Ann. Funct. Anal. 8 (2017), no. 4, 502–511
http://dx.doi.org/10.1215/20088752-2017-0014
ISSN: 2008-8752 (electronic)
http://projecteuclid.org/afa

THE WEAK HAAGERUP PROPERTY FOR C^*-ALGEBRAS

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Communicated by J.-C. Bourin

Abstract. We define and study the weak Haagerup property for C^*-algebras in this article. A C^*-algebra with the Haagerup property always has the weak Haagerup property. We prove that a discrete group has the weak Haagerup property if and only if its reduced group C^*-algebra also has that property. Moreover, we consider the permanence of the weak Haagerup property under a few canonical constructions of C^*-algebras.

1. Introduction

In order to study the relation between weak amenability and the Haagerup property, Knudby introduced the weak Haagerup property in [9].

Definition 1.1. Let G be a discrete group. Then G has the weak Haagerup property if there are a constant $C > 0$ and a net $\{u_\alpha\}_{\alpha \in I}$ in $B_2(G) \cap C_0(G)$ such that

(1) $\|u_\alpha\|_{B_2} \leq C$, for every $\alpha \in I$;
(2) $u_\alpha(g) \to 1$ as $\alpha \to \infty$, for every $g \in G$.

The weak Haagerup constant $\Lambda_{\mathrm{WH}}(G)$ is defined as the infimum of those cases of C for which such a net $\{u_\alpha\}_{\alpha \in I}$ exists; if no such net exists, then we write $\Lambda_{\mathrm{WH}}(G) = \infty$. In [10], Knudby introduced the weak Haagerup property for von Neumann algebras and proved that a discrete group has the weak Haagerup property if and only if its group von Neumann algebra has that property.

Received Nov. 25, 2016; Accepted Jan. 5, 2017.
First published online Jun. 22, 2017.
2010 *Mathematics Subject Classification.* Primary 46L05; Secondary 22D25.
Keywords. C^*-algebra, weak Haagerup property, tracial state.

Motivated by the above results, we define and study the weak Haagerup property for C^*-algebras in this article. Our main results appear in Section 2. We show that a C^*-algebra with the Haagerup property always has the weak Haagerup property. We also prove that a discrete group has the weak Haagerup property if and only if its reduced group C^*-algebra does. As a consequence of these two results, we have some new examples of C^*-algebras concerning the weak Haagerup property. Moreover, we prove the permanence of the weak Haagerup property under a few canonical constructions of C^*-algebras, such as direct sums, crossed products, and minimal tensor products.

Throughout this article, G is a discrete group, $C_0(G)$ is the space of functions vanishing at infinity, and $B_2(G)$ is the algebra of Herz–Schur multipliers. In fact, $B_2(G)$ is a unital Banach algebra when equipped with the Herz–Schur norm $\|\cdot\|_{B_2}$. It is known that $\|u\|_\infty \leq \|u\|_{B_2}$ for any $u \in B_2(G)$. (We advise the readers to consult [2, Appendix D], [10, Section 3], [1], and [7] for more information on the Herz–Schur multiplier.)

2. Main results

Throughout this section, let A be a unital C^*-algebra with a faithful tracial state τ. Since τ is a faithful tracial state on A, by the Gel'fand–Naimark–Segal (GNS) construction, τ defines an A-Hilbert bimodule, denoted by $L_2(A,\tau)$. We also denote by $\|\cdot\|_{2,\tau}$ the associated Hilbert norm. Thus, for each $a \in A$, we have

$$\|a\|_{2,\tau} = \big(\tau(a^*a)\big)^{\frac{1}{2}} \leq \|a\|.$$

Suppose that $\phi : A \to A$ is a completely bounded map and that there exists $K > 0$ such that $\|\phi(a)\|_{2,\tau} \leq K\|a\|_{2,\tau}$ for every $a \in A$. Then ϕ can be extended to a bounded operator on $L_2(A,\tau)$ with norm at most K. We say that ϕ is *L_2-compact* if ϕ can be extended to a compact operator on $L_2(A,\tau)$.

Let F be a bounded finite-rank operator on $L_2(A,\tau)$; then F takes the form $F(x) = \sum_{k=1}^{N}\langle x, y_k\rangle x_k$ for all $x \in L_2(A,\tau)$, where $x_k, y_k \in L_2(A,\tau)$. For any $\varepsilon > 0$, by replacing x_k with $a_k \in A$ that is close to x_k, we can get a finite-rank map $Q : A \to A$ such that Q is bounded with respect to the norm $\|\cdot\|_{2,\tau}$ and such that $\|F(a) - Q(a)\|_{2,\tau} \leq \varepsilon\|a\|_{2,\tau}$ for all $a \in A$. Hence, we can see that ϕ is L_2-compact if and only if, for any $\varepsilon > 0$, there exists a finite-rank map $Q : A \to A$ such that Q is bounded with respect to the norm $\|\cdot\|_{2,\tau}$ and such that

$$\big\|\phi(a) - Q(a)\big\|_{2,\tau} \leq \varepsilon\|a\|_{2,\tau}$$

for all $a \in A$.

Definition 2.1. Let A be a unital C^*-algebra with a faithful tracial state τ. Then (A,τ) has the weak Haagerup property if there are a constant $C > 0$ and a net $\{T_\alpha\}_{\alpha\in I}$ of completely bounded maps on A such that

(1) $\|T_\alpha\|_{\mathrm{cb}} \leq C$ for every $\alpha \in I$,
(2) $\|T_\alpha(a)\|_{2,\tau} \leq C\|a\|_{2,\tau}$ for every $a \in A$ and every $\alpha \in I$,
(3) each T_α is L_2-compact,
(4) $\|T_\alpha(a) - a\|_{2,\tau} \to 0$ for every $a \in A$.

The weak Haagerup constant $\Lambda_{\mathrm{WH}}(A,\tau)$ is defined as the infimum of those cases of C for which such a net $\{T_\alpha\}_{\alpha\in I}$ exists; if no such net exists, then we write $\Lambda_{\mathrm{WH}}(A,\tau)=\infty$. It is not hard to see that the infimum is actually a minimum and that $\Lambda_{\mathrm{WH}}(A,\tau)\geq 1$.

We will now show that the weak Haagerup property is indeed weaker than the Haagerup property. (The reader is advised to consult [4] and [12] for basic information on the Haagerup property; we use the definition of the Haagerup property in [12, Definition 2.1] here.)

Theorem 2.2. *If (A,τ) has the Haagerup property, then (A,τ) has the weak Haagerup property. In fact, $\Lambda_{\mathrm{WH}}(A,\tau)=1$.*

Proof. Since (A,τ) has the Haagerup property, then there is a net $\{\Phi_\alpha\}_{\alpha\in I}$ of unital completely positive maps from A to itself satisfying the following conditions:

(1) Each Φ_α decreases τ; that is, for all $a\in A^+$, we have $\tau(\Phi_\alpha(a))\leq\tau(a)$.
(2) For any $a\in A$, $\|\Phi_\alpha(a)-a\|_{2,\tau}\to 0$ as $\alpha\to\infty$.
(3) Each Φ_α is L_2-compact.

For any $a\in A$, we have

$$\big\|\Phi_\alpha(a)\big\|_{2,\tau}=\tau\big(\Phi_\alpha(a)^*\Phi_\alpha(a)\big)^{\frac{1}{2}}\leq\tau\big(\Phi_\alpha(a^*a)\big)^{\frac{1}{2}}\leq\tau(a^*a)^{\frac{1}{2}}=\|a\|_{2,\tau}.$$

Since unital completely positive maps have completely bounded norm 1, this shows that $\Lambda_{\mathrm{WH}}(A,\tau)\leq 1$. □

It is mentioned in [12] that the class of all C^*-algebras with the Haagerup property turns out to be quite large. It contains all nuclear C^*-algebras with a faithful tracial state, residually finite-dimensional C^*-algebras with a faithful tracial state, and many exact C^*-algebras. Hence, the class of all C^*-algebras with the weak Haagerup property is also quite large.

We now turn to discrete groups and their reduced group C^*-algebras. For the moment, fix a discrete group G. Let λ denote the left regular representation of G on $\ell^2(G)$. The reduced group C^*-algebra $C_r^*(G)$ is the C^*-algebra generated by $\lambda(G)$ inside $B(\ell^2(G))$. It is equipped with the faithful tracial state τ given by $\tau(x)=\langle x\delta_e,\delta_e\rangle$ for $x\in C_r^*(G)$, where δ_e is the characteristic function of the identity element.

Theorem 2.3. *Let G be a discrete group. The following conditions are equivalent.*

(1) *The group G has the weak Haagerup property.*
(2) *The reduced group C^*-algebra $C_r^*(G)$ has the weak Haagerup property with respect to τ.*

More precisely, $\Lambda_{\mathrm{WH}}(G)=\Lambda_{\mathrm{WH}}(C_r^*(G),\tau)$.

Proof. (1) ⇒ (2) Suppose that there is a net $\{u_\alpha\}_{\alpha\in I}$ of maps in $B_2(G)\cap C_0(G)$ witnessing the weak Haagerup property of G with $\|u_\alpha\|_{B_2}\leq C$ for every α. Let M_{u_α} be the corresponding multiplier on the group von Neumann algebra $L(G)$; that is,

$$M_{u_\alpha}\big(\lambda(g)\big)=u_\alpha(g)\lambda(g),\quad g\in G. \tag{2.1}$$

Let T_α be the restriction of M_{u_α} to $C_r^*(G)$; then $\|T_\alpha\|_{\mathrm{cb}} = \|u_\alpha\|_{B_2}$ (see [3], [2, Proposition D.6]). It follows from (2.1) that T_α extends to a diagonal operator $\widetilde{T}_\alpha$ on $L_2(C_r^*(G),\tau)$ when $L_2(C_r^*(G),\tau)$ has the standard basis $\{\lambda(g)\}_{g\in G}$. Since $\|u_\alpha\|_\infty \le C$, $\|\widetilde{T}_\alpha\| \le C$. The rest of the proof is similar to that of [10, Theorem B].

(2) $\Rightarrow$ (1) Suppose that the net $\{T_\alpha\}_{\alpha\in I}$ of maps on $C_r^*(G)$ witnesses the weak Haagerup property of $C_r^*(G)$ with $\|T_\alpha\|_{\mathrm{cb}} \le C$ and $\|T_\alpha(x)\|_{2,\tau} \le C\|x\|_{2,\tau}$ for every α. Let

$$u_\alpha(g) = \tau\big(\lambda(g)^* T_\alpha\big(\lambda(g)\big)\big), \quad g \in G.$$

For every $g \in G$, we have

$$\begin{aligned} \big|u_\alpha(g) - 1\big| &= \big|\tau\big(\lambda(g)^* T_\alpha\big(\lambda(g)\big)\big) - \tau\big(\lambda(g)^*\lambda(g)\big)\big| \\ &= \big|\tau\big(\lambda(g)^*\big(T_\alpha\big(\lambda(g)\big) - \lambda(g)\big)\big)\big| \\ &\le \big\|T_\alpha\big(\lambda(g)\big) - \lambda(g)\big\|_{2,\tau} \to 0. \end{aligned}$$

The rest of the proof is similar to that of [10, Theorem B]. □

The preceding theorem is important in the theory of the weak Haagerup property, since it can give many standard examples.

Example 2.4.

(1) Let Γ be a lattice in $\mathrm{Sp}(1,n)$, where $n \ge 2$ and $H = \mathbb{Z}/2 \wr \mathbb{F}_2$. It follows from [6, Theorem C] and [10] that $\Lambda_{\mathrm{WH}}(\Gamma\times H) = 2n-1$, but $\Gamma \times H$ does not have the Haagerup property. Hence, $\Lambda_{\mathrm{WH}}(C_r^*(\Gamma\times H),\tau) = 2n-1$, but $(C_r^*(\Gamma\times H),\tau)$ does not have the Haagerup property.
(2) It follows from [10, Section 9] that $\mathbb{Z}^2 \rtimes \mathrm{SL}_2(\mathbb{Z})$ does not have the weak Haagerup property. Hence, $\Lambda_{\mathrm{WH}}(C_r^*(\mathbb{Z}^2 \rtimes \mathrm{SL}_2(\mathbb{Z})),\tau) = \infty$.

From now on, we show several hereditary results for the weak Haagerup property of C^*-algebras.

Theorem 2.5. *Suppose that (A,τ) has the weak Haagerup property and that $B \subseteq A$ is a unital C^*-subalgebra. If there is a τ-preserving conditional expectation $\mathcal{E}$ from A onto B, then $(B,\tau|_B)$ has the weak Haagerup property and*

$$\Lambda_{\mathrm{WH}}(B,\tau|_B) \le \Lambda_{\mathrm{WH}}(A,\tau).$$

Proof. Suppose that there is a net $\{T_\alpha\}_{\alpha\in I}$ of maps on A witnessing the weak Haagerup property of A with $\|T_\alpha\|_{\mathrm{cb}} \le C$ and $\|T_\alpha(a)\|_{2,\tau} \le C\|a\|_{2,\tau}$ for every α. For each α, let

$$\phi_\alpha(b) = \mathcal{E}\circ T_\alpha(b)$$

for all $b \in B$. Then ϕ_α is a completely bounded map from B to itself such that

$$\|\phi_\alpha\|_{\mathrm{cb}} = \|\mathcal{E}\circ T_\alpha|_B\|_{\mathrm{cb}} \le \|T_\alpha\|_{\mathrm{cb}} \le C,$$

and, for all $b \in B$,

$$\big\|\phi_\alpha(b)\big\|_{2,\tau|_B} = \big\|\mathcal{E}\circ T_\alpha(b)\big\|_{2,\tau|_B} \le \big\|T_\alpha(b)\big\|_{2,\tau} \le C\|b\|_{2,\tau|_B}.$$

As α tends to infinity, we have

$$\begin{aligned}\big\|\phi_\alpha(b)-b\big\|_{2,\tau|_B}^2 &= \tau\big(\big(\phi_\alpha(b)-b\big)^*\big(\phi_\alpha(b)-b\big)\big)\\ &= \tau\big(\big(\mathcal{E}\circ T_\alpha(b)-\mathcal{E}(b)\big)^*\big(\mathcal{E}\circ T_\alpha(b)-\mathcal{E}(b)\big)\big)\\ &= \tau\big(\big(\mathcal{E}\big(T_\alpha(b)-b\big)\big)^*\mathcal{E}\big(T_\alpha(b)-b\big)\big)\\ &\le \tau\circ\mathcal{E}\big(\big(T_\alpha(b)-b\big)^*\big(T_\alpha(b)-b\big)\big)\\ &= \big\|T_\alpha(b)-b\big\|_{2,\tau}^2 \to 0\end{aligned}$$

for all $b \in B$. Since T_α is L_2-compact, then for any $\varepsilon > 0$ there exists a finite-rank map $Q : A \to A$ such that Q is bounded with respect to the norm $\|\cdot\|_{2,\tau}$ and

$$\big\|T_\alpha(a)-Q(a)\big\|_{2,\tau} \le \varepsilon\|a\|_{2,\tau}$$

for all $a \in A$. Hence, we have

$$\begin{aligned}\big\|\phi_\alpha(b)-\mathcal{E}\circ Q(b)\big\|_{2,\tau|_B} &= \big\|\mathcal{E}\circ T_\alpha(b)-\mathcal{E}\circ Q(b)\big\|_{2,\tau|_B}\\ &\le \big\|T_\alpha(b)-Q(b)\big\|_{2,\tau}\\ &\le \varepsilon\|b\|_{2,\tau|_B}\end{aligned}$$

for all $b \in B$. So ϕ_α is L_2-compact. Hence, $(B,\tau|_B)$ has the weak Haagerup property and $\Lambda_{\mathrm{WH}}(B,\tau|_B) \le \Lambda_{\mathrm{WH}}(A,\tau)$. □

Let Γ be a discrete group that acts on a unital C^*-algebra A through an action β. We use $A \rtimes_{\beta,r} \Gamma$ to denote the reduced crossed product of (A,Γ,β) (see [2, Definition 4.1.4]).

Corollary 2.6. *Let Γ be a discrete group that acts on a unital C^*-algebra A through an action β, let τ be a faithful β-invariant tracial state of A, and let τ' be the induced faithful tracial state of $A \rtimes_{\beta,r} \Gamma$. If $(A \rtimes_{\beta,r} \Gamma, \tau')$ has the weak Haagerup property, then (A,τ) has the weak Haagerup property. In fact,*

$$\Lambda_{\mathrm{WH}}(A,\tau) \le \Lambda_{\mathrm{WH}}(A \rtimes_{\beta,r} \Gamma, \tau').$$

Proof. Let $\mathcal{E} : A \rtimes_{\beta,r} \Gamma \to A$ be the canonical faithful conditional expectation such that $\tau' = \tau\circ\mathcal{E}$. Then $\mathcal{E}$ is a τ'-preserving conditional expectation from $A \rtimes_{\beta,r} \Gamma$ onto A. □

If $p \in A$ is a nonzero projection, then let τ_p denote the faithful tracial state on pAp given as $\tau_p(x) = \tau(p)^{-1}\tau(x)$.

Theorem 2.7. *Suppose that (A,τ) has the weak Haagerup property; then (pAp,τ_p) has the weak Haagerup property, and*

$$\Lambda_{\mathrm{WH}}(pAp,\tau_p) \le \Lambda_{\mathrm{WH}}(A,\tau).$$

Proof. Let $P : A \to pAp$ be the map $P(a) = pap$, $a \in A$. Then P is unital and completely positive. Given a net $\{T_\alpha\}_{\alpha\in I}$ witnessing the weak Haagerup property of A with $\|T_\alpha\|_{\mathrm{cb}} \le C$ and $\|T_\alpha(a)\|_{2,\tau} \le C\|a\|_{2,\tau}$ for every α, let $S_\alpha = P\circ T_\alpha|_{pAp}$. Clearly,

$$\|S_\alpha\|_{\mathrm{cb}} \le \|T_\alpha|_{pAp}\|_{\mathrm{cb}} \le \|T_\alpha\|_{\mathrm{cb}} \le C.$$

For any $x \in pAp$, we have

$$\begin{aligned}\tau_p\big(S_\alpha(x)^* S_\alpha(x)\big) &= \tau(p)^{-1}\tau\big(\big(P \circ T_\alpha(x)\big)^* P \circ T_\alpha(x)\big)\\ &\le \tau(p)^{-1}\tau\big(P\big(\big(T_\alpha(x)\big)^* T_\alpha(x)\big)\big)\\ &\le \tau(p)^{-1}\tau\big(\big(T_\alpha(x)\big)^* T_\alpha(x)\big)\\ &\le \tau(p)^{-1}C^2\tau(x^*x)\\ &= C^2\tau_p(x^*x).\end{aligned}$$

This shows that

$$\big\|S_\alpha(x)\big\|_{2,\tau_p} \le C\|x\|_{2,\tau_p}, \quad x \in pAp.$$

Let $V : L_2(pAp, \tau_p) \to L_2(A, \tau)$ be the map $V(x) = \tau(p)^{-\frac{1}{2}}x$. Then V is an isometry and $V^*(a) = \tau(p)^{\frac{1}{2}}pap$ for every $a \in A$. It follows that on pAp we have $S_\alpha = V^* T_\alpha V$. Hence, S_α extends to the compact operator

$$\widetilde{S}_\alpha = V^* \widetilde{T}_\alpha V$$

on $L_2(pAp, \tau_p)$, where $\widetilde{T}_\alpha$ is the extension of T_α on $L_2(A, \tau)$.

For every $x \in pAp$, we have

$$\begin{aligned}\big\|S_\alpha(x) - x\big\|_{2,\tau_p}^2 &= \tau_p\big(\big(S_\alpha(x) - x\big)^*\big(S_\alpha(x) - x\big)\big)\\ &= \tau_p\big(\big(P\big(T_\alpha(x) - x\big)\big)^*\big(P\big(T_\alpha(x) - x\big)\big)\big)\\ &\le \tau_p\big(P\big(\big(T_\alpha(x) - x\big)^*\big(T_\alpha(x) - x\big)\big)\big)\\ &\le \tau(p)^{-1}\tau\big(\big(T_\alpha(x) - x\big)^*\big(T_\alpha(x) - x\big)\big)\\ &= \tau(p)^{-1}\big\|T_\alpha(x) - x\big\|_{2,\tau}^2 \to 0.\end{aligned}$$

This shows that (pAp, τ_p) has the weak Haagerup property. Also, it is easy to see that $\Lambda_{\mathrm{WH}}(pAp, \tau_p) \le \Lambda_{\mathrm{WH}}(A, \tau)$. □

Theorem 2.8. *Let (A_1, τ_1) and (A_2, τ_2) be unital C^*-algebras with faithful tracial states.*

(1) *Both (A_1, τ_1) and (A_2, τ_2) have the weak Haagerup property if and only if $(A_1 \oplus A_2, \tau)$ has the weak Haagerup property for any tracial state τ of the form $\tau = \theta\tau_1 + (1-\theta)\tau_2$, where $0 < \theta < 1$. In fact,*

$$\Lambda_{\mathrm{WH}}(A_1 \oplus A_2, \tau) = \max\big\{\Lambda_{\mathrm{WH}}(A_1, \tau_1), \Lambda_{\mathrm{WH}}(A_2, \tau_2)\big\}.$$

(2) *Both (A_1, τ_1) and (A_2, τ_2) have the weak Haagerup property if and only if $(A_1 \otimes_{\min} A_2, \tau_1 \otimes \tau_2)$ has the weak Haagerup property. Moreover, we have*

$$\Lambda_{\mathrm{WH}}(A_1 \otimes_{\min} A_2, \tau_1 \otimes \tau_2) \le \Lambda_{\mathrm{WH}}(A_1, \tau_1)\Lambda_{\mathrm{WH}}(A_2, \tau_2). \tag{2.2}$$

Proof. (1) (⇒) The proof is similar to that of [10, Theorem C′(4)], and so we omit it.

(⇐) It follows from Theorem 2.7.

(2) (⇒) It is known that a tracial state is faithful if and only if its GNS representation is faithful. Let π_{τ_1}, π_{τ_2}, and $\pi_{\tau_1\otimes\tau_2}$ be the GNS representations

associated to τ_1, τ_2, and $\tau_1 \otimes \tau_2$, respectively. Then $\pi_{\tau_1} \otimes \pi_{\tau_2}$ and $\pi_{\tau_1 \otimes \tau_2}$ are unitarily equivalent. Hence, $\tau_1 \otimes \tau_2$ is a faithful tracial state.

Suppose that we are given a net $\{S_\alpha\}_{\alpha \in I}$ witnessing the weak Haagerup property of (A_1, τ_1) with $\|S_\alpha\|_{\text{cb}} \leq C_1$ and $\|S_\alpha(a_1)\|_{2,\tau_1} \leq C_1 \|a_1\|_{2,\tau_1}$ for every α, as well as a net $\{T_j\}_{j \in J}$ witnessing the weak Haagerup property of (A_2, τ_2) with $\|T_j\|_{\text{cb}} \leq C_2$ and $\|T_j(a_2)\|_{2,\tau_2} \leq C_2 \|a_2\|_{2,\tau_2}$ for every j. Let $\widetilde{S}_\alpha$ denote the extension of S_α on $L_2(A_1, \tau_1)$, and let $\widetilde{T}_j$ denote the extension of T_j on $L_2(A_2, \tau_2)$. For each $\gamma = (\alpha, j) \in I \times J$, it follows from [5, Proposition 8.1.5] and [5, Proposition 8.1.6] that there is a completely bounded map $R_\gamma = S_\alpha \otimes T_j : A_1 \otimes_{\min} A_2 \to A_1 \otimes_{\min} A_2$ such that

$$R_\gamma(a_1 \otimes a_2) = S_\alpha \otimes T_j(a_1 \otimes a_2) = S_\alpha(a_1) \otimes T_j(a_2)$$

for all $a_1 \in A_1$, $a_2 \in A_2$, and

$$\|R_\gamma\|_{\text{cb}} \leq \|S_\alpha\|_{\text{cb}} \|T_j\|_{\text{cb}} \leq C_1 C_2.$$

We give $I \times J$ the product order. If $V : L_2(A_1, \tau_1) \otimes L_2(A_2, \tau_2) \to L_2(A_1 \otimes_{\min} A_2, \tau_1 \otimes \tau_2)$ is the unitary operator which is the identity on $A_1 \otimes_{\text{alg}} A_2$, then we have

$$R_\gamma = V(S_\alpha \otimes T_j)V^*.$$

Since the tensor product of two compact operators is compact, R_γ extends to a compact operator

$$\widetilde{R}_\gamma = V(\widetilde{S}_\alpha \otimes \widetilde{T}_j)V^*$$

on $L_2(A_1 \otimes_{\min} A_2, \tau_1 \otimes \tau_2)$, and

$$\left\|R_\gamma(x)\right\|_{2,\tau_1 \otimes \tau_2} \leq C_1 C_2 \|x\|_{2,\tau_1 \otimes \tau_2}$$

for all $x \in A_1 \otimes_{\min} A_2$. By considering elementary tensors, we have

$$\left\|R_\gamma(x) - x\right\|_{2,\tau_1 \otimes \tau_2} \to 0 \quad \text{as } \gamma \to \infty$$

for every $x \in A_1 \otimes_{\min} A_2$. Hence $(A_1 \otimes_{\min} A_2, \tau_1 \otimes \tau_2)$ has the weak Haagerup property, and $\Lambda_{\text{WH}}(A_1 \otimes_{\min} A_2, \tau_1 \otimes \tau_2) \leq \Lambda_{\text{WH}}(A_1, \tau_1)\Lambda_{\text{WH}}(A_2, \tau_2)$.

($\Leftarrow$) Let $\mathcal{E}$ be the conditional expectation from $A_1 \otimes_{\min} A_2$ onto A_2 such that

$$\mathcal{E}(a_1 \otimes a_2) = \tau_1(a_1)a_2.$$

It is easy to see that $(\tau_1 \otimes \tau_2) \circ \mathcal{E} = \tau_1 \otimes \tau_2$. It follows from Theorem 2.5 that (A_2, τ_2) has the weak Haagerup property. Similarly, we can show that (A_1, τ_1) also has the weak Haagerup property. $\square$

It is not hard to see that if $\Lambda_{\text{WH}}(A_1, \tau_1) = 1$ or $\Lambda_{\text{WH}}(A_2, \tau_2) = 1$, then (2.2) is an equality.

Lemma 2.9. *Let A be a unital C^*-algebra with a faithful tracial state τ. Assume that there exists a net of unital C^*-algebras $\{A_i\}_{i \in I}$, each of which has the weak Haagerup property with respect to faithful tracial states τ_i, and assume that for every i there exist unital completely positive maps $S_i : A \to A_i$ and $T_i : A_i \to A$ such that $\tau_i \circ S_i \leq \tau$, $\tau \circ T_i \leq \tau_i$ and such that $\|T_i \circ S_i(a) - a\|_{2,\tau} \to 0$ for every*

$a \in A$. If there exists $C \in [1, +\infty)$ such that $\Lambda_{\mathrm{WH}}(A_i, \tau_i) \leq C$ for every i, then (A, τ) has the weak Haagerup property with constant at most C.

Proof. The proof follows more or less from the argument of [8, Theorem 2.3(ii)]. More precisely, let $F \subseteq A$ be a finite set, and let $\varepsilon > 0$ be given. There exists $i \in I$ such that

$$\big\|T_i \circ S_i(x) - x\big\|_{2,\tau} \leq \frac{\varepsilon}{2}, \quad \forall x \in F.$$

By assumption, there is a completely bounded map $\Phi : A_i \to A_i$ such that $\|\Phi\|_{\mathrm{cb}} \leq C$, $\|\Phi(a_i)\|_{2,\tau_i} \leq C\|a_i\|_{2,\tau_i}$ for every $a_i \in A_i$, Φ is L_2-compact, and

$$\big\|\Phi\big(S_i(x)\big) - S_i(x)\big\|_{2,\tau_i} \leq \frac{\varepsilon}{2}, \quad \forall x \in F.$$

Let $\Phi_\alpha = T_i \circ \Phi \circ S_i$, where $\alpha = (F, \varepsilon)$. Then Φ_α is a completely bounded map on A with $\|\Phi_\alpha\|_{\mathrm{cb}} \leq C$ and Φ_α is L_2-compact. For all $a \in A$, we have

$$\big\|\Phi_\alpha(a)\big\|_{2,\tau} = \big\|T_i \circ \Phi \circ S_i(a)\big\|_{2,\tau} \leq \big\|\Phi \circ S_i(a)\big\|_{2,\tau_i} \leq C\big\|S_i(a)\big\|_{2,\tau_i} \leq C\|a\|_{2,\tau}.$$

For all $x \in F$, we get

$$\begin{aligned}\big\|\Phi_\alpha(x) - x\big\|_{2,\tau} &\leq \big\|T_i \circ \Phi \circ S_i(x) - T_i \circ S_i(x)\big\|_{2,\tau} + \big\|T_i \circ S_i(x) - x\big\|_{2,\tau} \\ &\leq \big\|\Phi\big(S_i(x)\big) - S_i(x)\big\|_{2,\tau_i} + \frac{\varepsilon}{2} \leq \varepsilon.\end{aligned}$$

The net $\{\Phi_\alpha\}_{\alpha \in J}$ indexed by $J = \{(F, \varepsilon) \mid F \subseteq A \text{ finite}, \varepsilon > 0\}$ shows that (A, τ) has the weak Haagerup property with constant at most C. □

Theorem 2.10. *Let Γ be a discrete amenable group that acts on a unital C^*-algebra A through an action β, let τ be a faithful β-invariant tracial state of A, and let τ' be the induced faithful tracial state of $A \rtimes_{\beta,r} \Gamma$. If (A, τ) has the weak Haagerup property, then $(A \rtimes_{\beta,r} \Gamma, \tau')$ has the weak Haagerup property and*

$$\Lambda_{\mathrm{WH}}(A \rtimes_{\beta,r} \Gamma, \tau') = \Lambda_{\mathrm{WH}}(A, \tau).$$

Proof. Since Γ is amenable, there exists a Følner net $\{F_i\}_{i \in I}$. We define $S_i : A \rtimes_{\beta,r} \Gamma \to A \otimes_{\min} M_{F_i}(\mathbb{C})$ such that

$$S_i\big(a\lambda(s)\big) = \sum_{t \in F_i \cap sF_i} \beta_t^{-1}(a) \otimes e_{t,s^{-1}t}$$

and $T_i : A \otimes_{\min} M_{F_i}(\mathbb{C}) \to A \rtimes_{\beta,r} \Gamma$ such that

$$T_i(a \otimes e_{s,t}) = \frac{1}{|F_i|}\beta_s(a)\lambda(st^{-1}).$$

It follows from the argument of [4, Theorem 2.5] that S_i and T_i are unital completely positive maps such that $\tau_i \circ S_i \leq \tau'$, $\tau' \circ T_i \leq \tau_i$, where τ_i is the induced tracial state on $A \otimes_{\min} M_{F_i}(\mathbb{C})$, and such that $\|T_i \circ S_i(x) - x\|_{2,\tau'} \to 0$ for every $x \in A \rtimes_{\beta,r} \Gamma$. Since $M_{F_i}(\mathbb{C})$ is nuclear, [12, Theorem 3.6] shows that $M_{F_i}(\mathbb{C})$ has the Haagerup property. By Theorems 2.2 and 2.8(2), we have

$$\Lambda_{\mathrm{WH}}\big(A \otimes_{\min} M_{F_i}(\mathbb{C}), \tau_i\big) \leq \Lambda_{\mathrm{WH}}(A, \tau)$$

for every i. It follows from Lemma 2.9 that $A \rtimes_{\beta,r} \Gamma$ has the weak Haagerup property and that $\Lambda_{\mathrm{WH}}(A \rtimes_{\beta,r} \Gamma, \tau') \leq \Lambda_{\mathrm{WH}}(A, \tau)$. In fact, Corollary 2.6 shows that $\Lambda_{\mathrm{WH}}(A \rtimes_{\beta,r} \Gamma, \tau') = \Lambda_{\mathrm{WH}}(A, \tau)$. □

Theorem 2.11. *Let Γ be a discrete group that acts on a unital C^*-algebra A through an action β, let τ be a faithful β-invariant tracial state of A, and let τ' be the induced faithful tracial state of $A \rtimes_{\beta,r} \Gamma$. If $(A \rtimes_{\beta,r} \Gamma, \tau')$ has the weak Haagerup property, then Γ has the weak Haagerup property. In fact,*

$$\Lambda_{\mathrm{WH}}(\Gamma) \leq \Lambda_{\mathrm{WH}}(A \rtimes_{\beta,r} \Gamma, \tau').$$

Proof. Suppose that we are given a net $\{\Phi_\alpha\}_{\alpha \in I}$ witnessing the weak Haagerup property of $(A \rtimes_{\beta,r} \Gamma, \tau')$. For each α, let

$$\varphi_\alpha(g) = \tau'\big(\lambda(g)^* \Phi_\alpha\big(\lambda(g)\big)\big)$$

for all $g \in \Gamma$. We identify $A \rtimes_{\beta,r} \Gamma \subseteq B(L_2(A \rtimes_{\beta,r} \Gamma, \tau'))$. Then there exists a unique unit vector $\xi \in L_2(A \rtimes_{\beta,r} \Gamma, \tau')$ such that $\tau'(x) = \langle x\xi, \xi \rangle$ for all $x \in A \rtimes_{\beta,r} \Gamma$. It follows from the fundamental factorization theorem of completely bounded maps (see [11, Theorem 1.6]) that there are a Hilbert space $\mathcal{K}$, a representation $\pi : B(L_2(A \rtimes_{\beta,r} \Gamma, \tau')) \to B(\mathcal{K})$, and operators $V_1 : L_2(A \rtimes_{\beta,r} \Gamma, \tau') \to \mathcal{K}$, $V_2 : \mathcal{K} \to L_2(A \rtimes_{\beta,r} \Gamma, \tau')$ such that $\|V_1\|\|V_2\| = \|\Phi_\alpha\|_{\mathrm{cb}}$ and $\Phi_\alpha(x) = V_2 \pi(x) V_1$. Hence, we have

$$\begin{aligned}\varphi_\alpha(h^{-1}g) &= \tau'\big(\lambda(h^{-1}g)^* \Phi_\alpha\big(\lambda(h^{-1}g)\big)\big)\\ &= \big\langle \pi\big(\lambda(g)\big) V_1 \lambda(g)^* \xi, \pi\big(\lambda(h)\big) V_2^* \lambda(h)^* \xi \big\rangle\end{aligned}$$

for all $g, h \in \Gamma$. It follows from [10, Proposition 3.1] that $\varphi_\alpha \in B_2(\Gamma)$ and that

$$\|\varphi_\alpha\|_{B_2} \leq \Lambda_{\mathrm{WH}}(A \rtimes_{\beta,r} \Gamma, \tau').$$

Moreover, as $\alpha \to \infty$,

$$\big|\varphi_\alpha(g) - 1\big| = \big|\tau'\big(\lambda(g)^*\big(\Phi_\alpha\big(\lambda(g)\big) - \lambda(g)\big)\big)\big| \leq \big\|\Phi_\alpha\big(\lambda(g)\big) - \lambda(g)\big\|_{2,\tau'} \to 0$$

for any $g \in \Gamma$. It follows from the compactness of Φ_α that

$$\limsup_{g\to\infty} \big|\varphi_\alpha(g)\big| = \limsup_{g\to\infty} \big|\tau'\big(\lambda(g)^* \Phi_\alpha\big(\lambda(g)\big)\big)\big| \leq \limsup_{g\to\infty} \big\|\Phi_\alpha\big(\lambda(g)\big)\big\|_{2,\tau'} \to 0.$$

This proves the weak Haagerup property of Γ. □

Next we state an open question that arises naturally from our investigation.

Problem 2.12. Does the weak Haagerup property for C^*-algebras depend on the choice of a faithful tracial state?

It follows from [12, Theorem 4.18] that the Haagerup property for C^*-algebras does depend on the choice of a faithful tracial state. But the property of the Haagerup property does not occur in the context of von Neumann algebras (see [8, Proposition 2.4]). In [10], Knudby also proved that the weak Haagerup property for von Neumann algebras does not depend on the choice of a faithful normal tracial state, but the proof used several techniques which can be used only in the case of von Neumann algebras. Hence, we will ask if the weak Haagerup property for C^*-algebras depends on the choice of a faithful tracial state. We expect that it does but have not found an example so far.

Acknowledgments. The author appreciates the valuable comments made by the anonymous referee, and he also thanks his Ph.D. advisor, Chi-Keung Ng, for some useful suggestions.

The author's work was supported in part by the National Natural Science Foundation of China (NSFC) grants 11371222 and 11271224.

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