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MULTIPLICATIVE OPERATOR FUNCTIONS AND ABSTRACT CAUCHY PROBLEMS

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Abstract. We use the duality between functional and differential equations to solve several classes of abstract Cauchy problems related to special functions. As a general framework, we investigate operator functions which are multiplicative with respect to convolution of a hypergroup. This setting contains all representations of (hyper)groups, and properties of continuity are shown; examples are provided by translation operator functions on homogeneous Banach spaces and weakly stationary processes indexed by hypergroups. Then we show that the concept of a multiplicative operator function can be used to solve a variety of abstract Cauchy problems, containing discrete, compact, and noncompact problems, including C_0-groups and cosine operator functions, and more generally, Sturm–Liouville operator functions.

1. Introduction

It is well-known that C_0-semigroups obey the exponential law and solve first-order abstract Cauchy problems. Similarly, a theory of cosine operator functions exists—that is, operator functions which obey the cosine functional equation are used to solve second-order abstract Cauchy problems (see [1, Sections 3.14–3.16] for an introduction). This duality between operator-valued functional equations and abstract Cauchy problems is extended by the author in [11] to abstract

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Sturm–Liouville problems. Therefore, the underlying functional equation is provided by Sturm–Liouville hypergroups.

In the present paper we consider, more generally, operator functions which are multiplicative with respect to the convolution of an arbitrary hypergroup. We will see that our concept of a "multiplicative operator function" contains some further abstract Cauchy problems; that is, we establish a unifying framework for several classes of abstract Cauchy problems.

Let $(K, *)$ be a hypergroup with left Haar measure m (a *hypergroup* is a generalization of a locally compact group, where $t * s$, $t, s \in K$ stands for a probability measure instead of just a single element; for the convenience of the reader, some details are collected in Section 2). Throughout, let X be a complex Banach space, and let $\mathcal{L}(X)$ denote the Banach algebra of bounded linear operators on X with unit I.

Definition 1.1 (Strong version). A function $S : K \to \mathcal{L}(X)$ is called a [*strong*] *multiplicative operator function* if S is strongly continuous (i.e., $S(\cdot)x : K \to X$ is continuous for each $x \in X$) and if

(i) $S(e) = I$,
(ii) $S(t)S(s)x = S(t * s)x$ for all $t, s \in K$ and any $x \in X$,
(iii) $\lim_{t\to e} S(t)x = x$ for each $x \in X$,

where the right-hand side of the functional equation (ii) is defined by

$$S(t * s)x := \big(S(\cdot)x\big)(t * s) := \int_K S(\cdot)x \,\mathrm{d}(t * s).$$

In Section 3 we give two further, weak versions of Definition 1.1 and show that all three definitions are equivalent (Theorem 3.5). Thus, the notion of a multiplicative operator function contains all representations of (hyper)groups, and we obtain characterizations of weak, strong, and uniform continuity. In Section 4, natural examples of multiplicative operator functions are provided by translation operator functions on homogeneous Banach spaces; in particular, all weakly stationary processes indexed by hypergroups are contained in this setting.

Section 5, which is the heart of the present paper (and also of [11]), shows that abstract Cauchy problems (the solutions of the corresponding scalar problem are multiplicative with respect to convolution of some hypergroup) can be solved by corresponding multiplicative operator functions. Since K may be quite different, we consider abstract Cauchy problems in integral form.

Let us give a brief outline of the principle idea here. Let $\mathfrak{J}$ and $\{\delta_{\mathcal{J}}, \mathcal{J} \in \mathfrak{J}\}$ be families of bounded nonnegative nonzero (resp., bounded) measures with compact support on a commutative hypergroup K. For reasons of intuition, let us suppose that $\mathfrak{J}$ is parameterized by $t \in K$ and that "$\mathrm{supp}(\mathcal{J}_t) \to \{e\}$" as $t \to e$. Suppose that for each multiplicative function χ there exists a constant c_χ such that

$$\int_K \chi \,\mathrm{d}\delta_{\mathcal{J}_t} = c_\chi \int_K \chi \,\mathrm{d}\mathcal{J}_t$$

for all $t \in K$. Then a multiplicative operator function S on K solves the abstract Cauchy problem

$$\int_K S(\cdot)x \, \mathrm{d}\delta_{\mathcal{J}_t} = \mathbb{A}_0 \int_K S(\cdot)x \, \mathrm{d}\mathcal{J}_t,$$

given $x \in X$, for all $t \in K$, where

$$\mathbb{A}_0 y := \lim_{t \to e} \frac{\int_K S(\cdot)y \, \mathrm{d}\delta_{\mathcal{J}_t}}{\mathcal{J}_t(K)}$$

whenever this limit exists.

In Section 6 we demonstrate that this approach comprises a variety of abstract Cauchy problems, including C_0-groups, cosine, and Sturm–Liouville operator functions as well as problems of discrete type, arising from orthogonal polynomials, and compact Jacobi and Sturm–Liouville type. (The present article took shape originally in the author's doctoral dissertation [12].)

2. Preliminaries

In this section we give a short introduction to hypergroups and extend some basic theorems about convolution to vector-valued functions. The stated results are of a general nature and slightly stronger than needed in the present paper.

2.1. **Hypergroups.**

We present hypergroups via the axiomatic of Jewett (see [17]) which is widely accepted (see [2] for further details and notation). The following presentation of hypergroups is intended to emphasize the analogy to locally compact groups.

Let K be a nonvoid, locally compact Hausdorff space. Let $M^1(K)$ denote the space of probability, and let $M^b(K)$ be the space of all bounded complex Borel measures on K. Both $M^1(K)$ and $M^b(K)$ are endowed with the weak topology induced by $C_b(K)$, the space of continuous and bounded functions. A measure is usually regarded as a continuous linear functional on $C_c(K)$, the space of continuous functions with compact support endowed with the inductive limit topology. The support of a measure μ is denoted by $\mathrm{supp}(\mu)$.

Suppose that

$$* : K \times K \to M^1(K)$$

is a continuous mapping. Identifying $t \in K$ with its point measure ε_t, its bilinear positive-continuous extension

$$* : M^b(K) \times M^b(K) \to M^b(K),$$

called *convolution*, is given by

$$(\mu * \nu)(f) = \int_K \int_K (\varepsilon_t * \varepsilon_s)(f) \mu(\mathrm{d}t) \nu(\mathrm{d}s), \quad f \in C_c(K)$$

for $\mu, \nu \in M^b(K)$.

Definition 2.1. Let K be a nonvoid locally compact Hausdorff space with convolution as above. Then K, $(K,*)$, or more precisely the triple $(M^b(K),+,*)$, is called a *hypergroup* if the following conditions are satisfied.

(H1) The convolution $*$ is *associative*.
(H2) There exists a (unique) *neutral element*, that is, $e \in K$, such that $\varepsilon_e * \varepsilon_t = \varepsilon_t * \varepsilon_e = \varepsilon_t$ for all $t \in K$.
(H3) There exists a (unique) *involution*, that is, a self-inverse homeomorphism $^-$ on K, such that $e \in \operatorname{supp}(\varepsilon_t * \varepsilon_s)$ if and only if $t = s^-$, and $(\varepsilon_t * \varepsilon_s)^- = \varepsilon_{s^-} * \varepsilon_{t^-}$ for all $t, s \in K$, where μ^- denotes the image of μ under $^-$.
(H4) For every $t, s \in K$, $\operatorname{supp}(\varepsilon_t * \varepsilon_s)$ is compact, and the mapping $(t,s) \mapsto \operatorname{supp}(\varepsilon_t * \varepsilon_s)$ of $K \times K$ into $\mathscr{C}(K)$ is continuous, where $\mathscr{C}(K)$ denotes the collection of nonvoid compact subsets of K, endowed with the Michael topology (see (2.1)).

A hypergroup is called *commutative* if the algebra $(M^b(K),+,*)$ is commutative.

The measure algebra of a hypergroup is in fact a Banach $\sim$-algebra, which is often considered part of the definition.

Theorem 2.2. *Let $(K,*)$ be a hypergroup. Then $(M^b(K),+,*)$ is a Banach $\sim$-algebra with convolution $*$, involution $\mu^\sim := \overline{\mu^-}$, and unit ε_e.*

The collection of nonvoid compact subsets of K is denoted by $\mathscr{C}(K)$ and is given the *Michael topology*, that is, the topology generated by the subbasis of all

$$\mathscr{C}_U(V) := \big\{C \in \mathscr{C}(K) : C \cap U \neq \varnothing \text{ and } C \subset V\big\} \tag{2.1}$$

with U and V open subsets of K, which makes $\mathscr{C}(K)$ a locally compact Hausdorff space. A *neighborhood* of a point $t \in K$ is by definition any open set containing t. The closure of a subset A of K is denoted by $\operatorname{cl}(A)$. We denote by 1_A the function which is equal to 1 on A and is 0, otherwise. Let $(K,*)$ be a hypergroup. For a (measurable) function f and $t, s \in K$, we set

$$f(t*s) := \int_K f \,\mathrm{d}(\varepsilon_t * \varepsilon_s)$$

whenever this expression makes sense. The *left translate* is defined by

$$(T^t f)(s) := f(t*s)$$

and the *right translate* by

$$(T_t f)(s) := f(s*t).$$

In the following, we always suppose that if m is a left Haar measure on K, then m^- is a right Haar measure. It has long been known that a Haar measure exists for compact and commutative hypergroups (see [3] for a treatment of a general setting).

2.2. **Translations and convolutions of vector-valued functions.** Let $(K, *)$ be a hypergroup with left Haar measure m, and let X be a complex Banach space. In this section we extend some results for scalar-valued functions (see [17], also [2]) to vector-valued functions. The notation from above will also be used in the context of vector-valued functions whenever this makes sense.

We begin with continuous functions. We denote by $C(K, X)$, $C_b(K, X)$, $C_0(K, X)$, and $C_c(K, X)$ the space of continuous functions on K with values in X, the subspaces of bounded functions, the functions vanishing at infinity, and the functions with compact support, respectively. The following technical statements can easily be transferred from the scalar case, either directly (see Lemma 2.3) or by approximation, using partition of unity (see Lemma 2.4, Proposition 2.6; note that Corollary 2.5 is an immediate consequence of Lemma 2.4).

Lemma 2.3 (Urysohn)**.** *Suppose that $f \in C(K, X)$. Then, given a compact set C and an open set U with $C \subset U \subset K$, there exists $g \in C_c(K, X)$ with $\|g(r)\|_X \leq \|f(r)\|_X$ for all $r \in K$, $g = f$ on C and $g = 0$ on $K \backslash U$.*

Lemma 2.4. *Suppose that $\mu_\iota, \mu \in M^b(K)$, $\tau_v - \lim_\iota \mu_\iota = \mu$ vaguely, that is, $\lim_\iota \int_K \varphi \, \mathrm{d}\mu_\iota = \int_K \varphi \, \mathrm{d}\mu$ for each $\varphi \in C_c(K)$, and $\limsup_\iota \|\mu_\iota\| < \infty$. Then for each $f \in C_c(K, X)$,*

$$\lim_\iota \int_K f \, \mathrm{d}\mu_\iota = \int_K f \, \mathrm{d}\mu.$$

Corollary 2.5. *Suppose that $f \in C(K, X)$. Then the mapping*

$$\begin{aligned} K \times K &\to X \\ (t, s) &\mapsto f(t * s) \end{aligned}$$

is continuous.

Proposition 2.6. *Suppose that $f \in C_0(K, X)$. Then for any $t \in K$, $T^t f \in C_0(K, X)$, and the mapping*

$$\begin{aligned} K &\to C_0(K, X) \\ t &\mapsto T^t f \end{aligned}$$

is $\|\cdot\|_\infty$-continuous. If $f \in C_c(K, X)$, then $T^t f \in C_c(K, X)$.

Next we consider integrable functions. All integrals in this article are to be understood in the sense of Bochner (except when we explicitly refer to the Pettis integral). An appropriate introduction to the Bochner integral which fits with the theory of hypergroups can be found in [6, Chapter 1, Section 1]. We refer to [14, Chapter III], for the development of integration theory on locally compact spaces, which is also the measure theoretical background for hypergroups.

Definition 2.7. Let $L^p(K, m, X)$, $1 \leq p < \infty$, denote the space of p-integrable functions from K to X with respect to m and with norm $\|\cdot\|_p$. Furthermore, let $L^\infty(K, m, X)$ denote the space of locally m-almost everywhere bounded measurable functions from K to X with essential supremum norm $\|\cdot\|_\infty$. The space of *locally bounded measurable functions* $L^\infty_{\mathrm{loc}}(K, m, X)$ is defined as the space of (equivalence classes of) functions $f : K \to X$ such that every $t \in K$ has a neighborhood U with $1_U f \in L^\infty(K, m, X)$.

Proposition 2.8. *Suppose that* $1 \le p \le \infty$ *and that* $f \in L^p(K,m,X)$. *Then for any* $t \in K$, *we have* $T^t f \in L^p(K,m,X)$, *and*

$$\|T^t f\|_p \le \|f\|_p.$$

The proof runs as in the scalar case (see [17, Lemma 3.3B]; strong measurability can be deduced from the Pettis measurability theorem).

Corollary 2.9. *Suppose that* $1 \le p < \infty$ *and that* $f \in L^p(K,m,X)$. *Then the mapping*

$$\begin{aligned} K &\to L^p(K,m,X) \\ t &\mapsto T^t f \end{aligned}$$

is continuous.

Since $C_c(K,X)$ is dense in $L^p(K,m,X)$, this follows from Proposition 2.8 and Proposition 2.6.

Suppose that X, Y, and Z are Banach spaces, and that X operates on Y in the sense that $X \times Y \to Z$ is a bilinear continuous mapping such that $\|xy\|_Z \le \|x\|_X \|y\|_Y$. The following generalization of Hölder's inequality is immediate see [7, Section 2, paragraph 2.36(a)].

Proposition 2.10 (Hölder's inequality)**.** *Suppose that* $p, q \in [1,\infty]$ *are conjugate numbers; that is,* $1/p + 1/q = 1$, *and* $f \in L^p(K,m,X)$, $g \in L^q(K,m,Y)$. *Then* $f \cdot g \in L^1(K,m,Z)$, *where*

$$(f \cdot g)(t) := f(t)g(t) \in Z$$

and

$$\|f \cdot g\|_1 \le \|f\|_p \|g\|_q.$$

Proposition 2.11 (Young-type inequality)**.** *Suppose that* $p, q \in [1,\infty]$ *are conjugate numbers,* $f \in L^p(K,m,X)$, $g \in L^q(K,m,Y)$. *Then for any* $t \in K$,

$$(f * g^-)(t) := \int_K f(t * r) g(r) m(\mathrm{d}r) \in Z$$

is well defined and

$$\sup_{t \in K} \big\|(f * g^-)(t)\big\|_Z \le \|f\|_p \|g\|_q.$$

This is clear in view of Propositions 2.8 and 2.10.

Theorem 2.12. *Suppose that* $p, q \in [1,\infty]$ *are conjugate numbers and that* $f \in L^p(K,m,X)$, $g \in L^q(K,m,Y)$. *Then for any* $t \in K$,

$$\int_K T^t f(r) g(r) m(\mathrm{d}r) = \int_K f(r) T^{t^-} g(r) m(\mathrm{d}r). \tag{2.2}$$

According to Proposition 2.11, the left- and the right-hand side of (2.2) can be regarded as bilinear continuous mappings from $L^p(K,m,X) \times L^q(K,m,Y)$ to Z. Thus this identity is easily transferred from the scalar case (see Theorem 5.1D in [17]) by approximation (e.g., with step functions; if $q = \infty$ consider functions f with compact support first).

Theorem 2.13. *Suppose that $p, q \in [1, \infty]$ are conjugate numbers and that $f \in L^p(K, m, X)$, $g \in L^q(K, m, Y)$. Then $f * g^- \in C_b(K, Z)$. If $1 < p < \infty$, then $f * g^- \in C_0(K, Z)$.*

This is an immediate consequence of the preceding results. The scalar version of this theorem is stated in [17, Theorems 5.5D, 5.5P].

3. Multiplicative operator functions

Besides the strong definition from the Introduction, we give here two further, weak definitions of a multiplicative operator function, and we show that all three definitions are equivalent. We obtain that representations of hypergroups are a subclass of multiplicative operator functions. In Section 3.2, we prove that the weak formulation implies the strong one, and we derive criteria for uniform continuity.

3.1. **Weak definitions and the relation to representations.** Before we begin, let us review some partial results which are contained in the present setting. Hewitt and Ross established in their monumental treatise (see [14, Theorem 22.8]) that a weakly measurable representation of a locally compact group (on some reflexive Banach space) is strongly continuous. De Leeuw and Glicksberg gave a proof that weakly continuous representations of locally compact groups (on arbitrary Banach spaces) are strongly continuous (see [4, Theorem 2.8]). Concerning hypergroups, representation theory was initiated by Jewett in [17, Section 11.3] (see also [2, Section 2.1]). Here, representations of the convolution structure $(K, *)$ and the hypergroup $(M^b(K), *, \sim)$ critically differ (see the discussion below). That is the reason we call a representation of $(K, *)$ just a "multiplicative operator function." Operator functions of this type have occurred in the literature several times (see, e.g., [26, Section 2]).

The following Definition 3.2 of a multiplicative operator function rests upon the weak (Pettis) integral and is weaker than Definition 3.1.5 in the author's dissertation [12], which is based on the Bochner integral. The proof of Theorem 3.4 given here is simple and follows the ideas of Hewitt–Ross and de Leeuw–Glicksberg; this approach has recently been elaborated by Lasser in [22].

Let $(K, *)$ be a hypergroup with left Haar measure m.

Definition 3.1 (Weak version). A function $S : K \to \mathcal{L}(X)$ is called a [*weak*] *multiplicative operator function* if the following conditions are satisfied.

(i) $S(e) = I$.

(ii) For any $x \in X$, $x^* \in X^*$ (the dual space of X), the mapping $t \mapsto x^*(S(t)x)$ is continuous and

$$x^*\big(S(t)S(s)x\big) = \int_K x^*\big(S(r)x\big)(\varepsilon_t * \varepsilon_s)(\mathrm{d}r)$$

for all $t, s \in K$.

(iii) For any $x \in X$, $x^* \in X^*$,

$$\lim_{t \to e} x^*\big(S(t)x\big) = x^*(x).$$

Definition 3.2 (Weak-measurable version). A function $S : K \to \mathcal{L}(X)$ is called a [*weak-measurable*] *multiplicative operator function* if the following conditions are satisfied.

(i) $S(e) = I$.

(ii) For any $x \in X$, $x^* \in X^*$, it is $x^*(S(\cdot)x) \in L^\infty_{\text{loc}}(K, m)$ and for all $t \in K$,

$$x^*\big(S(t)S(s)x\big) = \int_K x^*\big(S(r)x\big)(\varepsilon_t * \varepsilon_s)(\mathrm{d}r)$$

for locally m-almost every $s \in K$.

(iii) For any $x \in X$, $x^* \in X^*$, there exists a local m-null set N^* such that

$$\lim_{\substack{t\to e\\ t\notin N^*}} x^*\big(S(t)x\big) = x^*(x).$$

Moreover, we suppose that for each $x \in X$ the function $S(\cdot)x$ is Pettis-integrable with respect to m in a neighborhood of e.

Remark 3.3. The additional assumption in Definition 3.2(iii) concerning the Pettis integral is superfluous if X is reflexive or separable. More generally this is the case if X has the Pettis integral property with respect to m in a neighborhood of e (see [25, Section 8] and the references therein).

At least formally, Definition 3.1 is stronger than Definition 3.2; the only nontrivial thing is to see that, for a [weak] multiplicative operator function S, $S(\cdot)x$ is Pettis-integrable with respect to m in a neighborhood of e. This follows from the Krein–Šmulian theorem stating that the closed convex hull of a weakly compact subset of a Banach space is weakly compact; this idea is also used in the proof of Theorem 2.8 in [4] (see [27, Appendix C, Corollary C.13] for full generality and details).

Theorem 3.4. *Let S be a multiplicative operator function in the sense of Definition 3.1 or Definition 3.2. Then S is strongly continuous; that is, for each $x \in X$, the mapping $S(\cdot)x : K \to X$ is continuous.*

We postpone the proof of Theorem 3.4 to Section 3.2.

Theorem 3.5. *Definition 1.1, Definition 3.1, and Definition 3.2 of a multiplicative operator function are equivalent.*

Proof of Theorem 3.5. This is a consequence of Theorem 3.4. Concerning the functional equation, note that the complement of any local m-null set is dense in K. □

For the rest of this section (and after the proof of Theorem 3.4), we use Theorem 3.5, and in particular the simple form of Definition 1.1, without further notice.

Remarks 3.6. The notion of a multiplicative operator function does not depend on whether we consider the left Haar measure m or the right Haar measure m^- since involution is a homeomorphism and null sets are preserved (more precisely, $m = \Delta m^-$, $\Delta : K \to \mathbb{R}_+^\times$ the right modular function). As one would expect, if

$X = \mathbb{C}$, then a multiplicative operator function can be identified with a multiplicative function and vice versa (we always suppose that multiplicative functions are continuous). If K is commutative, then $S(t)S(s) = S(s)S(t)$ for all $t, s \in K$.

Further, two multiplicative operator functions $S_1 : K \to \mathcal{L}(X)$ and $S_2 : K \to \mathcal{L}(Y)$, X, Y some Banach spaces, can be combined to a multiplicative operator function $\mathcal{S}$ from K to $\mathcal{L}(Z)$, $Z = X \times Y$, defined by

$$\mathcal{S}(t) := \begin{bmatrix} S_1(t) & 0 \\ 0 & S_2(t) \end{bmatrix}.$$

The definition of equivalent representations can be transferred to multiplicative operator functions: given a multiplicative operator function $S : K \to \mathcal{L}(X)$ and any isomorphism $T \in \mathcal{L}(X)$, the mapping $t \mapsto T^{-1}S(t)T$, $K \to \mathcal{L}(X)$ is also a multiplicative operator function.

In the remaining part of this section, we clarify the relationship between multiplicative operator functions and representation theory.

Definition 3.7. A *representation of convolution* of a hypergroup K on X is a mapping $D : M^b(K) \to \mathcal{L}(X)$ such that

(i) $D(\varepsilon_e) = I$,
(ii) D is a representation of the Banach algebra $(M^b(K), *)$,
(iii) for each $x \in X$, $x^* \in X^*$, $\mu \mapsto x^*(D(\mu)x)$ is continuous on $M^b_+(K)$ with respect to the weak topology (recall that this is the relative topology on $M^b_+(K)$ induced by the weak topology $\sigma(M^b(K), C_b(K))$).

For abbreviation, we write $D(t)$ for $D(\varepsilon_t)$.

Proposition 3.8. *Let D be a representation of convolution of a hypergroup K. Then D is*

(i) *uniformly bounded (i.e., there exists $M \geq 1$ such that $\|D(\mu)\| \leq M\|\mu\|$ for all $\mu \in M^b(K)$),*
(ii) *strongly continuous in the sense that, for each $x \in X$, the mapping $\mu \mapsto D(\mu)x$, $M^b_+(K) \to X$ is continuous where $M^b_+(K)$ bears the weak topology,*
(iii) *representable as*

$$D(\mu)x = \int_K D(t)x\mu(\mathrm{d}t) \tag{3.1}$$

for all $x \in X$ and $\mu \in M^b(K)$.

Proof. Assume that (i) is not true. Since D is linear, we find a sequence $(\mu_n)_{n\in\mathbb{N}}$ in $M^b_+(K)$ such that $\|\mu_n\| \to 0$ and $\|D(\mu_n)\| \to \infty$ as $n \to \infty$, which, however, contradicts Definition 3.7(iii) and the uniform boundedness principle.

Concerning (iii), for each $x \in X$, $x^* \in X^*$,

$$x^*\big(D(\mu)x\big) = \int_K x^*\big(D(t)x\big)\mu(\mathrm{d}t) \tag{3.2}$$

holds for all $\mu \in M^b(K)$ by linearity since the finitely supported measures are dense in $M^b_+(K)$ (see [17, Lemma 2.2A]). Now it is clear by definition of D that its restriction to $\{\varepsilon_t, t \in K\}$, identified with K, is a multiplicative operator function in the sense of Definition 3.1, and hence for each $x \in X$, $D(\cdot)x : K \to X$ is continuous by Theorem 3.4. Thus the Bochner integral in (3.1) exists and (iii) follows from (3.2).

It remains to justify (ii). Suppose that $\mu_\iota \to \mu$ in $M_+^b(K)$. Then (μ_ι) is a tight net (see [15, paragraph 1.2.20(2)]), thus (ii) follows from (iii) (where $D(\cdot)x \in C_b(K, X)$ by (i)), Lemma 2.3, and Lemma 2.4. □

Let H be a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$, and let $\mathcal{L}(H)$ be the Banach $\sim$-algebra of bounded linear operators on H. Recall that $M^b(K)$ is a Banach $\sim$-algebra with $\sim$ as in Theorem 2.2.

Definition 3.9. A *representation of a hypergroup* K on H is a mapping $D : M^b(K) \to \mathcal{L}(H)$ such that

(i) $D(\varepsilon_e) = I$,
(ii) D is a $\sim$-representation of the Banach $\sim$-algebra $(M^b(K), *, \sim)$,
(iii) for each $x, y \in H$, $\mu \mapsto \langle D(\mu)x, y \rangle$ is continuous on $M_+^b(K)$ with respect to the weak topology.

Remark 3.10. Here D is a $\sim$-representation and thus contractive, that is, $\|D(\mu)\| \leq \|\mu\|$ for all $\mu \in M^b(K)$ (see [14, Theorem 21.22]); indeed, contractivity also follows from the C^*-identity $\|D(\mu)\|^2 = \|D(\mu^\sim * \mu)\|$ and Proposition 3.8(i). This refinement of Proposition 3.8(i), however, is not possible for a representation of convolution in general: for K a hypergroup but not a group, consider for example (see Section 4) the translation operator function $S : K \to \mathcal{L}(H)$, where $H = B = L^2(K, g^2 m^-)$ with $g = 1 + \varphi$, $\varphi \in C_c(K)$, $\varphi \geq 0$, a function depending only on $(K, *)$. (For D a representation of a hypergroup, Proposition 3.8(ii) is shown in [17, Lemma 11.3B].)

Now the proof of the following two theorems is straightforward.

Theorem 3.11. *Suppose that $D : M^b(K) \to \mathcal{L}(X)$ is a representation of convolution. Then the restriction $D|_K = D|_{\{\varepsilon_t, t \in K\}}$ is a multiplicative operator function $S : K \to \mathcal{L}(X)$ such that*

(i) *S is uniformly bounded.*

Conversely, a multiplicative operator function $S : K \to \mathcal{L}(X)$ satisfying (i) can be extended uniquely to a representation of convolution $D : M^b(K) \to \mathcal{L}(X)$ via the formula

$$D(\mu)x = \int_K S(t)x\mu(\mathrm{d}t), \quad x \in X$$

for all $\mu \in M^b(K)$.

Theorem 3.12. *Suppose that $D : M^b(K) \to \mathcal{L}(H)$ is a representation of hypergroup. Then the restriction $D|_K = D|_{\{\varepsilon_t, t \in K\}}$ is a multiplicative operator function $S : K \to \mathcal{L}(H)$ such that*

(i) *S is uniformly bounded, and*
(ii) *$S(t^-) = (S(t))^\sim$ for all $t \in K$.*

Conversely, a multiplicative operator function $S : K \to \mathcal{L}(H)$ satisfying (i) and (ii) can be extended uniquely to a representation of hypergroup $D : M^b(K) \to$

$\mathcal{L}(H)$ *via the formula*

$$D(\mu)x = \int_K S(t)x\mu(\mathrm{d}t), \quad x \in H$$

for all $\mu \in M^b(K)$.

The following theorem shows that it is no loss of generality to suppose conditions (i) and (iii) in Definition 3.1 of a multiplicative operator function (see also Definition 2.1 in [26]). The proof runs as in [14, Theorem 21.2].

Theorem 3.13. *Let* $S : K \to \mathcal{L}(X)$ *be an operator function satisfying condition (ii) of Definition 3.1. Then* X *is the direct sum of closed invariant subspaces* X_0 *and* X_1 *such that*

(i) $S(t)X_0 = \{0\}$ *for all* $t \in K$,
(ii) $S_1 : K \to \mathcal{L}(X_1)$, $t \mapsto S(t)|_{X_1}$ *is a multiplicative operator function.*

Similar statements are true for Definition 3.7 and Definition 3.9; that is, the assumption $D(\varepsilon_e) = I$ is made without loss of generality, for the proofs use the fact that (3.2) in the proof of Proposition 3.8 does not depend on $D(\varepsilon_e) = I$.

3.2. **Strong and uniform continuity.** First of all, we deliver a proof of Theorem 3.4. Then we derive criteria for uniform continuity.

Lemma 3.14. *Suppose that* S *is a multiplicative operator function in the sense of Definition 3.2. Then* $\|S(\cdot)\|$ *is locally bounded.*

Proof. Let C be a compact subset of K, and let V_0 be a relatively compact neighborhood of e. Suppose that $x^* \in X^*$ and that $x \in X$.

(1) For all $t \in C$ and an arbitrary nonvoid open subset V of V_0, we have

$$\begin{aligned}
&\int_K \int_K \big|x^*\big(S(r)x\big)\big| (\varepsilon_t * \varepsilon_s)(\mathrm{d}r) \frac{1}{m(V)} 1_V(s) m(\mathrm{d}s) \\
&\quad = \int_K \big|x^*\big(S(s)x\big)\big| \frac{1}{m(V)} 1_V(t^- * s) m(\mathrm{d}s) \\
&\quad \le \big\|x^*\big(S(\cdot)x\big)1_{C*V_0}\big\|_{L^\infty(K,m)} \cdot \int_K \frac{1}{m(V)} 1_V(t^- * s) m(\mathrm{d}s) \\
&\quad = \big\|x^*\big(S(\cdot)x\big)1_{C*V_0}\big\|_{L^\infty(K,m)} =: M_*
\end{aligned}$$

using basic facts about hypergroups from [17]: the first integral exists by Lemma 3.3B, the first equality is the content of Theorem 5.1D, the support of the integrand is controlled by Lemma 4.1B, and m is an invariant measure on K.

Thus we find that, for each $t \in C$ and any nonvoid open subset V of V_0,

$$m\Big(\Big\{s \in V : \int_K \big|x^*\big(S(r)x\big)\big| (\varepsilon_t * \varepsilon_s)(\mathrm{d}r) \le M_*\Big\}\Big) > 0. \tag{3.3}$$

(2) Given $t \in C$, we choose V according to Definition 3.2(iii) such that

$$\big|x^*\big(S(t)S(s)x\big) - x^*\big(S(t)x\big)\big| \le 1$$

for m-almost everywhere $s \in V$. Using (3.3) we find $s \in V$ such that

$$\begin{aligned}\left|x^*\big(S(t)x\big)\right| &\le 1 + \left|x^*\big(S(t)S(s)x\big)\right| \\ &\le 1 + \int_K \left|x^*\big(S(r)x\big)\right|(\varepsilon_t * \varepsilon_s)(\mathrm{d}r) \\ &\le 1 + M_*.\end{aligned}$$

(3) So we have shown that, for each $x^* \in X^*$, $x \in X$, there exists a constant $M_* \ge 0$ such that

$$\left|x^*\big(S(t)x\big)\right| \le 1 + M_*$$

for all $t \in C$. Using the uniform boundedness principle twice, we obtain a constant $M \ge 0$ such that

$$\sup_{t \in C}\left\|S(t)\right\| \le M. \qquad \square$$

Proof of Theorem 3.4. Without loss of generality, we suppose that S is a [weak–measurable] multiplicative operator function in the sense of Definition 3.2 (see the lines preceding Theorem 3.4). Suppose that $x \in X$, without loss of generality $\|x\| \le 1$, and $t_0 \in K$. Let $W \subset K$ be a relatively compact neighborhood of t_0; set $M := \sup_{t \in W}\|S(t)\| < \infty$ (see Lemma 3.14). Suppose that $\varepsilon > 0$.

(1) Let U be a neighborhood of t_0 and V a neighborhood of e such that $U * V \subset W$ (see [17, Lemma 3.2D]). Referring to Definition 3.2(iii), we may assume without loss of generality that $S(\cdot)x$ is Pettis-integrable with respect to m on V. Thus for each $\varphi \in C_c(V) = \{\phi \in C_c(K) : \operatorname{supp}(\phi) \subset V\}$ define an element $T_\varphi x \in X$ through

$$x^*(T_\varphi x) = \int_K x^*\big(S(r)x\big)\varphi(r)m(\mathrm{d}r)$$

for all $x^* \in X^*$ (the existence of the Pettis integral is easily checked; it can also be regarded as an immediate consequence of Vitali's convergence theorem for the Pettis integral; see [25, Theorem 5.2]). We infer from Definition 3.2(iii) that x belongs to the weak closure of $\{T_\varphi x, \varphi \in C_c(V)\}$. Since $\{T_\varphi x, \varphi \in C_c(V)\}$ is convex its weak and norm closure coincide by Mazur's theorem (see e.g. [28, Theorem 3.12]). Thus we may choose $\varphi \in C_c(V)$ such that $\|T_\varphi x - x\| < \varepsilon$.

(2) Suppose that $t \in K$ and $x^* \in X^*$. Then

$$\begin{aligned}x^*\big(S(t)(T_\varphi x)\big) &= \int_K x^*\big(S(t)\big(S(r)x\big)\big)\varphi(r)m(\mathrm{d}r) \\ &= \int_K x^*\big(S(t*r)x\big)\varphi(r)m(\mathrm{d}r) \\ &= \int_K x^*\big(S(r)x\big)\varphi(t^- * r)m(\mathrm{d}r),\end{aligned}$$

where the last equality, once again, uses [17, Theorem 5.1D].

Then, after a possible reduction of U, we obtain for all $t \in U$

$$\begin{aligned}&\left|x^*\big(S(t)(T_\varphi x)\big) - x^*\big(S(t_0)(T_\varphi x)\big)\right| \\ &\le \int_K \left|x^*\big(S(r)x\big)\right|\left|\varphi(t^- * r) - \varphi(t_0^- * r)\right|m(\mathrm{d}r)\end{aligned}$$

$$\begin{aligned}&\leq \big\|x^*\big(S(\cdot)x\big)1_W\big\|_{L^\infty(K,m)} \cdot \big\|\varphi(t^- * \cdot) - \varphi(t_0^- * \cdot)\big\|_{L^1(K,m)}\\ &\leq M \cdot \varepsilon\end{aligned}$$

according to Corollary 2.9.

(3) Combining Step (1) and Step (2), we obtain

$$\begin{aligned}\big\|S(t)x - S(t_0)x\big\| &\leq \big\|S(t)(x - T_\varphi x)\big\| + \big\|S(t)T_\varphi x - S(t_0)T_\varphi x\big\|\\ &\quad + \big\|S(t_0)(T_\varphi x - x)\big\|\\ &\leq 3M \cdot \varepsilon\end{aligned}$$

for all $t \in U$. $\square$

The following theorem gives equivalent criteria for uniform continuity. The special case $K = \mathbb{R}_+$ is considered in Theorem 5.5 of [11], which in turn is motivated by corresponding investigations of cosine operator functions, dating back to Kurepa in 1962 [19, Theorem 1] and Sova in 1966 [29, Fundamental Theorem 3.4].

Theorem 3.15. *Let S be a multiplicative operator function. Then the following conditions are equivalent.*

(i) *S is uniformly continuous.*

(ii) *There exists a local m-null set N such that $\lim_{\substack{t\to e\\ t\notin N}} S(t) = I$ in uniform operator topology.*

(iii) *For each $L \in (\mathcal{L}(X))^*$ there exists a local m-null set N^* such that $\lim_{\substack{t\to e\\ t\notin N^*}} L(S(t)) = L(I)$, and $S : K \to \mathcal{L}(X)$ is Pettis-integrable with respect to m in a neighborhood of e.*

(iv) *$S : K \to \mathcal{L}(X)$ is locally m-measurable and somewhere invertible-integrable.*

We say a multiplicative operator function S is *somewhere invertible-integrable* if there exists a compact set $C \subset K$ with $m(C) > 0$ such that $(S(t))^{-1}$ exists for all $t \in C$ and $1_C(S(\cdot))^{-1} \in L^1(K, m, \mathcal{L}(X))$. This is a generalization of the notion "not locally null" in [2, Proposition 1.4.33].

Proof. Conditions (ii) and (iii) can be regarded as modifications of Definition 3.2(iii). Thus, the proof of Theorem 3.4 shows that (ii) and (iii) imply (i); in both cases it is easy to see that in Step (1) one can find $\varphi \in C_c(K)$ such that $\|T_\varphi - I\|_{\mathcal{L}(X)} < \varepsilon$.

Condition (iv) and the following proof are derived from [2, Proposition 1.4.33]. Suppose that S satisfies (iv); let $C \subset K$ be a compact set with $m(C) > 0$ such that $S(t)^{-1}$ exists for $t \in C$ and $1_C S(\cdot)^{-1} \in L^1(K, m, \mathcal{L}(X))$. Suppose that $t_0 \in K$ and U is a relatively compact neighborhood of t_0. Then $f := 1_{\mathrm{cl}(U)*C} S \in L^\infty(K, m, \mathcal{L}(X))$ and $g := m(C)^{-1} 1_C S^{-1} \in L^1(K, m, \mathcal{L}(X))$. For every $t \in U$

$$\begin{aligned}(f * g^-)(t) &= \int_K f(t * r) g(r) m(\mathrm{d}r)\\ &= \int_K S(t * r) m(C)^{-1} 1_C(r) S(r)^{-1} m(\mathrm{d}r)\end{aligned}$$

$$= \int_K S(t)S(r)m(C)^{-1}1_C(r)S(r)^{-1}m(\mathrm{d}r)$$
$$= S(t).$$

According to Theorem 2.13, $f * g^-$ is a continuous function from K to $\mathcal{L}(X)$.

It only remains to check that if $S : K \to \mathcal{L}(X)$ is uniformly continuous then S is somewhere invertible-integrable. In fact, there exists a neighborhood U of e such that $\|I - S(t)\| < \frac{1}{2}$ for all $t \in U$, thus $S(t)^{-1}$ exists, $t \mapsto S(t)^{-1}$, $U \to \mathcal{L}(X)$ is continuous, and $\|S(t)^{-1}\| \le 2$ for all $t \in U$. Since open sets are inner regular, we also find a compact set $C \subset U$ with $m(C) > 0$. □

Note that condition (iii) is always satisfied for all $L \in \{x^*(\,\cdot\, x) : x^* \in X^*, x \in X\} \subset (\mathcal{L}(X))^*$, which is a norming set for $\mathcal{L}(X)$. However, in contrast to the situation for cosine operator functions, measurability of $S : K \to \mathcal{L}(X)$ itself is in general not sufficient for uniform continuity (see Example 10.9 in [11]).

The situation is different if $K = G$ is a locally compact group; here the additional condition that S is somewhere invertible-integrable is superfluous since $S(t)^{-1} = S(t^{-1})$.

If X is finite-dimensional, then condition (ii) is always satisfied and we obtain the following corollary.

Corollary 3.16. *If X is finite-dimensional, then every (matrix-valued) multiplicative operator function $S : K \to \mathcal{L}(X)$ is uniformly continuous.*

4. Translation operator functions on homogeneous Banach spaces

We show that translation operator functions on homogeneous Banach spaces are examples of multiplicative operator functions. This setting contains weakly stationary processes indexed by hypergroups, see below, and elementary but important examples of C_0-groups, cosine operator functions, and more generally Sturm–Liouville operator functions (see [11, Section 9]).

First of all, we define the notion of a homogeneous Banach space with respect to an arbitrary hypergroup in the spirit of Katznelson [18]. Our notion is more general than the notion introduced by Fischer and Lasser in [9] for the dual Jacobi polynomial hypergroup. To obtain multiplicative operator functions, it is necessary to consider right translations. We use throughout that several results about left translations can be transferred to right translations by involution (and vice versa). So instead of the left Haar measure m we have to use the right Haar measure m^-.

Definition 4.1. A linear subspace $B \subset L^1_{\mathrm{loc}}(K, m^-)$ with norm $\|\cdot\|_B$ is called *homogeneous Banach space* if the following conditions are satisfied.

(i) B is complete with respect to $\|\cdot\|_B$ and for each compact set $C \subset K$ there exists $L \ge 0$ such that

$$\|f|_C\|_1 \le L\|f\|_B$$

for all $f \in B$.

(ii) For each $f \in B$, $t \in K$ it is $T_t f \in B$ and for each compact set $C \subset K$ there exists $M \geq 0$ such that

$$\|T_t f\|_B \leq M \|f\|_B$$

for all $f \in B$ and $t \in C$.

(iii) For each $f \in B$ the mapping $t \mapsto T_t f$, $K \to B$ is continuous.

Remark 4.2. A simple generalization of this definition is to allow vector-valued functions, that is, to consider $B \subset L^1_{\mathrm{loc}}(K, m^-, Y)$ with Y some Banach space (this is used in Example 4.9). All subsequent results and proofs concerning homogeneous Banach spaces can immediately be transferred to this more general setting, using the results of Section 2.2.

The following theorem introduces the aforementioned class of multiplicative operator functions. Its proof relies on Lemma 4.6 and will be conducted afterwards.

Theorem 4.3. *Let $X = B$ be a homogeneous Banach space. Then $S : t \mapsto T_t$, $K \to \mathcal{L}(B)$ is a multiplicative operator function.*

Definition 4.4. We call such a multiplicative operator function a *translation operator function.*

Proposition 4.5. *Given $t, s \in K$ and $f \in C_b(K)$, we have, for all $u \in K$,*

$$(T_t T_s f)(u) = \int_K T_r f(u)(\varepsilon_t * \varepsilon_s)(\mathrm{d}r).$$

This proposition is a simple consequence of associativity of convolution and is taken from [22, Proposition 1.1.8].

Lemma 4.6. *Suppose that $f \in L^1_{\mathrm{loc}}(K, m^-)$ and that $t, s \in K$. Then*

$$T_t T_s f = \int_K T_r f (\varepsilon_t * \varepsilon_s)(\mathrm{d}r)$$

in $L^1_{\mathrm{loc}}(K, m^-)$, where the right-hand side is to be read in the sense of distributions; that is,

$$\left\langle \int_K T_r f(\varepsilon_t * \varepsilon_s)(\mathrm{d}r), \varphi \right\rangle = \int_K \langle T_r f, \varphi \rangle (\varepsilon_t * \varepsilon_s)(\mathrm{d}r)$$

for all $\varphi \in C_c(K)$, where $\langle \cdot, \varphi \rangle = \int_K \cdot \, \varphi \mathrm{d}m^-$.

Proof. Suppose that $f \in L^1_{\mathrm{loc}}(K, m^-)$, $t, s \in K$, and choose an arbitrary $\varphi \in C_c(K)$. The space $L^1_{\mathrm{loc}}(K, m^-)$ is invariant under right translations; right translation is a continuous operation on $L^1(K, m^-)$ and thus the mapping $r \mapsto \langle T_r f, \varphi \rangle$, $K \to \mathbb{C}$ is continuous (see Proposition 2.8 and [17, Lemma 3.2B]). Note that functions in $L^1_{\mathrm{loc}}(K, m^-)$ are determined uniquely through $\langle \cdot, \varphi \rangle$, $\varphi \in C_c(K)$. So it remains to show that

$$\int_K (T_t T_s f)(u) \varphi(u) m^-(\mathrm{d}u) = \int_K \int_K (T_r f)(u) \varphi(u) m^-(\mathrm{d}u)(\varepsilon_t * \varepsilon_s)(\mathrm{d}r). \tag{4.1}$$

Therefore we may assume without loss of generality that $f \in L^1(K, m^-)$. If $f \in C_b(K)$, then (4.1) holds true by Proposition 4.5 and Fubini's theorem. Finally, use the fact that $C_c(K)$ is dense in $L^1(K, m^-)$. $\square$

Proof of Theorem 4.3. The only thing to prove is the functional equation. Suppose that $f \in B$ and that $t, s \in K$. Then $T_t T_s f \in B$, the B-valued Bochner integral $\int_K T_r f(\varepsilon_t * \varepsilon_s)(\mathrm{d}r) \in B$ exists, and we have to show that they are equal in $L^1_{\mathrm{loc}}(K, m^-)$. Therefore, note that $\langle \cdot, \varphi \rangle = \int_K \cdot \varphi \mathrm{d}m^- \in B^*$ for all $\varphi \in C_c(K)$ by Definition 4.1(i), and then apply Lemma 4.6. $\square$

Remark 4.7. Left translations do in general not form a multiplicative operator function. Consider the group G of automorphisms on a finite-dimensional Banach space X endowed with the uniform operator topology. Provided X is at least 2-dimensional, there exist $t, s \in G$ such that $ts \neq st$. Set $B = C_0(G)$ and choose $f \in C_0(G)$ with $f(ts) \neq f(st)$; it is $(T^t T^s f)(e) = f(st)$ and $(\int_K T^r f(\varepsilon_t * \varepsilon_s)(\mathrm{d}r))(e) = f(ts)$.

We give some examples of homogeneous Banach spaces. Let $C_{\mathrm{ub}}(K) \subset C_b(K)$ denote the set of uniformly continuous and bounded functions on K. By uniformly continuous we mean β-uniformly continuous in the sense of [2, Definition 1.2.26(ii)]; that is, for each $t_0 \in K$ and $\varepsilon > 0$ there exists a neighborhood U_{t_0} of t_0 such that $\|T_t f - T_{t_0} f\|_\infty < \varepsilon$ for all $t \in U_{t_0}$. Note that $C_{\mathrm{ub}}(K)$ endowed with $\|\cdot\|_\infty$ is a Banach space. Indeed, $C_{\mathrm{ub}}(K)$ is a closed linear subspace of $C_b(K)$ since $\|T_t f\|_\infty \leq \|f\|_\infty$ for all $f \in C_b(K)$.

Proposition 4.8. *The spaces $C_0(K)$, $C_{\mathrm{ub}}(K)$ with $\|\cdot\|_\infty$, and $L^p(K, m^-)$, $1 \leq p < \infty$ with $\|\cdot\|_p$ are homogeneous Banach spaces. In these cases translations are contractions.*

Proof. This is easily verified using the results of Section 2.2. We only stress that given $f \in C_{\mathrm{ub}}(K)$ and $t \in K$ it is $T_t f \in C_{\mathrm{ub}}(K)$. If K is a commutative hypergroup, translations commute and this is clear by definition. For the general case we note that for $s \in K$

$$T_s T_t f = \int_K T_u f(\varepsilon_s * \varepsilon_t)(\mathrm{d}u)$$

read in the Banach space $C_b(K)$. Indeed, the $C_b(K)$-valued Bochner integral on the right-hand side exists since the integrand is continuous by choice of f, and the equality holds pointwise by Proposition 4.5. In other words

$$T_s T_t f = g(s * t),$$

where $g := T_\bullet f \in C(K, Y)$ with $Y = C_b(K)$. Thus Corollary 2.5 yields $s \mapsto T_s(T_t f)$, $K \to Y$ continuous, that is $T_t f \in C_{\mathrm{ub}}(K)$ by definition. $\square$

Examples of translation operator functions are provided by K-weakly stationary processes, as introduced by Lasser and Leitner in [23] and [24] (see [16] for an exposition on random fields which can be treated by means of hypergroups).

Example 4.9. Let K be a hypergroup and let $(\Omega, \mathscr{F}, P)$ be a probability space. A family $(\mathcal{X}_t)_{t\in K} \subset L^2(\Omega, \mathscr{F}, P)$ is called a *K-weakly stationary process* if the following conditions are satisfied.

(i) The means are constant, i.e. there exists a constant $c \in \mathbb{C}$ such that $E[\mathcal{X}_t] = c$ for all $t \in K$.

(ii) The covariance function

$$\begin{aligned} d : K \times K &\to \mathbb{C} \\ (t, s) &\mapsto E\big[(\mathcal{X}_t - c)\overline{(\mathcal{X}_s - c)}\big] \end{aligned}$$

is continuous and bounded and satisfies

$$d(t, s) = \int_K d(r, e)(\varepsilon_t * \varepsilon_{s^-})(\mathrm{d}r)$$

for all $t, s \in K$.

In the following we always assume that K is a commutative hypergroup and any K-weakly stationary process is *centered* (i.e. $c = 0$).

Let $(\mathcal{X}_t)_{t\in K} \subset L^2(\Omega, \mathscr{F}, P)$ be a K-weakly stationary process. In [24, Section 2], Leitner introduces the notion of a *translation operator T_t^{ws} for $t \in K$ on $(\mathcal{X}_s)_{s\in K}$*. For a shortcut, his definition is equivalent to

$$T_t^{\mathrm{ws}} \mathcal{X}_s := \int_K \mathcal{X}_r (\varepsilon_t * \varepsilon_s)(\mathrm{d}r) = (T_t \mathcal{X})(s), \tag{4.2}$$

where

$$\begin{aligned} \mathcal{X} : K &\to L^2(\Omega, \mathscr{F}, P) \\ t &\mapsto \mathcal{X}_t \end{aligned} \tag{4.3}$$

is a continuous transformation (see [24, Theorem 2(8)], applying 6 to $\mathcal{X}_t = T_t^{\mathrm{ws}} \mathcal{X}_e$). Further, following [24, Theorem 2], $\|T_t^{\mathrm{ws}}\| \le 1$ for all $t \in K$, and thus the translation operators T_t^{ws} are extended to

$$H := \mathrm{cl}_{\|\cdot\|_{L^2(\Omega,\mathscr{F},P)}} \operatorname{lin}\{\mathcal{X}_s, s \in K\},$$

the closure of the linear span of $\{\mathcal{X}_s, s \in K\}$ in $L^2(\Omega, \mathscr{F}, P)$; then the mapping

$$t \mapsto T_t^{\mathrm{ws}}, \qquad K \to \mathcal{L}(H) \tag{4.4}$$

is strongly continuous,

$$\begin{aligned} T_e^{\mathrm{ws}} &= I, \\ T_t^{\mathrm{ws}} T_s^{\mathrm{ws}} \mathcal{X} &= \int_K T_r^{\mathrm{ws}} \mathcal{X} (\varepsilon_t * \varepsilon_s)(\mathrm{d}r) \end{aligned}$$

for all $t, s \in K$ and $\mathcal{X} \in H$, and

$$(T_t^{\mathrm{ws}})^{\sim} = T_{t^-}^{\mathrm{ws}} \tag{4.5}$$

for all $t \in K$.

In terms of our terminology,

$$\begin{aligned} S : K &\to \mathcal{L}(H) \\ t &\mapsto T_t^{\mathrm{ws}} \end{aligned}$$

is a uniformly bounded multiplicative operator function which can be extended to a $\sim$ representation by (4.5), see Theorem 3.12. Conversely, suppose that $H \subset L^2(\Omega, \mathscr{F}, P)$ is a Hilbert space such that $E[\mathcal{X}] = 0$ for all $\mathcal{X} \in H$, and let $S : K \to \mathcal{L}(H)$ be a uniformly bounded multiplicative operator function which is the restriction of a $\sim$ representation (in the sense of Theorem 3.12). Then for each $\mathcal{X} \in H$,

$$\mathcal{X}_t := S(t)\mathcal{X} \tag{4.6}$$

defines a K-weakly stationary process $(\mathcal{X}_t)_{t\in K}$ (cf. [24, p. 325]).

Finally, given a K-weakly stationary process $(\mathcal{X}_t)_{t\in K}$, associate $\mathcal{X}$ as defined in (4.3), then $\mathcal{X} \in C_{\mathrm{ub}}(K, L^2(\Omega, \mathscr{F}, P))$, which can easily be seen from (4.6). In particular, with Remark 4.2 in mind, each K-weakly stationary process $(\mathcal{X}_t)_{t\in K}$ can be identified with the orbit $S(\cdot)\mathcal{X}$ of the translation operator function S on the (generalized) homogeneous Banach space $B = C_{\mathrm{ub}}(K, L^2(\Omega, \mathscr{F}, P))$.

5. Abstract Cauchy problems

We show that under suitable conditions multiplicative operator functions solve abstract Cauchy problems. Concrete examples are discussed in the next section.

Assumptions. Let K be a commutative hypergroup with dual space $\widehat{K}$. Let $\mathfrak{J} \subset M_+^b(K)$ be a family of bounded nonnegative nonzero measures with compact support, and suppose that for any neighborhood U of e in K there exists $\mathcal{J} \in \mathfrak{J}$ such that $\mathrm{supp}(\mathcal{J}) \subset U$. Further, suppose that for each $\mathcal{J} \in \mathfrak{J}$ there exists $\delta_{\mathcal{J}} \in M^b(K)$ with compact support such that for each $\chi \in \mathrm{supp}(\pi) \subset \widehat{K}$ ($\mathrm{supp}(\pi)$ the support of the Plancherel measure) there exists a constant c_χ such that

$$\int_K \chi \,\mathrm{d}\delta_{\mathcal{J}} = c_\chi \int_K \chi \,\mathrm{d}\mathcal{J}$$

for all $\mathcal{J} \in \mathfrak{J}$. In this situation, we say K is a hypergroup with *associated integral equation.*

We remark in advance that every commutative hypergroup has an associated integral equation, namely, its functional equation (see Example 6.1). Recall that $\mathscr{C}(K)$ denotes the collection of nonvoid compact subsets of K endowed with the Michael topology (see (2.1)).

Definition 5.1. Let $S : K \to \mathcal{L}(X)$ be a multiplicative operator function on a hypergroup K with associated integral equation. Then the *universal generator* $\mathbb{A}_0$ is defined by

$$\mathbb{A}_0 x := \lim_{\substack{\mathrm{supp}(\mathcal{J}) \to \{e\} \text{ in } \mathscr{C}(K) \\ \mathcal{J} \in \mathfrak{J}}} \frac{\int_K S(\cdot)x \,\mathrm{d}\delta_{\mathcal{J}}}{\mathcal{J}(K)}$$

with domain

$$D(\mathbb{A}_0) := \{x \in X : \lim \ldots \text{ exists}\}.$$

The term *universal generator* emphasizes that its definition does not depend on the properties of a concrete integral equation or abstract Cauchy problem; however, for specific examples, the notion of an *adapted generator*, or *generator* for short, may be more convenient (see (6.1) and (6.8)).

Proposition 5.2. *Let $S : K \to \mathcal{L}(X)$ be a multiplicative operator function on a hypergroup K with associated integral equation, and let $\mathbb{A}_0$ be its universal generator. Suppose that $x \in D(\mathbb{A}_0)$. Then $S(t)x \in D(\mathbb{A}_0)$ and $\mathbb{A}_0 S(t)x = S(t)\mathbb{A}_0 x$ for all $t \in K$.*

Proof. The values of S commute since K is commutative. So the assertion is clear by Definition 5.1 and Hille's theorem (see, e.g., [5, pp. 47–48, Theorem 6]). ☐

Theorem 5.3. *Let $S : K \to \mathcal{L}(X)$ be a multiplicative operator function on a hypergroup K with associated integral equation, and let $\mathbb{A}_0$ be its universal generator. Suppose that $x \in X$. Then $\int_K S(\cdot)x \,\mathrm{d}\jmath \in D(\mathbb{A}_0)$ and*

$$\int_K S(\cdot)x \,\mathrm{d}\delta_\jmath = \mathbb{A}_0 \int_K S(\cdot)x \,\mathrm{d}\jmath$$

for all $\jmath \in \mathfrak{J}$.

Proof. (1) Suppose that $j, \jmath \in \mathfrak{J}$. Then for all $\chi \in \operatorname{supp}(\pi)$

$$\begin{aligned} c_\chi \int_K \chi \,\mathrm{d}j \int_K \chi \,\mathrm{d}\jmath &= \int_K \chi \,\mathrm{d}\delta_j \int_K \chi \,\mathrm{d}\jmath \\ &= \int_K \int_K \int_K \chi \,\mathrm{d}(\varepsilon_t * \varepsilon_s)\, \delta_j(\mathrm{d}t)\, \jmath(\mathrm{d}s) \end{aligned} \tag{5.1}$$

and

$$\begin{aligned} \int_K \chi \,\mathrm{d}j \, c_\chi \int_K \chi \,\mathrm{d}\jmath &= \int_K \chi \,\mathrm{d}\,j \int_K \chi \,\mathrm{d}\,\delta_\jmath \\ &= \int_K \int_K \int_K \chi \,\mathrm{d}(\varepsilon_t * \varepsilon_s)\, j(\mathrm{d}t)\, \delta_\jmath(\mathrm{d}s). \end{aligned} \tag{5.2}$$

The measures defined by the right-hand sides of (5.1) and (5.2) coincide since the Fourier–Stieltjes transform is injective (see [2, Theorem 2.2.24]).

(2) Suppose that $x \in X$. Reading Step (1) backwards, as far as possible, with the vector-valued function $S(\cdot)x$ in place of χ, we arrive at

$$\int_K S(\cdot) \,\mathrm{d}\delta_j \int_K S(\cdot)x \,\mathrm{d}\jmath = \int_K S(\cdot) \,\mathrm{d}j \int_K S(\cdot)x \,\mathrm{d}\delta_\jmath, \tag{5.3}$$

where we have used that S is multiplicative (and Hille's theorem); the operator-valued integrals are defined in the strong sense. Dividing (5.3) by $j(K) > 0$, and taking the limit $\operatorname{supp}(j) \to \{e\}$ in $\mathscr{C}(K)$, $j \in \mathfrak{J}$, the right-hand side gives

$$\begin{aligned} \frac{1}{j(K)} \int_K S(\cdot) \,\mathrm{d}j \int_K S(\cdot)x \,\mathrm{d}\delta_\jmath &= \int_K S(s) \frac{1}{j(K)} \int_K S(t)x\, j(\mathrm{d}t)\, \delta_\jmath(\mathrm{d}s) \\ &\to \int_K S(s)x \,\delta_\jmath(\mathrm{d}s), \end{aligned}$$

where we have used that $\jmath$ are nonnegative, nonzero measures. Thus the left-hand side of (5.3) yields $\int_K S(\cdot)x \, \mathrm{d}\jmath \in D(\mathbb{A}_0)$ and

$$\mathbb{A}_0 \int_K S(\cdot)x \, \mathrm{d}\jmath = \int_K S(\cdot)x \, \mathrm{d}\delta_\jmath. \qquad \square$$

The following conclusions of Theorem 5.3 are almost copies of those in the Sturm–Liouville setting (see [11]); in the cosine setting the ideas can be traced back to Sova and Kurepa. The proofs are included for the sake of completeness.

Remark 5.4. For $x \in D(\mathbb{A}_0)$, the universal generator $\mathbb{A}_0$ and the integral commute, that is

$$\mathbb{A}_0 \int_K S(\cdot)x \, \mathrm{d}\jmath = \int_K S(\cdot)\mathbb{A}_0 x \, \mathrm{d}\jmath.$$

Indeed, this can be seen from (5.3), using

$$\int_K S(\cdot) \, \mathrm{d}\delta_j \int_K S(\cdot)x \, \mathrm{d}\jmath = \int_K S(s) \int_K S(t)x \, \delta_j(\mathrm{d}t) \, \jmath(\mathrm{d}s).$$

Theorem 5.5. *Let $S : K \to \mathcal{L}(X)$ be a multiplicative operator function on a hypergroup K with associated integral equation. Then its universal generator $\mathbb{A}_0$ is densely defined and closed.*

Proof. To show that $\mathbb{A}_0$ is densely defined, choose an arbitrary $x \in X$ and $\varepsilon > 0$. Then there exists $j \in \mathfrak{J}$ close to $\{e\}$ in $\mathscr{C}(K)$ such that $\|x - x_j\| < \varepsilon$ where

$$x_j := \big(j(K)\big)^{-1} \int_K S(\cdot)x \, \mathrm{d}j.$$

Theorem 5.3 yields $x_j \in D(\mathbb{A}_0)$ and

$$\mathbb{A}_0 x_j = \big(j(K)\big)^{-1} \int_K S(\cdot)x \, \mathrm{d}\delta_j.$$

Hence $\mathbb{A}_0$ is densely defined.

To show that $\mathbb{A}_0$ is closed, assume $(x_n)_{n\in\mathbb{N}} \subset D(\mathbb{A}_0)$, $x, y \in X$ and $x_n \to x$, $\mathbb{A}_0 x_n \to y$ as $n \to \infty$. Applying Theorem 5.3 to x_n, $n \in \mathbb{N}$, using Remark 5.4, and taking the limit $n \to \infty$, we obtain for any $j \in \mathfrak{J}$

$$\int_K S(\cdot)x \, \mathrm{d}\delta_j = \int_K S(\cdot)y \, \mathrm{d}j.$$

It follows from Definition 5.1 that $x \in D(\mathbb{A}_0)$ and $\mathbb{A}_0 x = y$. $\square$

Remark 5.6. The proof above also shows, by iteration, that $D(\mathbb{A}_0^n)$ is dense in X for all $n \in \mathbb{N}$.

Theorem 5.7. *Let $S : K \to \mathcal{L}(X)$ be a multiplicative operator function on a hypergroup K with associated integral equation, and let $\mathbb{A}_0$ be its universal generator. Then $\lim_{t\to 0^+} S(t) = I$ in uniform operator topology if and only if S is uniformly continuous. In this case, $\mathbb{A}_0$ is bounded.*

Proof. The first equivalence is content of Theorem 3.15. So suppose that S is uniformly continuous. According to Theorem 5.3 it is sufficient to show that there exists $j \in \mathfrak{J}$ such that the operator $\int_K S(\cdot)\,\mathrm{d}j$, defined as Bochner integral in $\mathcal{L}(X)$, is invertible. Therefore take $j \in \mathfrak{J}$ close to $\{e\}$ in $\mathscr{C}(K)$ such that

$$\left\| I - \big(j(K)\big)^{-1} \int_K S(\cdot)\,\mathrm{d}j \right\| \leq \big(j(K)\big)^{-1} \int_K \big\| I - S(\cdot) \big\|\,\mathrm{d}j < \frac{1}{2}. \qquad \square$$

In the general setting of Theorem 5.7, the converse assertion is not true, that is if $\mathbb{A}_0$ is bounded then S may or may not be uniformly continuous, see Example 6.1.

6. Examples

The results of the previous section take quite different forms when applied to specific hypergroups. The following list reflects examples found in the literature as well as examples of current or possible future interest.

6.1. **Abstract functional equations, problems of discrete type.** The first example is somehow of pedagogical nature, stating that given a multiplicative operator function on an arbitrary commutative hypergroup the functional equation itself can be regarded as an abstract Cauchy problem. Interesting enough, the discrete setting contains a model for birth and death processes described by orthogonal polynomials, see the following subexample.

Example 6.1 (Abstract functional equation)*.* Let K be a commutative hypergroup and suppose that $t_0 \in K$. Set $\mathfrak{J} := \{\varepsilon_s, s \in K\}$ and $\delta_{\varepsilon_s} := \varepsilon_{t_0} * \varepsilon_s$ for each $\varepsilon_s \in \mathfrak{J}$. Further, given $\chi \in \widehat{K}$ set $c_\chi := \chi(t_0)$. This mimics the functional equation $\chi(t_0 * s) = \chi(t_0)\chi(s)$, i.e.

$$\int_K \chi\,\mathrm{d}\delta_{\varepsilon_s} = c_\chi \int_K \chi\,\mathrm{d}\varepsilon_s$$

for all $\varepsilon_s \in \mathfrak{J}$.

Let S be a multiplicative operator function on K. Then for each $x \in X$,

$$\mathbb{A}_0 x = \lim_{\substack{s \\ \{s\}\to\{e\} \text{ in } \mathscr{C}(K) \\ s\in K}} \frac{\int_K S(\cdot)x\,\mathrm{d}\delta_{\varepsilon_s}}{\varepsilon_s(K)} = \lim_{s\to e} \frac{S(t_0)S(s)x}{1} = S(t_0)x,$$

that is the universal generator is given by

$$\mathbb{A}_0 = S(t_0) \in \mathcal{L}(X),$$

and the corresponding abstract Cauchy problem states that for each $x \in X$

$$S(t_0 * s)x = \mathbb{A}_0 S(s)x$$

for all $s \in K$.

Example 6.2 (Abstract linear difference equations)*.* This is a special subexample of Example 6.1 in the discrete setting, $K = \mathbb{N}_0$ a polynomial hypergroup, extracted from [8]. We start with a quick introduction to polynomial hypergroups, see [20], [21]. Let $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$, and $(c_n)_{n\in\mathbb{N}}$ be sequences of nonnegative real numbers such that $a_n + b_n + c_n = 1$ for all $n \in \mathbb{N}$, $a_n, c_n > 0$ for all $n \in \mathbb{N}$, and suppose that $a_0 > 0$, $b_0 \in \mathbb{R}$, and $a_0 + b_0 = 1$. Let $(R_n)_{n \in \mathbb{N}_0}$ be the sequence of polynomials defined recursively by

$$R_0(t) = 1, \qquad R_1(t) = \frac{1}{a_0}(t - b_0) \qquad \text{and}$$
$$R_1(t)R_n(t) = a_n R_{n+1}(t) + b_n R_n(t) + c_n R_n(t), \quad n \in \mathbb{N}$$

with $t \in \mathbb{R}$. By construction $R_n(1) = 1$ for all $n \in \mathbb{N}_0$. According to Favard's theorem, $(R_n)_{n \in \mathbb{N}_0}$ is orthogonal with respect to some measure $\pi \in M^1(\mathbb{R})$. From the orthogonality one can deduce that

$$R_n(t)R_m(t) = \sum_{k=|n-m|}^{n+m} g(n,m;k)R_k(t)$$

for all $n, m \in \mathbb{N}_0$ and $t \in \mathbb{R}$, where $g(n,m;k) \in \mathbb{R}$ for all $k = |n-m|, \dots, n+m$ and $g(n,m;|n-m|) \neq 0$, $g(n,m;n+m) \neq 0$. Many important and well-known examples of orthogonal polynomials satisfy the crucial condition

$$g(n,m;k) \geq 0 \quad \text{for all } k = |n-m|, \dots, n+m.$$

In this case,

$$\varepsilon_n * \varepsilon_m = \sum_{k=|n-m|}^{n+m} g(n,m;k)\varepsilon_k, \quad n, m \in \mathbb{N}_0$$

defines the convolution of point measures of a hypergroup $K = \mathbb{N}_0$, called *polynomial hypergroup* with respect to $(R_n)_{n \in \mathbb{N}_0}$, and denoted by $(\mathbb{N}_0, *(R_n))$. The neutral element is 0, involution is the identity map, and the dual space $\widehat{\mathbb{N}_0}$ is homeomorphic to

$$D_S := \big\{ t \in \mathbb{R} : \big|R_n(t)\big| \leq 1 \text{ for all } n \in \mathbb{N} \big\} \subset [1 - 2a_0, 1].$$

Let $(\mathbb{N}_0, *(R_n))$ be a polynomial hypergroup. We consider

$$\begin{aligned} R_1(t) &= c_t R_0(t) \quad \text{for } n = 0, \\ a_n R_{n+1}(t) + b_n R_n(t) + c_n R_{n-1}(t) &= c_t R_n(t) \quad \text{for all } n \in \mathbb{N} \end{aligned}$$

as the associated integral equation; here $\mathfrak{J} = \{\varepsilon_n, n \in \mathbb{N}_0\}$, and $\delta_{\varepsilon_0} = \varepsilon_1$, $\delta_{\varepsilon_n} = a_n \varepsilon_{n+1} + b_n \varepsilon_n + c_n \varepsilon_{n-1}$ for all $n \in \mathbb{N}$, and $t \in D_S$, $c_t = R_1(t)$.

Let S be a multiplicative operator function on $(\mathbb{N}_0, *(R_n))$. Then its universal generator is given by

$$\mathbb{A}_0 = \frac{\int_{\mathbb{N}_0} S(\cdot)\, \mathrm{d}\delta_{\varepsilon_0}}{\varepsilon_0(\mathbb{N}_0)} = S(1) \in \mathcal{L}(X),$$

and the corresponding abstract Cauchy problem takes the form

$$\begin{aligned} S(1) &= \mathbb{A}_0 S(0) \quad \text{for } n = 0, \\ a_n S(n+1) + b_n S(n) + c_n S(n-1) &= \mathbb{A}_0 S(n) \quad \text{for all } n \in \mathbb{N}. \end{aligned}$$

In this setting it is convenient to define the *(adapted) generator* $\mathbb{A}$ through

$$\mathbb{A}_0 = R_1(\mathbb{A}); \tag{6.1}$$

that is

$$\mathbb{A} = a_0\mathbb{A}_0 + b_0 = a_0 S(1) + b_0.$$

Then it is easy to see that

$$S(n) = R_n(\mathbb{A})$$

for all $n \in \mathbb{N}_0$ (cf. [8, Theorem 1]).

6.2. **Problems of compact type.** It seems that only few is known about the following classes of problems. The abstract Jacobi problem below occurs in a paper by Weinmann and Lasser [30] in the context of duality of Lipschitz spaces with respect to Jacobi translations. It is a special example of abstract Sturm–Liouville problems of compact type, see the subsequent example.

Example 6.3 (Abstract Jacobi problems). This example is inspired by Section 3 of [30], where the case of translation operator functions on homogeneous Banach spaces is investigated. Here the associated integral equation arises from dual Jacobi polynomial hypergroups.

To begin with, let us collect some facts and notation for dual Jacobi polynomial hypergroups. Let $R_n^{(\alpha,\beta)}$, $n \in \mathbb{N}_0$, denote the Jacobi polynomials with parameters $(\alpha,\beta) \in J$, where $J = \{(\alpha',\beta') \in \mathbb{R}^2 : \alpha' \geq \beta' \geq -\frac{1}{2} \vee (\alpha' \geq \beta' > -1 \wedge \alpha' + \beta' \geq 0)\}$, normalized by $R_n^{(\alpha,\beta)}(1) = 1$. These are orthogonal with respect to $\pi^{(\alpha,\beta)}$, the probability measure on $\mathcal{S} = [-1,1]$ with Lebesgue density $w(s) = c_{\alpha,\beta}(1-s)^\alpha(1+s)^\beta$, $c_{\alpha,\beta} = 2^{-\alpha-\beta-1}\Gamma(\alpha+\beta+2)\Gamma(\alpha+1)^{-1}\Gamma(\beta+1)^{-1}$. It has been shown by Gasper in [13] that there exists a positive linearization formula on $\mathcal{S} = [-1,1]$, and Lasser in [20, Section 4] has shown that $\mathcal{S}$ becomes a hypergroup with dual space $\widehat{\mathcal{S}} = \{R_n^{(\alpha,\beta)}, n \in \mathbb{N}_0\}$. Its neutral element is 1, and involution is the identity map. In the remainder, let $(\alpha,\beta) \in J$ be fixed; we drop its notation.

It is well-known that Jacobi polynomials satisfy the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(w(t)(1-t^2)\frac{\mathrm{d}}{\mathrm{d}t}R_n(t)\big) = -n(n+\alpha+\beta+1)w(t)R_n(t), \tag{6.2}$$

$$R_n(1) = 1, \qquad R_n'(1) = -\frac{1}{2(\alpha+1)}\big[-n(n+\alpha+\beta+1)\big]. \tag{6.3}$$

Integration gives the integral equation

$$R_n(t) - 1 = -n(n+\alpha+\beta+1)\int_t^1 \frac{1}{w(s)(1-s^2)}\int_s^1 R_n(r)w(r)\,\mathrm{d}r\,\mathrm{d}s,$$

$t \in]-1,1]$; after integration by parts it takes the form

$$R_n(t) - 1 = -n(n+\alpha+\beta+1)\int_t^1 \theta(t,s)R_n(s)\pi(\mathrm{d}s),$$

where

$$\theta(t,s) := \int_t^s \frac{1}{w(r)(1-r^2)}\,\mathrm{d}r\,1_{(t,1)}(s)$$

and $t \in]-1,1]$ (see [30]).

Suppose now that S is a multiplicative operator function on $\mathcal{S} = [-1,1]$. It is easily checked that Theorem 5.3 is applicable, thus for every $x \in X$, $\int_t^1 \theta(t,s)S(s)x\,\pi(\mathrm{d}s) \in D(\mathbb{A}_0)$ and

$$S(t)x - x = \mathbb{A}_0 \int_t^1 \theta(t,s)S(s)x\,\pi(\mathrm{d}s)$$

for all $t \in]-1,1]$ (see Lemma 3.4 in [30], for the special case of a translation operator function on a homogeneous Banach space, and a proof using Fourier analysis).

The abstract Cauchy problem corresponding to (6.2), (6.3) takes the form

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\big(w(t)(1-t^2)\frac{\mathrm{d}}{\mathrm{d}t}S(t)x\big) = \mathbb{A}_0 w(t)S(t)x, \quad t \in]-1,1], \\ &S(1)x = x, \qquad S'(1)x = -\frac{1}{2(\alpha+1)}\mathbb{A}_0 x \end{aligned}$$

for $x \in D(\mathbb{A}_0)$.

Example 6.4 (Abstract Sturm–Liouville problems of compact type). Each dual Jacobi polynomial hypergroup from Example 6.3 is isomorphic to a Sturm–Liouville hypergroup of compact type. So, more generally, one could consider this class of hypergroups; examples are provided by Achour–Trimèche, Zeuner, and Fourier–Bessel hypergroups. (For the first statement, and the examples, see [2, Examples 3.5.80–3.5.88].)

6.3. **Problems of noncompact type.** These are of primary interest. The duality between functional equation and abstract Cauchy problem is well-known for C_0-semigroups and cosine operator functions. Our approach contains C_0-groups and cosine operator functions, and can be extended to Sturm–Liouville operator functions. The latter are investigated in detail by the author in [11].

Example 6.5 (C_0-groups). This is probably the most simple application of Theorem 5.3. Let $K = \mathbb{R}$ be the group of real numbers with addition. The multiplicative functions are given by $\{\exp(\lambda\cdot), \lambda \in \mathbb{C}\}$, consider the associated integral equation $\exp(\lambda t) - 1 = \lambda \int_0^t \exp(\lambda r)\,\mathrm{d}r$, $\lambda \in \mathbb{C}$. Then a multiplicative operator function S on $K = \mathbb{R}$ is a C_0-group of operators, Theorem 5.3 yields the abstract integral equation, and differentiation and Proposition 5.2 give

$$\begin{aligned} &S'(t)x = \mathbb{A}S(t)x, \quad t \in \mathbb{R}, \\ &S(0)x = x \end{aligned}$$

for any $x \in D(\mathbb{A})$ where $\mathbb{A} := \mathbb{A}_0$ is the usual generator of the C_0-group S; the (adapted) generator $\mathbb{A}$ and the universal generator $\mathbb{A}_0$ coincide.

Example 6.6 (Cosine operator functions). Let $K = \mathbb{R}_+$ be the cosine hypergroup on the nonnegative real line, that is the hypergroup with $\varepsilon_t * \varepsilon_s = \frac{1}{2}\varepsilon_{t+s} + \frac{1}{2}\varepsilon_{|t-s|}$ for all $t, s \in \mathbb{R}_+$ and multiplicative functions $\{\cos(\lambda\cdot), \lambda \in \mathbb{C}\}$. Consider the associated integral equation $\cos(\lambda t) - 1 = -\lambda^2 \int_0^t \int_0^s \cos(\lambda r)\,\mathrm{d}r\,\mathrm{d}s$, $\lambda \in \mathbb{C}$. Then a multiplicative operator function S on $K = \mathbb{R}_+$ is a cosine operator function, and, similarly as above, Theorem 5.3 yields

$$\begin{aligned} S''(t)x &= \mathbb{A}S(t)x, \quad t \in \mathbb{R}_+, \\ S(0)x &= x, \qquad S'(0)x = 0 \end{aligned}$$

for any $x \in D(\mathbb{A})$ where $\mathbb{A} := \mathbb{A}_0$ is the usual generator of the cosine operator function S. This example is contained in Example 6.7 as the special case where $A \equiv 1$.

The duality between the operator-valued cosine functional equation and the abstract Cauchy problem of the second order can be traced back to Sova [29] (see [1, Sections 3.14–3.16] for an introduction to cosine operator functions).

Example 6.7 (Abstract Sturm–Liouville problems on $\mathbb{R}_+$). Let us give a brief outline of the principal idea (see [11] for further details). Here $K = \mathbb{R}_+$ is a Sturm–Liouville hypergroup on the nonnegative real line as investigated by Zeuner in [31], [32] (see [10] for an asymptotic perspective).

A Sturm–Liouville hypergroup $(\mathbb{R}_+, *(A))$ is a hypergroup $K = \mathbb{R}_+$ together with a function $A : \mathbb{R}_+ \to \mathbb{R}_+$ (satisfying several conditions). Its multiplicative functions are exactly the solutions Φ_λ, $\lambda \in \mathbb{C}$ of

$$L\Phi_\lambda(t) = (\lambda^2 - \rho^2)\Phi_\lambda(t), \quad t > 0, \tag{6.4}$$

$$\Phi_\lambda(0) = 1, \qquad \Phi'_\lambda(0) = 0, \tag{6.5}$$

where $L = \mathrm{d}^2/\mathrm{d}t^2 + A'(t)/A(t) \cdot \mathrm{d}/\mathrm{d}t$ denotes the associated Sturm–Liouville operator and $\rho := \frac{1}{2}\lim_{t\to\infty} A'(t)/A(t) \geq 0$ is the index of $(\mathbb{R}_+, *(A))$.

Now integration of (6.4) and (6.5) gives an associated integral equation, Theorem 5.3 yields an abstract integral equation, and differentiating again one obtains the abstract Sturm–Liouville equation

$$LS(t)x = (\mathbb{A} - \rho^2)S(t)x, \quad t > 0, \tag{6.6}$$

$$S(0)x = x, \qquad S'(0)x = 0, \tag{6.7}$$

where $x \in D(\mathbb{A})$ and

$$\mathbb{A} := \rho^2 + \mathbb{A}_0 \tag{6.8}$$

is called the *adapted generator* or shortly the *generator* (see [11]). In particular, (6.6) and (6.7) contains cosine, Bessel, and Legendre operator functions as special cases.

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