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ON m-GENERALIZED INVERTIBLE OPERATORS ON BANACH SPACES

HAMID EZZAHRAOUI

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ABSTRACT. A bounded linear operator S on a Banach space X is called an *m-left generalized inverse of an operator T* for a positive integer m if

$$T\sum_{j=0}^{m}(-1)^j\binom{m}{j}S^{m-j}T^{m-j}=0,$$

and it is called an *m-right generalized inverse of T* if

$$S\sum_{j=0}^{m}(-1)^j\binom{m}{j}T^{m-j}S^{m-j}=0.$$

If T is both an m-left and an m-right generalized inverse of T, then it is said to be an *m-generalized inverse of T*.

This paper has two purposes. The first is to extend the notion of generalized inverse to m-generalized inverse of an operator on Banach spaces and to give some structure results. The second is to generalize some properties of m-partial isometries on Hilbert spaces to the class of m-left generalized invertible operators on Banach spaces. In particular, we study some cases in which a power of an m-left generalized invertible operator is again m-left generalized invertible.

1. Introduction and preliminaries

Throughout this paper, X shall denote a complex Banach space, and $\mathcal{L}(X)$ shall denote the algebra of all bounded linear operators on X. We denote X by

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H if it is a Hilbert space. For an arbitrary operator $T \in \mathcal{L}(X)$, we use $N(T)$ to denote its kernel, $R(T)$ its range, and T^* its adjoint.

An operator $T \in \mathcal{L}(X)$ is said to be *left invertible* if there is an operator $L \in \mathcal{L}(X)$ such that $LT = I$, and it is said to be *right invertible* if there is an operator $R \in \mathcal{L}(X)$ such that $TR = I$, where I denotes the identity operator.

The concept of *generalized inverses* of matrices was first proposed by E. H. Moore in the 1920s, and a generalization of his original idea to the bounded linear operators between Hilbert spaces with closed range was mainly due to his student Y.-Y. Tseng in the 1930s and 1940s via a series of papers (see [5] for more details). An operator $S \in \mathcal{L}(X)$ is said to be a *left generalized inverse* of $T \in \mathcal{L}(X)$ if $TST = T$, and it is said to be a *right generalized inverse* of T if $STS = S$. An operator $S \in \mathcal{L}(X)$ is said to be a *generalized inverse* of $T \in \mathcal{L}(X)$ if it is both a left and right generalized inverse of T; that is, $TST = T$ and $STS = S$. It is well known that an operator $T \in \mathcal{L}(X)$ has a generalized inverse if and only if $N(T)$ and $R(T)$ are closed and complemented subspaces of X (see, e.g., [6]). We notice that the equality $TST = T$ is a necessary and sufficient condition for T to have a generalized inverse. Indeed, it is clear that $S' = STS$ is a generalized inverse of T.

An operator $T \in \mathcal{L}(H)$ is said to be a *partial isometry* provided that $\|Tx\| = \|x\|$ for every $x \in N(T)^{\perp}$; that is, T^* is a generalized inverse of T (i.e., $TT^*T = T$). It is known that T is a partial isometry if and only if T^* is a partial isometry. Partial isometries have been investigated by several authors (see, e.g., [4], [9], [10]). In particular, M. Mbekhta and L. Suciu [10] gave some results related to the problems of C. Badea and M. Mbekhta [4] concerning the similarity to partial isometries using the generalized inverses.

The article [8] extends the notions of left and right invertibility to *m-left* and *m-right invertibility*, respectively, on Banach spaces.

Definition 1.1. For some integer $m \geq 1$, an operator $T \in \mathcal{L}(X)$ is called

(1) *m-left invertible* if there exists $S \in \mathcal{L}(X)$ for which

$$S^m T^m - \binom{m}{1} S^{m-1} T^{m-1} + \cdots + (-1)^{m-1} \binom{m}{m-1} ST + (-1)^m I = 0$$

(in this case, S is called an *m-left inverse* for T);

(2) *m-right invertible* if there exists $R \in \mathcal{L}(X)$ for which

$$T^m R^m - \binom{m}{1} T^{m-1} R^{m-1} + \cdots + (-1)^{m-1} \binom{m}{m-1} TR + (-1)^m I = 0.$$

In the latter case, R is called an *m-right inverse* for T, where $\binom{m}{j}$ is the binomial coefficient.

If $T \in \mathcal{L}(X)$ is both m-left and m-right invertible, we say that T is *m-invertible*.

Remark 1.2. An 1-left inverse (resp., 1-right inverse) for T is a left inverse (resp., right inverse) for T.

The set of all m-left invertible operators in $\mathcal{L}(X)$ will be denoted by $L^m(X)$. For $T \in L^m(X)$, we denote by $\mathfrak{L}^m(T)$ the set of all m-left inverses of T; that is,

$$\mathfrak{L}^m(T) = \left\{ S \in \mathcal{L}(X) : \sum_{j=0}^{m} (-1)^j \binom{m}{j} S^{m-j} T^{m-j} = 0 \right\}.$$

The set of all m-right invertible operators in $\mathcal{L}(X)$ will be denoted by $R^m(H)$. For $T \in R^m(X)$, we denote by $\mathfrak{R}^m(T)$ the set of all m-right inverses of T; that is,

$$\mathfrak{R}^m(T) = \left\{ R \in \mathcal{L}(X) : \sum_{j=0}^{m} (-1)^j \binom{m}{j} T^{m-j} R^{m-j} = 0 \right\}.$$

An operator $T \in \mathcal{L}(H)$ is called an *m-isometry* if

$$\sum_{j=0}^{m} (-1)^j \binom{m}{j} T^{*m-j} T^{m-j} = 0;$$

that is, $T \in L^m(H)$ and $T^* \in \mathfrak{L}^m(T)$. Evidently, an isometry (i.e., a 1-isometry) is an m-isometry for all integers $m \geq 1$. A detailed study of this class on Hilbert spaces has been the object of some intensive study, especially by J. Agler and M. Stankus in [1], [2], and [3], and by S. Shimorin in [13]. Also, we refer the reader to [11] for more information about 2-isometries.

In [12], A. Saddi and O. A. Mahmoud Sid Ahmed gave a generalization of partial isometries and m-isometries to m-partial isometries on Hilbert spaces. An operator $T \in \mathcal{L}(H)$ is called an *m-partial isometry* for some integer $m \geq 1$ if

$$T \sum_{k=0}^{m} (-1)^k \binom{m}{k} T^{*m-k} T^{m-k} = 0 \quad \text{in } \mathcal{L}(H).$$

The case when $m = 1$ represents the partial isometries class. It is easily seen that an injective m-partial isometry is an m-isometry. An elementary operator theory of m-partial isometries is discussed in [12].

For an operator $T \in \mathcal{L}(X)$, the *reduced minimum modulus* is defined by

$$\gamma(T) := \begin{cases} \inf\{\|Tx\| : \operatorname{dist}(x, N(T)) = 1\} & \text{if } T \neq 0, \\ +\infty & \text{if } T = 0. \end{cases}$$

It is well known that $\gamma(T) > 0$ if and only if $R(T)$ is closed. Moreover, we have $\gamma(T) = \gamma(T^*)$.

The present paper is organized as follows. In Section 2, we generalize the notions of all classes already mentioned to m-left generalized inverses and m-right generalized inverses. We also extend some well-known results. In Section 3, we study some cases in which a power of an m-left (resp., m-right) generalized invertible operator is again an m-left (resp., m-right) generalized invertible operator.

2. m-Generalized invertible operators

Inspired by the above definitions of left generalized inverse and right generalized inverse and the work of m-partial isometries on Hilbert spaces (see [12]) and the work on m-left inverses and m-right inverses on Banach spaces (see [8]), we introduce the notions of m-left generalized inverse and m-right generalized inverse.

Definition 2.1. Let $m \geq 1$ be an integer, and let $T \in \mathcal{L}(X)$.

(1) (i) An operator $B \in \mathcal{L}(X)$ is called an *m-left generalized inverse* of T if

$$T\sum_{j=0}^{m}(-1)^j\binom{m}{j}B^{m-j}T^{m-j} = 0,$$

(ii) $R \in \mathcal{L}(X)$ is called an *m-right generalized inverse* of T if

$$R\sum_{j=0}^{m}(-1)^j\binom{m}{j}T^{m-j}R^{m-j} = 0.$$

(2) An operator $S \in \mathcal{L}(X)$ is called an *m-generalized inverse* of T if S is both an m-left and m-right generalized inverse of T; that is,

$$T\sum_{j=0}^{m}(-1)^j\binom{m}{j}S^{m-j}T^{m-j} = 0$$

and

$$S\sum_{j=0}^{m}(-1)^j\binom{m}{j}T^{m-j}S^{m-j} = 0.$$

The set of all m-left generalized invertible operators in $\mathcal{L}(X)$ will be denoted by $L_G^m(X)$. For $T \in L_G^m(X)$, we denote by $\mathfrak{L}_G^m(T)$ the set of all m-left generalized inverses of T; that is,

$$\mathfrak{L}_G^m(T) = \left\{B \in \mathcal{L}(X) : T\sum_{j=0}^{m}(-1)^j\binom{m}{j}B^{m-j}T^{m-j} = 0\right\}.$$

The set of all m-right generalized invertible operators in $\mathcal{L}(X)$ will be denoted by $R_G^m(X)$. For $T \in R_G^m(X)$, we denote by $\mathfrak{R}_G^m(T)$ the set of all m-right generalized inverses of T; that is,

$$\mathfrak{R}_G^m(T) = \left\{R \in \mathcal{L}(X) : R\sum_{j=0}^{m}(-1)^j\binom{m}{j}T^{m-j}R^{m-j} = 0\right\}.$$

Remark 2.2. Let T be in $\mathcal{L}(X)$. Then

(1) a 1-left (resp., a 1-right) generalized inverse of T is a left (resp., a right) generalized inverse of T;
(2) a 1-generalized inverse of T is a generalized inverse of T;
(3) $T \in L_G^m(X)$ and $B \in \mathfrak{L}_G^m(T)$ if and only if $B \in R_G^m(X)$ and $T \in \mathfrak{R}_G^m(B)$.

It is clear that we have the following.

Proposition 2.3.

(1) *We have* $L^m(X) \subset L^m_G(X)$ *and* $R^m(X) \subset R^m_G(X)$. *More precisely, if* $T \in L^m(X)$ *(resp.,* $T \in R^m(X)$*), then* $\mathfrak{L}^m(T) \subset \mathfrak{L}^m_G(T)$ *(resp.,* $\mathfrak{R}^m(T) \subset \mathfrak{R}^m_G(T)$*).*

In particular,

(2) $T \in \mathcal{L}(H)$ *is an* m*-partial isometry if and only if* $T \in L^m_G(H)$ *and* $T^* \in \mathfrak{L}^m_G(T)$.

Example 2.4. Consider the operator $T = \left[\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\right]$ and an arbitrary operator $S = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ acting on $H = \mathbb{C}^2$. An easy computation shows that $S^2T^2 - 2ST + I \neq 0$ for all complex numbers a, b, c, and d. Thus $T \notin L^2(H)$. Now, for $S = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$, it is easy to see that $T(S^2T^2 - 2ST + I) = 0$. Thus $T \in L^2_G(H)$ and $S \in \mathfrak{L}^2_G(T)$. This justifies the definitions of $L^m_G(X)$ and $R^m_G(X)$.

It is clear that if $S \in \mathcal{L}(X)$ is a generalized inverse of $T \in \mathcal{L}(X)$, then $P = TS$ and $Q = ST$ are idempotents (i.e., $P^2 = P$ and $Q^2 = Q$), $R(T) = R(P)$, and $N(T) = N(Q) = R(I - Q)$.

In the remainder of this paper, if S is an m-left generalized inverse of T, then we set

$$Q_m = \sum_{j=0}^{m-1} (-1)^j \binom{m}{j} S^{m-j} T^{m-j}.$$

Moreover, if S is an m-right generalized inverse of T, then we set

$$P_m = \sum_{j=0}^{m-1} (-1)^j \binom{m}{j} T^{m-j} S^{m-j}.$$

Clearly, we have $TQ_m = (-1)^{m+1}T$. In particular, S is an m-left inverse of T if and only if $Q_m = (-1)^{m+1}I$.

Proposition 2.5. *If* $T \in L^m_G(X)$ *and* $S \in \mathfrak{L}^m_G(T)$*, then we have the following.*

(1) $N(Q_m) = N(T) = R((-1)^{m+1}I - Q_m)$. *In particular,* $Q_m^2 = (-1)^{m+1}Q_m$, *and if* m *is an odd integer, then* Q_m *is idempotent.*
(2) $R(Q_m) = N((-1)^{m+1}I - Q_m)$. *In particular,* $x \in R(Q_m)$ *if and only if* $Q_m x = (-1)^{m+1}x$.
(3) $N(Q_m)$ *and* $R(Q_m)$ *are algebraically complemented subspaces of* X*; that is,* $X = N(Q_m) \oplus R(Q_m)$.

Proof.

(1) It is clear that $N(T) \subset N(Q_m)$, and since $TQ_m = (-1)^{m+1}T$, we also have $R((-1)^{m+1}I - Q_m) \subseteq N(T)$. Now, let $x \in N(Q_m)$, and since $(-1)^{m+1}x = ((-1)^{m+1}I - Q_m)x \in R((-1)^{m+1}I - Q_m)$, we get $x \in R((-1)^{m+1}I - Q_m)$.
(2) The inclusion $N((-1)^{m+1}I - Q_m) \subseteq R(Q_m)$ is obvious. Now, suppose that $x \in R(Q_m)$. Then $x = Q_m u$ for some $u \in X$. We have $((-1)^{m+1}I - Q_m)x = ((-1)^{m+1}Q_m - Q_m^2)u = 0$, and hence $x \in N((-1)^{m+1}I - Q_m)$.

(3) It is easily seen that $X = R((-1)^{m+1}I - Q_m) + R(Q_m)$. But we have $R((-1)^{m+1}I - Q_m) = N(Q_m)$, and thus $X = N(Q_m) + R(Q_m)$. Since $N(Q_m) \cap R(Q_m) = N(Q_m) \cap N((-1)^{m+1}I - Q_m) = \{0\}$, we have the result. □

In the following proposition, we generalize Proposition 2.2 in [9].

Proposition 2.6. *Let $T \in \mathcal{L}(X)$, and let S be an m-generalized inverse of T. Then*

$$\frac{1}{m\|S\|(1+\|S\|\|T\|)^{m-1}} \leq \gamma(T) \leq \frac{\|TS\|\|Q_m\|}{\|Q_mS\|}.$$

Proof. Consider an arbitrary vector $x \in X$. We have

$$\begin{aligned}
\|Q_mx\| &= \left\|\sum_{j=0}^{m-1}(-1)^j\binom{m}{j}S^{m-j}T^{m-j}x\right\| \\
&\leq \|S\|\left\|\sum_{j=0}^{m-1}\binom{m}{j}\|S\|^{m-1-j}\|T\|^{m-1-j}\right\|\|Tx\| \\
&\leq m\|S\|\big(1+\|S\|\|T\|\big)^{m-1}\|Tx\|,
\end{aligned}$$

where the last inequality follows since $\binom{m}{j} \leq m\binom{m-1}{j}$ for $0 \leq j \leq m-1$. On the other hand, $(-1)^mx + Q_mx \in N(T)$, and thus

$$\operatorname{dist}\big(x, N(T)\big) = \operatorname{dist}\big(Q_mx, N(T)\big) \leq \|Q_mx\| \leq m\|S\|\big(1+\|S\|\|T\|\big)^{m-1}\|Tx\|.$$

Therefore,

$$\frac{1}{m\|S\|(1+\|S\|\|T\|)^{m-1}} \leq \gamma(T).$$

For the second inequality, let $v \in X$, and let $x = Q_mSv$. Since $SP_mv = (-1)^{m+1}Sv$, we have $TQ_mSP_mv = (-1)^{m+1}TQ_mSv = (-1)^{m+1}Tx$. But $SP_m = (-1)^{m+1}S$ and $TQ_m = (-1)^{m+1}T$, and thus $TSv = (-1)^{m+1}Tx$. On the other hand, for $\varepsilon > 0$, there exists $u \in N(T)$ such that $\operatorname{dist}(x, N(T)) \geq \|x+u\| - \varepsilon$. Therefore, it follows that

$$\|x+u\| \leq \operatorname{dist}\big(x, N(T)\big) + \varepsilon \leq \frac{1}{\gamma(T)}\|Tx\| + \varepsilon = \frac{1}{\gamma(T)}\|TSv\| + \varepsilon.$$

Now, since $x \in R(Q_m)$ and $N(Q_m) = N(T)$, from Proposition 2.5 we have $Q_m(x+u) = Q_mx = (-1)^{m+1}x$. Therefore,

$$\|Q_mSv\| = \|x\| = \big\|Q_m(x+u)\big\| \leq \|Q_m\|\|x+u\| \leq \|Q_m\|\Big\{\frac{1}{\gamma(T)}\|TS\|\|v\| + \varepsilon\Big\}.$$

Because $\varepsilon > 0$ is arbitrary, for every $v \in X$, we obtain

$$\|Q_mSv\| \leq \frac{1}{\gamma(T)}\|TS\|\|Q_m\|\|v\|.$$

The result is proved. □

For $m = 1$, S is a generalized inverse of T, $Q_m = Q = ST$, $P_m = P = TS$, $Q_m S = S$, and $m\|S\|(1+\|S\|\|T\|)^{m-1} = \|S\|$. Therefore, we retrieve the following result given in [9].

Corollary 2.7 ([9, Proposition 2.2])**.** *Let $T \in \mathcal{L}(X)$, and let S be a generalized inverse of T. Then*

$$\frac{1}{\|S\|} \leq \gamma(T) \leq \frac{\|P\|\|Q\|}{\|S\|}.$$

Corollary 2.8. *If T is m-invertible and S is an m-left inverse of T, then*

$$\frac{1}{m\|S\|(1+\|S\|\|T\|)^{m-1}} \leq \gamma(T) \leq \frac{\|TS\|}{\|S\|}.$$

Proof. Since $Q_m = (-1)^{m+1}I$, we have $\frac{\|TS\|\|Q_m\|}{\|Q_m S\|} = \frac{\|TS\|}{\|S\|}$. $\square$

Corollary 2.9. *If $T \in L^m_G(X)$, then $\gamma(T) > 0$. In particular, if $T \in L^m_G(X)$, then $R(T)$ is closed.*

Recall that $T \in \mathcal{L}(H)$ is an m-isometry if and only if it is an injective m-partial isometry. In the following we extend this property.

Proposition 2.10. *If $T \in \mathcal{L}(X)$, then the following assertions are equivalent:*

(1) *$T \in L^m_G(X)$ and T is injective,*
(2) *$T \in L^m(X)$.*

Proof. (1) $\Longrightarrow$ (2): Suppose that $T \in L^m_G(X)$ is injective, and let S be an m-left generalized inverse of T. By Proposition 2.5, we have $R((-1)^{m+1}I - Q_m) = N(T) = \{0\}$. This implies that $Q_m = (-1)^{m+1}I$, and thus $T \in L^m(X)$.

(2) $\Longrightarrow$ (1): Let T be in $L^m(X)$. Since $L^m(X) \subset L^m_G(X)$, it suffices to show that T is injective. Since $Q_m = (-1)^{m+1}I$, according to Proposition 2.5, we have $N(T) = N(Q_m) = N(I) = \{0\}$. The proof is completed. $\square$

The following result extends Theorem 3.1 given in [12].

Theorem 2.11. *If $T, S \in \mathcal{L}(H)$ such that $N(T)^{\perp}$ is an invariant subspace for both T and S, then the following properties are equivalent:*

(1) *$T \in L^m_G(H)$ and $S \in \mathfrak{L}^m_G(T)$,*
(2) *$T_{|N(T)^{\perp}} \in L^m(H)$ and $S_{|N(T)^{\perp}} \in \mathfrak{L}^m(T)$.*

Proof. (1) $\Longrightarrow$ (2): Suppose that $T \in L^m_G(H)$, let $S \in \mathfrak{L}^m_G(T)$, and let x be in $N(T)^{\perp}$. Since by assumption $N(T)^{\perp}$ is an invariant subspace for both T and S, we have

$$\sum_{j=0}^{m} \binom{m}{j} (-1)^j S^{m-j} T^{m-j} x \in N(T)^{\perp}.$$

On the other hand,

$$\sum_{j=0}^{m} \binom{m}{j} (-1)^j S^{m-j} T^{m-j} x \in N(T);$$

thus,

$$\sum_{j=0}^{m}\binom{m}{j}(-1)^j S^{m-j}T^{m-j}x = 0$$

for all $x \in N(T)^{\perp}$, and so $T_{|N(T)^{\perp}} \in L^m(H)$ and $S_{|N(T)^{\perp}} \in \mathfrak{L}^m(T)$.

(2) $\Longrightarrow$ (1): Let $x \in H$ such that $x = x_1 + x_2$ with $x_1 \in N(T)$ and $x_2 \in N(T)^{\perp}$. We have

$$T\sum_{j=0}^{m}\binom{m}{j}(-1)^j S^{m-j}T^{m-j}x = T\sum_{j=0}^{m}\binom{m}{j}(-1)^j S^{m-j}T^{m-j}x_2.$$

But by assumption we have

$$\sum_{j=0}^{m}\binom{m}{j}(-1)^j S^{m-j}T^{m-j}x_2 = 0,$$

and thus

$$T\sum_{j=0}^{m}\binom{m}{j}(-1)^j S^{m-j}T^{m-j}x = 0.$$

Since $x \in H$ is arbitrary, $T \in L_G^m(H)$ and $S \in \mathfrak{L}_G^m(T)$. The result is obtained. □

Corollary 2.12. *If $T, R \in \mathcal{L}(H)$ such that $N(R)^{\perp}$ is an invariant subspace for both T and R, then the following properties are equivalent:*

(1) *$T \in R_G^m(H)$ and $R \in \mathfrak{R}_G^m(T)$,*
(2) *$T_{|N(R)^{\perp}} \in R^m(H)$ and $R_{|N(R)^{\perp}} \in \mathfrak{R}^m(T)$.*

From the Theorem 2.11, we conclude Theorem 3.1 from [12] alternatively.

Corollary 2.13 ([12, Theorem 3.1])**.** *If $T \in \mathcal{L}(H)$ and $N(T)$ is a reducing subspace for T, then the following properties are equivalent:*

(1) *T is an m-partial isometry,*
(2) *$T_{|N(T)^{\perp}}$ is an m-isometry.*

Proof. (1) $\Longrightarrow$ (2): Since $N(T)$ is reducing for T, we see that $(T_{|N(T)^{\perp}})^* = T^*_{|N(T)^{\perp}} = S_{|N(T)^{\perp}}$ where $S = T^*$. Moreover, $N(T)^{\perp}$ is an invariant subspace for both T and S. Since T is an m-partial isometry, by Proposition 2.3, we have $T \in L_G^m(H)$ and $S \in \mathfrak{L}_G^m(T)$. Now, by Theorem 2.11, we get $T_{|N(T)^{\perp}} \in L^m(H)$ and $(T_{|N(T)^{\perp}})^* = S_{|N(T)^{\perp}} \in \mathfrak{L}^m(T)$. Hence $T_{|N(T)^{\perp}}$ is an m-isometry.

(2) $\Longrightarrow$ (1): Suppose that $T_{|N(T)^{\perp}}$ is an m-isometry. Then $T_{|N(T)^{\perp}} \in L^m(H)$ and $(T_{|N(T)^{\perp}})^* \in \mathfrak{L}^m(T)$. But $(T_{|N(T)^{\perp}})^* = T^*_{|N(T)^{\perp}}$, and thus $T^*_{|N(T)^{\perp}} \in \mathfrak{L}^m(T)$. From Theorem 2.11, $T \in L_G^m(H)$ and $T^* \in \mathfrak{L}_G^m(T)$, and by Proposition 2.3 we infer that T is an m-partial isometry. □

3. Power of m-left and m-right generalized invertible operators

In [11, Theorem 2.1], S. M. Patel showed that a power of 2-isometry is again a 2-isometry. This result was extended in [8] for 2-left and 2-right invertible operators. Another result for m-partial operators is given in [7, Theorem 2.16]. In the following, we extend this result more generally for 2-left generalized and 2-right generalized invertible operators.

Theorem 3.1. *Let $T \in L_G^2(H)$, and let $B \in \mathfrak{L}_G^2(T)$. If $N(T)^\perp$ is an invariant subspace for both T and S, then $T^n \in L_G^2(H)$ and $B^n \in \mathfrak{L}_G^2(T^n)$ for all $n \in \mathbb{N}$.*

Proof. Let $n \geq 0$ be an integer. From Theorem 2.11, $T_{|N(T)^\perp} \in L^2(H)$ and $B_{|N(T)^\perp} \in \mathfrak{L}^2(T)$. According to [8, Proposition 3.1], we have $T^n_{|N(T)^\perp} \in L^2(H)$ and $B^n_{|N(T)^\perp} \in \mathfrak{L}^2(T^n)$. Now, by Theorem 2.11, we derive that $T^n \in L_G^2(H)$ and $B^n \in \mathfrak{L}_G^2(T^n)$. $\square$

Lemma 3.2.

(1) *Let $T \in L_G^2(X)$, and let $B \in \mathfrak{L}_G^2(T)$ such that $BT = TB$. Then*

$$TB^kT^k = kTBT - (k-1)T, \quad k = 0, 1, \ldots.$$

(2) *If $T \in R_G^2(X)$ and $R \in \mathfrak{R}_G^2(T)$ such that $RT = TR$, then*

$$RT^kR^k = kRTR - (k-1)R, \quad k = 0, 1, \ldots.$$

Proof.

(1) We will proceed by induction on k. For $k = 0, 1$ there is nothing to prove. Since $B \in \mathfrak{L}_G^2(T)$, we have

$$T(B^2T^2 - 2BT + I) = 0,$$

and thus

$$TB^2T^2 = 2TBT - T.$$

Then the equation is verified for $k = 2$. Now, suppose that $TB^kT^k = kTBT - (k-1)T$ for some k. We have

$$\begin{aligned}
TB^{k+1}T^{k+1} &= TBB^kT^kT \\
&= BTB^kT^kT \quad (\text{since } BT = TB) \\
&= B\big(kTBT - (k-1)T\big)T \quad (\text{by assumption}) \\
&= kTB^2T^2 - (k-1)BT^2 \quad (\text{since } BT = TB) \\
&= k(2TBT - T) - (k-1)BT^2 \quad (TB^2T^2 = 2TBT - T) \\
&= 2kTBT - kT - (k-1)TBT \quad (\text{since } BT = TB) \\
&= (2k - k + 1)TBT - kT \\
&= (k+1)TBT - kT.
\end{aligned}$$

(2) Since R is a 2-right generalized inverse of T, then T is a 2-left generalized inverse of R, and the result follows from the first part. $\square$

Theorem 3.3. *Let T be in $L_G^2(X)$, and let m be an integer such that $m \geq 1$. If there exists an operator $B \in \mathfrak{L}_G^2(T)$ such that $BT = TB$, then $T^n \in L_G^m(X)$ and $B^n \in \mathfrak{L}_G^m(T^n)$ for all integers n.*

Proof. Suppose that $B \in \mathfrak{L}_G^2(T)$ is such that $BT = TB$ and $m \geq 1$ is an integer. Since $\binom{m}{k}k = m\binom{m-1}{k-1}$ for $k = 1, 2, \ldots, m$, we have

$$\begin{aligned}
\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}k &= \sum_{k=1}^{m}(-1)^{m-k}\binom{m}{k}k \\
&= \sum_{k=0}^{m-1}(-1)^{m-1-k}\binom{m-1}{k}m \\
&= (-1+1)^{m-1}m \\
&= 0.
\end{aligned}$$

On the other hand, from Lemma 3.2 we have $TB^{nk}T^{nk} = nkTBT - (nk-1)T$ for $k = 0, 1, 2, \ldots$. Thus, for all $n \geq 1$, we have

$$\begin{aligned}
T^n \sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}B^{nk}T^{nk} &= T^{n-1}\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}TB^{nk}T^{nk} \\
&= T^{n-1}\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\big(nkTBT - (nk-1)T\big) \\
&= n\underbrace{\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}k}_{=0}\,T^n(BT - I) \\
&\quad + \underbrace{\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}}_{=0}T^n \\
&= 0.
\end{aligned}$$

□

Corollary 3.4. *Let T be a 2-right generalized invertible operator, and let m be an integer such that $m \geq 1$. If there exists an operator $R \in \mathfrak{R}_G^2(T)$ such that $RT = TR$, then $T^n \in R_G^m(X)$ and $R^n \in \mathfrak{R}_G^m(T^n)$ for all integers n.*

It is well known from [7, Proposition 2.2] that if T is an m-partial isometry such that T^k is a partial isometry for $k = 1, \ldots, m-1$, for some integer $m \geq 2$, then the power T^m is a partial isometry. In the following, we generalize this result.

Proposition 3.5. *Let $T \in \mathcal{L}(X)$ be in $L_G^m(X)$, and let $B \in \mathfrak{L}_G^m(T)$ for some integer $m \geq 2$. If $T^k \in L_G^1(X)$ and $B^k \in \mathfrak{L}_G^1(T^k)$ for $k = 0, 1, \ldots, m-1$, then $T^m \in L_G^1(X)$ and $B^m \in \mathfrak{L}_G^1(T^m)$.*

Proof. Since T is an m-left generalized invertible operator and $B \in \mathfrak{L}_G^m(T)$, we have

$$T\sum_{k=0}^{m}(-1)^k\binom{m}{k}B^{m-k}T^{m-k} = 0.$$

Multiplying the above equation from the left by T^{m-1}, we get

$$T^mB^mT^m + \sum_{k=1}^{m}(-1)^k\binom{m}{k}T^mB^{m-k}T^{m-k} = 0.$$

But by assumption we have $T^kB^kT^k = T^k$ for $k = 1, \dots, m-1$, and thus

$$\begin{aligned} T^mB^{m-k}T^{m-k} &= T^kT^{m-k}B^{m-k}T^{m-k} \\ &= T^kT^{m-k} \\ &= T^m. \end{aligned}$$

Therefore,

$$T^mB^mT^m + \sum_{k=1}^{m}(-1)^k\binom{m}{k}T^m = 0.$$

Since $\sum_{k=1}^{m}(-1)^k\binom{m}{k} = -1$, we get

$$T^mB^mT^m = T^m.$$

Therefore, $T^m \in L^1_G(X)$ and $B^m \in \mathfrak{L}^1_G(T^m)$. □

Theorem 3.6. *Let $T \in \mathcal{L}(X)$ be in $L^m_G(X)$, and let $B \in \mathfrak{L}^m_G(T)$ for some integer $m \geq 1$. If $S \in L^1(X)$ and $A \in \mathfrak{L}^1(S)$ are such that S and A commute with both T and B, then $TS \in L^m_G(X)$ and $AB \in \mathfrak{L}^m_G(ST)$.*

Proof. We have

$$\begin{aligned} &TS\sum_{k=0}^{m}(-1)^k\binom{m}{k}(AB)^{m-k}(TS)^{m-k} \\ &= TS\sum_{k=0}^{m}(-1)^k\binom{m}{k}B^{m-k}T^{m-k}(A^{m-k}S^{m-k}) \\ &= \underbrace{T\sum_{k=0}^{m}(-1)^k\binom{m}{k}B^{m-k}T^{m-k}}_{=0}S \\ &= 0, \end{aligned}$$

and the result is obtained. □

Corollary 3.7 ([12, Proposition 3.2])**.** *Let $T, S \in \mathcal{L}(H)$ be such that T is an m-partial isometry and S is an isometry with $TS = ST$ and $TS^* = S^*T$. Then TS is an m-partial isometry.*

Proof. Since T is an m-partial isometry, $T \in L^m_G(X)$ and $T^* \in \mathfrak{L}^m_G(T)$. Let $A = S^*$, and let $B = T^*$. It is clear that all the conditions of Theorem 3.6 are satisfied, and thus we have the result. □

Proposition 3.8 ([8, Proposition 3.3])**.** *We have the following inclusions.*

1. *If $T \in L^m(X)$, then $\mathfrak{L}^m(T) \subset \mathfrak{L}^{m+k}(T), k \in \mathbb{N}$.*
2. *If $T \in R^m(X)$, then $\mathfrak{R}^m(T) \subset \mathfrak{R}^{m+k}(T), k \in \mathbb{N}$.*

The following result is given in [12].

Proposition 3.9 ([12, Proposition 3.5])**.** *Let $T \in \mathcal{L}(H)$ be an m-partial isometry such that $N(T)$ is a reducing subspace for T. Then T is an $(m+n)$-partial isometry for $n = 0, 1, 2, \dots$.*

In the following, we generalize the previous result for m-left generalized invertible operators.

Theorem 3.10. *Let $T \in \mathcal{L}(H)$ be in $L_G^m(H)$, and let $S \in \mathfrak{L}_G^m(T)$. If $N(T)^\perp$ is an invariant subspace for both T and S, then $T \in L_G^{m+n}(H)$ and $S \in \mathfrak{L}_G^{m+n}(T)$ for $n = 0, 1, 2, \dots$.*

Proof. Since $T \in L_G^m(H)$ and $S \in \mathfrak{L}_G^m(T)$, from Theorem 2.11 we have $T_{|N(T)^\perp} \in L^m(H)$ and $S_{|N(T)^\perp} \in \mathfrak{L}^m(T)$. Now, by Proposition 3.8, we get $T_{|N(T)^\perp} \in L^{m+n}(H)$ and $S_{|N(T)^\perp} \in \mathfrak{L}^{m+n}(T)$ for $n = 0, 1, 2, \dots$. From Theorem 2.11 again we get the desired result. ☐

Corollary 3.11. *Let $T \in \mathcal{L}(H)$ be an m-right generalized invertible operator, and let $R \in \mathfrak{R}_G^m(T)$. If $N(R)^\perp$ is an invariant subspace for both T and R, then $T \in R_G^{m+n}(H)$ and $R \in \mathfrak{R}_G^{m+n}(T)$ for $n = 0, 1, 2, \dots$.*

Proof. Since $T \in R_G^m(H)$ and $R \in \mathfrak{R}_G^m(T)$, we have $R \in L_G^m(H)$ and $T \in \mathfrak{L}_G^m(R)$ and the result follows from Theorem 3.10. ☐

S. Hamidou Jah proved in [7, Theorem 2.12] that if T is an m-partial isometry such that T is an m-isometry on $R(T)$, then T is an $(m+1)$-partial isometry. In the following, we generalize this result.

Theorem 3.12. *Let $T \in \mathcal{L}(X)$ be an m-left generalized invertible operator, and let $B \in \mathfrak{L}_G^m(T)$. If T is m-left invertible on $R(T)$ and B is an m-left inverse of T on $R(T)$, then $T \in L_G^{m+1}(X)$ and $B \in \mathfrak{L}_G^{m+1}(T)$.*

Proof. The proof outlines the one of Theorem 2.12 given in [7]:

$$
\begin{aligned}
&T\sum_{k=0}^{m+1}(-1)^k\binom{m+1}{k}B^{m+1-k}T^{m+1-k}\\
&\quad= T\Big(B^{m+1}T^{m+1} + \sum_{k=1}^{m}(-1)^k\Big\{\binom{m}{k} + \binom{m}{k-1}\Big\}B^{m+1-k}T^{m+1-k} - (-1)^m I\Big)\\
&\quad= T\Big(B^{m+1}T^{m+1} + \sum_{k=1}^{m}(-1)^k\binom{m}{k}B^{m+1-k}T^{m+1-k}\Big)\\
&\qquad + T\Big(\sum_{k=1}^{m}(-1)^k\binom{m}{k-1}B^{m+1-k}T^{m+1-k} - (-1)^m I\Big)\\
&\quad= TB\Big(B^mT^m + \sum_{k=1}^{m}(-1)^k\binom{m}{k}B^{m-k}T^{m-k}\Big)T - \underbrace{T\sum_{k=0}^{m}(-1)^k\binom{m}{k}B^{m-k}T^{m-k}}_{=0}
\end{aligned}
$$

$$= TB \underbrace{\sum_{k=0}^{m} (-1)^k \binom{m}{k} B^{m-k} T^{m-k} T}_{=0} \quad \big(B \text{ is an } m\text{-left inverse of } T \text{ on } R(T)\big)$$
$$= 0. \qquad \square$$

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Hamid Ezzahraoui, Faculty of Sciences, Mohamed V University, BP 1014 Rabat, Morocco.

E-mail address: ezzahraouifsr@gmail.com