Contact structures on plumbed 3-manifolds

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Abstract We show that the Ozsváth–Szabó contact invariant $c^+(\xi) \in HF^+(-Y)$ of a contact 3-manifold (Y, ξ) can be calculated combinatorially if Y is the boundary of a certain type of plumbing X and if ξ is induced by a Stein structure on X. Our technique uses an algorithm of Ozsváth and Szabó to determine the Heegaard–Floer homology of such 3-manifolds. We discuss two important applications of this technique in contact topology. First, we show that it simplifies the calculation of the Ozsváth–Stipsicz–Szabó obstruction to admitting a planar open book for a certain class of contact structures. We also define a numerical invariant of contact manifolds that respects a partial ordering induced by Stein cobordisms. Using this technique, we do a sample calculation showing that the invariant can get infinitely many distinct values.

1. Introduction

The last decade was the scene of many achievements in 3-dimensional contact topology. In his seminal work [11], Giroux established a one-to-one correspondence between contact structures and open book decompositions of a closed oriented 3-manifold. This allowed Ozsváth and Szabó [22] to find a Heegaard–Floer homology class that reflects certain properties of a given contact structure. In another direction, based on Giroux's work, Ozbagci and Etnyre [8] defined an invariant, the support genus, which is the minimal page genus of an open book decomposition compatible with a fixed contact structure. Previously, Etnyre [5] had proved that being supported by a genus zero open book puts some restrictions on intersection forms of symplectic fillings of a contact structure. His result was later improved by Ozsváth, Stipsicz, and Szabó, who showed that the image of the Ozsváth–Szabó contact invariant in the reduced version of Heegaard–Floer homology is actually an obstruction to being supported by a planar open book. More precisely, they proved the following.

THEOREM 1.1 ([17, THEOREM 1.2])

Suppose that the contact structure ξ on Y is compatible with a planar open book decomposition. Then its contact invariant $c^+(\xi) \in HF^+(-Y)$ is contained in $U^d \cdot HF^+(-Y)$ for all $d \in \mathbb{N}$.

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In spite of having useful corollaries, this theorem may not be easy to apply in all cases because one needs to calculate the group HF^+ and identify the contact invariant in this group. The former problem can be solved if we restrict our attention to a certain class of manifolds. In [19], Ozsváth and Szabó gave a purely combinatorial description of Heegaard–Floer homology groups HF^+ of some 3-manifolds which are given as the boundary of certain plumbings of disk bundles over 2-sphere. The present work is about pinning down the contact invariant within this combinatorial object.

To state our main results, we shall assume that G is a negative definite plumbing tree with at most one bad vertex (see Section 3 for the definition). Let $X(G)$ and $Y(G)$ be the 4- and 3-manifolds determined by the plumbing diagram, respectively. Denote by $\mathrm{Char}(G)$ the set of all characteristic covectors of the lattice $H^2(X(G),\mathbb{Z})$. We form the group $\mathbb{K}^+(G) = (\mathbb{Z}^{n\geq 0} \times \mathrm{Char}(G))/\sim$. The relation $\sim$ is defined in Section 3. Recall that the Heegaard–Floer homology group HF^+ of any 3-manifold is equipped with an endomorphism U. In [19] (see also (3.1) below), Ozsváth and Szabó established the following isomorphism:

$$\mathrm{Hom}\Big(\frac{\mathbb{K}^+(G)}{\mathbb{Z}^{>0} \times \mathrm{Char}(G)}, \mathbb{F}\Big) \simeq \mathrm{Ker}(U) \subset HF^+\big(-Y(G)\big). \tag{1.1}$$

Recall from [22] that if ξ is a contact structure, its Ozsváth–Szabó contact invariant $c^+(\xi)$ is a homogeneous element in $\mathrm{Ker}(U) \subset HF^+(-Y(G))$. It is also known that $c^+(\xi)$ is nonzero if ξ is induced by a Stein filling. The following proposition pins down the image of contact invariant under the above isomorphism. See Remark 4.1 for generalizations.

PROPOSITION 1.2

Let J be a Stein structure on $X(G)$, and let ξ be the induced contact structure on $Y(G)$. Under the identification described in (1.1), the contact invariant $c^+(\xi)$ is represented by the dual of the first Chern class $c_1(J) \in H^2(X,\mathbb{Z})$.

When combined with Theorem 1.1, this proposition allows us to determine whether certain contact structures admit planar open books. Recall from [18] that the correction term for any spin^c structure $\mathfrak{t}$ of a rational homology 3-sphere Y is the minimal degree of any nontorsion class in $HF^+(Y,\mathfrak{t})$ coming from $HF^\infty(Y,\mathfrak{t})$.

THEOREM 1.3

Let J be a Stein structure on $X(G)$, and let ξ be the induced contact structure on $Y(G)$. Denote the correction term of the induced spin^c *structure $\mathfrak{t}$ on $-Y(G)$ by d. Also, let $d_3(\xi)$ be the 3-dimensional invariant of the contact structure ξ. Suppose that we have either $d_3(\xi) \neq -d-1/2$ or $\mathrm{rank}(HF^+_d(-Y(G),\mathfrak{t})) > 1$. Then ξ cannot be supported by a planar open book.*

By [17] (see also Section 3 below), checking the conditions stated in this theorem is simply a combinatorial matter. Corollary 1.7 of [17], which holds for arbitrary rational homology 3-spheres, implies the above statement when $d_3(\xi) \neq -d - 1/2$. This could be taken as evidence to conjecture that Theorem 1.3 also holds for every rational homology 3-sphere.

REMARK 1.4

There is another version of the Ozsváth–Szabó contact invariant $c(\xi)$ which lives in $\widehat{HF}(-Y)$ and is related to $c^+(\xi)$ by $\iota(c(\xi)) = c^+(\xi)$, where ι is the natural map $\iota : \widehat{HF}(-Y) \to HF^+(-Y)$. The invariant $c(\xi)$ can be calculated combinatorially as shown in [25] and [1]. However, for the present applications the usage of c^+ is essential.

The techniques of this paper can also be used to study a natural partial ordering on contact 3-manifolds up to some equivalence. Following [6] and [9], we write $(Y_1, \xi_1) \preceq (Y_2, \xi_2)$ if there is a Stein cobordism from (Y_1, ξ_1) to (Y_2, ξ_2). Moreover, we write $(Y_1, \xi_1) \sim (Y_2, \xi_2)$ if these contact manifolds satisfy $(Y_1, \xi_1) \preceq (Y_2, \xi_2)$ and $(Y_2, \xi_2) \preceq (Y_1, \xi_1)$. Clearly, the relation $\sim$ is an equivalence relation on the set of contact manifolds and $\preceq$ is a partial ordering on the set of equivalence classes. One can define a numerical invariant of contact manifolds respecting this partial ordering. Let

$$\sigma(Y, \xi) = -\max\{d : c^+(\xi) \in U^d \cdot HF^+(-Y)\}.$$

Note that the σ-invariant can be infinite. In fact by Theorem 1.1, if (Y, ξ) admits a planar open book, then $\sigma(Y, \xi) = -\infty$. The key property of σ is the following.

THEOREM 1.5

We have $\sigma(Y_1, \xi_1) \leq \sigma(Y_2, \xi_2)$ *whenever* $(Y_1, \xi_1) \preceq (Y_2, \xi_2)$.

Clearly, if two contact manifolds have different σ-invariants, they lie in different $\sim$ equivalence classes. The following theorem tells us that there are infinitely many such equivalence classes.

THEOREM 1.6

Any negative integer can be realized as the σ-invariant of a contact manifold.

In fact, we are going to construct contact manifolds with distinct σ-invariants by doing Legendrian surgery on certain stabilizations of certain torus knots (see Theorem 7.1 below). After completing the first draft of this paper, the author [13] found an explicit formula for the σ-invariant of a contact manifold obtained by contact surgery on a Legendrian knot in the 3-sphere if the knot has smooth L-space surgery. The formula depends only on the Alexander polynomial, the Thurston–Bennequin number, and the rotation number of the knot, and it generalizes Theorem 7.1. The technique, however, is different from the one used here.

REMARK 1.7

Recently Latschev and Wendl [14] defined an analogous invariant of contact manifolds, which they call *algebraic torsion*, in arbitrary odd dimension within the context of symplectic field theory. In dimension 3, both invariants provide obstructions to exact symplectic cobordisms, so one may wonder if they are somehow related. So far, the author cannot see an obvious relation. One reason is that [14, Theorem 1.1] says that contact manifolds with algebraic torsion are not strongly fillable, whereas the author's examples are all Stein fillable.

This paper is organized as follows. In Section 2, basic properties of Heegaard–Floer homology and the contact invariant are briefly reviewed. Section 3 is devoted to the algorithm of Ozsváth and Szabó to determine the generators of Heegaard–Floer homology of 3-manifolds given by plumbing diagrams. Remarks given at the end of the section allow us to find relations easily by combinatorial means. We prove Proposition 1.2 and Theorem 1.3 in Section 4. Some simple examples are given in Section 5. We discuss the planar obstruction in Section 6. Theorem 1.6 is proved in Section 7. The proof of Theorem 1.5 is given in Section 8.

2. Heegaard–Floer homology and the contact invariant

Let Y be a closed oriented 3-manifold, and let $\mathfrak{t}$ be a spin^c structure on Y. In [20] and [21], Ozsváth and Szabó define four versions of *Heegaard–Floer homology groups* $\widehat{HF}(Y,\mathfrak{t})$, $HF^+(Y,\mathfrak{t})$, $HF^-(Y,\mathfrak{t})$, and $HF^\infty(Y,\mathfrak{t})$. These groups are all smooth invariants of $(Y,\mathfrak{t})$. When Y is a rational homology sphere, they admit absolute $\mathbb{Q}$-gradings. The groups HF^+, HF^-, and HF^∞ are also $\mathbb{Z}[U]$ modules where multiplication by U decreases the degree by 2. Any spin^c cobordism $(X,\mathfrak{s})$ between $(Y_1,\mathfrak{t}_1)$ and $(Y_2,\mathfrak{t}_2)$ induces a homomorphism well defined up to sign

$$F^\circ_{X,\mathfrak{s}}: HF^\circ(Y_1,\mathfrak{t}_1) \to HF^\circ(Y_2,\mathfrak{t}_2).$$

Here HF° stands for any one of $\widehat{HF}$, HF^+, HF^-, or HF^∞. We work with $\mathbb{F}=\mathbb{Z}/2\mathbb{Z}$ coefficients in order to avoid sign ambiguities. Also, we drop the spin^c structure from the notation when we directly sum over all spin^c structures.

Given any contact structure ξ on Y, Ozsváth and Szabó [22] associate an element $c(\xi)\in\widehat{HF}(-Y)$ which is an invariant of the isotopy class of ξ. In this paper we are interested in the image $c^+(\xi)$ of $c(\xi)$ in $HF^+(Y)$ under the natural map. We list some of the properties of this element below.

(1) $c^+(\xi)$ lies in the summand $HF^+(-Y,\mathfrak{t})$ where $\mathfrak{t}$ is the spin^c structure induced by ξ.

(2) $c^+(\xi)=0$ if ξ is overtwisted.

(3) $c^+(\xi)\neq 0$ if ξ is Stein fillable.

(4) $c^+(\xi)\in \mathrm{Ker}(U)$.

(5) $c^+(\xi)$ is homogeneous. When Y is a rational homology sphere, it has degree $-d_3(\xi)-1/2$, where $d_3(\xi)$ is the 3-dimensional invariant of ξ.

(6) $c^+(\xi)$ is natural under Stein 2-handle attachments: Suppose that the contact manifold (Y', ξ') is obtained from (Y, ξ) by Legendrian surgery. Let (W, J) be the associated Stein cobordism, and let $\mathfrak{s}$ be the canonical spin^c structure. Then the induced homomorphism $F^+_{W,\mathfrak{s}} : (-Y', \mathfrak{t}_{\xi'}) \to (-Y, \mathfrak{t}_\xi)$ satisfies $F^+_{W,\mathfrak{s}}(c^+(\xi')) = c^+(\xi)$.

The contact invariant $c^+(\xi)$ is studied by Plamenevskaya [24]. The following result is going to be used in this paper when we prove our main theorem. We state it in a slightly more general form than in [24], but Plamanevskaya's proof is valid for our case as well.

THEOREM 2.1 ([24, THEOREM 4])

Let X be a smooth compact 4-manifold with boundary $Y = \partial X$. Let J be a Stein structure on X that induces a spin^c *structure $\mathfrak{s}_1$ on X and contact structure ξ_1 on Y. Let $\mathfrak{s}_2$ be another* spin^c *structure on X that does not necessarily come from a Stein structure. Suppose that $\mathfrak{s}_1|_Y = \mathfrak{s}_2|_Y$ but the* spin^c *structures $\mathfrak{s}_1$, $\mathfrak{s}_2$ are not isomorphic. We puncture X and regard it as a cobordism from $-Y$ to S^3. Then*

(1) $F^+_{X,\mathfrak{s}_2}(c^+(\xi_1)) = 0$;
(2) $F^+_{X,\mathfrak{s}_1}(c^+(\xi_1))$ *is a generator of* $HF^+_0(S^3)$.

Note that in this theorem if the spin^c structures $\mathfrak{s}_1|_Y$ and $\mathfrak{s}_2|_Y$ are not the same, then conclusion (1) follows trivially.

REMARK 2.2

Theorem 2.1 was later generalized by Ghiggini in [10] where he requires J to be only an ω-tame almost complex structure for some symplectic structure ω on $X(G)$ that gives a strong filling for the boundary contact structure. In this paper, we work with rational homology spheres. For these manifolds, any weak filling can be perturbed into a strong filling (see [16]).

3. The algorithm

In this section, we review Ozsváth and Szabó's combinatorial description of the Heegaard–Floer homology of plumbed 3-manifolds given in [19] to set our notation. The proof of our main theorem heavily relies on understanding the algebraic structure of their combinatorial description. Particularly one should understand the U-action in this combinatorial object. We describe this action in (3.2). Strictly speaking, Ozsváth and Szabó's algorithm determines only the part of the Heegaard–Floer homology group that lies in the kernel of the U-map. In order to determine the full group, one should find all the minimal relations. Although these relations can be found in some special cases, no general technique is known to find all of them. Toward the end of the section we discuss a systematic method to find some (not necessarily minimal) relations. These relations turn out to be minimal in the cases of interest (Example 6.4).

Let G be a weighted graph. For every vertex v of G, let $m(v)$ denote the weight of v, and let $d(v)$ denote the number of edges connected to v. A vertex v is said to be a *bad vertex* if $m(v)+d(v)>0$. Enumerating all vertices of G, one can form the *intersection matrix* whose ith diagonal entry is $m(v_i)$ and (i,j)th entry is 1 if there is an edge between v_i and v_j, and is 0 otherwise. Throughout, we assume that G satisfies the following conditions.

(1) G is a connected tree.
(2) The intersection matrix of G is negative definite.
(3) G has at most one bad vertex.

There is a 4-manifold $X(G)$ obtained by plumbing together disk bundles D_i, $i=1,\dots,|G|$, over 2-sphere where D_i is plumbed to D_j whenever there is an edge connecting v_i to v_j. Let $Y(G)$ be the boundary of $X(G)$. Ozsváth and Szabó [19] give a purely combinatorial description of Heegaard–Floer homology group $HF^+(-Y(G))$. From now on, we identify spin^c structures on 4-manifolds with their first Chern classes. Since all our 4-manifolds are simply connected and have nonempty boundary, this does not cause any ambiguity. However, we should be careful in the 3-manifold level when 2-torsion exists in the first homology. We deal with such an example in Section 6 (see Remark 6.5).

The second homology $H^2(X(G),\mathbb{Z})$ is a free module generated by vertices of G. Let $\mathrm{Char}(G)$ be the set all characteristic (co)vectors of this module; that is, every element K of $\mathrm{Char}(G)$ satisfies $\langle K,v\rangle = m(v) \pmod 2$ for every vertex v. Let $\mathcal{T}^+$ be the graded algebra $\mathbb{F}[U,U^{-1}]/U\mathbb{F}[U]$, where the formal variable U has degree -2. Form the set $\mathbb{H}^+(G)\subset \mathrm{Hom}(\mathrm{Char}(G),\mathcal{T}^+)$ where any element ϕ of $\mathbb{H}^+(G)$ satisfies the following property: if K is a characteristic vector, v is a vertex, and n is an integer such that

$$\langle K,v\rangle + m(v) = 2n,$$

we have that

$$U^{m+n}\phi\big(K+2\,\mathrm{PD}(v)\big) = U^m\phi(K) \quad \text{if } n>0$$

or

$$U^{m}\phi\big(K+2\,\mathrm{PD}(v)\big) = U^{m-n}\phi(K) \quad \text{if } n<0.$$

The set of spin^c structures on $Y(G)$ gives rise to a natural splitting for $\mathbb{H}^+(G)$. If $\mathfrak{t}$ is a spin^c structure on $Y(G)$, one can consider the subset $\mathrm{Char}_{\mathfrak{t}}(G)$ consisting of those characteristic vectors whose restriction on $Y(G)$ is $\mathfrak{t}$. The set $\mathbb{H}^+(G,\mathfrak{t})$ is the set of all maps in $\mathbb{H}^+(G)$ with support $\mathrm{Char}_{\mathfrak{t}}(G)$. It is easy to see that $\mathbb{H}^+(G)=\bigoplus_{\mathfrak{t}}\mathbb{H}^+(G,\mathfrak{t})$.

The group $\mathbb{H}^+(G)$ is graded in the following way. An element $\phi\in\mathbb{H}^+(G)$ is said to be *homogeneous of degree d* if, for every characteristic vector K with

$\phi(K) \neq 0$, $\phi(K) \in \mathcal{T}^+$ is a homogeneous element with

$$\deg\bigl(\phi(K)\bigr) - \frac{K^2 + |G|}{4} = d.$$

We are ready to describe the isomorphism relating $\mathbb{H}^+(G)$ to the Heegaard–Floer homology of $Y(G)$. Fix a spin^c structure $\mathfrak{t}$ on $-Y(G)$. Let K be a characteristic vector on $\mathrm{Char}_{\mathfrak{t}}(G)$. Puncture $X(G)$ and regard it as a cobordism from $-Y(G)$ to S^3. It is known that $X(G)$ and K induce a homomorphism

$$F_{X(G),K} : HF^+\bigl(-Y(G), \mathfrak{t}\bigr) \to HF^+(S^3) \simeq \mathcal{T}^+.$$

Now the map $T^+ : HF^+(-Y(G), \mathfrak{t}) \to \mathbb{H}^+(G, \mathfrak{t})$ is defined by the rule $T^+(\xi)(K) = F_{X(G),K}(\xi)$

THEOREM 3.1 ([19, THEOREM 2.1])

We have that T^+ is a U-equivariant isomorphism preserving the absolute $\mathbb{Q}$-grading.

To simplify the calculations, we work with the dual of $\mathbb{H}^+(G)$. Let $\mathbb{K}^+$ be the quotient set $\mathbb{Z}^{\geq 0} \times \mathrm{Char}(G)/\sim$, where the equivalence relation $\sim$ is defined as follows. Denote a typical element of $\mathbb{Z}^{\geq 0} \times \mathrm{Char}(G)$ by $U^m \otimes K$. (We drop $U^m \otimes$ from our notation if $m = 0$.) Let v be a vertex, and let n be an integer such that

$$2n = \langle K, v\rangle + m(v).$$

Then we have that

$$U^{m+n} \otimes \bigl(K + 2\,\mathrm{PD}(v)\bigr) \sim U^m \otimes K \quad \text{if } n \geq 0$$

or

$$U^{m} \otimes \bigl(K + 2\,\mathrm{PD}(v)\bigr) \sim U^{m-n} \otimes K \quad \text{if } n < 0.$$

Define a pairing $\mathbb{K}^+(G) \times \mathbb{H}^+(G) \to \mathbb{Z}$ by $(\phi, U^m \otimes K) \to (U^m\phi(K))_0$, where $(\cdot)_0$ denotes the projection to the degree 0 subspace of $\mathcal{T}^+$. It is possible to show that this pairing is well defined and nondegenerate, and hence, it defines an isomorphism between $\mathbb{H}^+(G)$ and $\mathrm{Hom}(\mathbb{K}^+(G), \mathbb{Z})$. Using the duality map and isomorphism T^+ one can identify $\ker U^{n+1} \subset HF^+(-Y(G))$ as a quotient of $\mathbb{K}^+(G)$ for every $n \geq 0$.

LEMMA 3.2 ([19, LEMMA 2.3])

Let B_n denote the set of characteristic vectors $B_n = \{K \in \mathrm{Char}(G) : \forall v \in G, |\langle K, v\rangle| \leq -m(v) + 2n\}$. The quotient map induces a surjection from

$$\bigcup_{i=0}^{n} U^i \otimes B_{n-i}$$

onto the quotient space

$$\frac{\mathbb{K}^+(G)}{\mathbb{Z}^{>n} \times \mathrm{Char}(G)}.$$

In turn, we have an identification

$$\mathrm{Hom}\Big(\frac{\mathbb{K}^+(G)}{\mathbb{Z}^{>n}\times \mathrm{Char}(G)},\mathbb{F}\Big)\simeq \ker U^{n+1}\subset \mathbb{H}^+(G). \tag{3.1}$$

One should regard the above isomorphism as one between $\mathbb{F}[U]$ modules where the U-action on the left-hand side of (3.1) is defined by the following relation:

$$U.(U^p\otimes K)^*(U^r\otimes K')=\begin{cases}1 & \text{if } U^p\otimes K\sim U^{r+1}\otimes K',\\ 0 & \text{if } U^p\otimes K\nsim U^{r+1}\otimes K',\end{cases} \tag{3.2}$$

where $(U^p\otimes K)^*$ denotes the dual of $U^p\otimes K$.

Lemma 3.2 gives us a finite model for $\ker U^{n+1}$ for every $n\geq 0$. It is known that these groups stabilize to give HF^+. Therefore, one can understand HF^+ by studying the quotients $\mathbb{K}^+(G)/\mathbb{Z}^{\geq n}\times \mathrm{Char}(G)$ for all $n\geq 0$. The first quotient is well understood thanks to an algorithm of Ozsváth and Szabó. Below, we describe the algorithm and discuss a possible extension.

A characteristic vector K is called an *initial vector* if, for every vertex v, we have that

$$m(v)+2\leq \langle K,v\rangle \leq -m(v). \tag{3.3}$$

Start with an initial vector K_0. Form a sequence $(K_0,K_1,\dots,K_n)$ of characteristic vectors as follows: K_{i+1} is obtained from K_i by adding $2\,\mathrm{PD}(v)$ where v is a vertex with $\langle K_i,v\rangle=-m(v)$. The terminal vector K_n satisfies one of the following:

(1) $m(v)\leq \langle K_n,v\rangle\leq -m(v)-2$ for all v.
(2) $\langle K_n,v\rangle>-m(v)$ for some v.

The sequence $(K_0,K_1,\dots,K_n)$ is called a *full path*, and characteristic vector K_n is called the *terminal vector* of the full path. We say that a full path is called *good* if its terminal vector satisfies property (1) above and *bad* if the terminal vector satisfies (2). We list some of the properties of full paths; the reader can consult [19] (especially [19, Proposition 3.1]) for proofs.

- Two characteristic vectors in B_0 are equivalent in $\mathbb{K}^+(G)$ if and only if there is a full path containing both of them where the set B_0 is defined as in Lemma 3.2.
- If an initial vector K_0 has a good full path, then any other full path starting with K_0 is good.
- If K_0 and K_0' are initial vectors having good full paths and $K_0\neq K_0'$, then $K_0\nsim K_0'$ in $\mathbb{K}^+(G)$.
- A terminal vector K_n of a bad full path is equivalent to $U^m\otimes K'$ in $\mathbb{K}^+(G)$ for some $m>0$ and $K'\in \mathrm{Char}(G)$. A terminal vector of a good full path cannot be equivalent to such an element of $\mathbb{H}^+(G)$.

Note that these properties allow us to find the generators of $\ker U$. They are simply the initial vectors having good full paths. In other words, we know the generators of the lowest grade subgroup of $HF^+(-Y(G))$. Recall from [18] that the lowest degree $d(Y,\mathfrak{t})$ of nontorsion elements in $HF^+(Y,\mathfrak{t})$ is called the *correction term* for a spin^c manifold $(Y,\mathfrak{t})$. The algorithm above provides us an efficient method to calculate the correction term $d(-Y(G),\mathfrak{t})$ for any spin^c structure $\mathfrak{t}$ (see [19, Corollary 1.5]):

$$d\big(-Y(G),\mathfrak{t}\big) = \min -\frac{K^2+|G|}{4}, \tag{3.4}$$

where the minimum is taken over all characteristic vectors admitting good full paths which induce the spin^c structure $\mathfrak{t}$.

The whole group $\mathbb{H}^+(G) \simeq HF^+(-Y(G))$ is determined by the relations amongst the generators of $\mathrm{Ker}(U)$. Given two characteristic vectors K_i, K_j admitting good full paths and inducing the same spin^c structure on $Y(G)$, a *relation* between K_1 and K_2 is a pair of integers (n,m) satisfying $U^n \otimes K_1 \sim U^m \otimes K_2$. If the nonnegative integers (n,m) are minimal with that property, we call the corresponding relation *minimal*. Here we describe a systematic method to find relations. Say that K is a characteristic vector, and say that n is a positive integer. We want to understand the equivalence class in $\mathbb{K}^+(G)$ containing $U^n \otimes K$. We define three operations that do not change this equivalence class:

(R1) $U^n \otimes K \to U^n \otimes K'$, where K' is obtained from K by applying the algorithm to find full paths;

(R2) $U^n \otimes K \to U^{n-1} \otimes (K + 2\,\mathrm{PD}(v))$, where v is a vertex with $\langle K, v\rangle + m(v) = -2$;

(R3) $U^n \otimes K \to U^{n+1} \otimes (K + 2\,\mathrm{PD}(v))$, where v is a vertex with $\langle K_n, v\rangle + m(v) = 2$.

Now assume that K is a characteristic initial vector which admits a good full path. In order to find particular representatives with small U-depths for the equivalence class containing $U^n \otimes K$, we apply (R1), and then apply (R2) if possible, else (R3). Then we repeat the same procedure as necessary until it terminates at an element $U^r \otimes K'$, where one cannot apply any of the moves (R1), (R2), or (R3) any more. (Recall that we do not allow the exponent of U to be negative.) We call the vector K' a *root vector* of $U^m \otimes K$. (The exponent r is automatically determined by m and degrees of K and K'.) A root vector is not unique; it depends upon choices we made along the way, like the choice of the vertex at which we apply (R2) or (R3). However, the set of root vectors is a finite set which can be found easily, and it can be used to establish relations amongst the generators of $\mathrm{Ker}(U)$. This simple observation will be useful when we do our calculations.

PROPOSITION 3.3

Let K_1 and K_2 be two characteristic initial vectors admitting good full paths. Suppose that n and m are nonnegative integers such that the root vector sets of $U^n \otimes K_1$ and $U^m \otimes K_2$ intersect nontrivially. Then we have $U^n \otimes K_1 \sim U^m \otimes K_2$.

Proof
This follows from the definitions. □

4. Main theorem

Proof of Proposition 1.2
Let $\mathfrak{s}$ be the canonical spin^c structure, and let $\mathfrak{s}'$ be any other spin^c structure on $X(G)$. Note that $c_1(\mathfrak{s}) \nsim c_1(\mathfrak{s}')$. Recall that the isomorphism $\mathrm{Ker}(U) \simeq \mathrm{Hom}([\mathbb{K}^+(G)]/[\mathbb{Z}^{>0} \times \mathrm{Char}(G)], \mathbb{F})$ is given by means of the pairing

$$P : \mathrm{Ker}(U) \times \mathrm{Hom}\Big(\frac{\mathbb{K}^+(G)}{\mathbb{Z}^{>0} \times \mathrm{Char}(G)}, \mathbb{F}\Big) \to \mathbb{F},$$
$$P(a, L) = \big(F^+_{X(G),L}(a)\big)_0.$$

In view of this observation, it is enough to show that the following two equations hold:

$$\big(F^+_{X(G),\mathfrak{s}}\big(c(\xi)\big)\big)_0 = 1, \tag{4.1}$$

$$\big(F^+_{X(G),\mathfrak{s}'}\big(c(\xi)\big)\big)_0 = 0. \tag{4.2}$$

These are simply the conclusions of Theorem 2.1. □

REMARK 4.1
The following is a list of possible ways to generalize Proposition 1.2. In each case the proof is similar to the one given above. Unfortunately, it is not possible to combine all these generalizations at the same time.

- The graph G may have two bad vertices instead of one. In this case, the group on the left-hand side of (1.1) gives only even-degree elements in $\mathrm{Ker}(U) \subset HF^+(-Y(G))$.
- We may work with a negative semidefinite graph instead of a negative definite graph. This implies that $b_1(Y) = 1$. For the proof, we use the generalization of the Ozsváth–Szabó algorithm given in [27].
- Instead of Stein fillings, we can use weak symplectic fillings of the contact structure. In the proof, we should use Ghiggini's generalization of Plamenevskaya's theorem (see Remark 2.2).

Proof of Theorem 1.3
Let $K = c_1(J)$. By Theorem 1.1 and Proposition 1.2, it is enough to show that $K^* \notin \mathrm{Im}(U^k)$ for some $k \in \mathbb{N}$. To do that we will use the identification in (3.1), keeping in mind that the U-action is determined by (3.2). Let $\{K_1, K_2, \dots, K_r\}$ be the set of characteristic initial vectors admitting good paths such that $\deg(K_i^*) \le \deg(K^*) = -d_3(\xi) - 1/2$ and $K_i|_{Y(G)} = \mathfrak{t}$ for all $i = 1, \dots, r$. Basic properties of the contact invariant imply that this set is not empty if one of the assumptions is satisfied. It is known that, on any rational homology sphere and for any spin^c structure, the Heegaard–Floer homology decomposes as $HF^+ = \mathcal{T}^+ \oplus HF_{\mathrm{red}}$.

This decomposition tells us that in large even degrees the Heegaard–Floer homology is generated by a single element. So, one can find integers $n_0, n_1, \dots, n_r$ such that

$$U^{n_0} \otimes K \sim U^{n_1} \otimes K_1 \sim \cdots \sim U^{n_r} \otimes K_r.$$

Moreover, by choosing these numbers large enough, we can guarantee that the dual of $U^{n_0} \otimes K$ is the unique generator of the degree $-d_3(\xi) - 1/2 + 2n_0$ subspace of $HF^+(-Y)$. Then by (3.2),

$$U^{n_0}(U^{n_0} \otimes K)^* = K^* + (U^{n_1 - n_0} \otimes K_1)^* + \cdots + (U^{n_r - n_0} \otimes K_r)^*.$$

Therefore, $K \notin \mathrm{Im}(U^{n_0})$. □

5. Examples

In this section, we discuss two examples. These examples have no particular importance on their own, but they are simple enough to give a clear explanation of the ideas used in this paper.

EXAMPLE 5.1

Let G be the graph indicated in Figure 1. Index the vertices so that the central one comes first. Our aim is to find all the characteristic covectors in the intersection lattice of $X(G)$ which admit good full path. We will denote each $K \in H^2(X(G))$ as a row vector $[\langle K, v_1\rangle, \dots, \langle K, v_4\rangle]$, where v_i is the homology class of the sphere corresponding to the ith vertex for all $i = 1, \dots, 4$. If K is characteristic and satisfies (3.3), then $\langle K, v_i\rangle = 0$ or 2 for every i. Therefore, we need to find out which of the possible 16 covectors admit good full paths. To represent full paths, we indicate the index of the vertex whose twice Poincaré dual is added to the characteristic vector. The algorithm terminates at the very first step for $K_1 = [0,0,0,0]$. For $K_2 = [0,2,0,0]$, we have the following good full path: $2,1,3,4,1,2$. By symmetry, $K_3 = [0,0,2,0]$ and $K_4 = [0,0,0,2]$ also admit good full paths. For $[2,0,0,0]$, the full path $1,2,3,4$ terminates at a bad vector. Also, it is easy to show that if $\langle K, v_i\rangle = 2$ for more than one i values, then K admits a bad full path. Therefore, $K_1, \dots, K_4$ are the only characteristic covectors admitting good full paths.

Next we claim that each one of $K_1, \dots, K_4$ restricts to a different spin^c structure $\mathfrak{t}_1, \dots, \mathfrak{t}_4$ on the boundary. One way of seeing this is to apply the criterion mentioned in Remark 6.5. Another way is the following: Recall that the set of spin^c structures on any 3-manifold can be identified with its first homology. In this case the first homology of $Y(G)$ is given by $\mathbb{Z}^4/\mathrm{Im}\, I(G)$, where $I(G)$ is the

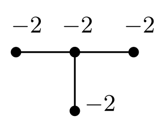

Figure 1

intersection matrix. Observe that $\det(I(G)) = 4$, so $-Y(G)$ has 4 spin^c structures. Each one of these spin^c structures is torsion, so the Heegaard–Floer homology of $-Y(G)$ is nontrivial in the corresponding component. Since we have exactly 4 covectors contributing to the Heegaard–Floer homology, they must lie in different spin^c components. This shows that $-Y(G)$ (and hence $Y(G)$) is an L-space (i.e., its Heegaard–Floer homology is the same as a Lens space).

Let us calculate the degree of each K_i. In the formula $\deg(K) = (K^2 + |G|)/4$, the inverse of the intersection matrix should be used when squaring K. We see that $\deg(K_1) = 1$ and $\deg(K_j) = 0$ for $j = 2, 3, 4$. Since the isomorphism given in (3.1) is given in terms of dual covectors, we should take the negative of the degrees when we think of K_i's as elements of the Heegaard–Floer homology. As a result, $HF^+(-Y(G), \mathfrak{t}_1) = \mathcal{T}^+_{(-1)}$ and $HF^+(-Y(G), \mathfrak{t}_i) = \mathcal{T}^+_{(0)}$, for $i = 2, 3, 4$.

Having calculated the Heegaard–Floer homology of the boundary, we now want to see how the Ozsváth–Szabó invariant of a contact structure sits in this group. We equip $X(G)$ with the obvious Stein structure J: First make the attaching circles of handles corresponding to the vertices Legendrian unknot with $tb = -1$ (see Figure 2). Since each handle is attached with framing $tb - 1$, the unique Stein structure on the 4-ball extends across these handles (see [4]). To identify the contact invariant, we need to determine the Chern class of J. The value of $c_1(J)(v_i)$ is given by the rotation number of the corresponding Legendrian unknot. In this case the rotation numbers are all 0, so $c_1(J) = K_1$. Hence, the Ozsváth–Szabó invariant of the induced contact structure is the unique generator of $HF^+(-Y(G), \mathfrak{t}_1)$ in degree -1. Note that the invariant is in the image of U^k for every k, so we do not get any obstruction to planarity.

EXAMPLE 5.2

This example is a follow-up of the calculation of the Heegaard–Floer homology of the Brieskorn sphere $\Sigma(3, 5, 7)$ given in [19]. This 3-manifold is given by the plumbing graph G which we indicate in Figure 3. We order the vertices so that

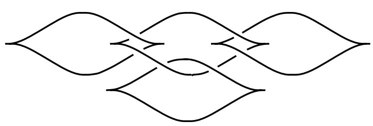

Figure 2.

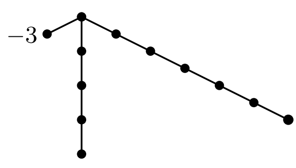

Figure 3. Plumbing graph for $\Sigma(3, 5, 7)$. Unlabeled vertices have weight -2.

the central node comes first, the -3-sphere second, then the four vertices in the middle, and finally, the six vertices on the right. It is shown in [19] that only the following characteristic covectors admit good full paths:

$$K_1 = (0,-1,0,0,0,0,0,0,0,0,0,0),$$

$$K_2 = (0,1,0,0,0,0,0,0,0,0,0,0),$$

$$K_3 = (0,1,0,0,0,0,0,0,0,0,0,-2),$$

$$K_4 = (0,1,0,0,0,-2,0,0,0,0,0,0).$$

We have that $\deg(K_1) = \deg(K_2) = 0$, and $\deg(K_3) = \mathrm{K}_4 = 2$. So the correction term for the unique spin^c structure is -2. Next we consider the Stein structures on $X(G)$. We make each unknot Legendrian as before, but this time we should do a stabilization for the -3 framed unknot to get the framing $tb-1$. Depending on how we do the stabilization, we obtain two Stein structures J_1, J_2 whose Chern classes are given by K_1 and K_2. Let ξ_1 and ξ_2, respectively, denote the induced contact structures on the boundary. Since $-\deg(K_i)$ is not minimal, neither contact structure is compatible with a planar open book by Theorem 1.3. This also can be seen by using simple criteria found by Ozsváth, Stipsicz, and Szabó (see Theorems 6.2 and 6.3).

Finally, we would like to show why $c^+(\xi_i)$ is not in the image of U, for $i = 1, 2$. By Theorem 1.2 the contact invariant $c^+(\xi_i)$ is represented by the dual K_i^*. It was shown in [19] that the minimal relations are given as follows:

$$U \otimes K_3 \sim U \otimes K_4,$$

$$U^2 \otimes K_3 \sim U \otimes K_1 \sim U \otimes K_2.$$

Therefore, $(U^2 \otimes K_3)^*$ is the unique generator of degree 2 and $U(U^2 \otimes K_3)^* = (U \otimes K_3)^* + K_1^* + K_2^*$, by (3.2). So neither K_1^* nor K_2^* is in the image of U.

6. Planar obstruction

In this section, we illustrate an application of Theorem 1.3 and show that certain Stein fillable contact structures do not admit planar open books. Obstructions to being supported by planar open books were known to exist before. Some of these obstructions can be checked by using simple criteria. The importance of our examples is that no other simple criterion is sufficient to prove their nonplanarity. Before discussing our examples we give a brief exposition on what is known about obstructions to planarity.

The first known obstruction to planarity was found by Etnyre [5]. It puts some restrictions on intersection forms of symplectic fillings of planar open books.

THEOREM 6.1 ([5, THEOREM 4.1])

If X is a symplectic filling of a contact 3-manifold (Y, ξ) which is compatible with a planar open book decomposition, then $b_+^2(X) = b_0^2(X) = 0$, the boundary of X is connected, and the intersection form Q_X embeds into a diagonalizable matrix over integers.

Ozsváth, Szabó, and Stipsicz found another obstruction in [17]. Their obstruction is a consequence of Theorem 1.1 above, though its statement has no reference to Floer homology.

THEOREM 6.2 ([17, COROLLARY 1.5])
Suppose that the contact 3*-manifold* (Y, ξ) *with* $c_1(s(\xi)) = 0$ *admits a Stein filling* (X, J) *such that* $c_1(X, J) \neq 0$*. Then* ξ *is not supported by a planar open book decomposition.*

Yet another criterion is stated in [17]. It partially implies Theorem 1.3 above.

THEOREM 6.3 ([17, COROLLARY 1.7])
Suppose that Y *is a rational homology* 3*-sphere. The number of homotopy classes of* 2*-plane fields which admit contact structures which are both symplectically fillable and compatible with planar open book decompositions is bounded above by the number of elements in* $H_1(Y; \mathbb{Z})$*. More precisely, each* spin^c *structure* $\mathfrak{s}$ *is represented by at most one such* 2*-plane field, and moreover, the Hopf invariant of the corresponding* 2*-plane field must coincide with the correction term* $d(-Y, s)$*.*

Below, we give examples of nonplanar Stein fillable contact structures on a Seifert fibered space. The nonplanarity of some of our examples does not follow from Theorem 6.1, 6.2, or 6.3.

EXAMPLE 6.4
Consider the star-shaped plumbing graph consisting of eight vertices where the central vertex has weight -4, a neighboring vertex has weight -3, and all the others are of weight -2 (see Figure 4). The boundary 3-manifold Y is the Seifert fibered space $M(-4, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{3})$. The reason why we have so many self-intersection -2 spheres is that we want to avoid L-spaces where Theorem 1.1 does not provide an obstruction to admitting a planar open book (compare with Example 5.1). For the topological characterization of L-spaces among Seifert fibered spaces see [15].

The intersection form is negative definite and has determinant 128. Moreover, it can be embedded into a symmetric matrix which is diagonalizable over integers. To see this, index the vertices so that the central one comes first and the weight -3 vertex is the last. Let $e_1, e_2, \ldots, e_{11}$ be a basis for $\mathbb{R}^{11}$ such that $e_i \cdot e_i = -1$

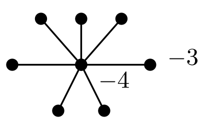

Figure 4. Plumbing description of Y. All unmarked vertices have weight -2.

for all $i = 1, 2, \ldots, 11$. The embedding is defined by the following set of equations:

$$
\begin{aligned}
&v_1 \to -e_1 - e_2 - e_3 - e_4,\\
&v_2 \to e_2 - e_7,\\
&v_3 \to e_2 + e_7,\\
&v_4 \to e_3 - e_8,\\
&v_5 \to e_3 + e_8,\\
&v_6 \to e_4 - e_9,\\
&v_7 \to e_4 + e_9,\\
&v_8 \to e_1 + e_{10} + e_{11}.
\end{aligned}
$$

First, we calculate $HF^+(-Y, \mathfrak{t})$ for every spin^c structure $\mathfrak{t}$. For similar calculations, see [3], [26], [27], and [19, Section 3.2]. As before, we write any characteristic vector K in the form $K = [\langle K, v_1\rangle, \ldots, \langle K, v_8\rangle]$. There are 768 characteristic vectors satisfying (3.3), and 138 of them have good full paths. When we distribute these to spin^c structures of Y, we see that for 10 spin^c structures $\mathrm{Ker}(U)$ has rank 2, and for the rest it has rank 1. Table 1 shows HF^+ for these 10 spin^c structures.

REMARK 6.5

As pointed out in [19], the set of spin^c structures on Y can be identified with $2H^2(X(G), \partial X(G))$ orbits in $\mathrm{Char}(G)$. Therefore, two characteristic vectors K_1, K_2 induce the same spin^c structure on the boundary if and only if all the entries of $(1/2)I(G)^{-1}(K_1 - K_2)$ are integers where $I(G)$ is the intersection matrix.

We would like to point out how we do the Heegaard–Floer homology calculation shown in Table 1 by utilizing Proposition 3.3. Let us consider the first spin^c structure indicated in the table which contains two initial vectors $K_1 = [2, 0, 0, 0, 0, 0, 0, -1]$ and $K_2 = [-2, 0, 0, 0, 0, 0, 0, 3]$, each admitting a good full path. These vectors are inequivalent because their terminal vectors are different. We claim that $U \otimes K_1 \sim U \otimes K_2$. To see this, we shall show that both elements have a common root vector. Indeed, if we take $U \otimes K_1$ and apply the following moves in the given order: $(\mathrm{R}_2(v_1), \mathrm{R}_1(v_2), \mathrm{R}_1(v_3), \ldots, \mathrm{R}_1(v_7), \mathrm{R}_3(v_1), \mathrm{R}_1(v_8), \mathrm{R}_2(v_i))$, where i is any element of $\{2, \ldots, 7\}$, we get the root vector $[2, 0, \ldots, 0, \underbrace{-4}_{i}, 0, \ldots, 0, -3]$. The same vector can be obtained from $U \otimes K_2$ by applying $(\mathrm{R}_1(v_8), \mathrm{R}_2(v_i))$. Having proved that $U \otimes K_1 \sim U \otimes K_2$, we next note that this relation is necessarily minimal, which proves that the Heegaard–Floer homology in this spin^c is as indicated in Table 1. Other spin^c structures can be handled similarly.

Table 1. HF^+ of $M(-4, \frac{1}{2}, \dots, \frac{1}{2}, \frac{1}{3})$ for 10 spin^c structures

Spin^c	Characteristic vectors	Degree	Relation	$HF^+(-Y)$
1	$[2,0,0,0,0,0,0,-1]$ $[-2,0,0,0,0,0,0,3]$	$7/8$ $7/8$	$U \otimes K_1 = U \otimes K_2$	$\mathcal{T}^+_{-7/8} \oplus \mathbb{F}_{-7/8}$
2	$[-2,0,0,0,0,0,0,1]$ $[0,0,0,0,0,0,0,3]$	$7/8$ $7/8$	$U \otimes K_1 = U \otimes K_2$	$\mathcal{T}^+_{-7/8} \oplus \mathbb{F}_{-7/8}$
3	$[-2,0,0,0,0,0,0,-1]$ $[0,0,0,0,0,0,0,1]$	$-1/8$ $15/8$	$U \otimes K_1 = U^2 \otimes K_2$	$\mathcal{T}^+_{-15/8} \oplus \mathbb{F}_{1/8}$
4	$[2,0,0,0,0,0,0,1]$ $[0,0,0,0,0,0,0,-1]$	$-1/8$ $15/8$	$U \otimes K_1 = U^2 \otimes K_2$	$\mathcal{T}^+_{-15/8} \oplus \mathbb{F}_{1/8}$
$5+j$	$[-2,0,\dots,\underbrace{2}_{j+2},\dots,0,-1]$ $[0,0,\dots,\underbrace{2}_{j+2},\dots,0,1]$ $j=0,\dots,5$	$3/4$ $3/4$	$U \otimes K_1 = U \otimes K_2$	$\mathcal{T}^+_{-3/4} \oplus \mathbb{F}_{-3/4}$

Spin^c	Root vectors
1	$[2,0,\dots,0,\underbrace{-4}_{i},0,\dots,0,-3]$ $i=2,\dots,7$
2	$[0,0,0,0,0,0,0,-5]$, $[0,0,\dots,0,\underbrace{-4}_{i},0,\dots,0,1]$, $i=2,\dots,7$
3	$[0,0,\dots,0,\underbrace{-4}_{i},0,\dots,0,-1]$, $i=2,\dots,7$
4	$[-2,0,0,0,0,0,0,-3]$
$5+j$	$[2,0,\dots,\underbrace{-4}_{i},\dots,\underbrace{-2}_{j+2},\dots,0,-1]$ $i=2,\dots,7$ $j=0,\dots,5$

Next, we consider the obvious Stein structures that arise from the handlebody diagram associated to G. Following Eliashberg, we isotope the attaching circles of 2-handles into Legendrian position so that their framing becomes one less than the Thurston–Bennequin framing. For -2 framed unknots, there is a unique way to do that. For the other unknots which correspond to v_1 and v_8, take Legendrian isotopes with rotation numbers i and j, respectively, where $i=-2,0,2$ and $j=-1,1$. Denote the resulting Stein structure as $J_{i,j}$, and denote the induced contact structure by $\xi_{i,j}$ (see Figure 5 for a picture of $J_{2,-1}$). Note that the first Chern class of $J_{i,j}$ is given by the characteristic vector $K_{i,j}=[i,0,0,0,0,0,0,j]$. It is easy to verify that $d_3(\xi_{i,j})+1/2=(K_{i,j}^2+|G|)/4=\deg(K_{i,j})$. According to Theorem 1.3, the contact structures $\xi_{\pm 2,\pm 1}$ do not admit planar open books. By the algorithm given in [7] these contact structures do admit genus one open

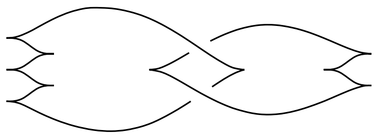

Figure 5. Legendrian handlebody diagram giving $J_{2,-1}$. The curve on the left corresponds to v_1, and the other represents v_8. They are both oriented counterclockwise. We omit the other unknots linking to v_1 in order to simplify the picture.

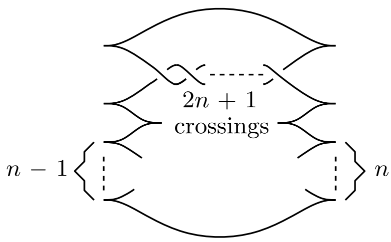

Figure 6. L_n.

books, so their support genera are all one. One cannot use Theorem 6.2 directly to get this conclusion because the Chern classes of the corresponding spin^c structures are all of order 4. Though Theorem 6.3 also implies our conclusion for $\xi_{2,1}$ and $\xi_{-2,-1}$, it does not apply to $\xi_{2,-1}$ or $\xi_{-2,1}$. So the latter two are the contact structures we promised at the beginning of the example.

REMARK 6.6

The main result of [8] implies that the support genera of plumbings with at most two bad vertices are at most one. On the other hand, the algorithm of Ozsváth and Szabó does not work if the number of bad vertices is greater than two. Therefore, the techniques used in this paper do not seem to be sufficient to find an example of a contact structure with support genus strictly greater than one. We are planning to return to this problem in a future project using a different approach.

7. Calculation of σ

In this section we prove Theorem 1.6 by calculating explicitly the σ-invariant of a family of contact 3-manifolds. Our argument is based on a previous work of Rustamov [26].

For every positive integer n, consider the contact manifold (Y_n, ξ_n) obtained from $(S^3, \xi_{\mathrm{std}})$ by doing Legendrian surgery on the $(2, 2n+1)$ torus knot L_n stabilized $2n-1$ times as in Figure 6. Observe that the Thurston–Bennequin invariant of L_n is zero, so the topological surgery coefficient is negative one. In fact, the 3-manifold Y_n is the Brieskorn homology sphere $\Sigma(2, 2n+1, 4n+3)$.

THEOREM 7.1
We have that $\sigma(Y_n,\xi_n) = -(p_n - 1)$, *where* p_n *is the* n*th element of the sequence*

$$1,1,2,2,3,3,\ldots.$$

Clearly, Theorem 7.1 implies Theorem 1.6. Another immediate application of Theorem 7.1 is that (Y_n,ξ_n) cannot be supported by a planar open book. This was first pointed out in [17]. Finally, combining this theorem with the fact that the σ-invariant respects the partial ordering coming from Stein cobordisms, we have the following corollary.

COROLLARY 7.2
There is no Stein cobordism from (Y_n,ξ_n) *to* (Y_m,ξ_m) *if* $m > n+1$. *In particular, one cannot obtain* (Y_m,ξ_m) *from* (Y_n,ξ_n) *via Thurston–Bennequin minus one* $(tb-1)$ *surgery on a Legendrian link.*

The above corollary should be compared to a classical result of Ding and Geiges in [2], where it was proved that any two contact manifolds can be obtained from each other via a sequence of $(tb-1)$ or $(tb+1)$ contact surgeries. In fact, one can always choose such a sequence which contains at most one $tb+1$ surgery. Therefore, the corollary tells us that the existence of $(tb+1)$ surgery is essential even though the contact manifolds at both ends are Stein fillable.

Proof of Theorem 7.1
Let V be the 4-manifold obtained by attaching a Weinstein 2-handle to a 4-ball along L_n. Eliashberg's theorem [4, Theorem 1.3.1] states that V admits a Stein structure. Let $\mathfrak{s}$ be the canonical spin^c structure on V, and denote the homology class determined by the 2-handle (for some orientation of L_n) by h. The way we stabilize L_n ensures that

$$c_1(\mathfrak{s})(h) = \mathrm{rot}(L_n) = \pm 1. \tag{7.1}$$

Here $\mathrm{rot}(L_n)$ stands for the rotation number of L_n. Note that the sign of the rotation number depends on how we orient L_n. Next, V is blown up $n+2$ times, and we do the handleslides indicated in Figure 8. We see that the resulting 4-manifold is given by the plumbing graph G in Figure 7. The manifold $X(G)$ is no longer Stein, but it does admit a symplectic structure. Let $\mathfrak{s}'$ be the canonical spin^c structure on this symplectic manifold. Let e_i denote the homology class of the ith exceptional sphere. We have that

$$c_1(\mathfrak{s}')(e_i) = 1, \quad i = 1,2,\ldots,n+2. \tag{7.2}$$

Order the vertices of G so that first four are the ones with weight -1, -2, -3, and $-4n-3$, respectively, and all the remaining ones corresponding to -2's on the right are ordered according to the distance from the root, starting with

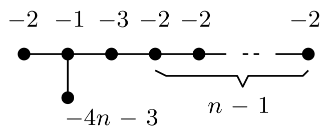

Figure 7. Plumbing graph for Brieskorn homology sphere $\Sigma(2, 2n+1, 4n+3)$.

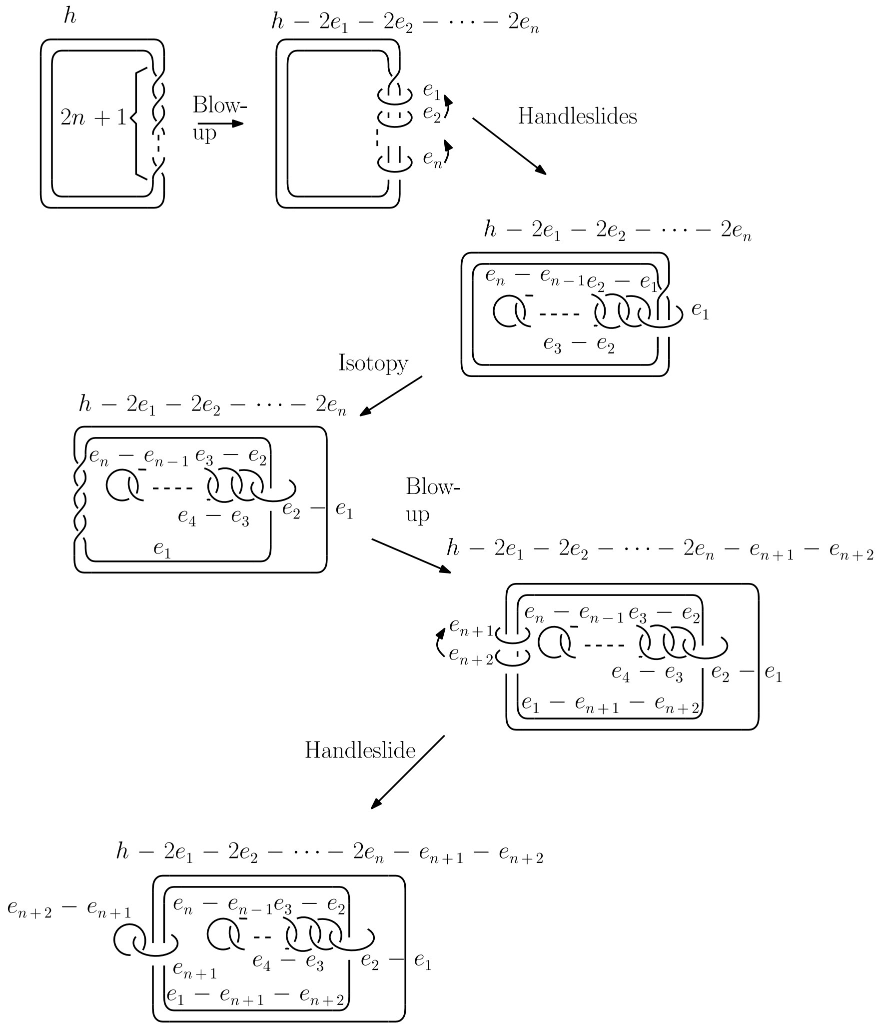

Figure 8. Sequence of blowups from Y_n to plumbing.

the closest one. Rustamov [26] proves that

$$HF^+(-Y_n) = \mathcal{T}_0^+ \oplus \mathbb{F}_{(0)}^{p_n} \oplus \bigoplus_{i=1}^{n-1} (\mathbb{F}_{q_{n-i}}^{p_i} \oplus \mathbb{F}_{q_{n-i}}^{p_i}),$$

where $q_i = i(i+1)$, $\mathbb{F}^r_{(k)} = \mathbb{F}[U]/U^r\mathbb{F}[U]$, and U^{r-1} lies in degree k. More precisely, he shows that $\operatorname{Ker} U \subset HF^+(-Y_n)$ is generated by the characteristic vectors

$$K_i = (1, 0, -1, -4n-3+2i, 0, 0, \dots, 0), \quad i = 1, 2, \dots, 2n.$$

He also proves that the minimal relations are given as follows:

$$U^{p_i} \otimes K_i \sim U^{p_i + q_{n-i}/2} \otimes K_{n+1}, \tag{7.3}$$

$$U^{p_i} \otimes K_{2n+1-i} \sim U^{p_i + q_{n-i}/2} \otimes K_n, \tag{7.4}$$

where $i = 1, 2, \dots, n$. Note that the characteristic vectors K_n and K_{n+1} are in the bottom level, and their degree is zero.

Our aim is to pin down the contact invariant $c^+(\xi_n)$ in $HF^+(-Y_n)$. Note that Proposition 1.2 in the stated form cannot be applied directly as it concerns Stein fillings of plumbed manifolds. However, as indicated in Remark 4.1, it is also true for strong symplectic fillings. The only difference in the proof is that one uses Ghiggini's generalization [10] of Plamenevskaya's theorem [24, Theorem 4]. Alternatively, one can use the blowup formula and handleslide invariance for this particular case to see that (4.1) and (4.2) hold. In any case, we see that the contact invariant $c^+(\xi_n)$ is represented by the first Chern class $c_1(\mathfrak{s}')$ of the canonical spin^c structure. In Figure 8, we keep track of the homology classes in order to pin down the first Chern class. By (7.1) and (7.2), we have $c_1(\mathfrak{s}') = K_n$ or K_{n+1}, depending on the orientation of L_n. Then the contact invariant is not in the image of U^{p_n} because of the relation $U^{p_n} \otimes K_n \sim U^{p_n} \otimes K_{n+1}$. It is in the image of $U^{p_n - 1}$, since all of the relations given in (7.3) and (7.4) involve higher powers of U on their right-hand side and these relations are minimal. □

8. Monotonicity of σ

In this section, we establish the monotonicity of the σ-invariant under Stein cobordisms. We start by proving a lemma whose proof is purely algebraic.

LEMMA 8.1

Suppose that (Y_1, ξ_1) and (Y_2, ξ_2) are contact manifolds, and suppose that there is a U-equivariant homomorphism

$$f : HF^+(-Y_2, \mathfrak{t}_{\xi_2}) \to HF^+(-Y_1, \mathfrak{t}_{\xi_1})$$

with $f(c^+(\xi_2)) = c^+(\xi_1)$. Then $\sigma(\xi_1) \le \sigma(\xi_2)$.

Proof

Suppose that $c^+(\xi_2) \in U^d \cdot HF^+(-Y_2, \mathfrak{t}_{\xi_2})$. Let $\alpha \in HF^+(-Y_2, \mathfrak{t}_{\xi_2})$ such that $U^d \alpha = c^+(\xi_2)$. Then $U^d f(\alpha) = c^+(\xi_1)$, implying that $c^+(\xi_1) \in U^d \cdot HF^+(-Y_1, \mathfrak{t}_{\xi_1})$. This implies that

$$\max\{d : c^+(\xi_2) \in U^d \cdot HF^+(-Y_2)\} \le \max\{d : c^+(\xi_1) \in U^d \cdot HF^+(-Y_1)\}.$$

The lemma follows from the definition of the σ-invariant. □

The following statement establishes the naturality of the contact invariant under Stein 1-handle cobordisms. Clearly this was known to the experts in the area, but the author does not know if it was proved elsewhere. See [10, Lemma 2.11] for a similar statement for Stein 2-handle cobordisms.

LEMMA 8.2

Let (Y,ξ) be a contact manifold, and let $W : Y \to Y\#S^1 \times S^2$ be a 1-handle cobordism. Equip $Y\#S^1 \times S^2$ with the contact structure $\xi' = \xi\#\xi_0$, where ξ_0 is the unique tight contact structure on $S^1 \times S^2$, and regard W as a Stein cobordism. Let $\mathfrak{s}$ be the canonical spin^c *structure. Then the induced homomorphism*

$$F_{W,\mathfrak{s}} : \big(-(Y\#S^1 \times S^2), \mathfrak{t}_{\xi'}\big) \to (-Y, \mathfrak{t}_\xi)$$

satisfies $F_{W,\mathfrak{s}}(c^+(\xi')) = c^+(\xi)$.

Proof

The proof follows by combining well-known facts about the contact invariant and 1-handle cobordisms. In [12], Honda, Kazez, and Matić explicitly describe the cycle representing the contact invariant, starting with an open book supporting the contact structure. In what follows we review their description and prove that the homomorphism associated to a 1-handle attachment sends the contact invariant to the contact invariant in the chain level. Henceforth, we assume that the reader is familiar with the definition of the Heegaard–Floer chain complex defined using Heegaard diagrams (see [20, Theorem 4]).

Let (Σ,ϕ) be an open book supporting (Y,ξ). Let $a_1, a_2, \ldots, a_g$ be properly embedded arcs in Σ whose complement is a disk. Let $b_1, b_2, \ldots, b_g$ be small translates of $a_1, a_2, \ldots, a_g$ in the direction determined by the orientation of $\partial\Sigma$. Note that each a_i intersects b_i at a unique point x_i, for $i = 1, \ldots, g$. Let $S = \Sigma_1 \bigcup_{\partial\Sigma_1 \sim \partial\Sigma_2} -\Sigma_2$ be the closed surface obtained by doubling the page Σ. Define α_i to be the circle on S obtained by doubling a_i. The circle β_i is defined by gluing the arc $b_i \subset \Sigma_1$ to the arc $\phi(b_i) \subset \Sigma_2$ along their common boundary. Put the base point z to the big region in Σ_1. The quadruple $(S, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ is a Heegaard diagram for Y. It turns out that the intersection point $\{x_1, x_2, \ldots, x_g\} \in T_{\boldsymbol{\beta}} \cap T_{\boldsymbol{\alpha}}$ gives rise to a cycle $[\{x_1, x_2, \ldots, x_g\}, 0]$ in the chain complex $CF^+(\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha}, z) = CF^+(-Y)$ (see [12] for details).

An open book (Σ', ϕ') supporting $(Y\#S^1 \times S^2, \xi')$ can be obtained by attaching a 2-dimensional 1-handle to the page Σ and extending the monodromy ϕ trivially over this 1-handle. Next we repeat the procedure described in the previous paragraph for this new open book. We now have an extra pair of arcs a_{g+1} and b_{g+1} corresponding to the additional handle. Denote the resulting Heegaard diagram by $(S', \boldsymbol{\alpha'}, \boldsymbol{\beta'}, z)$, where $S' = S\#T^2$, $\boldsymbol{\alpha'} = \boldsymbol{\alpha} \cup \{\alpha_{g+1}\}$, and $\boldsymbol{\beta'} = \boldsymbol{\beta} \cup \{\beta_{g+1}\}$. The circles α_{g+1} and β_{g+1} intersect at two points x_{g+1} and y_{g+1}. Note that $(T^2, \{\alpha_{g+1}\}, \{\beta_{g+1}\})$ is the standard Heegaard diagram of $S^1 \times S^2$ in the sense of [23, Definition 2.8]. Only one of x_{g+1} and y_{g+1} is contained in Σ'_1; say that it is x_{g+1}. Then the contact invariant $c^+(\xi')$ is represented by $[\{x_1, x_2, \ldots, x_{g+1}\}, 0]$.

Now regard the intersection points x_{g+1} and y_{g+1} as the generators of the chain complex $\widehat{CF}(T^2, \{\beta_{g+1}\}, \{\alpha_{g+1}\})$. There is a Whitney disk u connecting y_{g+1} to x_{g+1} whose domain is one of the two small bigons. The Maslov index of u is 1, so $\deg(y_{g+1}) > \deg(x_{g+1})$.

One can describe the chain map $F_W^+ : CF^+(S', \boldsymbol{\beta}', \boldsymbol{\alpha}', z) \to CF^+(S, \boldsymbol{\beta}, \boldsymbol{\alpha}, z)$ following [23, Section 4.3]. Note that if $\boldsymbol{r} \in T_{\boldsymbol{\beta}'} \cap T_{\boldsymbol{\alpha}'}$, then the component of r on β_{g+1} is x_{g+1} or y_{g+1}. The chain map then is given by

$$F_W^+([\boldsymbol{r}, n]) = \begin{cases} [\boldsymbol{r} - \{x_{g+1}\}, n] & \text{if } x_{g+1} \in \boldsymbol{r}, \\ 0 & \text{if } y_{g+1} \in \boldsymbol{r}. \end{cases}$$

Therefore, $F_W([\{x_1, \ldots, x_g, x_{g+1}\}, 0]) = [\{x_1, \ldots, x_g\}, 0]$. Passing to the homology, we see that $F_W^+(c^+(\xi')) = c^+(\xi)$. Finally, we observe that $\mathfrak{s}$ is the only spin^c on W which extends $\mathfrak{t}_{\xi'}$, so $F_{W,\mathfrak{s}}^+(c^+(\xi')) = F_W^+(c^+(\xi'))$. ☐

Proof of Theorem 1.5
Suppose that there is a Stein cobordism $(W, J) : (Y_1, \xi_1) \to (Y_2, \xi_2)$. It suffices to construct a U-equivariant homomorphism $f : HF^+(-Y_2) \to HF^+(-Y_1)$ sending $c^+(Y_2, \xi_2)$ to $c^+(Y_2, \xi_2)$. Then the theorem follows from Lemma 8.1.

By [4], W can be decomposed as a union of subcobordisms

$$W = W_1 \cup W_2 \cup \cdots \cup W_n,$$

where each W_i is either a 1-handle attachment or a $(tb - 1)$ framed 2-handle attachment. Let $\mathfrak{s}_i$ be the restriction of the canonical spin^c structure on W_i, for all $i = 1, \ldots, n$. Let $f = F_{W_n,\mathfrak{s}_n} \circ F_{W_{n-1},\mathfrak{s}_{n-1}} \circ \cdots \circ F_{W_1,\mathfrak{s}_1}$. Each $F_{W_i,\mathfrak{s}_i}$ is U-equivariant and respects contact invariants by Lemma 8.2 and [10, Lemma 2.11], so f is the required map. ☐

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