - \mathcal{M}_n S_{2^N,2^N} f\|_{H_{2/3}}^{2/3} \\
&\quad + \|\mathcal{M}_n S_{2^N,2^N} f - S_{2^N,2^N} f\|_{H_{2/3}}^{2/3} + \|S_{2^N,2^N} f - f\|_{H_{2/3}}^{2/3} \\
&= \big\|\mathcal{M}_n (S_{2^N,2^N} f - f)\big\|_{H_{2/3}}^{2/3} \\
&\quad + \|\mathcal{M}_n S_{2^N,2^N} f - S_{2^N,2^N} f\|_{H_{2/3}}^{2/3} + \|S_{2^N,2^N} f - f\|_{H_{2/3}}^{2/3} \\
&\le c\big(\log(n+1)+1\big)\omega^{2/3}\Big(\frac{1}{2^N}, f\Big)_{H_{2/3}} \\
&\quad + \|\mathcal{M}_n S_{2^N,2^N} f - S_{2^N,2^N} f\|_{H_{2/3}}^{2/3}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\mathcal{M}_n S_{2^N,2^N} f - S_{2^N,2^N} f \\
&\quad = \frac{1}{n}\sum_{k=0}^{2^N} S_{k,k} S_{2^N,2^N} f + \frac{1}{n}\sum_{k=2^N+1}^{n} S_{k,k} S_{2^N,2^N} f - S_{2^N,2^N} f \\
&\quad = \frac{1}{n}\sum_{k=0}^{2^N} S_{k,k} f + \frac{(n-2^N) S_{2^N,2^N} f}{n} - S_{2^N,2^N} f \\
&\quad = \frac{2^N}{n}(\mathcal{M}_{2^N} f - S_{2^N,2^N} f) \\
&\quad = \frac{2^N}{n}(S_{2^N,2^N}\mathcal{M}_{2^N} f - S_{2^N,2^N} f) \\
&\quad = \frac{2^N}{n} S_{2^N,2^N}(\mathcal{M}_{2^N} f - f).
\end{aligned}
$$

Combining (4) and Theorem G, and following the steps of estimation (6) we get

$$
\begin{aligned}
\|\mathcal{M}_n S_{2^N,2^N} f - S_{2^N,2^N} f\|_{H_{2/3}}^{2/3} &\le \Big(\frac{2^N}{n}\Big)^{2/3} \big\|S_{2^N,2^N}(\mathcal{M}_{2^N} f - f)\big\|_{H_{2/3}}^{2/3} \\
&\le \|\mathcal{M}_{2^N} f - f\|_{H_{2/3}}^{2/3} \to 0, \quad \text{while } n \to \infty.
\end{aligned} \tag{7}
$$

We immediately have that if

$$\omega\Big(\frac{1}{2^n},f\Big)_{H_{2/3}}=o\Big(\frac{1}{n^{3/2}}\Big),\quad \text{as } n\to\infty,$$

then

$$\|\mathcal{M}_n f-f\|_{H_{2/3}}\to 0,\quad \text{while } n\to\infty.$$

It completes the proof of the first part of our theorem.

Now, we prove the second part of Theorem 1. We set

$$a_i(x^1,x^2)=2^{2^i}\big(D_{2^{2^i+1}}(x^1)-D_{2^{2^i}}(x^1)\big)\big(D_{2^{2^i+1}}(x^2)-D_{2^{2^i}}(x^2)\big)$$

and

$$f_{A,A}(x^1,x^2)=\sum_{i=1}^{A}\frac{a_i(x^1,x^2)}{2^{3i/2}}.$$

Since

$$S_{2^A,2^A}a_k(x^1,x^2)=\begin{cases}a_k(x^1,x^2) & \text{if } 2^k\le A,\\ 0 & \text{if } 2^k>A,\end{cases}$$

and

$$\operatorname{supp} a_k=I^2_{2^k},$$
$$\int_{I^2_{2^k}}a_k\,d\mu=0,$$
$$\|a_k\|_\infty\le\mu(\operatorname{supp} a_k)^{-3/2},$$

by Theorem W we conclude that $f\in H_{2/3}$. We write that

$$\begin{aligned}
&f-S_{2^n,2^n}f\\
&=(f^{(1)}-S_{2^n,2^n}f^{(1)},\dots,f^{(n)}-S_{2^n,2^n}f^{(n)},\dots,f^{(n+k)}-S_{2^n,2^n}f^{(n+k)},\dots)\\
&=(0,\dots,0,f^{(n+1)}-f^{(n)},\dots,f^{(n+k)}-f^{(n)},\dots)\\
&=\Big(0,\dots,0,\dots,\sum_{i=\log n+1}^{\log n+k}\frac{a_i(x)}{2^{3i/2}},\dots\Big),\quad k\in\mathbb{N}_+.
\end{aligned}$$

Hence

$$\omega\Big(\frac{1}{2^n},f\Big)_{H_{2/3}}\le\sum_{i=[\log n]}^{\infty}\frac{1}{2^{3i/2}}=O\Big(\frac{1}{n^{3/2}}\Big),$$

where $[\log n]$ denotes the integer part of $\log n$.

Set $n_{2^{A-2}}=2^{4\cdot 2^{A-2}}+2^{4\cdot 2^{A-2}-4}+\dots+2^4+2^0=2^{2^A}+2^{2^A-4}+\dots+2^4+2^0$ as in Lemma 1:

$$\mathcal{M}_{n_{2^{k-2}}}(f)-f=\frac{2^{2^k}\mathcal{M}_{2^{2^k}}(f)}{n_{2^{k-2}}}+\frac{1}{n_{2^{k-2}}}\sum_{j=2^{2^k}+1}^{n_{2^{k-2}}}S_{j,j}(f)-\frac{2^{2^k}f}{n_{2^{k-2}}}-\frac{n_{2^{k-2}-1}f}{n_{2^{k-2}}}.\tag{8}$$

It is easy to show that

$$\widehat{f}(i,j)=\begin{cases}\frac{2^{2^k}}{2^{3k/2}} & \text{if } (i,j)\in\{2^{2^k},\dots,2^{2^k+1}-1\}^2, k=0,1,\dots,\\ 0 & \text{if } (i,j)\notin\bigcup_{k=0}^{\infty}\{2^{2^k},\dots,2^{2^k+1}-1\}^2.\end{cases}\tag{9}$$

Let $2^{2^k}<j\le n_{2^{k-2}}$. Since $w_{v+2^{2^k}}=w_{2^{2^k}}w_v$, when $v<2^{2^k}$ using (9) we have

$$\begin{aligned}
&S_{j,j}f(x^1,x^2)\\
&=S_{2^{2^k},2^{2^k}}f(x^1,x^2)+\sum_{v=2^{2^k}}^{j-1}\sum_{s=2^{2^k}}^{j-1}\widehat{f}(v,s)w_{v,s}(x^1,x^2)\\
&=S_{2^{2^k},2^{2^k}}f(x^1,x^2)+\frac{2^{2^k}}{2^{3k/2}}\sum_{v=0}^{j-2^{2^k}-1}\sum_{s=0}^{j-2^{2^k}-1}w_{v+2^{2^k}}(x^1)w_{s+2^{2^k}}(x^2)\\
&=S_{2^{2^k},2^{2^k}}f(x^1,x^2)+\frac{2^{2^k}w_{2^{2^k}}(x^1)w_{2^{2^k}}(x^2)}{2^{3k/2}}\sum_{v=0}^{j-2^{2^k}-1}\sum_{s=0}^{j-2^{2^k}-1}w_v(x^1)w_s(x^2)\\
&=S_{2^{2^k},2^{2^k}}f(x^1,x^2)+\frac{2^{2^k}w_{2^{2^k}}(x^1)w_{2^{2^k}}(x^2)D_{j-2^{2^k},j-2^{2^k}}(x^1,x^2)}{2^{3k/2}}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\frac{1}{n_{2^{k-2}}}\sum_{j=2^{2^k}+1}^{n_{2^{k-2}}}S_{j,j}f(x^1,x^2)\\
&=\frac{n_{2^{k-2}-1}S_{2^{2^k},2^{2^k}}f(x^1,x^2)}{n_{2^{k-2}}}+\frac{2^{2^k}w_{2^{2^k}}(x^1)w_{2^{2^k}}(x^2)}{n_{2^{k-2}}2^{3k/2}}\sum_{j=1}^{n_{2^{k-2}-1}}D_{j,j}(x^1,x^2)\\
&=\frac{n_{2^{k-2}-1}S_{2^{2^k},2^{2^k}}f(x^1,x^2)}{n_{2^{k-2}}}+\frac{2^{2^k}w_{2^{2^k}}(x^1)w_{2^{2^k}}(x^2)n_{2^{k-2}-1}K_{n_{2^{k-2}-1}}(x^1,x^2)}{n_{2^{k-1}}2^{3k/2}}.
\end{aligned}$$

Equality (8) yields

$$\begin{aligned}
\left\|\mathcal{M}_{n_{2^{k-2}}}(f)-f\right\|_{2/3}^{2/3}\ge{}&\frac{c}{2^k}\|n_{2^{k-2}-1}K_{n_{2^{k-2}-1}}\|_{2/3}^{2/3}\\
&-\left(\frac{2^{2^k}}{n_{2^{k-2}}}\right)^{2/3}\left\|\mathcal{M}_{2^{2^k}}(f)-f\right\|_{2/3}^{2/3}\\
&-\left(\frac{n_{2^{k-2}-1}}{n_{2^{k-2}}}\right)^{2/3}\|S_{2^{2^k},2^{2^k}}f-f\|_{2/3}^{2/3}.
\end{aligned}\tag{10}$$

Let

$$x^1\in I_{2^{k-2}}^{m,l}:=I_{2^{k-2}}(0,\dots,0,x_{4m}^1=1,0,\dots,0,x_{4l}^1=1,x_{4l+1}^1,\dots,x_{2^{k-2}-1}^1)$$

and

$$x^2\in J_{2^{k-2}}^{l,q}:=I_{2^{k-2}}(0,\dots,0,x_{4l}^2=1,x_{4l+1}^1,\dots,x_{4q-1}^1,1-x_{4q}^1,x_{4q+1}^2,\dots,x_{2^{k-2}-1}^2).$$

Applying Lemma 1 we have

$$n_{2^{k-2}-1}\left|K_{n_{2^{k-2}-1}}(x^1,x^2)\right| \geq 2^{4q+4l+4m-3}.$$

Hence, we can write that

$$\begin{aligned}
&\int_G \left(n_{2^{k-2}-1}\left|K_{n_{2^{k-2}-1}}(x^1,x^2)\right|\right)^{2/3} d\mu(x^1,x^2) \\
&\geq c \sum_{m=1}^{2^{k-2}-3} \sum_{l=m+1}^{2^{k-2}-2} \sum_{q=l+1}^{2^{k-2}-1} \sum_{x^1_{4l+1}=0}^{1} \cdots \sum_{x^1_{2^{k-2}-1}=0}^{1} \sum_{x^2_{4q+1}=0}^{1} \cdots \sum_{x^2_{2^{k-2}-1}=0}^{1} \\
&\quad \int_{I^{m,l}_{2^{k-2}} \times J^{l,q}_{2^{k-2}}} \left(n_{2^{k-2}-1}\left|K_{n_{2^{k-2}-1}}(x^1,x^2)\right|\right)^{2/3} d\mu(x^1,x^2) \\
&\geq c \sum_{m=1}^{2^{k-2}-3} \sum_{l=m+1}^{2^{k-2}-2} \sum_{q=l+1}^{2^{k-2}-1} \sum_{x^1_{4l+1}=0}^{1} \cdots \sum_{x^1_{2^{k-2}-1}=0}^{1} \sum_{x^2_{4q+1}=0}^{1} \cdots \sum_{x^2_{2^{k-2}-1}=0}^{1} \\
&\quad \mu(I^{m,l}_{2^{k-2}} \times J^{l,q}_{2^{k-2}}) 2^{(8q+8l+8m)/3} \\
&\geq c \sum_{m=1}^{2^{k-2}-3} \sum_{l=m+1}^{2^{k-2}-2} \sum_{q=l+1}^{2^{k-2}-1} 2^{(8q+8l+8m)/3} 2^{2^{k-2}-4l} 2^{2^{k-2}-4q} \left(\frac{1}{2^{2^{k-2}}}\right)^2 \\
&\geq c \sum_{m=1}^{2^{k-2}-3} 2^{8m/3} \sum_{l=m+1}^{2^{k-2}-2} 2^{-4l/3} \sum_{q=l+1}^{2^{k-2}-1} 2^{-4q/3} \geq c \sum_{m=1}^{2^{k-2}-3} 1 \geq c2^k.
\end{aligned}$$

Using (10) we have

$$\limsup_{k\to\infty} \left\|\mathcal{M}_{n_{2^{k-2}}}(f) - f\right\|_{2/3} \geq c > 0.$$

The proof of Theorem 1 is complete. □

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