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THE CONVEX HULL-LIKE PROPERTY AND SUPPORTED IMAGES OF OPEN SETS

B. RICCERI

Dedicated to Professor Anthony To-Ming Lau, with esteem and friendship

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Abstract. In this note, as a particular case of a more general result, we obtain the following theorem.

Let $\Omega \subseteq \mathbf{R}^n$ be a nonempty bounded open set, and let $f : \overline{\Omega} \to \mathbf{R}^n$ be a continuous function which is C^1 in Ω. Then, at least one of the following assertions holds:

(a) $f(\Omega) \subseteq \mathrm{conv}(f(\partial\Omega))$.
(b) There exists a nonempty open set $X \subseteq \Omega$, with $\overline{X} \subseteq \Omega$, satisfying the following property: for every continuous function $g : \Omega \to \mathbf{R}^n$ which is C^1 in X, there exists $\tilde{\lambda} \geq 0$ such that, for each $\lambda > \tilde{\lambda}$, the Jacobian determinant of the function $g + \lambda f$ vanishes at some point of X.

As a consequence, if $n = 2$ and $h : \Omega \to \mathbf{R}$ is a nonnegative function, for each $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ satisfying in Ω the Monge–Ampère equation

$$u_{xx}u_{yy} - u_{xy}^2 = h,$$

one has

$$\nabla u(\Omega) \subseteq \mathrm{conv}\big(\nabla u(\partial\Omega)\big).$$

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1. Introduction and preliminaries

Here and in what follows, Ω is a nonempty relatively compact and open set in a topological space E, with $\partial\Omega \neq \emptyset$, and Y is a real locally convex Hausdorff topological vector space. $\overline{\Omega}$ and $\partial\Omega$ denote the closure and the boundary of Ω, respectively. Since $\overline{\Omega}$ is compact, $\partial\Omega$, being closed, is compact, too.

Let us first recall some well-known definitions.

Let S be a subset of Y, and let $y_0 \in S$. As usual, we say that S is supported at y_0 if there exists $\varphi \in Y^* \setminus \{0\}$ such that $\varphi(y_0) \leq \varphi(y)$ for all $y \in S$. If this happens, of course, $y_0 \in \partial S$.

Further, extending a maximum principle definition for real-valued functions, a continuous function $f : \overline{\Omega} \to Y$ is said to satisfy the convex hull property in $\overline{\Omega}$ (see [1], [2] and references therein) if

$$f(\Omega) \subseteq \overline{\mathrm{conv}}\big(f(\partial\Omega)\big),$$

$\overline{\mathrm{conv}}(f(\partial\Omega))$ being the closed convex hull of $f(\partial\Omega)$.

When $\dim(Y) < \infty$, since $f(\partial\Omega)$ is compact, $\mathrm{conv}(f(\partial\Omega))$ is compact too and so $\overline{\mathrm{conv}}(f(\partial\Omega)) = \mathrm{conv}(f(\partial\Omega))$.

A function $\psi : Y \to \mathbf{R}$ is said to be *quasiconvex* if, for each $r \in \mathbf{R}$, the set $\psi^{-1}(]-\infty, r])$ is convex.

Notice the following proposition.

Proposition 1.1. *For each pair A, B of nonempty subsets of Y, the following assertions are equivalent:*

(a_1) *$A \subseteq \overline{\mathrm{conv}}(B)$.*

(a_2) *For every continuous and quasiconvex function $\psi : Y \to \mathbf{R}$, one has*

$$\sup_A \psi \leq \sup_B \psi.$$

Proof. Let (a_1) hold. Fix any continuous and quasiconvex function $\psi : Y \to \mathbf{R}$. Fix $\tilde{y} \in A$. Then, there is a net $\{y_\alpha\}$ in $\mathrm{conv}(B)$ converging to $\tilde{y}$. So, for each α, we have $y_\alpha = \sum_{i=1}^k \lambda_i z_i$, where $z_i \in B$, $\lambda_i \in [0,1]$ and $\sum_{i=1}^k \lambda_i = 1$. By quasiconvexity, we have

$$\psi(y_\alpha) = \psi\Big(\sum_{i=1}^k \lambda_i z_i\Big) \leq \max_{1 \leq i \leq k} \psi(z_i) \leq \sup_B \psi$$

and so, by continuity,

$$\psi(\tilde{y}) = \lim_\alpha \psi(y_\alpha) \leq \sup_B \psi$$

which yields (a_2).

Now, let (a_2) hold. Let $x_0 \in A$. If $x_0 \notin \overline{\mathrm{conv}}(B)$, by the standard separation theorem, there would be $\psi \in Y^* \setminus \{0\}$ such that $\sup_{\overline{\mathrm{conv}}(B)} \psi < \psi(x_0)$, against (a_2). So, (a_1) holds. □

Clearly, applying Proposition 1.1, we obtain the following.

Proposition 1.2. *For any continuous function $f : \overline{\Omega} \to Y$, the following assertions are equivalent:*

(b_1) *f satisfies the convex hull property in $\overline{\Omega}$.*

(b_2) *For every continuous and quasiconvex function $\psi : Y \to \mathbf{R}$, one has*

$$\sup_{x\in\Omega} \psi\big(f(x)\big) = \sup_{x\in\partial\Omega} \psi\big(f(x)\big).$$

In view of Proposition 1.2, we now introduce the notion of the convex hull-like property for functions defined in Ω only.

Definition 1.3. A continuous function $f : \Omega \to Y$ is said to satisfy the convex hull-like property in Ω if, for every continuous and quasiconvex function $\psi : Y \to \mathbf{R}$, there exists $x^* \in \partial\Omega$ such that

$$\limsup_{x\to x^*} \psi\big(f(x)\big) = \sup_{x\in\Omega} \psi\big(f(x)\big).$$

We have the following.

Proposition 1.4. *Let $g : \overline{\Omega} \to Y$ be a continuous function, and let $f = g_{|\Omega}$.*

Then, the following assertions are equivalent:

(c_1) *f satisfies the convex hull-like property in Ω.*

(c_2) *g satisfies the convex hull property in $\overline{\Omega}$.*

Proof. Let (c_1) hold. Let $\psi : Y \to \mathbf{R}$ be any continuous and quasiconvex function. Then, by Definition 1.3, there exists $x^* \in \partial\Omega$ such that

$$\limsup_{x\to x^*} \psi\big(f(x)\big) = \sup_{x\in\Omega} \psi\big(f(x)\big).$$

But

$$\limsup_{x\to x^*} \psi\big(f(x)\big) = \psi\big(g(x^*)\big),$$

and hence

$$\sup_{x\in\partial\Omega} \psi\big(g(x)\big) = \sup_{x\in\Omega} \psi\big(g(x)\big).$$

So, by Proposition 1.2, (c_2) holds.

Now, let (c_2) hold. Let $\psi : Y \to \mathbf{R}$ be any continuous and quasiconvex function. Then, by Proposition 1.2, one has

$$\sup_{x\in\partial\Omega} \psi\big(g(x)\big) = \sup_{x\in\Omega} \psi\big(g(x)\big).$$

Since $\partial\Omega$ is compact and $\psi \circ g$ is continuous, there exists $x^* \in \partial\Omega$ such that

$$\psi\big(g(x^*)\big) = \sup_{x\in\partial\Omega} \psi\big(g(x)\big).$$

But

$$\psi\big(g(x^*)\big) = \lim_{x\to x^*} \psi\big(f(x)\big),$$

and, by continuity again,

$$\sup_{x\in\Omega} \psi\big(g(x)\big) = \sup_{x\in\overline{\Omega}} \psi\big(g(x)\big)$$

and so

$$\lim_{x\to x^*} \psi\big(f(x)\big) = \sup_{x\in\Omega} \psi\big(f(x)\big),$$

which yields (c_1). □

After the above preliminaries, we can declare the aim of this short note: to establish Theorem 1.5 below jointly with some of its consequences.

Theorem 1.5. *For any continuous function $f : \Omega \to Y$, at least one of the following assertions holds:*

(i) *f satisfies the convex hull-like property in Ω.*

(ii) *There exists a nonempty open set $X \subseteq \Omega$, with $\overline{X} \subseteq \Omega$, satisfying the following property: for every continuous function $g : \Omega \to Y$, there exists $\tilde{\lambda} \geq 0$ such that, for each $\lambda > \tilde{\lambda}$, the set $(g + \lambda f)(X)$ is supported at one of its points.*

2. Proof of Theorem 1.5

Assume that (i) does not hold. So, we are assuming that there exists a continuous and quasiconvex function $\psi : Y \to \mathbf{R}$ such that

$$\limsup_{x \to z} \psi\big(f(x)\big) < \sup_{x \in \Omega} \psi\big(f(x)\big) \tag{2.1}$$

for all $z \in \partial\Omega$.

In view of (2.1), for each $z \in \partial\Omega$, there exists an open neighborhood U_z of z such that

$$\sup_{x \in U_z \cap \Omega} \psi\big(f(x)\big) < \sup_{x \in \Omega} \psi\big(f(x)\big).$$

Since $\partial\Omega$ is compact, there are finitely many $z_1, \ldots, z_k \in \partial\Omega$ such that

$$\partial\Omega \subseteq \bigcup_{i=1}^{k} U_{z_i}. \tag{2.2}$$

Put

$$U = \bigcup_{i=1}^{k} U_{z_i}.$$

Hence

$$\sup_{x \in U \cap \Omega} \psi\big(f(x)\big) = \max_{1 \leq i \leq k} \sup_{x \in U_{z_i} \cap \Omega} \psi\big(f(x)\big) < \sup_{x \in \Omega} \psi\big(f(x)\big).$$

Now, fix a number r so that

$$\sup_{x \in U \cap \Omega} \psi\big(f(x)\big) < r < \sup_{x \in \Omega} \psi\big(f(x)\big), \tag{2.3}$$

and set

$$K = \big\{x \in \Omega : \psi\big(f(x)\big) \geq r\big\}.$$

Since f, ψ are continuous, K is closed in Ω. But, since $K \cap U = \emptyset$ and U is open, in view of (2.2), K is closed in E. Hence, K is compact since $\overline{\Omega}$ is so. By (2.3), we can fix $\bar{x} \in \Omega$ such that $\psi(f(\bar{x})) > r$. Notice that the set $\psi^{-1}(]-\infty, r])$ is closed and convex. So, thanks to the standard separation theorem, there exists a nonzero continuous linear functional $\varphi : Y \to \mathbf{R}$ such that

$$\varphi\big(f(\bar{x})\big) < \inf_{y \in \psi^{-1}(]-\infty, r])} \varphi(y). \tag{2.4}$$

Then, from (2.4), it follows that

$$\varphi\big(f(\bar{x})\big) < \inf_{x\in\Omega\setminus K} \varphi\big(f(x)\big).$$

Now, choose ρ so that

$$\varphi\big(f(\bar{x})\big) < \rho < \inf_{x\in\Omega\setminus K} \varphi\big(f(x)\big),$$

and set

$$X = \big\{x \in \Omega : \varphi\big(f(x)\big) < \rho\big\}.$$

Clearly, X is a nonempty open set contained in K. Now, let $g : \Omega \to Y$ be any continuous function. Set

$$\tilde{\lambda} = \inf_{x\in X} \frac{\varphi(g(x)) - \inf_{z\in K} \varphi(g(z))}{\rho - \varphi(f(x))}.$$

Fix $\lambda > \tilde{\lambda}$. So, there is $x_0 \in X$ such that

$$\frac{\varphi(g(x_0)) - \inf_{z\in K} \varphi(g(z))}{\rho - \varphi(f(x_0))} < \lambda.$$

From this, we get

$$\varphi\big(g(x_0)\big) + \lambda\varphi\big(f(x_0)\big) < \lambda\rho + \inf_{z\in K} \varphi\big(g(z)\big). \tag{2.5}$$

By continuity and compactness, there exists $\hat{x} \in K$ such that

$$\varphi\big(g(\hat{x}) + \lambda f(\hat{x})\big) \le \varphi\big(g(x)\big) + \lambda f(x)) \tag{2.6}$$

for all $x \in K$. Let us prove that $\hat{x} \in X$. Arguing by contradiction, assume that $\varphi(f(\hat{x})) \ge \rho$. Then, taking (2.5) into account, we would have

$$\varphi\big(g(x_0)\big) + \lambda\varphi\big(f(x_0)\big) < \lambda\varphi\big(f(\hat{x})\big) + \varphi\big(g(\hat{x})\big),$$

contradicting (6). So, it is true that $\hat{x} \in X$, and, by (2.6), the set $(g + \lambda f)(X)$ is supported at its point $g(\hat{x}) + \lambda f(\hat{x})$.

3. Applications

The first application of Theorem 1.5 shows a strongly bifurcating behavior of certain equations in $\mathbf{R}^n$.

Theorem 3.1. *Let Ω be a nonempty bounded open subset of $\mathbf{R}^n$, and let $f : \Omega \to \mathbf{R}^n$ a continuous function.*

Then, at least one of the following assertions holds:

- (d_1) *f satisfies the convex hull-like property in Ω.*
- (d_2) *There exists a nonempty open set $X \subseteq \Omega$, with $\overline{X} \subseteq \Omega$, satisfying the following property: for every continuous function $g : \Omega \to \mathbf{R}^n$, there exists $\tilde{\lambda} \ge 0$ such that, for each $\lambda > \tilde{\lambda}$, there exist $\hat{x} \in X$ and two sequences $\{y_k\}$, $\{z_k\}$ in $\mathbf{R}^n$, with*

$$\lim_{k\to\infty} y_k = \lim_{k\to\infty} z_k = g(\hat{x}) + \lambda f(\hat{x}),$$

such that, for each $k \in \mathbf{N}$, one has

(j) *the equation*

$$g(x) + \lambda f(x) = y_k$$

has no solution in X*;*

(jj) *the equation*

$$g(x) + \lambda f(x) = z_k$$

has two distinct solutions u_k, v_k *in* X *such that*

$$\lim_{k\to\infty} u_k = \lim_{k\to\infty} v_k = \hat{x}.$$

Proof. Apply Theorem 1.5 with $E = Y = \mathbf{R}^n$. Assume that (d_1) does not hold. Let $X \subseteq \Omega$ be an open set as in (ii) of Theorem 1.5. Fix any continuous function $g : \Omega \to \mathbf{R}^n$. Then, there is some $\tilde{\lambda} \geq 0$ such that, for each $\lambda > \tilde{\lambda}$, there exists $\hat{x} \in X$ such that the set $(g + \lambda f)(X)$ is supported at $g(\hat{x}) + \lambda f(\hat{x})$. As we observed at the beginning, this implies that $g(\hat{x}) + \lambda f(\hat{x})$ lies in the boundary of $(g+\lambda f)(X)$. Therefore, we can find a sequence $\{y_k\}$ in $\mathbf{R}^n \setminus (g+\lambda f)(X)$ converging to $g(\hat{x}) + \lambda f(\hat{x})$. So, such a sequence satisfies (j). For each $k \in \mathbf{N}$, denote by B_k the open ball of radius $\frac{1}{k}$ centered at $\hat{x}$. Let k be such that $B_k \subseteq X$. The set $(g + \lambda f)(B_k)$ is not open since its boundary contains the point $g(\hat{x}) + \lambda f(\hat{x})$. Consequently, by the invariance of domain theorem (see [3, p. 705]), the function $g + \lambda f$ is not injective in B_k. So, there are $u_k, v_k \in B_k$, with $u_k \neq v_k$ such that

$$g(u_k) + \lambda f(u_k) = g(v_k) + \lambda f(v_k).$$

Hence, if we take

$$z_k = g(u_k) + \lambda f(u_k),$$

the sequences $\{u_k\}, \{v_k\}, \{z_k\}$ satisfy (jj) and the proof is complete. □

Remark 3.2. Notice that, in general, Theorem 3.1 is no longer true when $f : \Omega \to \mathbf{R}^m$ with $m > n$. In this connection, consider the case $n = 1$, $m = 2$, $\Omega =]0, \pi[$ and $f(\theta) = (\cos\theta, \sin\theta)$ for $\theta \in [0, \pi]$. So, for each $\lambda > 0$, on the one hand, the function λf is injective, while, on the other hand, $\lambda f(]0, \pi[)$ is not contained in $\mathrm{conv}(\{f(0), f(\pi)\})$.

If $S \subseteq \mathbf{R}^n$ is a nonempty open set, $x \in S$, and $h : S \to \mathbf{R}^n$ is a C^1 function, then we denote by $\det(J_h(x))$ the Jacobian determinant of h at x.

Another important consequence of Theorem 1.5 is as follows.

Theorem 3.3. *Let* Ω *be a nonempty bounded open subset of* $\mathbf{R}^n$*, and let* $f : \Omega \to \mathbf{R}^n$ *be a* C^1*-function.*

Then, at least one of the following assertions holds:

(e_1) f *satisfies the convex hull-like property in* Ω*.*

(e_2) *There exists a nonempty open set* $X \subseteq \Omega$*, with* $\overline{X} \subseteq \Omega$*, satisfying the following property: for every continuous function* $g : \Omega \to \mathbf{R}^n$ *which is* C^1 *in* X*, there exists* $\tilde{\lambda} \geq 0$ *such that, for each* $\lambda > \tilde{\lambda}$*, one has*

$$\det\big(J_{g+\lambda f}(\hat{x})\big) = 0$$

for some $\hat{x} \in X$*.*

Proof. Assume that (e_1) does not hold. Let X be an open set as in (ii) of Theorem 1.5. Let $g : \Omega \to \mathbf{R}^n$ be a continuous function which is C^1 in X. Then, there is some $\tilde{\lambda} \geq 0$ such that, for each $\lambda > \tilde{\lambda}$, there exists $\hat{x} \in X$ such that the set $(g+\lambda f)(X)$ is supported at $g(\hat{x})+\lambda f(\hat{x})$. By remarks already made, we infer that the function $g+\lambda f$ is not a local homeomorphsim at $\hat{x}$, and so $\det(J_{g+\lambda f}(\hat{x})) = 0$ in view of the classical inverse function theorem. $\square$

In turn, here is a consequence of Theorem 3.3 when $n = 2$.

Theorem 3.4. *Let Ω be a nonempty bounded open set of $\mathbf{R}^2$, let $h : \Omega \to \mathbf{R}$ be a continuous function, and let $\alpha, \beta : \Omega \to \mathbf{R}$ be two C^1-functions such that $|\alpha_x\beta_y - \alpha_y\beta_x| + |h| > 0$ and $(\alpha_x\beta_y - \alpha_y\beta_x)h \geq 0$ in Ω.*

Then, any C^1-solution (u, v) in Ω of the system

$$\begin{cases} u_x v_y - u_y v_x = h, \\ \beta_y u_x - \beta_x u_y - \alpha_y v_x + \alpha_x v_y = 0 \end{cases} \tag{3.1}$$

satisfies the convex hull-like property in Ω.

Proof. Arguing by contradiction, assume that (u, v) does not satisfy the convex hull-like property in Ω. Then, by Theorem 3.3, applied taking $f = (u, v)$ and $g = (\alpha, \beta)$, there exist $\lambda > 0$ and $(\hat{x}, \hat{y}) \in \Omega$ such that

$$\det\bigl(J_{g+\lambda f}(\hat{x}, \hat{y})\bigr) = 0.$$

On the other hand, for each $(x, y) \in \Omega$, we have

$$\begin{aligned} \det\bigl(J_{g+\lambda f}(x, y)\bigr) = {} & (u_x v_y - u_y v_x)(x, y)\lambda^2 + (\beta_y u_x - \beta_x u_y - \alpha_y v_x + \alpha_x v_y)(x, y)\lambda \\ & + (\alpha_x\beta_y - \alpha_y\beta_x)(x, y) \end{aligned}$$

and hence

$$h(\hat{x}, \hat{y})\lambda^2 + (\alpha_x\beta_y - \alpha_y\beta_x)(\hat{x}, \hat{y}) = 0,$$

which is impossible in view of our assumptions. $\square$

We conclude by highlighting two applications of Theorem 3.4.

Theorem 3.5. *Let Ω be a nonempty bounded open subset of $\mathbf{R}^2$, let $h : \Omega \to \mathbf{R}$ be a continuous nonnegative function, and let $w \in C^2(\Omega)$ be a function satisfying in Ω the Monge–Ampère equation*

$$w_{xx}w_{yy} - w_{xy}^2 = h.$$

Then, the gradient of w satisfies the convex hull-like property in Ω.

Proof. It is enough to observe that (w_x, w_y) is a C^1-solution in Ω of the system (3.1) with $\alpha(x, y) = -y$ and $\beta(x, y) = x$ and that such α, β satisfy the assumptions of Theorem 3.4. $\square$

Theorem 3.6. *Let Ω be a nonempty bounded open subset of $\mathbf{R}^2$, and let $\beta : \Omega \to \mathbf{R}$ be a C^1-function. Assume that there exists another C^1-function $\alpha : \Omega \to \mathbf{R}$ so that the function $\alpha_x\beta_y - \alpha_y\beta_x$ vanishes at no point of Ω.*

Then, for any function $u \in C^1(\Omega) \cap C^0(\overline{\Omega})$ *satisfying in* Ω *the equation*

$$\beta_y u_x - \beta_x u_y = 0,$$

one has

$$\sup_{\Omega} u = \sup_{\partial\Omega} u$$

and

$$\inf_{\Omega} u = \inf_{\partial\Omega} u.$$

Proof. Observe that the function $(u, 0)$ satisfies the system (3.1) with $h = 0$ and that the assumptions of Theorem 3.4 are fulfilled. So, $(u, 0)$ satisfies the convex hull-like property in Ω. Since $u \in C^0(\overline{\Omega})$, the conclusion follows from Proposition 1.4. □

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Department of Mathematics, University of Catania, Viale A. Doria 6, 95125 Catania, Italy.

E-mail address: ricceri@dmi.unict.it