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THE HANKEL OPERATORS AND NONCOMMUTATIVE BMO SPACES

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Abstract. Let $\mathcal{M}$ be a von Neumann algebra with a faithful normal semifinite trace τ. The noncommutative Hardy space $H^p(\mathcal{M})$ associates with $\mathcal{A}$, which is a subdiagonal algebra of $\mathcal{M}$. We define the Hankel operator H_t on $H^p(\mathcal{M})$, and we obtain that the norm $\|H_t\|$ is equal to $d(t;\mathcal{A})$ and is also the equivalent of the BMO($\mathcal{M}^{\mathrm{sa}}$) norm of t for every $t \in \mathcal{M}$, where $\mathcal{M}^{\mathrm{sa}}$ are the self-adjoint operators in $\mathcal{M}$.

1. Introduction and preliminaries

In [1], Arverson introduced the subdiagonal algebras as noncommutative analogues of weak-$*$ Dirichlet algebras $\mathcal{A}$ of $\mathcal{M}$ for the von Neumann algebra $\mathcal{M}$ with a faithful normal finite trace. The noncommutative $H^p(\mathcal{M})$ spaces associated with such algebras are studied by several authors in [2], [3], [6], [7], and [11]. In particular, Nehari's problem of a noncommutative Hankel operator associated with a finite and σ-finite subdiagonal algebra is considered, and the noncommutative analogue of the classical results is shown to be valid (see [5], [6], [10]). The distance formulas for Toeplitz and Hankel operators associated with a subdiagnoal algebra are established in [10] and [12]. We now consider the Hankel operator on a noncommutative Hardy space associated with a semifinite von Neumann algebra.

Throughout the present article, $\mathcal{M}$ will denote a semifinite von Neumann algebra possessing a normal semifinite faithful trace τ. Let $x = u|x|$ be the polar decomposition of x. Let $r(x) = u^*u$, and let $\ell(x) = uu^*$. We call $r(x)$ and $\ell(x)$

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the *left* and *right* supports of x, respectively. If x is self-adjoint, then $\ell(x) = r(x)$. This common projection is then called the *support* of x, and is denoted by $s(x)$. Let $\mathcal{S}_+(\mathcal{M}) = \{x \in M_+ : \tau(s(x)) < \infty\}$ and $\mathcal{S}(\mathcal{M})$ be the linear span of $\mathcal{S}_+(\mathcal{M})$. A projection e is said to be of *finite trace* if $\tau(e) < \infty$. The elements of $\mathcal{S}(\mathcal{M})$ are said to be *supported by projection* of finite trace. We will often denote $\mathcal{S}(\mathcal{M})$ simply as $\mathcal{S}$. $L^p(\mathcal{M})$ is the noncommutative L^p-space associated with $(\mathcal{M}, \tau)$ (see [9], [11]). For $X \subset L^p(\mathcal{M})$, $[X]_p$ denotes the closure of X in $L^p(\mathcal{M})$, and $\mathrm{J}(X)$ is the family of the adjoints of the elements of X. Let $\mathcal{A}$ be a weak-$*$ closed unital subalgebra of $\mathcal{M}$ with semifinite subdiagonal $\mathcal{D} = \mathcal{A} \cap \mathrm{J}(\mathcal{A})$, and let $\mathcal{E}$ be the faithful normal conditional expectation from $\mathcal{M}$ onto $\mathcal{D}$. In addition, $\mathcal{A}$ is a subdiagonal subalgebra of $\mathcal{M}$ with respect to $\mathcal{E}$ if:

- $\mathcal{A} + \mathrm{J}(\mathcal{A})$ is weak-$*$-dense in $\mathcal{M}$,
- $\mathcal{E}$ is multiplicative on $\mathcal{A}$, and
- $\tau \circ \mathcal{E} = \tau$.

We say that $\mathcal{A}$ is a *maximal subdiagonal algebra* in $\mathcal{M}$ with respect to $\mathcal{E}$ in the case that $\mathcal{A}$ is not properly contained in any other subalgebra of $\mathcal{M}$ which is subdiagonal with respect to $\mathcal{E}$. It was proved by Ji in [4] that a semifinite subdiagonal algebra $\mathcal{A}$ is automatically maximal. For $p \in [1, \infty)$, the closure $[\mathcal{A} \cap \mathcal{S}]_p$ in $L^p(\mathcal{M})$ is denoted by $H^p(\mathcal{M})$. Let the closure $[\mathcal{A}_0 \cap \mathcal{S}]_p$ be denoted by $H^p(\mathcal{M})$, where $\mathcal{A}_0 = \{x \in \mathcal{A} : \mathcal{E}(x) = 0\}$. There exists an increasing family $\{e_\lambda\}_{\lambda \in \Lambda}$ of the projections of $\mathcal{D}$ such that $\tau(e_\lambda) < \infty$ for every $\lambda \in \Lambda$, and e_λ converges in the strong operator topology to the unit element 1 of $\mathcal{M}$. Let $\mathcal{M}_{e_\lambda} = e_\lambda \mathcal{M} e_\lambda$, let $\mathcal{A}_{e_\lambda} = e_\lambda \mathcal{A} e_\lambda$, and let $\mathcal{D}_{e_\lambda} = e_\lambda \mathcal{D} e_\lambda$ for any $\lambda \in \Lambda$. Then $\mathcal{A}_{e_\lambda}$ is a maximal subdiagonal subalgebra of $\mathcal{M}_{e_\lambda}$. Since $e_\lambda \mathcal{M} e_\lambda \subset \mathcal{S}$, it follows that $L^p(\mathcal{M}_\lambda) \subset L^p(\mathcal{M})$ for all $\lambda \in \Lambda$. Moreover, for any $x \in L^p(\mathcal{M})$, we have $xe_\lambda \xrightarrow{\|\cdot\|_p} x$; hence, $L^p(\mathcal{M}) = [\bigcup_{\lambda \in \Lambda} L^p(\mathcal{M}_\lambda)]_p$, $H^p(\mathcal{M}) = [\bigcup_{\lambda \in \Lambda} H^p(\mathcal{M}_\lambda)]_p$, and $H_0^p(\mathcal{M}) = [\bigcup_{\lambda \in \Lambda} H_0^p(\mathcal{M}_\lambda)]_p$ (for details, see [2]).

Let e be a τ finite projection in $\mathcal{D}$. It is well known that we have the following decomposition:

$$\mathcal{A}_e + \mathrm{J}(\mathcal{A})_e = e\mathcal{A}_0 e \oplus \mathcal{D}_e \oplus e\mathrm{J}(\mathcal{A}_0)e.$$

The notion of Hilbert transforms (or conjugate operators) plays an important role for studying function spaces, and noncommutative settings have been considered by several authors. We recall that the Hilbert transform or the conjugate operator is defined as

$$\mathrm{H}(a + d + b^*) = -i(a - b^*), \quad a, b \in e\mathcal{A}_0 e, d \in \mathcal{D}_e$$

(see [2], [7], [9]). It is clear that $x + i\mathrm{H}(x) \in \mathcal{A}$ and $\mathrm{H}(x) \in \mathrm{Ker}(\mathcal{E})$ for all $x \in \mathcal{A}_e + \mathrm{J}(\mathcal{A})_e$. The Hilbert transform H can be extended to a bounded map on $L^p(\mathcal{M})$ with $1 < p < +\infty$ (see [2]), and the Riesz projection from $L^p(\mathcal{M})$ onto $H^p(\mathcal{M})$ is

$$\mathrm{P} = \frac{1}{2}(\mathrm{I} + i\mathrm{H} + \mathcal{E}).$$

Let X be a Banach space. For $x \in X$ and $Y \subset X^*$, which is the dual space of X, the notation $x \perp Y$ means that $f(x) = 0$ for all $f \in Y$, and the real dual space of X is denoted by $X^{\mathrm{re}*}$.

Proposition 1.1 ([2, Proposition 3.2])**.** *Let $\mathcal{A}$ be a subdiagonal algebra of $\mathcal{M}$.*

(1) *If $1 \le p \le \infty, \frac{1}{p} + \frac{1}{q} = 1$, then*

$$H^p(\mathcal{M}) = \big\{x \in L^p(\mathcal{M}) : x \perp \mathrm{J}\big(H_0^q(\mathcal{M})\big)\big\},$$
$$H_0^p(\mathcal{M}) = \big\{x \in L^p(\mathcal{M}) : x \perp \mathrm{J}\big(H^q(\mathcal{M})\big)\big\}.$$

(2) *If $1 < p < \infty$, then*

$$\begin{aligned} L^p(\mathcal{M}) &= H^p(\mathcal{M}) \oplus \mathrm{J}\big(H_0^p(\mathcal{M})\big) \\ &= H_0^p(\mathcal{M}) \oplus L^p(\mathcal{D}) \oplus \mathrm{J}\big(H_0^p(\mathcal{M})\big). \end{aligned}$$

(3) *If $1 < p < \infty, \frac{1}{p} + \frac{1}{q} = 1$, then $H^p(\mathcal{M})^* = H^q(\mathcal{M})$ isometrically, with associated duality bracket given by $\langle x, y\rangle = \tau(xy^*)$ for $x \in H^p(\mathcal{M})$, $y \in H^q(\mathcal{M})$.*

For $1 < p < \infty$, the Hilbert transform is used to establish the decomposition $L^p(\mathcal{M}) = H_0^p(\mathcal{M}) \oplus L^p(\mathcal{D}) \oplus \mathrm{J}(H_0^p(\mathcal{M}))$. Given $1 < p < \infty$, the following properties of the Hilbert transform are valid. If $x \in L^p(\mathcal{M})$ and $y \in L^q(\mathcal{M})$, then $\tau(x\mathrm{H}(y)) = -\tau(\mathrm{H}(x)y)$ (see [2]). In the $p = 1$ case, the Hilbert transform is unbounded and the decomposition is invalid, but we use the same method as in [2] and [8] to obtain the following results. For $1 \le p \le \infty$, $x \in \mathrm{Re}(H^p(\mathcal{M}))$, we have

- if $x \in \mathrm{Re}(H_0^p(\mathcal{M}))$, then $x + i\mathrm{H}(x) \in H_0^p(\mathcal{M})$;
- $x - \mathcal{E}(x) \in \mathrm{Re}(H_0^p(\mathcal{M}))$ and $\mathrm{H}(x) = \mathrm{H}(x - \mathcal{E}(x))$;
- $\mathrm{H}(\mathrm{H}(x)) = -(I - \mathcal{E})x$.

2. Toeplitz and Hankel operators

Let $\mathcal{A}$ be a subdiagonal algebra of $\mathcal{M}$, let $1 < p < \infty$, and let P be the Riesz projection from $L^p(\mathcal{M})$ to $H^p(\mathcal{M})$ and $t \in \mathcal{M}$. We respectively define the (left) Toelitz and Hankel operators with symbol t by $T_t = \mathrm{P}L_t\mathrm{P}$ and $H_t = (\mathrm{I} - \mathrm{P})L_t\mathrm{P}$, where the (left) multiplication operator L_t is defined as $L_t f = tf$ for all $f \in L^p(\mathcal{M})$. If the domain is $H^p(\mathcal{M})$, then

$$\begin{aligned} T_t : H^p(\mathcal{M}) &\to H^p(\mathcal{M}), \\ h &\mapsto \mathrm{P}(th) \end{aligned}$$

and

$$\begin{aligned} H_t : H^p(\mathcal{M}) &\to \mathrm{J}\big(H_0^p(\mathcal{M})\big), \\ h &\mapsto (\mathrm{I} - \mathrm{P})(th). \end{aligned}$$

Let $\xi \in H^p(\mathcal{M})$ and let $\eta \in H^q(\mathcal{M})$. Then $\langle T_{t^*}\xi, \eta\rangle = \langle \mathrm{P}(t^*\xi), \eta\rangle = \langle t^*\xi, \mathrm{P}(\eta)\rangle = \langle t^*\xi, \eta\rangle = \langle \xi, \mathrm{P}(t\eta)\rangle = \langle \xi, T_t\eta\rangle = \langle T_t^*\xi, \eta\rangle$. Furthermore, we have

$$\begin{aligned} \langle T_{t_1}T_{t_2}\xi, \eta\rangle &= \big\langle \mathrm{P}L_{t_1}\mathrm{P}(t_2\xi), \eta\big\rangle = \big\langle L_{t_1}\mathrm{P}(t_2\xi), \mathrm{P}(\eta)\big\rangle \\ &= \big\langle L_{t_1}\mathrm{P}(t_2\xi), \eta\big\rangle = \big\langle \mathrm{P}(t_2\xi), L_{t_1^*}\eta\big\rangle = \big\langle \mathrm{P}(t_2\xi), t_1^*\eta\big\rangle. \end{aligned}$$

If $t_2 \in \mathcal{A}$, then $t_2\xi \in H^p(\mathcal{M})$. Thus $\langle \mathrm{P}(t\xi), s^*\eta\rangle = \langle t\xi, s^*\eta\rangle$. On the other hand, if $t_1 \in \mathcal{A}^*$, then $t_1^*\eta \in H^q(\mathcal{M})$, and so

$$\big\langle \mathrm{P}(t\xi), t_1^*\eta\big\rangle = \big\langle t_2\xi, \mathrm{P}(t_1^*\eta)\big\rangle = \langle t_2\xi, t_1^*\eta\rangle.$$

Therefore,

$$\begin{aligned}\langle T_{t_1}T_{t_2}\xi,\eta\rangle &= \big\langle \mathrm{P}(t_2\xi), {t_1}^*\eta\big\rangle = \langle t_2\xi, t_1^*\eta\rangle\\ &= \langle t_1t_2\xi,\eta\rangle = \big\langle \mathrm{P}(t_1t_2\xi),\eta\big\rangle\\ &= \langle T_{t_1t_2}\xi,\eta\rangle \quad \text{for all } \xi\in H^p(\mathcal{M}), \eta\in H^q(\mathcal{M});\end{aligned}$$

that is, $T_{t_1}T_{t_2} = T_{t_1t_2}$. Summing up, we get the following properties about Toeplitz operators.

Proposition 2.1. *Let $1<p<\infty$. The we have*

- $(T_t)^* = T_{t^*}$, *for all $t\in\mathcal{M}$;*
- $T_{t_1}T_{t_2} = T_{t_1t_2}$, *where $t_2\in\mathcal{A}$ or $t_1\in\mathcal{A}^*$.*

Let $\sigma(x)$ be the spectral set of x. Using the method in [12], we get our Hartman–Wintner spectral inclusion properties in the general case.

Theorem 2.2. *Let $\mathcal{M}$ be a semifinite von Neumann algebra and let $1<p<\infty$. Suppose that $t\in\mathcal{M}$. Then $\sigma(t)=\sigma(L_t)\subset\sigma(T_t)$.*

Since $\mathrm{J}(H_0^p(\mathcal{M}))^* = \mathrm{J}(H_0^q(\mathcal{M}))(\frac{1}{p}+\frac{1}{q}=1)$, it follows that

$$\begin{aligned}\|H_t\| &= \sup\Big\{\frac{|\langle H_tx,y\rangle|}{\|x\|_p\|y\|_q} : 0\neq y\in \mathrm{J}\big(H_0^q(\mathcal{M})\big), 0\neq x\in H^p(\mathcal{M})\Big\}\\ &= \sup\Big\{\frac{|\langle tx,y\rangle|}{\|x\|_p\|y\|_q} : 0\neq y\in \mathrm{J}\big(H_0^q(\mathcal{M})\big), 0\neq x\in H^p(\mathcal{M})\Big\}\\ &= \sup\Big\{\frac{|\tau(txy^*)|}{\|x\|_p\|y*\|_q} : 0\neq y\in \mathrm{J}\big(H_0^q(\mathcal{M})\big), 0\neq x\in H^p(\mathcal{M})\Big\}\\ &= \sup\Big\{\frac{|\tau(txh)|}{\|x\|_p\|h\|_q} : 0\neq h\in H_0^q(\mathcal{M}), 0\neq x\in H^p(\mathcal{M})\Big\}.\end{aligned}$$

Thus

$$\|H_t\| = \sup\big\{\big|\tau(tgh)\big| : g\in H^p(\mathcal{M}), h\in H_0^q(\mathcal{M}), \|g\|_p\le 1, \|h\|_q\le 1\big\}.$$

Since $\mathcal{A}$ is a weak-$*$ closed subalgebra of $\mathcal{M}$, by the Hahn–Banach theorem we know that

$$\begin{aligned}d(t,\mathcal{A}) &= \sup\big\{\big|\tau(tx)\big| : x\in L^1(\mathcal{M}), \|x\|_1\le 1, \tau(xa)=0, \forall a\in\mathcal{A}\big\}\\ &= \sup\big\{\big|\tau(tx)\big| : x\in H_0^1(\mathcal{M}), \|x\|_1\le 1\big\},\end{aligned}$$

where $t\in\mathcal{M}$.

Now, we discuss Nehari's problem.

Theorem 2.3. *Let $1<p<\infty$. If $\mathcal{A}$ is a subdiagonal subalgebra of $\mathcal{M}$ and $t\in\mathcal{M}$, then $\|H_t\| = d(t;\mathcal{A})$.*

Proof. By the discussion above, we have

$$\begin{aligned}\|H_t\| &= \sup\big\{\big|\tau(tgh)\big| : g\in H^p(\mathcal{M}), h\in H_0^q(\mathcal{M}), \|g\|_p\le 1, \|h\|_q\le 1\big\}\\ &\le \sup\big\{\big|\tau(tf)\big| : f\in H_0^1(\mathcal{M}), \|f\|_1\le 1\big\}\\ &= d(t;\mathcal{A}).\end{aligned}$$

Conversely, for an arbitrary $\varepsilon > 0$, there exists $f \in H_0^1(\mathcal{M})$ such that $\|f\|_1 \le 1$ and $|\tau(tf)| \ge d(t;\mathcal{A}) - \varepsilon$. Since $fe_\lambda \xrightarrow{\|\cdot\|_p} f$, there exists some $\lambda_0 \in \Lambda$ such that

$$\bigl|\tau(tfe_\lambda)\bigr| \ge d(t;\mathcal{A}) - 2\varepsilon, \quad \forall \lambda \ge \lambda_0.$$

On the other hand, $fe_\lambda \in H_0^1(\mathcal{M}_{e_\lambda})$, where $\mathcal{M}_{e_\lambda}$ is a finite von Neumann algebra and $\mathcal{A}_{e_\lambda}$ is a subalgebra of $\mathcal{M}_{e_\lambda}$. By the noncommutative Riesz factorization theorem (Theorem 3.4 of [3]), there exist $g_\lambda \in H^p(\mathcal{M}_{e_\lambda})$ and $h_\lambda \in H_0^q(\mathcal{M}_{e_\lambda})$ such that $fe_\lambda = g_\lambda h_\lambda$, $\|g_\lambda\|_p \le \sqrt{1+\varepsilon}$, and $\|h_\lambda\|_q \le \sqrt{1+\varepsilon}$. This implies that

$$\|H_t\| \ge \Bigl|\tau\Bigl(t\frac{g_\lambda}{\sqrt{1+\varepsilon}}\frac{h_\lambda}{\sqrt{1+\varepsilon}}\Bigr)\Bigr| = \frac{1}{1+\varepsilon}\bigl|\tau(tfe_\lambda)\bigr| \ge \frac{1}{1+\varepsilon}d(t;\mathcal{A}) - \varepsilon.$$

Letting $\varepsilon \to 0$, we obtain

$$\|H_t\| \ge d(t;\mathcal{A}),$$

and hence

$$\|H_t\| = d(t;\mathcal{A}). \qquad \square$$

Corollary 2.4. *Let $1 < p < \infty$. If $\mathcal{A}$ is a subdiagonal subalgebra of $\mathcal{M}$ and $t \in \mathcal{M}$, then*

$$\|H_t\| = \sup\bigl\{\bigl|\tau(tf)\bigr| : f \in H_0^1(\mathcal{M}), \|f\|_1 \le 1\bigr\}.$$

3. Noncommutative BMO space

Let $\mathrm{Re}(f) = \frac{f+f^*}{2}$, and let $\mathrm{Im}(f) = \frac{f-f^*}{2i}$ where $f \in L^1(\mathcal{M})$. We denote $H^1_{\mathrm{Re}}(\mathcal{M}) = \{\mathrm{Re}(f) : f \in H^1(\mathcal{M})\}$ and $H^1_{\mathrm{Im}}(\mathcal{M}) = \{\mathrm{Im}(f) : f \in H^1(\mathcal{M})\}$. If $x \in H^1(\mathcal{M})$, then $\mathrm{Re}\, x \in L^1(\mathcal{M})^{\mathrm{sa}}$ and $\mathrm{H}(\mathrm{Re}(x)) \in L^1(\mathcal{M})$. On the other hand, if $x \in L^1(\mathcal{M})^{\mathrm{sa}}$ and $\mathrm{H}(x) \in L^1(\mathcal{M})$, then we have $x = f + g^*$ for some $f, g \in H^1(\mathcal{M})$. Since $x = x^*$, we have

$$x = \Bigl(\frac{f+g}{2}\Bigr) + \Bigl(\frac{f+g}{2}\Bigr)^* \in H^1_{\mathrm{Re}}(\mathcal{M}).$$

We have that $H^1_{\mathrm{Re}}(\mathcal{M})$ is a normed real vector space with a graph norm, as follows:

$$\bigl\|\mathrm{Re}(f)\bigr\|_{H^1_{\mathrm{Re}}} = \bigl\|\mathrm{Re}(f)\bigr\|_1 + \bigl\|\mathrm{H}\bigl(\mathrm{Re}(f)\bigr)\bigr\|_1 = \bigl\|\mathrm{Re}(f)\bigr\|_1 + \bigl\|(I-\mathcal{E})\,\mathrm{Im}(f)\bigr\|_1.$$

For $f \in L^1(\mathcal{M})$, we have $\|f\|_1 \le \|\mathrm{Re}(f)\|_{H^1_{\mathrm{Re}}} \le \|2f\|_1$ since $\mathrm{H}(\mathrm{Re}(f)) = \mathrm{Im}(f)$ and $f \in H_0^1(\mathcal{M})$, and so

$$\begin{aligned} \mathrm{Re} : H_0^1(\mathcal{M}) &\to H^1_{\mathrm{Re}}(\mathcal{M}), \\ f &\mapsto \mathrm{Re}(f) \end{aligned}$$

is a real linear isomorphic injection. Let $\mathcal{M}^{\mathrm{sa}} = \{x \in \mathcal{M} : x = x^*\}$, and let $(H_0^1(\mathcal{M}))^{\mathrm{re}*}$ be the real dual space of $H_0^1(\mathcal{M})$. For $F_1 \in (H_0^1(\mathcal{M}))^{\mathrm{re}*}$ then, by the real Hahn–Banach theorem, we can extend with norm preservation to $F_1 \in (L^1(\mathcal{M})^{\mathrm{sa}} \oplus L^1(\mathcal{M})^{\mathrm{sa}})^{\mathrm{re}*} = \mathcal{M}^{\mathrm{sa}} \oplus \mathcal{M}^{\mathrm{sa}}$; thus, there exist $x, y \in \mathcal{M}^{\mathrm{sa}}$ such that $F_1(w) = \tau(\mathrm{Re}(w)(x + \mathrm{H}(y)))$ for all $w \in H_0^2(\mathcal{M})$. Since $H_0^2(\mathcal{M}) \cap \mathcal{S}$ is dense in $H_0^1(\mathcal{M})$, it follows that F is represented by $x + \mathrm{H}(y)$ via the pairing

$$\bigl\langle f, x + \mathrm{H}(y)\bigr\rangle = \tau\bigl(\mathrm{Re}(f)x\bigr) - \tau\bigl(\mathrm{Im}(f)y\bigr).$$

On $H_0^2(\mathcal{M})$ the pairing is simply $\tau(\operatorname{Re}(f)(x+\mathrm{H}(y)))$, and every operator $x+\mathrm{H}(y)$ represents such a functional, where $x, y \in \mathcal{M}^{\mathrm{sa}}$.

Proposition 3.1. *Suppose that $F \in (H_0^1(\mathcal{M}))^{\mathrm{re}*}$ represented by $x+\mathrm{H}(y)$ with $x, y \in \mathcal{M}^{\mathrm{sa}}$ is uniquely determined up to perturbation by the elements of $\mathcal{D}$.*

Proof. Suppose that $\tau(\operatorname{Re}(f)x)-\tau(\operatorname{Im}(f)y)=0$ for all $f \in H_0^1(\mathcal{M})$. Now $\operatorname{Im}(f)=\operatorname{Re}(-if)$ and $\operatorname{Re}(f)=-\operatorname{Im}(-if)$, and so $\tau(\operatorname{Re}(f)y)+\tau(\operatorname{Im}(f)x)=0$. Then we get $\tau(f(x+iy))=0$. Since $f \in H_0^1(\mathcal{M})$ was arbitrary, we get $x+iy \in H^2(\mathcal{M}) \cap \mathcal{M} \subset \mathcal{A}$. We next show that for $x, y \in \mathcal{M}^{\mathrm{sa}}$, $x+iy \in \mathcal{A}$ if and only if $x+\mathrm{H}(y) \in \mathcal{D}$. We consider $x, y \in L^2(\mathcal{M})^{\mathrm{sa}}$. If $x+\mathrm{H}(y) \in \mathcal{D}$, then $\mathrm{H}(x)+\mathcal{E}(y)-y=\mathrm{H}(x+\mathrm{H}(y))=0$ and $x+\mathrm{H}(y)-\mathcal{E}(x)=x+\mathrm{H}(y)-\mathcal{E}(x+\mathrm{H}(y))=0$. Hence

$$\begin{aligned}
2(x+iy) &= x+x+iy+iy \\
&= x+\mathcal{E}(x)-\mathrm{H}(y)+iy+i\big(\mathrm{H}(x)+\mathcal{E}(y)\big) \\
&= x+i\big(\mathrm{H}(x)\big)+i\big(y+i\mathrm{H}(y)\big)+\mathcal{E}(x+iy) \\
&\in H^2(\mathcal{M}) \cap \mathcal{M} \\
&\subset \mathcal{A}.
\end{aligned}$$

Conversely, if $x+iy \in \mathcal{A}$, then $\mathrm{H}(y)=\mathrm{H}(\operatorname{Im}(x+iy))=-x+\mathcal{E}(x)$. So $x+H(y)=\mathcal{E}(x) \in \mathcal{D}$. $\square$

We identify the (complex) dual of $H_0^1(\mathcal{M})$. For fixed $F \in (H_0^1(\mathcal{M}))^*$, there exists a $F_1 \in (H_0^1(\mathcal{M}))^{\mathrm{re}*}$ such that $F(w)=F_1(w)-iF_1(iw)$ for all $w \in H_0^1(\mathcal{M})$.

Let e be a τ-finite projection in $\mathcal{D}$. Then $\mathcal{M}_e^{\mathrm{sa}}=e\mathcal{M}^{\mathrm{sa}}e$ and $\mathcal{D}_e=e\mathcal{D}e$ are finite von Neumann algebras. We define noncommutative $\mathrm{BMO}(\mathcal{M}_e^{\mathrm{sa}})$ as the set

$$\big\{x+\mathrm{H}(y) : x, y \in \mathcal{M}_e^{\mathrm{sa}}\big\}$$

with norm

$$\begin{aligned}
&\big\|x+\mathrm{H}(y)\big\|_{\mathrm{BMO}(\mathcal{M}_e^{\mathrm{sa}})} \\
&\quad=\inf\big\{\|u\|_\infty+\|v\|_\infty : x+\mathrm{H}(y)-u-\mathrm{H}(v) \in \mathcal{D}_e, u, v \in \mathcal{M}_e^{\mathrm{sa}}\big\}.
\end{aligned}$$

Lemma 3.2. *Given $\mu, \nu \in \Lambda$ and $\mu \leq \nu$, we have*

$$\big\|x+\mathrm{H}(y)\big\|_{\mathrm{BMO}(\mathcal{M}_{e_\mu}^{\mathrm{sa}})}=\big\|x+\mathrm{H}(y)\big\|_{\mathrm{BMO}(\mathcal{M}_{e_\nu}^{\mathrm{sa}})}$$

for all $x+\mathrm{H}(y) \in \mathrm{BMO}(\mathcal{M}_{e_\mu}^{\mathrm{sa}})$.

Proof. Let $x+\mathrm{H}(y) \in \mathrm{BMO}(\mathcal{M}_{e_\mu}^{\mathrm{sa}})$. First, we have from the fact that $\mathcal{M}_{e_\mu}^{\mathrm{sa}} \subset \mathcal{M}_{e_\nu}^{\mathrm{sa}}$ that

$$\big\|x+\mathrm{H}(y)\big\|_{\mathrm{BMO}(\mathcal{M}_{e_\mu}^{\mathrm{sa}})} \geq \big\|x+\mathrm{H}(y)\big\|_{\mathrm{BMO}(\mathcal{M}_{e_\nu}^{\mathrm{sa}})}.$$

Second, if $x+\mathrm{H}(y)=u+\mathrm{H}(v)+d$, $u, v \in \mathcal{M}_{e_\nu}^{\mathrm{sa}}$, and $d \in \mathcal{D}_{e_\nu}$, then

$$x+\mathrm{H}(y)=e_\mu\big(x+\mathrm{H}(y)\big)e_\mu=e_\nu u e_\nu+\mathrm{H}(e_\nu v e_\nu)+e_\nu d e_\nu.$$

We know that

$$\|u\|_\infty+\|v\|_\infty \geq \|e_\mu u e_\mu\|_\infty+\|e_\mu v e_\mu\|_\infty.$$

This allows us to deduce that $\|x+\mathrm{H}(y)\|_{\mathrm{BMO}(\mathcal{M}_{e_\mu}^{\mathrm{sa}})} \leq \|x+\mathrm{H}(y)\|_{\mathrm{BMO}(\mathcal{M}_{e_\nu}^{\mathrm{sa}})}$. Summing up the above, we get the conclusion. $\square$

The noncommutative $\mathrm{BMO}(\mathcal{M}^{\mathrm{sa}})$ space is defined as the completion of the set norm

$$\{x+\mathrm{H}(y) : x, y \in \mathcal{M}^{\mathrm{sa}}\}$$

with BMO norm $\|x+\mathrm{H}(y)\|_{\mathrm{BMO}(\mathcal{M}^{\mathrm{sa}})}$ equal to the infimum of $\|u\|_\infty + \|v\|_\infty$, where $e_\lambda x e_\lambda + \mathrm{H}(e_\lambda y e_\lambda) - e_\lambda u e_\lambda - \mathrm{H}(e_\lambda v e_\lambda) \in \mathcal{D}, u, v \in \mathcal{M}^{\mathrm{sa}}, \lambda \in \Lambda$. We define multiplication by i on this space by $i(x+\mathrm{H}(y)) = \mathrm{H}(x+\mathrm{H}(y))$. With this definition of multiplication by i, the space becomes a complex Banach space. For all $a \in H_0^1(\mathcal{M})$, and $x, y \in \mathcal{M}$, the dual pairing of x and $x+\mathrm{H}(y)$ is defined as the following:

$$\langle a, x+\mathrm{H}(y)\rangle = \lim_\lambda \tau\big(a\big(e_\lambda x e_\lambda + \mathrm{H}(e_\lambda y e_\lambda)\big)^*\big).$$

Remark 3.3. For $w \in \mathcal{M}$, we have

$$\begin{aligned} w &= w_1 + iw_2 = w_1 + i\big(w_2 + \mathrm{H}(0)\big) \\ &= w_1 + H\big(w_2 + \mathrm{H}(0)\big) = w_1 + \mathrm{H}(w_2), \end{aligned}$$

where w_1 and w_2 are in $\mathcal{M}^{\mathrm{sa}}$. We immediately get

$$\|w\|_{\mathrm{BMO}(\mathcal{M}^{\mathrm{sa}})} \le \|w_1\|_\infty + \|w_2\|_\infty \le 2\|w\|_\infty. \tag{3.1}$$

Theorem 3.4. *The dual space of $H_0^1(\mathcal{M})$ can be isomorphically identified with $\mathrm{BMO}(\mathcal{M}^{\mathrm{sa}})$ under the dual pairing above.*

Proof. Let $x+\mathrm{H}(y) \in \mathrm{BMO}(\mathcal{M}^{\mathrm{sa}})$. There exist $u, v \in \mathcal{M}^{\mathrm{sa}}$. For $a \in H_0^2(\mathcal{M}) \cap H_0^1(\mathcal{M})$ and $\mu \in \Lambda$, we have

$$\begin{aligned} &\big|\tau\big(a\big(e_\mu x e_\mu + H(e_\mu y e_\mu)\big)^*\big)\big| \\ &\quad = \big|\tau\big(e_\mu a e_\mu\big(e_\mu u e_\mu + H(e_\mu v e_\mu)\big)^*\big)\big| \\ &\quad = \big|\tau(e_\mu a e_\mu e_\mu u^* e_\mu) + \tau\big(e_\mu a e_\mu H(e_\mu v e_\mu)^*\big)\big| \\ &\quad = \big|\tau(e_\mu a e_\mu e_\mu u^* e_\mu) - \tau\big(H(e_\mu a e_\mu)(e_\mu v e_\mu)^*\big)\big| \\ &\quad \le \|e_\mu a e_\mu\|_1 \|e_\mu u^* e_\mu\|_\infty + \big\|\mathrm{H}(e_\mu a e_\mu)\big\|_1 \big\|(e_\mu v^* e_\mu)\big\|_\infty \\ &\quad = \|e_\mu a e_\mu\|_1 \|e_\mu u^* e_\mu\|_\infty + \big\|-i(I-\mathcal{E})(e_\mu a e_\mu)\big\|_1 \big\|(e_\mu v^* e_\mu)\big\|_\infty \\ &\quad \le 2\|e_\mu a e_\mu\|_1 \big(\|e_\mu u e_\mu\|_\infty + \|e_\mu v e_\mu\|_\infty\big) \\ &\quad \le 2\|a\|_1 \big(\|u\|_\infty + \|v\|_\infty\big). \end{aligned}$$

By the definition of BMO, we have

$$\big|\tau\big(a\big(e_\mu x e_\mu + \mathrm{H}(e_\mu y e_\mu)\big)^*\big)\big| \le 2\|a\|_1 \big\|x+\mathrm{H}(y)\big\|_{\mathrm{BMO}(\mathcal{M}^{\mathrm{sa}})} \quad \text{for all } \mu \in \Lambda.$$

Given $\mu, \nu \in \Lambda$. Using the inequality above, we have

$$\begin{aligned} &\big|\tau\big(a\big(e_\mu x e_\mu + \mathrm{H}(e_\mu y e_\mu)\big)^*\big) - a\big(e_\nu x e_\nu + \mathrm{H}(e_\nu y e_\nu)\big)^*\big| \\ &\quad = \big|\tau\big((e_\mu a e_\mu - e_\nu a e_\nu)\big(e_\lambda x e_\lambda + \mathrm{H}(e_\lambda y e_\lambda)\big)^*\big)\big| \\ &\quad \le 2\|e_\mu a e_\mu - e_\nu a e_\nu\|_1 \big\|x+\mathrm{H}(y)\big\|_{\mathrm{BMO}(\mathcal{M}^{\mathrm{sa}})}, \end{aligned}$$

where $\lambda \in \Lambda$ and $\lambda \ge \mu, \nu$.

Since $e_\lambda a e_\lambda \xrightarrow{\|\cdot\|_p} a$, we conclude that $\{\tau(a(e_\mu x e_\mu + \mathrm{H}(e_\mu y e_\mu))^*)\}$ is a Cauchy net; hence, we can define a bounded function on $H_0^1(\mathcal{M})$:

$$F_{x+\mathrm{H}(y)} : H_0^1(\mathcal{M}) \to \mathbb{C},$$
$$a \mapsto \tau\big(a\big(x + \mathrm{H}(y)\big)^*\big),$$

where $\tau(a(x + \mathrm{H}(y))^*) = \lim_\lambda(\tau(e_\lambda x e_\lambda + \mathrm{H}(e_\lambda y e_\lambda)))$, and the norm

$$\|F_{x+\mathrm{H}(y)}\|_{H_0^1(\mathcal{M})^*} \leq 2\big\|x + \mathrm{H}(y)\big\|_{\mathrm{BMO}(\mathcal{M}^{\mathrm{sa}})}.$$

On the other hand, if $F \in H_0^1(\mathcal{M})^*$, then by the Hahn–Banach theorem we can extend F to some F_w of the form $F_w(a) = \tau(aw^*)$, for all $a \in H_0^1(\mathcal{M})$, with $w^* \in L^1(\mathcal{M})^* = \mathcal{M}$ and $\|F\|_{H_0^1(\mathcal{M})^*} = \|w\|_\infty$, and so we obtain

$$F(a) = \tau(aw^*) = \lim \tau\big(a\big(e_\lambda(w_1 + iw_2)e_\lambda\big)^*\big) = \lim \tau\big(a\big(e_\lambda w_1 e_\lambda + \mathrm{H}(e_\lambda w_2 e_\lambda)\big)^*\big)$$

for all $a \in H_0^1(\mathcal{M})$, $w = w_1 + iw_2$, and $w_1, w_2 \in \mathcal{M}^{\mathrm{sa}}$. And by Remark 3.3 we get

$$\|F\|_{H_0^1(\mathcal{M})^*} = \|w\|_\infty \geq \frac{1}{2}\|w\|_{\mathrm{BMO}(\mathcal{M}^{\mathrm{sa}})}. \qquad \square$$

Now we immediately deduce the equivalent relationship between the norm of the Hankel operator and the $\mathrm{BMO}(\mathcal{M}^{\mathrm{sa}})$ norm of the norm of symbol t.

Theorem 3.5. *Given $1 < p < \infty$ and $t \in \mathcal{M}$, the Hankel operator H_t is defined on the $H^p(\mathcal{M})$ space. Then we have $\|H_t\| \approx \|t\|_{\mathrm{BMO}(\mathcal{M}^{\mathrm{sa}})}$.*

Proof. By Theorems 2.3 and 3.4, we get the result immediately. $\square$

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