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APPROXIMATIVE COMPACTNESS IN MUSIELAK–ORLICZ FUNCTION SPACES OF BOCHNER TYPE

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Abstract. In this article, we give the criteria for approximative compactness of every proximinal convex subset of Musielak–Orlicz–Bochner function spaces equipped with the Orlicz norm. As a corollary, we give the criteria for approximative compactness of Musielak–Orlicz–Bochner function spaces equipped with the Orlicz norm.

1. Introduction and preliminaries

Let X be a Banach space, and let X^* be the dual space of X. Denote by $B(X)$ and $S(X)$ the closed unit ball and the unit sphere of X. Let $C \subset X$ be a nonempty subset of X. Then the set-valued mapping $P_C : X \to C$

$$P_C(x) = \left\{ z \in C : \|x - z\| = \operatorname{dist}(x, C) = \inf_{y \in C} \|x - y\| \right\}$$

is called the metric projection operator from X onto C.

A subset C of X is said to be *proximinal* if $P_C(x) \neq \emptyset$ for all $x \in X$(see [5]). It is well known that X is reflexive if and only if each closed convex subset of X is proximinal (see [5]).

Definition 1.1. A nonempty subset C of X is said to be *approximatively compact* if for any $\{y_n\}_{n=1}^{\infty} \subset C$ and any $x \in X$ satisfying $\|x - y_n\| \to \inf_{y \in C} \|x - y\|$ as $n \to \infty$, there exists a subsequence of $\{y_n\}_{n=1}^{\infty}$ converging to an element in C.

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A Banach space X is called approximatively compact if every nonempty closed convex subset of X is approximatively compact.

Let us present the history of approximative compactness and related notions. The notion of approximative compactness was introduced by Efimov and Stečkin in [1] as a property of Banach spaces, which guarantees the existence of the best approximation element in a nonempty closed convex set C for any $x \in X$. Ošman [2] established that if X is approximative compact and rotund, then the projector operator P_C is continuous. In 1998, Hudzik and Wang proved that an Orlicz function space is approximatively compact if and only if it is reflexive (see [3]). In 2014, Shang and Cui gave a criterion for approximative compactness of every weakly* closed convex set in an Orlicz function space (see [4]). Chen et al. [5] proved that a Banach space X is approximative compact if and only if X is reflexive and it has the H-property. In this article, we give the criteria for approximative compactness of every proximinal convex subset of Musielak–Orlicz–Bochner function spaces equipped with the Orlicz norm. As a corollary, we give the criteria for approximative compactness of Musielak–Orlicz function spaces equipped with the Orlicz norm.

Definition 1.2. A Banach space X is said to have the *Radon–Nikodym property* whenever the following holds. If (T, Σ, μ) is a nonatomic measure space and v is a vector measure on Σ with values in X which is absolutely continuous with respect to μ and has a bounded variation, then there exists $f \in L_1(X)$ such that for any $A \in \Sigma$,

$$v(A) = \int_A f(t)\, dt.$$

Let (T, Σ, μ) be a nonatomic measurable space. Suppose that a function $M : T \times \mathbb{R} \to [0, \infty]$ satisfies the following conditions.

(1) For μ-almost everywhere $t \in T$, $M(t, 0) = 0$, $\lim_{u\to\infty} M(t, u) = \infty$ and $M(t, u') < \infty$ for some $u' > 0$.
(2) For μ-almost everywhere $t \in T$, $M(t, u)$ is convex on $[0, \infty)$ and even on $\mathbb{R}$ with respect to u.
(3) For each $u \in [0, \infty)$, $M(t, u)$ is a Σ-measurable function of t on T.

Let $p(t, u)$ denote the right derivative of $M(t, \cdot)$ at $u \in R^+$ (where if $M(t, u) = \infty$, then $p(t, u) = \infty$), and let $q(t, \cdot)$ be the generalized inverse function of $p(t, \cdot)$ defined on R^+ by

$$q(t, v) := \sup_{u\geq 0}\big\{u \geq 0 : p(t, u) \leq v\big\}.$$

Then $N(t, v) = \int_0^v q(t, s)\, ds$ for any $v \in R$, and μ-almost everywhere $t \in T$ is called the Musielak–Orlicz function complementary to $M(t, u)$ in the sense of Young. It is well known that there holds the Young inequality $uv \leq M(t, u) + N(t, v)$ for μ-almost everywhere $t \in T$ and all $u, v \in R$. Moreover, $uv = M(t, u) + N(t, u) \Leftrightarrow u = q(t, v)$ or $v = q(t, u)$. Let

$$e(t) = \sup\big\{u > 0 : M(t, u) = 0\big\} \qquad \text{and} \qquad E(t) = \sup\big\{u > 0 : M(t, u) < \infty\big\}.$$

For fixed $t \in T$ and $v \geq 0$, if there exists $\varepsilon \in (0,1)$ such that

$$M(t,v) = \frac{1}{2}M(t,v+\varepsilon) + \frac{1}{2}M(t,v-\varepsilon) < \infty,$$

then we call v a *nonstrictly convex point* of $M(t,\cdot)$. The set of all nonstrictly convex points of $M(t,\cdot)$ is denoted by K_t. For a fixed $t \in T$, if $K_t = \emptyset$, then we say that $M(t,\cdot)$ is *strictly convex*.

Definition 1.3 (see [6])**.** We say that M *satisfies* condition $\Delta (M \in \Delta)$ if there exist $K \geq 1$ and a measureable nonnegative function $\delta(t)$ on T such that $\int_T M(t,\delta(t))\,dt < \infty$ and $M(t,2u) \leq KM(t,u)$ for almost all $t \in T$ and all $u \geq \delta(t)$.

Moreover, for a given Banach space $(X, \|\cdot\|)$, we denote by X_T the set of all strongly Σ-measurable functions from T to X, and for each $u \in X_T$, we define the modular of u by

$$\rho_M(u) = \int_T M\big(t, \|u(t)\|\big)\,dt.$$

Put

$$L_M(X) = \big\{u \in X_T : \rho_M(\lambda u) < \infty \text{ for some } \lambda > 0\big\},$$
$$E_M(X) = \big\{u \in X_T : \rho_M(\lambda u) < \infty \text{ for all } \lambda > 0\big\}.$$

It is well known that Musielak–Orlicz–Bochner function spaces $L_M(X)$ and $E_M(X)$ are Banach spaces if they are equipped with the Luxemburg norm

$$\|u\| = \inf\Big\{\lambda > 0 : \rho_M\Big(\frac{u}{\lambda}\Big) \leq 1\Big\},$$

or the Orlicz norm

$$\|u\|^0 = \inf_{k>0} \frac{1}{k}\big[1 + \rho_M(ku)\big].$$

In particular, $L_M(R)$ and $L_M^0(R)$ are said to be *Musielak–Orlicz function spaces.* Moreover, by [9], we know that $\|u\| \leq \|u\|^0 \leq 2\|u\|$. Set

$$K(u) = \Big\{k > 0 : \frac{1}{k}\big(1 + \rho_M(ku)\big) = \|u\|^0\Big\}.$$

In particular, the set $K(u)$ can be empty or nonempty. To show that, we give some propositions.

Proposition 1.4 (see [7, p. 3])**.** *If* $\lim_{u\to\infty} M(t,u)/u = \infty$ *μ-almost everywhere* $t \in T$*, then* $K(v) \neq \emptyset$ *for any* $v \in L_M^0(X)$.

Proposition 1.5 (see [7, p. 4])**.** *If* $K(v) = \emptyset$*, then* $\|v\|^0 = \int_T A(t) \cdot \|v(t)\|\,dt$*, where* $A(t) = \lim_{u\to\infty} M(t,u)/u$.

2. Main results

Theorem 2.1. *Suppose that X^* has the Radon–Nikodym property. Then every proximinal convex subset of $L^0_M(X)$ is approximatively compact if and only if*

(a) *for any $v \in L^0_M(X) \setminus \{0\}$, the set $K(v)$ consists of one element from $(0, +\infty)$;*
(b) *$M \in \Delta$;*
(c) *$M(t, u)$ is strictly convex with respect to u for almost all $t \in T$;*
(d) *every proximinal convex subset of X is approximatively compact and X is round.*

In order to prove the theorem, we first give some lemmas.

Lemma 2.2 (see [6, p. 177]). *The following are equivalent:*

(a) *$M \notin \Delta$;*
(b) *for each $\varepsilon \in (0, 1)$, there exists $u \in L_M(X)$ such that $\rho_M(u) = \varepsilon$, $\|u\| = 1$, and $\|u(t)\| < E(t)$ μ-almost everywhere on T, where $E(t) = \sup\{u > 0 : M(t, u) < \infty\}$.*

Lemma 2.3 (see [8, p. 481]). *If $M \in \Delta$, then any $u \in L^0_M(X)$ has absolutely continuous norm.*

Lemma 2.4 (see [6, p. 183]). *Suppose that $M \in \Delta$ and $e(t) = 0$ μ-almost everywhere on T. Then*

$$\rho_M(u_n) \to 0 \Leftrightarrow \|u_n\| \to 0 \qquad \text{and} \qquad \rho_M(u_n) \to 1 \Leftrightarrow \|u_n\| \to 1.$$

Lemma 2.5. *The following are equivalent:*

(a) *every proximinal convex subset of X is approximatively compact;*
(b) *if $x^* \in S(X^*)$ is norm attainable and $x^*(x_n) \to 1$, where $\{x_n\}_{n=1}^\infty \subset S(X)$, then $\{x_n\}_{n=1}^\infty$ is relatively compact.*

Proof. For the necessary part, it is well known that if $x^* \in S(X^*)$ is norm attainable, then $H_{x^*} = \{x \in X : x^*(x) = 1\}$ is a proximinal convex subset of X. Then there exists $y_n \in H_{x^*}$ such that $\mathrm{dist}(x_n, H_{x^*}) = \|x_n - y_n\|$. Since

$$\lim_{n\to\infty} \|x_n - y_n\| = \lim_{n\to\infty} \mathrm{dist}(x_n, H_{x^*}) = \lim_{n\to\infty} \big|x^*(x) - x^*(x_n)\big| = 0,$$

we obtain that

$$\mathrm{dist}(0, H_{x^*}) = 1 = \lim_{n\to\infty} \|x_n\| = \lim_{n\to\infty} \|y_n\| = \lim_{n\to\infty} \|0 - y_n\|.$$

This implies that the sequence $\{y_n\}_{n=1}^\infty$ is relatively compact. Hence the sequence $\{x_n\}_{n=1}^\infty$ is relatively compact.

For the sufficient part, suppose that A is a proximinal convex subset of X and that $\|x - y_n\| \to \mathrm{dist}(0, A)$ as $n \to \infty$. We will next prove that $\{y_n\}_{n=1}^\infty$ is relatively compact. We may assume without loss of generality that $x = 0$. Let $r = \mathrm{dist}(0, A)$. Since $\mathrm{int} B(0, r) \cap A = \emptyset$, by the separation theorem, there exists $f \in S(X^*)$ such that

$$\sup\big\{f(x) : x \in B(0, r)\big\} = \sup\big\{f(x) : x \in \mathrm{int}\, B(0, r)\big\} \le \inf\big\{f(x) : x \in A\big\},$$

where $B(0,r)=\{x\in X:\|x\|\le r\}$. Pick $y_0\in P_A(0)$. Since $B(0,r)\cap A=P_A(0)$, we have $f(y_0)=\|y_0\|=r$. Hence

$$\|y_0\|=f(y_0)\le f(y_n)\le\|0-y_n\|\to \operatorname{dist}(0,A)=\|y_0\|.$$

Then $f(y_n)\to\|y_0\|$. Therefore, by $\|y_n\|\to\|y_0\|$ and $f(y_0)=\|y_0\|$, we have

$$\lim_{n\to\infty} f\Big(\frac{y_n}{\|y_n\|}\Big)=1 \qquad\text{and}\qquad f\Big(\frac{y_0}{\|y_0\|}\Big)=1.$$

Hence f is norm attainable. This implies that $\{y_n/\|y_n\|\}_{n=1}^{\infty}$ is relatively compact. Hence $\{y_n\}_{n=1}^{\infty}$ is also relatively compact. This implies that the set A is approximatively compact. □

Lemma 2.6. *Suppose that every proximinal convex subset of X is approximatively compact. Then, if $x^*\in S(X^*)$ is norm attainable and $x^*(x_n)\to 1$, where $\{x_n\}_{n=1}^{\infty}\subset S(X)$, then there exists $y\in\{x\in S(X):x^*(x)=1\}$ such that $y\in\overline{\{x_n\}_{n=1}^{\infty}}$.*

Proof. By Lemma 2.5, there exists a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ such that $\{x_{n_k}\}_{k=1}^{\infty}$ is a Cauchy sequence. Let $x_{n_k}\to y$ as $k\to\infty$. Then $y\in\overline{\{y_n\}_{n=1}^{\infty}}$. Moreover, by $\{x_n\}_{n=1}^{\infty}\subset S(X)$ and $x^*(x_n)\to 1$, we obtain that $y\in S(X)$ and $x^*(y)=1$. □

Lemma 2.7. *Suppose that every proximinal convex subset of X is approximatively compact. Then, if $x=\sum_{n=1}^{\infty}t_nx_n$, then the sequence $\{x_n\}_{n=1}^{\infty}$ is relatively compact, where $x\in S(X)$, $x_n\in B(X)$, $t_n\in(0,1)$ for all $n\in N$ and $\sum_{n=1}^{\infty}t_n=1$.*

Proof. Suppose that $x=\sum_{n=1}^{\infty}t_nx_n$, where $x\in S(X)$, $x_n\in B(X)$, $t_n\in(0,1)$ for any $n\in N$, and $\sum_{n=1}^{\infty}t_n=1$. Then, by the Hahn–Banach theorem, there exists $f\in S(X^*)$ such that $f(x)=1$. Hence

$$f(x)=f\Big(\sum_{n=1}^{\infty}t_nx_n\Big)=\sum_{n=1}^{\infty}t_nf(x_n)=1\Rightarrow f(x_n)=1.$$

This implies that $f(x_n)=1$ for all $n\in N$. Therefore, by Lemma 2.5, we obtain that $\{x_n\}_{n=1}^{\infty}$ is relatively compact. □

Lemma 2.8 (see [8, p. 3013])**.** *Suppose that X^* has the Radon–Nikodym property. Then $(E_M(X))^*=L_N^0(X^*)$ and $(E_M^0(X))^*=L_N(X^*)$.*

Proof of Theorem 2.1. (2) ⇒ (3). We will first prove that condition (a) is true. Suppose that $M\notin\Delta$. Then, by Lemma 2.2, there exists $u\in L_M^0(X)$ such that $\rho_M(u)<1/2$, $\|u\|=1$ and $\|u(t)\|<E(t)$ μ-almost everywhere on T. Then for any $L>1$, we have $\rho_M(Lu)=\infty$. Indeed, suppose that there exists $L_1>1$ such that $\rho_M(L_1u)<\infty$. We know that the function $F(k)=\int_T M(t,k\|u(t)\|)\,dt$ is continuous on $[1,L_1]$. Then there exists $L_2>1$ such that $\rho_M(L_2u)=1$. This implies that $\|u\|\le 1/L_2$, which contradicts the condition $\|u\|=1$.

Decompose T into E_1 and G_1 such that $\mu E_1=\mu G_1$. Then, for any $L>1$, we obtain that $\int_{E_1}M(t,L\|u(t)\|)\,dt=\infty$ or $\int_{G_1}M(t,L\|u(t)\|)\,dt=\infty$. We may assume without loss of generality that $\int_{E_1}M(t,L\|u(t)\|)\,dt=\infty$. Decompose E_1 into E_2 and G_2 such that $\mu E_2=\mu G_2$. Then, for any $L>1$, we obtain that

$\int_{E_2} M(t, L\|u(t)\|)\,dt = \infty$ or $\int_{G_2} M(t, L\|u(t)\|)\,dt = \infty$. We may assume without loss of generality that $\int_{E_2} M(t, L\|u(t)\|)\,dt = \infty$. Generally, decompose E_n into E_{n+1} and G_{n+1} such that $\mu E_{n+1} = \mu G_{n+1}$. Then, for any $L > 1$, we obtain that $\int_{E_{n+1}} M(t, L\|u(t)\|)\,dt = \infty$ or $\int_{G_{n+1}} M(t, L\|u(t)\|)\,dt = \infty$. We may assume without loss of generality that $\int_{E_{n+1}} M(t, L\|u(t)\|)\,dt = \infty$. Hence

$$E_1 \supset E_2 \supset E_3 \supset \cdots, \qquad \mu E_i = \frac{1}{2}\mu E_{i+1} \qquad \text{and} \qquad \|u\chi_{E_i}\| = 1, \quad i = 1, 2, \ldots.$$

Pick $u_0 \in S(E_M^0(X))$ such that $\{t \in T : u_0(t) \neq 0\} \subset T\backslash E_2$. Then, for any $\varepsilon > 0$, pick $k \in R^+$ such that $\|u_0\|^0 + \varepsilon \geq (1/k)[1 + \rho_M(ku_0)]$. Define

$$u_n(t) = u_0(t) + u(t)\chi_{E_n}(t)$$

for all $n \in N$. Moreover, we have $(1/k)\int_T M(t, k\|u(t)\|\chi_{E_n}(t))\,dt < \varepsilon$, when n is large enough. Hence

$$\begin{aligned}
\|u_0\|^0 &\leq \|u_n\|^0 \\
&\leq \frac{1}{k}\Big[1 + \int_T M\big(t, \big\|ku_n(t)\big\|\big)\,dt\Big] \\
&\leq \frac{1}{k}\Big[1 + \int_T M\big(t, \big\|ku_0(t)\big\|\big)\,dt + \int_T M\big(t, k\big\|u(t)\big\|\chi_{E_n}(t)\big)\,dt\Big] \\
&= \frac{1}{k}\Big[1 + \int_T M\big(t, \big\|ku_0(t)\big\|\big)\,dt\Big] + \frac{1}{k}\int_T M\big(t, k\big\|u(t)\big\|\chi_{E_n}(t)\big)\,dt \\
&\leq \|u_0\|^0 + 2\varepsilon.
\end{aligned}$$

This implies that $\|u_n\|^0 \to \|u_0\|^0 = 1$. Then, by the Hahn–Banach theorem, there exists $v_0 \in S(L_N(X^*))$ such that $(u_0, v_0) = 1$. Noting that $\{t \in T : u_0(t) \neq 0\} \subset T\backslash E_2$, we have $\{t \in T : v_0(t) \neq 0\} \subset T\backslash E_2$. Hence, if $(u_0', v_0) = 1$, then $\{t \in T : u_0'(t) \neq 0\} \subset T\backslash E_2$, where $u_0' \in S(E_M^0(X))$. Since

$$0 \leq \Big|\int_T \big(u(t)\chi_{E_n}(t), v_0(t)\big)\,dt\Big| \leq \Big[\int_{E_n} M\big(t, \big\|u(t)\big\|\big)\,dt + \int_{E_n} N\big(t, v_0(t)\big)\,dt\Big] \to 0,$$

we obtain that

$$\int_T \big(u_n(t), v_0(t)\big)\,dt = \int_T \big(u_0(t), v_0(t)\big)\,dt + \int_T \big(u(t)\chi_{E_n}(t), v_0(t)\big)\,dt \to 1.$$

Noting that $\|u\chi_{E_n}\| = 1$ and $\{t \in T : u_0'(t) \neq 0\} \subset T\backslash E_2$, we obtain that $\|u_n - u_0'\|^0 \geq \|u\chi_{E_i}\| = 1$, which contradicts Lemma 2.6. Hence $M \in \Delta$.

We next prove that (a) and (c) are true. (a1) We will prove that for any $\|u\|^0 > \|e\|^0$, we have $K(u) \neq \emptyset$, where e denotes the function $e(t) = \sup\{u > 0 : M(t, u) = 0\}$. Suppose that there exists $u \in L_M^0(X)$ such that $\|u\|^0 > \|e\|^0$ and $K(u) = \emptyset$. Then, by Proposition 1.5, we have $A(t) < +\infty$ μ-almost everywhere on T. Moreover, there exists $\eta_1 > \eta_2 > 0$ such that $\mu T^0 > 0$, where

$$T^0 = \big\{t \in T : \big\|u(t)\big\| > \big\|e(t)\big\|, \eta_2 \leq \big\|u(t)\big\| \leq \eta_1\big\}.$$

Therefore, by Lemma 2.3 and $M \in \Delta$, there exist $\eta > 0$, $\eta' > 0$, and $\eta'' > 0$ such that $\mu T_0 > 0$ and $\|u\chi_{T_0}\|^0 < 1$, where

$$T_0 = \{t \in T^0 : M(t, \|u(t)\|) > \eta, \eta' < A(t) < \eta''\}.$$

Since $K(u) = \emptyset$, by Proposition 1.5, we obtain that $\|u\|^0 = \int_T A(t)\|u(t)\|\,dt$. Decompose T_0 into T_1^1, T_2^1 such that $T_1^1 \cap T_2^1 = \emptyset$, $T_1^1 \cup T_2^1 = T_0$ and $\int_{T_1} A(t) \times \|u(t)\|\,dt = \int_{T_2} A(t)\|u(t)\|\,dt$. Decompose T_1^1 into T_1^2, T_2^2 such that $T_1^2 \cap T_2^2 = \emptyset$, $T_1^2 \cup T_2^2 = T_1^1$, and $\int_{T_1^2} A(t)\|u(t)\|\,dt = \int_{T_2^2} A(t)\|u(t)\|\,dt$. Decompose T_2^1 into T_3^2, T_4^2 such that $T_3^2 \cap T_4^2 = \emptyset$, $T_3^2 \cup T_4^2 = T_2^1$, and $\int_{T_3^2} A(t)\|u(t)\|\,dt = \int_{T_4^2} A(t)\|u(t)\|\,dt$. Generally, decompose T_i^{n-1} into T_{2i-1}^n, T_{2i}^n such that

$$T_{2i-1}^n \cap T_{2i}^n = \emptyset, \qquad T_{2i-1}^n \cup T_{2i}^n = T_i^{n-1} \qquad \text{and}$$
$$\int_{T_{2i-1}^n} A(t)\|u(t)\|\,dt = \int_{T_{2i}^n} A(t)\|u(t)\|\,dt,$$

where $n = 1, 2, \ldots, i = 1, 2, \ldots, 2^{n-1}$. Define

$$u_n(t) = \begin{cases} u(t), & t \in T\backslash T_0, \\ u(t) - \frac{1}{2}u(t), & t \in T_1^n, \\ u(t) + \frac{1}{2}u(t), & t \in T_2^n, \\ \cdots & \cdots \\ u(t) - \frac{1}{2}u(t), & t \in T_{2^n-1}^n, \\ u(t) + \frac{1}{2}u(t), & t \in T_{2^n}^n, \end{cases} \qquad u_n'(t) = \begin{cases} u(t), & t \in T\backslash T_0, \\ u(t) + \frac{1}{2}u(t), & t \in T_1^n, \\ u(t) - \frac{1}{2}u(t), & t \in T_2^n, \\ \cdots & \cdots \\ u(t) + \frac{1}{2}u(t), & t \in T_{2^n-1}^n, \\ u(t) - \frac{1}{2}u(t), & t \in T_{2^n}^n, \end{cases}$$

and

$$(y_n(t))_{n=1}^\infty = (u_1(t), u_1'(t), u_2(t), u_2'(t), \ldots, u_n(t), u_n'(t), \ldots).$$

Then

$$\begin{aligned} \|u_n\|^0 &\le \int_T A(t) \cdot \|u_n(t)\|\,dt \\ &= \int_{T_0} A(t)\|u(t)\|\,dt + \int_{T_1^n} A(t)\Big\|u(t) - \frac{1}{2}u(t)\Big\|\,dt \\ &\quad + \int_{T_2^n} A(t)\Big\|u(t) + \frac{1}{2}u(t)\Big\|\,dt \\ &\quad + \cdots + \int_{T_{2^n-1}^n} A(t) \cdot \Big\|u(t) - \frac{1}{2}u(t)\Big\|\,dt + \int_{T_{2^n}^n} A(t) \cdot \Big\|u(t) + \frac{1}{2}u(t)\Big\|\,dt \\ &= \int_{T_0} A(t)\|u(t)\|\,dt + \int_{T_1^n} A(t)\Big(\|u(t)\| - \Big\|\frac{1}{2}u(t)\Big\|\Big)\,dt \\ &\quad + \int_{T_2^n} A(t)\Big(\|u(t)\| + \Big\|\frac{1}{2}u(t)\Big\|\Big)\,dt \\ &\quad + \cdots + \int_{T_{2^n-1}^n} A(t) \cdot \|u(t)\| + \Big\|\frac{1}{2}u(t)\Big\|\,dt \end{aligned}$$

$$+\int_{T^n_{2^n}} A(t)\cdot\Big(\big\|u(t)\big\| + \Big\|\frac{1}{2}u(t)\Big\|\Big)\,dt$$
$$= \int_T A(t)\cdot\big\|u(t)\big\|\,dt = \|u\|^0.$$

Similarly, we obtain that $\|u'_n\|^0 \le \|u\|^0$. Hence $\|y_n\|^0 \le \|u\|^0$. This implies that $y_n \in \|u\|^0 B(L_M(X))$. On the other hand, we have

$$\sum_{n=1}^{\infty}\Big(\frac{1}{2}\cdot\frac{1}{2^n}u_n(t)+\frac{1}{2}\cdot\frac{1}{2^n}u'_n(t)\Big) = \sum_{n=1}^{\infty}\frac{1}{2^{n+1}}\big(u_n(t)+u'_n(t)\big) = \sum_{n=1}^{\infty}\frac{2}{2^{n+1}}u(t) = u(t)$$

and

$$\sum_{n=1}^{\infty}\Big(\frac{1}{2}\cdot\frac{1}{2^n}+\frac{1}{2}\cdot\frac{1}{2^n}\Big) = \sum_{n=1}^{\infty}\Big(\frac{1}{2^n}\Big) = 1.$$

We next prove that $(y_n(t))_{n=1}^{\infty}$ is not relatively compact. For clarity, we will divide the proof into two cases.

Case I. Let $k(u_n - u_m) = \emptyset$. Then, by Proposition 1.5, we obtain that

$$\|u_n - u_m\|^0 = \int_T A(t)\big\|u_n(t)-u_m(t)\big\|\,dt = \int_{T_{n,m}} A(t)\big\|u(t)\big\|\,dt$$
$$= \frac{1}{2}\int_{T_0} A(t)\big\|u(t)\big\|\,dt,$$

where $T_{n,m} = \{t \in T_0 : u_n(t) \ne u_m(t)\}$.

Case II. Let $k(u_n - u_m) \ne \emptyset$. By the definition of T_0, there exists $\delta > 0$ such that $\mu T_{n,m} > \delta$. Pick $k_{n,m} \in k(u_n - u_m)$. Then, by $\|u\chi_{T_0}\|^0 < 1$, we have $\|u_n - u_m\|^0 < 1$. Hence, $k_{n,m} > 1$, and so

$$\begin{aligned}\|u_n - u_m\|^0 &= \frac{1}{k_{n,m}}\big[1+\rho_M\big(k_{n,m}(u_n-u_m)\big)\big]\\ &= \frac{1}{k_{n,m}}\Big[1+\int_{T_{n,m}} M\big(t, k_{n,m}\big\|u(t)\big\|\big)\,dt\Big]\\ &\ge \int_{T_{n,m}} \frac{M(t,k_{n,m}\|u(t)\|)}{k_{n,m}}\,dt \ge \int_{T_{n,m}} \frac{k_{n,m}M(t,\|u(t)\|)}{k_{n,m}}\,dt\\ &\ge \int_{T_{n,m}} \eta\,dt \ge \eta\delta.\end{aligned}$$

Therefore, by Cases I and II, we obtain that $(y_n(t))_{n=1}^{\infty}$ is not relatively compact, which is a contradiction. Hence, for any $\|u\|^0 > \|e\|^0$, we have $K(u) \ne \emptyset$.

We next prove that (c) is true. (c1) Note that $\|e\|^0 \le 3/2$ for any $u \in 2S(L^0_M(X))$. Hence $K(u) \ne \emptyset$. First, we will prove that for any $u \in 2S(L^0_M(X))$, we have $\mu\{t \in T : k\|u(t)\| \in K_t\} = 0$, where $k \in K(u)$. Suppose that there exists $n_0 \in N$ such that $\mu G > 0$, where

$$\begin{aligned}G = \Big\{t \in T : &M\big(t, k\big\|u(t)\big\|\big)\\ &= \frac{1}{2}M\Big(t, \Big(1+\frac{1}{n_0}\Big)k\big\|u(t)\big\|\Big) + \frac{1}{2}M\Big(t, \Big(1-\frac{1}{n_0}\Big)k\big\|u(t)\big\|\Big) < \infty\Big\}.\end{aligned}$$

It is easy to see that there exist $\lambda > 0$ and $\eta > 0$ such that $\mu H > 0$, where

$$H = \Big\{t \in G : \lambda < \Big\|\frac{1}{n_0}u(t)\Big\| < \eta, A(t) \cdot \frac{1}{n_0}\|u(t)\| > \lambda\Big\}.$$

Decompose H into E_1^1, E_2^1 such that

$$E_1^1 \cap E_2^1 = \emptyset, \qquad E_1^1 \cup E_2^1 = H \qquad \text{and}$$
$$\int_{E_1^1} p\Big(t, k\frac{1}{n_0}\|u(t)\|\Big)\, dt = \int_{E_2^1} p\Big(t, k\frac{1}{n_0}\|u(t)\|\Big)\, dt.$$

Decompose E_1^1 into E_1^2,E_2^2 such that

$$E_1^2 \cap E_2^2 = \emptyset, \qquad E_1^2 \cup E_2^2 = E_1^1 \qquad \text{and}$$
$$\int_{E_1^2} p\Big(t, k\frac{1}{n_0}\|u(t)\|\Big)\, dt = \int_{E_2^2} p\Big(t, k\frac{1}{n_0}\|u(t)\|\Big)\, dt.$$

Decompose E_2^1 into E_3^2,E_4^2 such that

$$E_3^2 \cap E_4^2 = \emptyset, \qquad E_3^2 \cup E_4^2 = E_2^1 \qquad \text{and}$$
$$\int_{E_3^2} p\Big(t, k\frac{1}{n_0}\|u(t)\|\Big)\, dt = \int_{E_4^2} p\Big(t, k\frac{1}{n_0}\|u(t)\|\Big)\, dt.$$

Generally, decompose E_i^{n-1} into E_{2i-1}^n, E_{2i}^n such that $E_{2i-1}^n \cap E_{2i}^n = \emptyset, E_{2i-1}^n \cup E_{2i}^n = E_i^{n-1}$, and

$$\int_{E_{2i-1}^n} p\Big(t, k\frac{1}{n_0}\|u(t)\|\Big)\, dt = \int_{E_{2i}^n} p\Big(t, k\frac{1}{n_0}\|u(t)\|\Big)\, dt,$$

where $n = 1, 2, \ldots, i = 1, 2, \ldots, 2^{n-1}$. Define

$$u_n(t) = \begin{cases} u(t), & t \in T\backslash H, \\ (1 - \frac{1}{n_0})u(t), & t \in E_1^n, \\ (1 + \frac{1}{n_0})u(t), & t \in E_2^n, \\ \cdots & \cdots \\ (1 - \frac{1}{n_0})u(t), & t \in E_{2^n-1}^n, \\ (1 + \frac{1}{n_0})u(t), & t \in E_{2^n}^n, \end{cases} \qquad u_n'(t) = \begin{cases} u(t), & t \in T\backslash H, \\ (1 + \frac{1}{n_0})u(t), & t \in E_1^n, \\ (1 - \frac{1}{n_0})u(t), & t \in E_2^n, \\ \cdots & \cdots \\ (1 + \frac{1}{n_0})u(t), & t \in E_{2^n-1}^n, \\ (1 - \frac{1}{n_0})u(t), & t \in E_{2^n}^n, \end{cases}$$

and

$$\big(y_n(t)\big)_{n=1}^{\infty} = \big(u_1(t), u_1'(t), u_2(t), u_2'(t), \ldots, u_n(t), u_n'(t), \ldots\big).$$

Then

$$\begin{aligned} \|u_n\|^0 &\leq \frac{1}{k}\big[1 + \rho_M(ku_n)\, dt\big] \\ &= \frac{1}{k}\Big[1 + \rho_M(ku \cdot \chi_H) + \rho_M\Big(k\Big(1 - \frac{1}{n_0}\Big)u \cdot \chi_{E_1^n}\Big) + \rho_M\Big(k\Big(1 + \frac{1}{n_0}\Big)u \cdot \chi_{E_2^n}\Big) \\ &\quad + \cdots + \rho_M\Big(k\Big(1 - \frac{1}{n_0}\Big)u \cdot \chi_{E_{2^n-1}^n}\Big) + \rho_M\Big(k\Big(1 + \frac{1}{n_0}\Big)u \cdot \chi_{E_{2^n}^n}\Big)\Big] \\ &= \frac{1}{k}\Big[1 + \rho_M(ku \cdot \chi_H) + \rho_M(ku\chi_{E_1^n}) \end{aligned}$$

$$
\begin{aligned}
&- \int_{E_1^n} p\Big(t, k\frac{1}{n_0}\|u(t)\|\Big)\,dt + \rho_M(ku\cdot\chi_{E_2^n})\\
&+ \int_{E_2^n} p\Big(t, k\frac{1}{n_0}\|u(t)\|\Big)\,dt + \cdots + \rho_M(ku\cdot\chi_{E_{2^n-1}^n})\\
&- \int_{E_{2^n-1}^n} p\Big(t, k\frac{1}{n_0}\|u(t)\|\Big)\,dt\\
&+ \rho_M(ku\cdot\chi_{E_{2^n}^n}) + \int_{E_{2^n-1}^n} p\Big(t, k\frac{1}{n_0}\|u(t)\|\Big)\,dt\Big]\\
= {}& \frac{1}{k}\big[1 + \rho_M(ku\cdot\chi_H) + \rho_M(ku\cdot\chi_{E_1^n}) + \rho_M(k\cdot u\chi_{E_2^n})\\
&+ \cdots + \rho_M(k\cdot u\chi_{E_{2^n-1}^n}) + \rho_M(k\cdot u\chi_{E_{2^n}^n})\big]\\
= {}& \frac{1}{k}\big[1+\rho_M(ku)\big] = \|u\|^0 = 1.
\end{aligned}
$$

Similarly, $\|u_n'\|^0 \le 1$. Hence $\|y_n\|^0 \le 1$ for any $n \in N$. On the other hand, we have

$$
\sum_{n=1}^{\infty}\Big(\frac{1}{2}\cdot\frac{1}{2^n}u_n(t) + \frac{1}{2}\cdot\frac{1}{2^n}u_n'(t)\Big) = \sum_{n=1}^{\infty}\frac{1}{2^{n+1}}\big(u_n(t)+u_n'(t)\big) = \sum_{n=1}^{\infty}\frac{2}{2^{n+1}}u(t) = u(t)
$$

and

$$
\sum_{n=1}^{\infty}\Big(\frac{1}{2}\cdot\frac{1}{2^n} + \frac{1}{2}\cdot\frac{1}{2^n}\Big) = \sum_{n=1}^{\infty}\Big(\frac{1}{2^n}\Big) = 1.
$$

By absolute continuity of the integral, we can find $\delta > 0$ such that $\mu E < \delta$ implies that

$$
\int_E p\Big(t, \frac{1}{n_0}\|u(t)\|\Big)\,dt \le \frac{1}{4}\int_H p\Big(t, \frac{1}{n_0}\|u(t)\|\Big)\,dt \qquad \text{and}
$$
$$
\int_E A(t)\|u(t)\|\,dt < \frac{1}{4}\lambda\delta.
$$

Set $T_{n,m} = \{t \in H : u_n(t) \neq u_m(t)\}$. Then it is easy to see that $\mu T_{n,m} > \delta$, where $m \neq n$. We may assume without loss of generality that $\int_H A(t)\|u(t)\|\,dt < \infty$ or $A(t) = \infty, t \in H$. We will derive a contradiction for each of the following three cases.

Case I. Let $K(u_n - u_m) \neq \emptyset$ and $\int_H A(t)\|u(t)\|\,dt < \infty$. Pick $k_{n,m} \in K(u_n - u_m)$. Then, by $\lim_{u\to\infty} M(t,u)/u = A(t)$, we have

$$
\lim_{n\to\infty}\frac{M(t, n\|\frac{2}{n_0}u(t)\|)}{n\|\frac{2}{n_0}u(t)\|}\cdot\Big\|\frac{2}{n_0}u(t)\Big\| = A(t)\Big\|\frac{2}{n_0}u(t)\Big\|
$$

μ-almost everywhere on H. Therefore, by Egorov's theorem, there exists $\beta > 0$ such that

$$
\Big|\frac{M(t, n\|\frac{2}{n_0}u(t)\|)}{n\|\frac{2}{n_0}u(t)\|}\cdot\Big\|\frac{2}{n_0}u(t)\Big\| - A(t)\Big\|\frac{2}{n_0}u(t)\Big\|\Big| < \frac{1}{4\mu T}\lambda\delta, \quad t \in H\backslash F
$$

whenever $n > \beta$, where $F \subset H$ and $\mu F < \delta/4$. Hence, if $k_{n,m} > \beta > 0$, then

$$\Big|\frac{M(t, k_{m,n}\|\frac{2}{n_0}u(t)\|)}{k_{m,n}\|\frac{2}{n_0}u(t)\|} \cdot \Big\|\frac{2}{n_0}u(t)\Big\| - A(t) \cdot \Big\|\frac{2}{n_0}u(t)\Big\|\Big| < \frac{1}{4\mu T}\lambda\delta, \quad t \in H\backslash F.$$

This implies that

$$\begin{aligned}
\|u_n - u_m\|^0 &= \frac{1}{k_{n,m}}\big[1 + \rho_M\big(k_{n,m}(u_n - u_m)\big)\big] \\
&\geq \int_{T_{n,m}} \frac{M(t, k_{n,m}\|\frac{2}{n_0}u(t)\|)}{k_{n,m}}\, dt \\
&\geq \int_{T_{m,n}\backslash F} \frac{M(t, k_{m,n}\|\frac{2}{n_0}u(t)\|)}{k_{m,n}\|\frac{2}{n_0}u(t)\|}\Big\|\frac{2}{n_0}u(t)\Big\|\, dt \\
&\geq \int_{T_{m,n}\backslash F} \Big[A(t) \cdot \Big\|\frac{2}{n_0}u(t)\Big\| - \frac{1}{4\mu T}\lambda\delta\Big]\, dt \\
&\geq \int_{T_{m,n}\backslash F} A(t) \cdot \Big\|\frac{2}{n_0}u(t)\Big\|\, dt - \int_{T_{m,n}\backslash F} \frac{1}{4\mu T}\lambda\delta\, dt \\
&\geq \frac{3}{4}\lambda\delta - \frac{1}{4}\lambda\delta = \frac{1}{2}\lambda\delta.
\end{aligned}$$

Moreover, if $k_{n,m} \leq \beta > 0$, then $\|u_n - u_m\|^0 = [1 + \rho_M(k_{n,m}(u_n - u_m))]/k_{n,m} \geq 1/\beta$.

Case II. Let $K(u_n - u_m) \neq \emptyset$ and $A(t) = \infty$, $t \in H$. Then, by

$$H = \bigcup_{n=2}^{\infty}\Big\{t \in H : \frac{M(t, n\lambda)}{n\lambda} \geq 1 > \frac{M(t, (n-1)\lambda)}{(n-1)\lambda}\Big\} \cup \Big\{t \in H : \frac{M(t, \lambda)}{\lambda} \geq 1\Big\},$$

there exists $\alpha > 0$ such that $\mu L < \delta/4$, where

$$L = H\backslash\Big\{t \in H : \frac{M(t, \alpha\lambda)}{\alpha\lambda} \geq 1\Big\}.$$

Hence, if $k_{n,m} > \alpha$, then

$$\begin{aligned}
\|u_n - u_m\|^0 &= \frac{1}{k_{n,m}}\big[1 + \rho_M\big(k_{n,m}(u_n - u_m)\big)\big] \geq \int_{T_{n,m}} \frac{M(t, k_{n,m}\|\frac{2}{n_0}u(t)\|)}{k_{n,m}}\, dt \\
&\geq \int_{T_{m,n}\backslash L} \frac{M(t, k_{m,n}\|\frac{2}{n_0}u(t)\|)}{k_{m,n}\|\frac{2}{n_0}u(t)\|} \cdot \Big\|\frac{2}{n_0}u(t)\Big\|\, dt \geq \int_{T_{m,n}\backslash L} 1 \cdot \lambda\, dt \geq \frac{3}{4}\delta\lambda,
\end{aligned}$$

and if $k_{n,m} \leq \alpha$, then $\|u_n - u_m\|^0 = [1 + \rho_M(k_{n,m}(u_n - u_m))]/k_{n,m} \geq 1/\alpha$.

Case III. Let $K(u_n - u_m) = \emptyset$. Then

$$\|u_n - u_m\|^0 = \int_T A(t) \cdot \big\|u_n(t) - u_m(t)\big\|\, dt = \int_{T_{n,m}} A(t) \cdot \Big\|\frac{2}{n_0}u(t)\Big\|\, dt \geq \lambda\delta.$$

Therefore, $(y_n)_{n=1}^{\infty}$ is not relatively compact, which is a contradiction. This implies that for any $u \in 2S(L_M^0(X))$, we obtain that $\mu\{t \in T : k\|u(t)\| \in K_t\} = 0$, where $k \in K(u)$.

(c2) Pick a dense set $\{r_i\}_{i=1}^{\infty}$ in $(0,\infty)$. Then, for each $n, i \in N$, we define measurable sets

$$G_{i,n} = \Big\{t \in T : 2M(t, r_i) = M\Big(t, \Big(1+\frac{1}{n}\Big)r_i\Big) + M\Big(t, \Big(1-\frac{1}{n}\Big)r_i\Big) < \infty\Big\}.$$

Then by the convexity of $M(t,u)$ with respect to u, we have

$$\bigcup_{i=1}^{\infty}\bigcup_{n=1}^{\infty} G_{i,n} = \{t \in T : K_t \neq \emptyset\}.$$

Hence, if (c) does not hold, then $\mu G_{i,n} > 0$ for some $i, n \in N$. Since

$$2M(t, r_i) = M\Big(t, \Big(1+\frac{1}{n}\Big)r_i\Big) + M\Big(t, \Big(1-\frac{1}{n}\Big)r_i\Big) < \infty,$$

then $p(t, r_i) < \infty$ μ-almost everywhere on $G_{i,n}$. Noting that $r_i p(t, r_i) = M(t, r_i) + N(t, p(t, r_i))$, we obtain that $N(t, p(t, r_i)) < \infty$ μ-almost everywhere on $G_{i,n}$. Therefore we can choose $B \subset G_{i,n}$ such that $\mu B > 0$ and $\int_B N(t, p(t, r_i))\,dt < 1$. Pick $v(t) \in L_M^0(X)$. Then there exists $d > 0$ such that $dv(t)\cdot\chi_{T\setminus B}(t) \in S(L_M^0(X))$. It is easy to see that there exists $k_0 > 0$ such that

$$\int_{T\setminus B} N(t, p\big(t, k_0\|dv(t)\|\big)\,dt = \int_T N(t, p\big(t, k_0\|dv(t)\cdot\chi_{T\setminus B}(t)\|\big)\big)\,dt \geq 1.$$

Since $M \in \Delta$, then $E(t) = \infty$ μ-almost everywhere on T. This implies that $p(t, k_0\|dv(t)\|) < \infty$ and $M(t, k_0\|dv(t)\|) < \infty$ μ-almost everywhere on T. Hence

$$N\big(t, p\big(t, k_0\|dv(t)\|\big)\big) = k_0\|dv(t)\| \cdot p\big(t, k_0\|dv(t)\|\big) - M\big(t, k_0\|dv(t)\|\big) < \infty$$

μ-almost everywhere on T. Therefore, we can choose $D \subset T \setminus B$ such that

$$\int_B N\big(t, p(t, r_i)\big)\,dt + \int_D N(t, p\big(t, k_0\|dv(t)\cdot\chi_E(t)\|\big)\,dt = 1.$$

Define $u(t) = r_i \cdot x \cdot \chi_B(t) + d \cdot k_0 \cdot v(t) \cdot \chi_D(t)$, where $x \in S(X)$. Then $\rho_N(p(u)) = 1$. Let $w(t)$ be a nonnegative real measurable function, and let $\rho_N(w) \leq 1$. Then, for any $k > 0$, we have

$$\begin{aligned}\int_T \|u(t)\| \cdot w(t)\,dt &= \frac{1}{k}\int_T k\|u(t)\| \cdot w(t)\,dt \\ &\leq \frac{1}{k}\Big[\int_T M\big(t, k\|u(t)\|\big)\,dt + \int_T N\big(t, w(t)\big)\,dt\Big] \\ &\leq \frac{1}{k}\Big[\int_T M\big(t, k\|u(t)\|\big)\,dt + 1\Big].\end{aligned}$$

This means that $\int_T \|u(t)\| \cdot w(t)\,dt \leq \inf_{k>0}\frac{1}{k}[\rho_M(ku)+1]$. Hence

$$\sup\Big\{\int_T \|u(t)\| \cdot w(t)\,dt : \rho_N(w) \leq 1, w(t) \geq 0\Big\} \leq \inf_{k>0}\frac{1}{k}\big[\rho_M(ku)+1\big].$$

Moreover, we have

$$\begin{aligned}\int_T \|u(t)\| \cdot p\big(t, \|u(t)\|\big)\, dt &= \int_T M\big(t, \|u(t)\|\big)\, dt + \int_T N\big(t, p\big(t, \|u(t)\|\big)\big)\, dt \\ &= \int_T M\big(t, \|u(t)\|\big)\, dt + 1.\end{aligned}$$

This implies that $\inf_{k>0} \frac{1}{k}[\rho_M(ku) + 1] = \rho_M(u) + 1$, that is, $\|u\|^0 = \rho_M(u) + 1$. Hence

$$\bigg\| \frac{u}{\frac{1}{2}\|u\|^0} \bigg\|^0 = \frac{1}{\frac{1}{2}\|u\|^0} \bigg[\rho_M\bigg(\frac{1}{2}\|u\|^0 \cdot \frac{u}{\frac{1}{2}\|u\|^0}\bigg) + 1\bigg].$$

Therefore, by (c1), we obtain that

$$\mu\Big\{t \in T : \frac{1}{2}\|u\|^0 \cdot \frac{\|u(t)\|}{\frac{1}{2}\|u\|^0} \in K_t\Big\} = \mu\big\{t \in T : \|u(t)\| \in K_t\big\} = 0,$$

which is a contradiction. Hence (c) is true.

(a2) Since $M(t, u)$ is strictly convex with respect to u for almost all $t \in T$, then $e(t) = 0$ for almost all $t \in T$. Therefore, for any $u \in L^0_M(X)\backslash\{0\}$, we obtain that $K(u) \neq \emptyset$.

(a3) Suppose that there exist $k_1, k_2 \in K(u)$ satisfying $k_1 \neq k_2$, where $u \in L^0_M\backslash\{0\}$. Define $k = k_1k_2/(k_1 + k_2)$. Then

$$\begin{aligned}2\|u\|^0 &= \|u\|^0 + \|u\|^0 \\ &= \frac{k_1 + k_2}{k_1k_2}\Big[1 + \frac{k_2}{k_1 + k_2}\rho_M(k_1u) + \frac{k_1}{k_1 + k_2}\rho_M(k_2u)\Big] \\ &= \frac{k_1 + k_2}{k_1k_2}\Big[1 + \frac{k_2}{k_1 + k_2}\int_T M\big(t, \|k_1u(t)\|\big)\, dt \\ &\qquad + \frac{k_1}{k_1 + k_2}\int_T M\big(t, \|k_2u(t)\|\big)\, dt\Big] \\ &\geq \frac{k_1 + k_2}{k_1k_2}\Big[1 + \int_T M\Big(t, \frac{k_2}{k_1 + k_2}\|k_1u(t)\| + \frac{k_1}{k_1 + k_2}\|k_2u(t)\|\Big)\, dt\Big] \\ &= \frac{k_1 + k_2}{k_1k_2}\Big[1 + \int_T M\Big(t, \Big\|\frac{2k_1k_2}{k_1 + k_2}u(t)\Big\|\Big)\, dt\Big] \\ &= 2\frac{1}{2k}\big[1 + \rho_M(2ku)\big] \\ &\geq 2\|u\|^0 \\ &= 2.\end{aligned}$$

This implies that

$$\|u\|^0 = \frac{1}{2k}\big[1 + \rho_M(2ku)\big]$$

(i.e., $2k \in K(u)$) and

$$\frac{k_2}{k_1 + k_2}M\big(t, k_1\|u(t)\|\big) + \frac{k_1}{k_1 + k_2}M\big(t, k_2\|u(t)\|\big) = M\big(t, 2k\|u(t)\|\big)$$

μ-almost everywhere on $\{t \in T : \|u(t)\| \neq 0\}$. Since $k_1\|u(t)\| \neq k_2\|u(t)\|$ on $\{t \in T : \|u(t)\| \neq 0\}$, then $2k\|u(t)\| \in K_t$ on $\{t \in T : \|u(t)\| \neq 0\}$, which is a contradiction. Hence condition (a) is true.

(d1) Suppose that X is not rotund. Then there exist $x, y, z \in S(X)$ with $2x = y + z$ and $y \neq z$. By the Hahn–Banach theorem, there exists $x^* \in S(X^*)$ such that $x^*(x) = 1$. Hence $x^*(y) = x^*(z) = x^*(x) = 1$. Pick $h(t) \in S(L^0_M(X))$. Then there exists $d > 0$ such that $\mu D > 0$, where $D = \{t \in T : \|h(t)\| \geq d\}$. Moreover, there exists $r > 0$ such that $\mu H > 0$, where $H = \{t \in D : M(t, \|y - z\|) > r\}$. Put $h_1(t) = d \cdot x \cdot \chi_H(t)$. Then it is easy to see that $h_1(t) \in L^0_M(X)\backslash\{0\}$. Hence there exists $l > 0$ such that $l \cdot h_1(t) \in S(L^0_M(X))$. By the Hahn–Banach theorem and $(E^0_M(R))^* = L_N(R)$, there exists $h_2(t) \in S(L_N(R))$ such that $\int_T ld \cdot \chi_H(t) \cdot h_2(t)\, dt = 1$.

Decompose H into H^1_1, H^1_2 such that $H^1_1 \cap H^1_2 = \emptyset$, $H^1_1 \cup H^1_2 = H$, and $\mu H^1_1 = \mu H^1_2$. Decompose H^1_1 into H^2_1, H^2_2 such that $H^2_1 \cap H^2_2 = \emptyset$, $H^2_1 \cup H^2_2 = H^1_1$, and $\mu H^2_1 = \mu H^2_2$. Decompose H^1_2 into H^2_3, H^2_4 such that $H^2_3 \cap H^2_4 = \emptyset$, $H^2_3 \cup H^2_4 = H^1_2$, and $\mu H^2_3 = \mu H^2_4$. Generally, decompose H^{n-1}_i into H^n_{2i-1}, H^n_{2i} such that

$$\begin{aligned} &H^n_{2i-1} \cap H^n_{2i} = \emptyset, \qquad H^n_{2i-1} \cup H^n_{2i} = H^{n-1}_i, \qquad \text{and} \\ &\quad \mu H^n_{2i-1} = \mu H^n_{2i}, \end{aligned}$$

where $n = 1, 2, \ldots, i = 1, 2, \ldots, 2^{n-1}$. Set

$$u_n(t) = \begin{cases} 0, & t \in T\backslash H, \\ y, & t \in H^n_1, \\ z, & t \in H^n_2, \\ \cdots & \cdots \\ y, & t \in H^n_{2^n-1}, \\ z, & t \in H^n_{2^n}, \end{cases} \qquad u(t) = \begin{cases} 0, & t \in T\backslash H, \\ y, & t \in H^n_1, \\ y, & t \in H^n_2, \\ \cdots & \cdots \\ y, & t \in H^n_{2^n-1}, \\ y, & t \in H^n_{2^n}, \end{cases}$$

and $v(t) = h_2(t) \cdot x^*$. Then it is easy to see that $\|u_n\| = 1/(ld)$, $\|u\| = 1/(ld)$, and $\|v\| = 1$. Therefore, by $x^*(y) = x^*(z) = x^*(x) = 1$, we obtain that

$$\int_T \big(u_n(t) \cdot v(t)\big)\, dt = \int_T \chi_H(t) \cdot h_2(t)\, dt = \frac{1}{ld} \int_T ld \cdot \chi_H(t) \cdot h_2(t)\, dt = \frac{1}{ld}$$

and

$$\int_T \big(u(t) \cdot v(t)\big)\, dt = \int_T \chi_H(t) \cdot h_2(t)\, dt = \frac{1}{ld} \int_T ld \cdot \chi_H(t) \cdot h_2(t)\, dt = \frac{1}{ld}.$$

This implies that $(u_n, v) = 1/(ld)$ and that v is norm attainable. Since every proximinal convex subset of $L^0_M(X)$ is approximatively compact, by Lemma 2.5, we obtain that $\{u_n\}^\infty_{n=1}$ is relatively compact. However, picking $k_{n,m} \in K(u_n - u_m)$, if $k_{n,m} \leq 1$, then we get

$$\|u_n - u_m\|^0 \geq \frac{1}{k_{n,m}} \big[1 + \rho_M\big(k_{n,m}(u_n - u_m)\big)\big] \geq 1.$$

If $k_{n,m} > 1$, then

$$\begin{aligned}\|u_n - u_m\|^0 &\geq \frac{1}{k_{n,m}}\big[1 + \rho_M\big(k_{n,m}(u_n - u_m)\big)\big]\\ &\geq \int_{H_{n,m}} \frac{M(t, k_{n,m}\|y-z\|)}{k_{n,m}}\,dt\\ &\geq \int_{H_{n,m}} \frac{k_{n,m}M(t, \|y-z\|)}{k_{n,m}}\,dt = \int_{H_{n,m}} M\big(t, \|y-z\|\big)\,dt\\ &\geq r\cdot\mu H_{n,m} = \frac{1}{2}r\cdot\mu H,\end{aligned}$$

where $H_{n,m} = \{t \in T : u_n(t) \neq u_m(t)\}$. This means that the sequence $\{u_n\}_{n=1}^{\infty}$ is not relatively compact, which is a contradiction.

(d) Pick $h \in S(L_M^0(X))$. Then there exists $d > 0$ such that $\mu E > 0$, where $E = \{t \in T : \|h(t)\| \geq d\}$. Put $h_1(t) = d\cdot x_0\cdot\chi_E(t)$, where $x_0 \in S(X)$. It is easy to see that $h_1(t) \in L_M^0(X)\backslash\{0\}$. Hence there exists $l > 0$ such that $l\cdot h_1(t) \in S(L_M^0(X))$. We next prove that X is isometrically embedded into $L_M^0(X)$. We define the operator $I\colon X \to L_M^0(X)$ by

$$I(x) = ld\cdot x\cdot\chi_E(t), \quad x \in X.$$

It is easy to see that $I(x_0) \in S(L_M^0(X))$. Hence, for any $x \in X\backslash\{0\}$, we have

$$\begin{aligned}\big\|I(x)\big\|^0 &= \inf_{k>0}\frac{1}{k}\big[1 + \rho_M\big(k\cdot I(x)\big)\big]\\ &= \inf_{k>0}\frac{1}{k}\Big[1 + \int_E M\big(t, k\cdot ld\|x\|\big)\,dt\Big]\\ &= \inf_{k>0}\frac{1}{k}\Big[1 + \int_E M\big(t, k\cdot\|x\|ld\|x_0\|\big)\,dt\Big] = \inf_{k>0}\frac{1}{k}\big[1 + \rho_M\big(k\cdot\|x\|I(x_0)\big)\big]\\ &= \big\|\|x\|\cdot I(x_0)\big\|^0 = \|x\|\cdot\big\|I(x_0)\big\|^0 = \|x\|.\end{aligned}$$

This implies that every proximinal convex subset of X is approximatively compact.

For the sufficient part, let $u_n, u \in S(L_M^0(X))$, $v \in S(L_N(X^*))$, $(u, v) = 1$, and $(u_n, v) \to 1$ as $n \to \infty$. Then it is easy to see $(u_n + u, v) \to 2$ as $n \to \infty$. The proof requires the consideration of few cases separately.

Case I. Let $\sup\{k_n\} < \infty$, where $k_n = K(u_n)$. Then we may assume without loss of generality that $k_n \to l$. We will prove that $\|u_n(t)\| \xrightarrow{\mu} \|u(t)\|$ in measure. Otherwise, we may assume without loss of generality that for each $n \in N$, there exists $E_n \subseteq T$, $\varepsilon_0 > 0$, and $\sigma_0 > 0$ such that $\mu E_n \geq \varepsilon_0$, where

$$E_n = \big\{t \in T : \big|\|u_n(t)\| - \|u(t)\|\big| \geq \sigma_0\big\}.$$

We define the sets

$$A_n = \Big\{t \in T : M\big(t, \big\|k_n u_n(t)\big\|\big) > \frac{8}{\varepsilon_0}\Big\} \qquad \text{and}$$

$$B = \Big\{t \in T : M\big(t, \big\|ku(t)\big\|\big) > \frac{8}{\varepsilon_0}\Big\},$$

where $k \in K(u)$. Then

$$1 = \int_T M\big(t, \|k_n u_n(t)\|\big)\, dt \geq \int_{A_n} M\big(t, \|k_n u_n(t)\|\big)\, dt \geq \frac{8}{\varepsilon_0} \mu A_n.$$

This implies that $\mu A_n \leq \varepsilon_0/8$. Similarly, we have $\mu B \leq \varepsilon_0/8$. For μ-almost everywhere $t \in T$, we define a bounded closed set

$$C_t = \Big\{(u,v) \in R^2 : M(t,u) \leq \frac{8}{\varepsilon_0}, M(t,v) \leq \frac{8}{\varepsilon_0}, |u-v| \geq \frac{1}{4}\sigma_0\Big\}$$

in 2-dimensional space. Since C_t is compact, we obtain that for μ-almost everywhere $t \in T$, there exists $(u_t, v_t) \in C_t$ such that

$$1 > \frac{M(t, (\frac{k}{k+l}u_t + \frac{l}{k+l}v_t))}{\frac{k}{k+l}M(t,u_t) + \frac{l}{k+l}M(t,v_t)} \geq \frac{M(t, (\frac{k}{k+l}u + \frac{l}{k+l}v))}{\frac{k}{k+l}M(t,u) + \frac{l}{k+l}M(t,v)} \tag{2.1}$$

for any $(u,v) \in C_t$. We define a function

$$1 - \delta(t) = \frac{M(t, (\frac{k}{k+l}u_t + \frac{l}{k+l}v_t))}{\frac{k}{k+l}M(t,u_t) + \frac{l}{k+l}M(t,v_t)}. \tag{2.2}$$

Then $\delta(t)$ is μ-measurable. In fact, pick a dense set $\{r_i\}_{i=1}^{\infty}$ in $[0,\infty)$. We define a function

$$1 - \delta_{r_i,r_j}(t) = \begin{cases} \frac{M(t,(\frac{k}{k+l}r_i + \frac{l}{k+l}r_j))}{\frac{k}{k+l}M(t,r_i) + \frac{l}{k+l}M(t,r_j)}, & M(t,r_i) \leq \frac{8}{\varepsilon_0} \text{ and } M(t,r_j) \leq \frac{8}{\varepsilon_0}, \\ 0, & M(t,r_i) > \frac{8}{\varepsilon_0} \text{ or } M(t,r_j) > \frac{8}{\varepsilon_0}. \end{cases}$$

By the definition of $M(t,u)$, it is easy to see that $1 - \delta_{r_i,r_j}(t)$ is μ-measurable and

$$1 - \delta(t) \geq \sup\Big\{1 - \delta_{r_i,r_j}(t) : |r_i - r_j| \geq \frac{1}{4}\sigma_0\Big\}.$$

On the other hand, since $\{r_i\}_{i=1}^{\infty}$ is dense in $[0,\infty)$, then $\{(r_i,r_j)\}_{i=1,j=1}^{\infty}$ is dense in $[0,\infty) \times [0,\infty)$. By definition of the function $1-\delta(t)$, we obtain that for μ-almost everywhere $t \in T$ and $\varepsilon > 0$, there exists $(r_i, r_j) \in C_t$ such that

$$1 - \delta(t) - \varepsilon < 1 - \delta_{r_i,r_j}(t) \leq \sup\Big\{1 - \delta_{r_i,r_j}(t) : |r_i - r_j| \geq \frac{1}{4}\sigma_0\Big\}$$

μ-almost everywhere on T. Since ε is arbitrary, we have

$$1 - \delta(t) \leq \sup\Big\{1 - \delta_{r_i,r_j}(t) : |r_i - r_j| \geq \frac{1}{4}\sigma_0\Big\}$$

μ-almost everywhere on T. Then $1 - \delta(t) = \sup\{1 - \delta_{r_i,r_j}(t) : |r_i - r_j| \geq \sigma_0/4\}$ μ-almost everywhere on T. This implies that $\delta(t)$ is μ-measurable. By formulas (2.1) and (2.2), we have

$$\delta(t) \leq 1 - \frac{M(t, (\frac{k}{k+l}u + \frac{l}{k+l}v))}{\frac{k}{k+l}M(t,u) + \frac{l}{k+l}M(t,v)}, \quad u, v \in C_t$$

for μ-almost everywhere $t \in T$. We know that

$$T \supset \bigcup_{n=1}^{\infty} \Big\{ t \in T : \frac{1}{n+1} < \delta(t) \le \frac{1}{n} \Big\}.$$

Since $M(t,u)$ is strictly convex with respect to u for almost all $t \in T$, there exists $2\delta_0 \in (0,1)$ such that $\mu G < \varepsilon_0/16$, where

$$G = \big\{ t \in T : \delta(t) \le 2\delta_0 \big\}.$$

We have $W_n(t) - Q_n(t) \to 0$ μ-almost everywhere on T, where

$$\begin{aligned} W_n(t) &= \frac{M(t, \frac{k}{k+k_n}\|k_n u_n(t)\| + \frac{k_n}{k+k_n}\|ku(t)\|)}{\frac{k}{k+k_n}M(t,\|k_n u_n(t)\|) + \frac{k_n}{k+k_n}M(t,\|ku(t)\|)} \cdot \chi_{E_n \setminus (A_n \cup B)}(t), \\ Q_n(t) &= \frac{M(t, \frac{k}{k+l}\|k_n u_n(t)\| + \frac{l}{k+l}\|ku(t)\|)}{\frac{k}{k+l}M(t,\|k_n u_n(t)\|) + \frac{l}{k+l}M(t,\|ku(t)\|)} \cdot \chi_{E_n \setminus (A_n \cup B)}(t). \end{aligned}$$

By Egorov's theorem, there exists N such that $|W_n(t) - Q_n(t)| < \delta_0/4, t \in E$, whenever $n > N$, where $E \subset T$ and $\mu(T \backslash E) < \varepsilon_0/16$. Let $E_{n1} = E_n \backslash (G \cup (T \backslash E))$. Hence, if $E_{n1} \backslash (A_n \cup B)$, then

$$\begin{aligned} \frac{3}{2}\delta_0 &= 2\delta_0 - \frac{1}{2}\delta_0 \\ &\le 1 - \frac{M(t, \frac{k}{k+l}\|k_n u_n(t)\| + \frac{l}{k+l}\|ku(t)\|)}{\frac{k}{k+l}M(t,\|k_n u_n(t)\|) + \frac{l}{k+l}M(t,\|ku(t)\|)} - \frac{1}{2}\delta_0 \\ &\le 1 - \frac{M(t, \frac{k}{k+k_n}\|k_n u_n(t)\| + \frac{k_n}{k+k_n}\|ku(t)\|)}{\frac{k}{k+k_n}M(t,\|k_n u_n(t)\|) + \frac{k_n}{k+k_n}M(t,\|ku(t)\|)}, \end{aligned}$$

when n is large enough. This implies that

$$\begin{aligned} M\Big(t, \frac{k}{k+k_n}\big\|k_n u_n(t)\big\| + \frac{k_n}{k+k_n}\big\|ku(t)\big\|\Big) \le (1-\delta_0)\Big[&\frac{k}{k+k_n} M\big(t, \big\|k_n u_n(t)\big\|\big) \\ &+ \frac{k_n}{k+k_n} M\big(t, \big\|ku(t)\big\|\big)\Big] \end{aligned}$$

on $E_{n1} \setminus (A_n \cup B_n)$. We know that $M(t, \frac{1}{\overline{k}+k}\sigma_0) > 0$ μ-almost everywhere on T, where $\overline{k} = \sup\{k_n\}$. Since

$$T \supset \bigcup_{i=1}^{\infty} \Big\{ t \in T : \frac{1}{i+1} < M\Big(t, \frac{1}{\overline{k}+k}\sigma_0\Big) \le \frac{1}{i} \Big\},$$

there exists $a > 0$ such that $\mu C < \varepsilon_0/8$, where

$$C = \Big\{ t \in T : M\Big(t, \frac{1}{\overline{k}+k}\sigma_0\Big) \le a \Big\}.$$

Let $H_n = E_n \backslash (A_n \cup B \cup G \cup (T\backslash E))$. Then $\mu H_n \geq \varepsilon_0/4$. Hence

$$\begin{aligned}
&\|u_n\|^0 + \|u\|^0 - \|u_n + u\|^0 \\
&\quad \geq \frac{1}{k_n}\big[1 + \rho_M(k_n u_n)\big] + \frac{1}{k}\big[1 + \rho_M(ku)\big] - \frac{k_n + k}{k_n k}\Big(1 + \rho_M\Big(\frac{k_n k}{k_n + k}(u_n + u)\Big)\Big) \\
&\quad \geq \frac{k_n + k}{k_n k}\int_{H_n}\Big[\frac{k}{k_n + k}M\big(t, \|k_n u_n(t)\|\big) + \frac{k_n}{k_n + k}M\big(t, \|ku(t)\|\big) \\
&\qquad - M\Big(t, \Big\|\frac{k_n k}{k_n + k}\big(u_n(t) + u(t)\big)\Big\|\Big)\Big]\,dt \\
&\quad \geq \frac{k_n + k}{k_n k}\int_{H_n}\Big[\frac{k}{k_n + k}M\big(t, \|k_n u_n(t)\|\big) + \frac{k_n}{k_n + k}M\big(t, \|ku(t)\|\big) \\
&\qquad - M\Big(t, \frac{k}{k_n + k}\|k_n u_n(t)\| + \frac{k_n}{k_n + k}\|ku(t)\|\Big)\Big]\,dt \\
&\quad \geq \frac{k_n + k}{k_n k}\int_{H_n}\delta_0\Big[\frac{k}{k_n + k}M\big(t, \|k_n u_n(t)\|\big) + \frac{k_n}{k_n + k}M\big(t, \|ku(t)\|\big)\Big]\,dt \\
&\quad \geq \frac{k_n + k}{k_n k}\int_{H_n}\delta_0\Big[M\Big(t, \frac{k}{k_n + k}\|k_n u_n(t)\| + \frac{k_n}{k_n + k}\|ku(t)\|\Big)\Big]\,dt \\
&\quad \geq \frac{k_n + k}{k_n k}\int_{H_n}\delta_0\Big[M\Big(t, \frac{kk_n}{k_n + k}\big|\|u_n(t)\| - \|u(t)\|\big|\Big)\Big]\,dt \\
&\quad \geq \frac{k_n + k}{k_n k}\int_{H_n}\delta_0\Big[M\Big(t, \frac{1}{\bar{k} + k}\sigma_0\Big)\Big]\,dt \\
&\quad \geq \frac{2}{\bar{k}k}\delta_0 a \cdot \frac{1}{4}\varepsilon_0,
\end{aligned}$$

when n large enough. By $(u_n + u, v) \to 2$, we obtain that $\|u_n + u\|^0 \to 2$. Hence $\|u_n\|^0 + \|u\|^0 - \|u_n + u\|^0 \to 0$ as $n \to \infty$, which is a contradiction. Hence $\|u_n(t)\| \to^{\mu} \|u(t)\|$ in measure. By the Riesz theorem, there exists a subsequence $\{n\}$ of $\{n\}$ such that $\|u_n(t)\| \to \|u(t)\|$ μ-almost everywhere on T. Noting that

$$\big|\big(u_n(t), v(t)\big)\big| \leq \|u_n(t)\| \cdot \|v(t)\|, \qquad \int_T \big(u_n(t), v(t)\big)\,dt \to 1$$

and

$$\int_T \|u_n(t)\| \cdot \|v(t)\|\,dt \leq \|u_n\|^0 \cdot \|v\| \leq 1,$$

we obtain that $\int_T \|u_n(t)\| \cdot \|v(t)\|\,dt \to 1$ and $\int_T [\|u_n(t)\| \cdot \|v(t)\| - (u_n(t), v(t))]\,dt \to 0$, that is, $\int_T |\|u_n(t)\| \cdot \|v(t)\| - (u_n(t), v(t))|\,dt \to 0$. This implies that $\|u_n(t)\| \cdot \|v(t)\| - (u_n(t), v(t)) \to^{\mu} 0$ in measure. Therefore, by the Riesz theorem, there exists a subsequence $\{n\}$ of $\{n\}$ such that $\|u_n(t)\| \cdot \|v(t)\| - (u_n(t), v(t)) \to 0$ μ-almost everywhere on T. By $\|u_n(t)\| \to \|u(t)\|$ μ-almost everywhere on T, it follows that $(u_n(t), v(t)) \to \|u(t)\| \cdot \|v(t)\|$ μ-almost everywhere on T. We may assume without loss of generality that

$$\Big(\frac{u_n(t)}{\|u(t)\|}, \frac{v(t)}{\|v(t)\|}\Big) \to 1 \quad \text{on } \big\{t \in T : \|u(t)\| \cdot \|v(t)\| \neq 0\big\}.$$

Then $\mu T_1 = 0$, where $T_1 = \{t \in T : \|v(t)\| = 0\} \cap \{t \in T : \|u(t)\| \neq 0\}$. In fact, if $\mu T_1 > 0$, then

$$\|u\|^0 = \frac{1}{k}\big[1 + \rho_M(ku)\big] > \frac{1}{k}\big[1 + \rho_M(ku\chi_{T\setminus T_1})\big] \geq \|u\chi_{T\setminus T_1}\|^0,$$

where $k \in K(u)$. Hence,

$$1 = \int_T (u, v)\, dt = \int_T (u\chi_{T\setminus T_1}, v)\, dt \leq \|u\chi_{T\setminus T_1}\|^0 \cdot \|v\| < 1,$$

which is a contradiction. We may assume without loss of generality that

$$\Big(\frac{u_n(t)}{\|u(t)\|}, \frac{v(t)}{\|v(t)\|}\Big) \to 1 \quad \text{on } \big\{t \in T : \big\|u(t)\big\| \neq 0\big\}.$$

Noting that $\|u_n(t)\| \to \|u(t)\|$ μ-almost everywhere on T, we may assume without loss of generality that $(u(t)/\|u(t)\|, v(t)/\|v(t)\|) = 1$. Since

$$\Big(\frac{u(t)}{\|u(t)\|}, \frac{v(t)}{\|v(t)\|}\Big) = 1, \qquad \frac{u_n(t)}{\|u(t)\|} \to 1 \qquad \text{and} \qquad \frac{v(t)}{\|v(t)\|} \in S(X^*),$$

by Lemma 2.5, we obtain that $\{u_n(t)/\|u(t)\|\}_{n=1}^{\infty}$ is relatively compact. Since X is rotund, we obtain that the sequence $\{u_n(t)/\|u(t)\|\}_{n=1}^{\infty}$ is convergent. In fact, suppose that there exists $t_0 \in \{t \in T : \|u(t)\| \neq 0\}$ such that $\{u_n(t_0)/\|u(t_0)\|\}_{n=1}^{\infty}$ is not convergent. Then there exist subsequences $\{n_i\}$ and $\{n_j\}$ of $\{n\}$ such that

$$\frac{u_{n_i}(t_0)}{\|u(t_0)\|} \to x_1, \qquad \frac{u_{n_j}(t_0)}{\|u(t_0)\|} \to x_2, \qquad \text{and} \qquad x_1 \neq x_2.$$

Hence

$$\Big(x_1, \frac{v(t_0)}{\|v(t_0)\|}\Big) = \Big(x_2, \frac{v(t_0)}{\|v(t_0)\|}\Big).$$

This implies that $x_1 = x_2$, which is a contradiction. Hence there exists $x(t) \in S(X)$ such that $u_n(t)/\|u(t)\| \to x(t), t \in \{t \in T : \|u(t)\| \neq 0\}$. Let

$$u_0(t) = \begin{cases} \|u(t)\|x(t), & t \in \{t \in T : \|u(t)\| \neq 0\}, \\ 0, & t \in \{t \in T : \|u(t)\| = 0\}. \end{cases}$$

Then it is easy to see that $\|u_0\|^0 = 1$ and $u_n(t) \to u_0(t)$ μ-almost everywhere on T. We next prove that $l = h$, where $h \in K(u_0)$ and $l = \lim_{n\to\infty} k_n$. In fact, by Fatou's lemma, it follows that

$$\frac{1}{h}\big[1+\rho_M(hu_0)\big] = \|u_0\|^0 = \lim_{n\to\infty} \|u_n\|^0 = \lim_{n\to\infty} \frac{1}{k_n}\big[1+\rho_M(k_n u_n)\big] \geq \frac{1}{l}\big[1+\rho_M(lu_0)\big],$$

so $l = h$. By the convexity of M, we have

$$\frac{M(t, \|k_n u_n(t)\|) + M(t, \|hu_0(t)\|)}{2} - M\Big(t, \frac{\|k_n u_n(t) - hu_0(t)\|}{2}\Big) \geq 0$$

for μ-almost everywhere $t \in T$. Moreover, we have $\rho_M(k_n u_n) = k_n\|u_n\|^0 - 1 \to h\|u_0\|^0 - 1 = \rho_M(hu_0)$. Therefore, by Fatou's lemma, we obtain the following:

$$\begin{aligned}\rho_M(hu_0) &= \int_T \lim_{n\to\infty}\Big[\frac{M(t,\|k_n u_n(t)\|) + M(t,\|hu_0(t)\|)}{2} \\ &\qquad - M\Big(t, \frac{\|k_n u_n(t) - hu_0(t)\|}{2}\Big)\Big]\, dt \\ &\le \liminf_{n\to\infty} \int_T \Big[\frac{M(t,\|k_n u_n(t)\|) + M(t,\|hu_0(t)\|)}{2} \\ &\qquad - M\Big(t, \frac{\|k_n u_n(t) - hu_0(t)\|}{2}\Big)\Big]\, dt \\ &= \rho_M(hu_0) - \limsup_{n\to\infty} \rho_M\Big[\frac{1}{2}(k_n u_n - hu_0)\Big].\end{aligned}$$

This implies that $\rho_M(\frac{1}{2}(k_n u_n - hu_0)) \to 0$ as $n \to \infty$. By Lemma 2.4, we obtain that $\|k_n u_n - hu_0\| \to 0$. Then $\|k_n u_n - hu_0\|^0 \le 2\|k_n u_n - hu_0\| \to 0$ as $n \to \infty$. Using the equalities $\lim_{n\to\infty} k_n = l = h$, we obtain $\|u_n - u_0\|^0 \to 0$ as $n \to \infty$. So $\{u_n\}_{n=1}^\infty$ is relatively compact.

Case II. Let $\sup\{k_n\} = \infty$, where $k_n = K(u_n)$. Then we consider the sequence $2u'_n = (u_n + u)$ in place of $\{u_n\}_{n=1}^\infty$, because $\|u_n - u\|^0 \to 0$ as $n \to \infty$ if and only if $\|u'_n - u\|^0 \to 0$ as $n \to \infty$. Moreover, we have

$$\Big\|\frac{1}{2}(u_n + u)\Big\|^0 \le \frac{1}{2}\big(\|u_n\|^0 + \|u\|^0\big)$$

for every $n \in N$. Hence $\limsup_{n\to\infty} \|(u_n + u)\|^0 \le 2$. Since

$$\int_T \Big(\frac{1}{2}(u_n + u), v\Big)\, dt = \frac{1}{2}\int_T (u_n, v)\, dt + \frac{1}{2}\int_T (u, v)\, dt \to 1,$$

we obtain that $\liminf_{n\to\infty} \|(u_n + u)\|^0 \ge 2$. This implies that $\lim_{n\to\infty} \|(u_n + u)\|^0 \to 2$ as $n \to \infty$. Define $w_n = (2k_n k)/(k_n + k)$, where $k \in K(u)$. Then the sequence $\{w_n\}_{n=1}^\infty$ is bounded. Moreover,

$$\begin{aligned}\Big\|\frac{1}{2}(u_n + u)\Big\|^0 &\le \frac{1}{w_n}\Big[1 + \rho_M\Big(w_n \cdot \frac{u_n + u}{2}\Big)\Big] \\ &= \frac{k_n + k}{2k_n k}\Big[1 + \rho_M\Big(\frac{k_n k}{k_n + k}(u_n + u)\Big)\Big] \\ &\le \frac{k_n + k}{2k_n k}\Big[1 + \frac{k}{k_n + k}\rho_M\big((k_n u_n)\big) + \frac{k_n}{k_n + k}\rho_M\big((ku)\big)\Big] \\ &\le \frac{1}{2}\Big[\frac{1}{k_n}\big(1 + \rho_M(k_n u_n)\big) + \frac{1}{k}\big(1 + \rho_M(ku)\big)\Big] \\ &= \frac{1}{2}\big[\|u_n\|^0 + \|u\|^0\big] \to 1,\end{aligned}$$

whence it follows that

$$\frac{k_n + k}{2k_n k}\Big[1 + \rho_M\Big(\frac{2k_n k}{k_n + k} \cdot \frac{1}{2}(u_n + u)\Big)\Big] \to 1 \quad \text{as } n \to \infty.$$

By $(u,v)=1$ and $(u_n,v)\to 1$, we have $(u'_n,v)\to 1$. Therefore, we can prove in the same way as in Case I that $\|u'_n-u\|^0\to 0$. So $\{u_n\}_{n=1}^{\infty}$ is relatively compact. This completes the proof. □

Corollary 2.9. *We have that $L^0_M(X)$ is approximatively compact if and only if*

(a) *for any $v\in L^0_M(X)\setminus\{0\}$, the set $K(v)$ consists of one element from $(0,+\infty)$;*
(b) *$M\in\Delta$ and $N\in\Delta$;*
(c) *$M(t,u)$ is strictly convex with respect to u for almost all $t\in T$;*
(d) *X is approximatively compact and round.*

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