Poisson deformations of affine symplectic varieties, II

Yoshinori Namikawa

Abstract This article is a continuation of previous work, which has the same title. Let Y be an affine symplectic variety with a $\mathbf{C}^*$-action with positive weights, and let $\pi : X \to Y$ be its crepant resolution. Then π induces a natural map $\mathrm{PDef}(X) \to \mathrm{PDef}(Y)$ of Kuranishi spaces for the Poisson deformations of X and Y. In Part I, we proved that $\mathrm{PDef}(X)$ and $\mathrm{PDef}(Y)$ are both nonsingular, and this map is a finite surjective map. In this article (Part II), we prove that it is a Galois covering. Markman already obtained a similar result in the compact case, which was a motivation for this article. As an application, we construct explicitly the universal Poisson deformation of the normalization $\tilde{O}$ of a nilpotent orbit closure $\bar{O}$ in a complex simple Lie algebra when $\tilde{O}$ has a crepant resolution.

Introduction

Let Y be an affine symplectic variety of dimension $2n$, and let $\pi : X \to Y$ be a crepant resolution. By the definition, there is a symplectic 2-form $\bar{\sigma}$ on the smooth part $Y_{\mathrm{reg}} \cong \pi^{-1}(Y_{\mathrm{reg}})$, and it extends to a 2-form σ on X. Since π is crepant, σ is a symplectic 2-form on X. The symplectic structures on X and Y define Poisson structures on them in a natural manner. One can define a Poisson deformation of X (resp., Y) (see [Na5]). A Poisson deformation of X is equivalent to a symplectic deformation, namely, a deformation of the pair (X, σ). Let

$$\mathrm{PD}_Y : (\mathrm{Art})_{\mathbf{C}} \to (\mathrm{Set})$$

be the Poisson deformation functor from the category of local Artinian $\mathbf{C}$-algebras with residue field $\mathbf{C}$ to the category of sets (cf. [Na6, Section 1.1]). In [Na6] we studied a morphism of functors

$$\pi_* : \mathrm{PD}_X \to \mathrm{PD}_Y$$

induced by π (cf. [Na6, Section 5]). In particular, PD_X and PD_Y are both unobstructed, and π_* is a *finite covering*. To apply these results to geometric situations, we need the algebraizations of various formal objects. For this

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purpose, we start with an affine symplectic variety Y with a $\mathbf{C}^*$-action. More precisely, we assume that this $\mathbf{C}^*$-action has a unique fixed point $0 \in Y$ and the cotangent space $m_{Y,0}/m_{Y,0}^2$ is decomposed into 1-dimensional eigenspaces with *positive* weights. Moreover, $\bar{\sigma}$ is assumed to be positively weighted (cf. [Na5, (Appendix 1)]). This $\mathbf{C}^*$-action extends to a $\mathbf{C}^*$-action on X (see [Na5, Proposition Appendix 7, Step 1]). By [Na6], one can construct a $\mathbf{C}^*$-equivariant commutative diagram

$$\begin{array}{ccc} \mathcal{X} & \longrightarrow & \mathcal{Y} \\ \downarrow & & \downarrow \\ \mathbf{A}^d & \longrightarrow & \mathbf{A}^d \end{array} \tag{1}$$

where $\mathcal{X} \to \mathbf{A}^d$ (resp., $\mathcal{Y} \to \mathbf{A}^d$) is a Poisson deformation of $X = \mathcal{X}_0$ (resp., $Y = \mathcal{Y}_0$), and both of them are universal at $0 \in \mathbf{A}^d$. Let $\mathrm{PDef}(X)$ (resp., $\mathrm{PDef}(Y)$) be a small open neighborhood of zero in the first affine space $\mathbf{A}^d$ (resp., second affine space $\mathbf{A}^d$). We call them the Kuranishi spaces for the Poisson deformations of X and Y, respectively. The map $\mathbf{A}^d \to \mathbf{A}^d$ restricts to the map $f : \mathrm{PDef}(X) \to \mathrm{PDef}(Y)$; f is a finite surjective map between smooth varieties of the same dimension.

The main result of this article claims that f is a Galois covering (cf. Theorem 1.1). Section 1 is devoted to the proof of Theorem 1.1. For a projective symplectic variety Y and its crepant resolution X, Markman [Ma] has already proved the same result for the usual Kuranishi spaces, where he pointed out the Weyl groups of the folded Dynkin diagrams appear as the Galois group. A main motivation for this article was to generalize his result for a Poisson deformation of an affine symplectic variety. While trying to understand his result, the author realized that his result can be proved very naturally in terms of Poisson deformations. This point of view also enables us to re-prove his original result in a slightly different manner (see Section 1.3).

In Section 1, we apply Theorem 1.1 to the Poisson deformations of an affine symplectic variety related to a nilpotent orbit in a complex simple Lie algebra. Let $\mathfrak{g}$ be a complex simple Lie algebra, and let G be the adjoint group. For a parabolic subgroup P of G, denote by $T^*(G/P)$ the cotangent bundle of G/P. The image of the Springer map $s : T^*(G/P) \to \mathfrak{g}$ is the closure $\bar{O}$ of a nilpotent (adjoint) orbit O in $\mathfrak{g}$. Then the normalization $\tilde{O}$ of $\bar{O}$ is an affine symplectic variety with the Kostant-Kirillov 2-form. If s is birational onto its image, then the Stein factorization $T^*(G/P) \to \tilde{O} \to \bar{O}$ of s gives a crepant resolution of $\tilde{O}$. In this situation, we prove that the Brieskorn-Slodowy diagram (cf. Section 2)

$$\begin{array}{ccc} G \times^P r(\mathfrak{p}) & \longrightarrow & \widetilde{G \cdot r(\mathfrak{p})} \\ \downarrow & & \downarrow \\ \mathfrak{k}(\mathfrak{p}) & \longrightarrow & \mathfrak{k}(\mathfrak{p})/W' \end{array} \tag{2}$$

coincides with the $\mathbf{C}^*$-equivariant commutative diagram of the universal Poisson deformations of $T^*(G/P)$ and $\tilde{O}$. The precise definitions and notation for the Brieskorn-Slodowy diagram can be found in Section 2. Here the group W' appears as the Galois group of the finite covering of Kuranishi spaces. The group W' is defined as $N_W(L)/W(L)$ (see Section 2), where L is a Levi subgroup of P, $W(L)$ is the Weyl group of L, and $N_W(L)$ is the normalizer group of L in the Weyl group W of G. This W' coincides with the W' in [Ho]. Howlett [Ho] has extensively studied W'. According to [Ho], W' is *almost* a reflection group. But in our situation, where the Springer map has degree 1, we can give a geometric proof that W' is a reflection group (see Lemma 2.2).

Terminologies

(i) A *symplectic variety* (X,ω) is a pair of a normal algebraic variety X defined over $\mathbf{C}$ and a symplectic 2-form ω on the regular part X_{reg} of X such that, for any resolution $\mu: \tilde{X} \to X$, the 2-form ω on $\mu^{-1}(X_{\mathrm{reg}})$ extends to a closed regular 2-form on $\tilde{X}$. We also have a similar notion of a symplectic variety in the complex analytic category (e.g., the germ of a normal complex space, a holomorphically convex, normal, complex space). The symplectic 2-form ω defines a bivector $\Theta \in \wedge^2 \Theta_{X_{\mathrm{reg}}}$ by the identification $\Omega^2_{X_{\mathrm{reg}}} \cong \wedge^2 \Theta_{X_{\mathrm{reg}}}$ by ω. Define a Poisson structure $\{\ ,\ \}$ on X_{reg} by $\{f,g\} := \Theta(df \wedge dg)$. Since X is normal, the Poisson structure on X_{reg} uniquely extends to a Poisson structure on X.

(ii) Let X be a Poisson variety, and let $f: \mathcal{X} \to T$ be a Poisson deformation of X such that $\mathcal{X}_0 = X$ with $0 \in T$. In this article, we say that f is *universal* (or, more precisely, *formally universal*) at $0 \in T$ if, for any Poisson deformation $\mathcal{X}' \to S$ of X ($\mathcal{X}'_0 = X,\ 0 \in S$) with a local Artinian base S, there is a unique map $(S,0) \to (T,0)$ such that $\mathcal{X}' \cong \mathcal{X} \times_T S$ as the Poisson deformations of X over S. In this case, for a small open neighborhood V of $0 \in T$, the family $f|_{f^{-1}(V)}: f^{-1}(V) \to V$ is called the Kuranishi family for the Poisson deformations of X, and V is called the Kuranishi space for the Poisson deformations of X.

1. The Kuranishi spaces for Poisson deformations and Galois coverings

Let (X,σ) and $(Y,\bar{\sigma})$ be the same as in the introduction. Let Σ be the singular locus of Y. According to [Ka], Σ is stratified into symplectic varieties. In particular, each stratum has even dimension. If $\mathrm{Codim}_Y \Sigma = 2$, then the maximal strata parameterize ADE singularities. More precisely, there is a closed subset $\Sigma_0 \subset \Sigma$ with $\mathrm{Codim}_Y \Sigma_0 \geq 4$, and Y is locally isomorphic to $(S,0) \times (\mathbf{C}^{2n-2}, 0)$ at every point $p \in \Sigma - \Sigma_0$, where S is an ADE surface singularity. Let $\mathcal{B}$ be the set of connected components of $\Sigma - \Sigma_0$. Let $B \in \mathcal{B}$. Pick a point $b \in B$, and take a transversal slice $S_B \subset Y$ of B passing through b. In other words, Y is locally isomorphic to $S_B \times (B,b)$ around b. Note that S_B is a surface with an ADE singularity. Put $\tilde{S}_B := \pi^{-1}(S_B)$. Then $\tilde{S}_B$ is a minimal resolution of S_B. Put $T_B := S_B \times (B,b)$ and $\tilde{T}_B := \pi^{-1}(T_B)$. Note that $\tilde{T}_B = \tilde{S}_B \times (B,b)$. We put $\bar{\sigma}_B := \bar{\sigma}|_{(T_B)_{\mathrm{reg}}}$ and $\sigma_B := \sigma|_{\tilde{T}_B}$. Then $(T_B, \bar{\sigma}_B)$ is a singular symplectic variety, and $(\tilde{T}_B, \sigma_B)$ is a smooth symplectic variety. Let C_i $(1 \leq i \leq r)$ be the (-2)-curves contained in $\tilde{S}_B$, and let $[C_i] \in H^2(\tilde{S}_B, \mathbf{R})$ be their classes in the second cohomology group. Then

$$\Phi := \{\Sigma a_i[C_i]; a_i \in \mathbf{Z}, (\Sigma a_i[C_i])^2 = -2\}$$

is a root system of the same type as that of the ADE singularity S_B. Let W be the Weyl group of Φ. Let $\{E_i(B)\}_{1\leq i\leq \bar{r}}$ be the set of irreducible exceptional divisors of π lying over B, and let $e_i(B) \in H^2(X, \mathbf{Z})$ be their classes. Clearly, $\bar{r} \leq r$. If $\bar{r} = r$, then we define $W_B := W$. If $\bar{r} < r$, the Dynkin diagram of Φ has a nontrivial graph automorphism. When Φ is of type A_r with $r > 1$, $\bar{r} = [r + 1/2]$ and the Dynkin diagram has a graph automorphism τ of order 2. When Φ is of type D_r with $r \geq 5$, $\bar{r} = r - 1$ and the Dynkin diagram has a graph automorphism τ of order 2. When Φ is of type D_4, the Dynkin diagram has two different graph automorphisms of orders 2 and 3. There are two possibilities for $\bar{r}$: $\bar{r} = 2$ or $\bar{r} = 3$. In the first case, let τ be the graph automorphism of order 3. In the latter case, let τ be the graph automorphism of order 2. Finally, when Φ is of type E_6, $\bar{r} = 4$ and the Dynkin diagram has a graph automorphism τ of order 2. In all these cases, we define

$$W_B := \{w \in W; \tau w \tau^{-1} = w\}.$$

THEOREM 1.1
The map $f : \mathrm{PDef}(X) \to \mathrm{PDef}(Y)$ *is a Galois covering with* $G = \prod_{B\in\mathcal{B}} W_B$.

Proof
We divide the proof into four steps.

Step 1: Outline of the proof. In Step 2(ii) we construct the Kuranishi space $\mathrm{PDef}(T_B)$ for the Poisson deformation of T_B and the Kuranishi space $\mathrm{PDef}(\tilde{T}_B)$ for the Poisson deformation of $\tilde{T}_B$. By the construction, there is a finite Galois map $f_B : \mathrm{PDef}(\tilde{T}_B) \to \mathrm{PDef}(T_B)$ whose Galois group is the Weyl group of the root system corresponding to the ADE singularity S_B. Since T_B (resp., $\tilde{T}_B$) is an open set of Y (resp., X), any Poisson deformation of Y (resp., X) over a local Artinian base induces a Poisson deformation of T_B (resp., $\tilde{T}_B$) over the same base. Thus we have a morphism of functors $\mathrm{PD}_Y \to \mathrm{PD}_{T_B}$ (resp., $\mathrm{PD}_X \to \mathrm{PD}_{\tilde{T}_B}$). Since $R^1\pi_*\mathcal{O}_X = 0$ and $\pi_*\mathcal{O}_X = \mathcal{O}_Y$, the crepant resolution $\pi : X \to Y$ induces a morphism of functors $\mathrm{PD}_X \to \mathrm{PD}_Y$ (cf. [Na6, Theorem 5.1, Proof(i)]). Similarly, we have a morphism of functors $\mathrm{PD}_{\tilde{T}_B} \to \mathrm{PD}_{T_B}$. These morphisms form a commutative diagram

$$\begin{array}{ccc} \mathrm{PD}_X & \longrightarrow & \mathrm{PD}_{\tilde{T}_B} \\ \downarrow & & \downarrow \\ \mathrm{PD}_Y & \longrightarrow & \mathrm{PD}_{T_B} \end{array} \tag{3}$$

For a complex space W with an origin $0 \in W$, we denote by $\hat{W}$ the formal completion of W at zero. The commutative diagram above induces a commutative diagram of formal spaces

$$\begin{array}{ccc} \widehat{\mathrm{PDef}(X)} & \longrightarrow & \widehat{\mathrm{PDef}(\tilde{T}_B)} \\ \downarrow & & \downarrow \\ \widehat{\mathrm{PDef}(Y)} & \longrightarrow & \widehat{\mathrm{PDef}(T_B)} \end{array} \tag{4}$$

By using the period maps (cf. Step 2(i)), one can see that this diagram is actually the formal completions of a commutative diagram of complex spaces (cf. Step 4(i))

$$\begin{array}{ccc} \mathrm{PDef}(X) & \xrightarrow{\varphi_B} & \mathrm{PDef}(\tilde{T}_B) \\ \downarrow & & \downarrow f_B \\ \mathrm{PDef}(Y) & \longrightarrow & \mathrm{PDef}(T_B) \end{array} \tag{5}$$

Put $V_B := \mathrm{Im}(\varphi_B)$. We prove that

(a) V_B and $f_B(V_B)$ are both nonsingular, and

(b) $f_B|_{V_B} : V_B \to f_B(V_B)$ is a W_B-Galois covering (cf. Step 3 and the final part of Step 4(ii)).

Then we get the commutative diagram

$$\begin{array}{ccc} \mathrm{PDef}(X) & \xrightarrow{\varphi_B} & \prod_{B\in\mathcal{B}} V_B \\ \downarrow & & \downarrow \\ \mathrm{PDef}(Y) & \longrightarrow & \prod_{B\in\mathcal{B}} f_B(V_B) \end{array} \tag{6}$$

We finally prove that the induced map

$$\iota :\ \mathrm{PDef}(X) \to \mathrm{PDef}(Y) \times_{\prod f_B(V_B)} \prod V_B$$

is an isomorphism (cf. Step 4(iii)).

Step 2: Poisson deformations and period maps.

(i) Let us consider the commutative diagram of universal Poisson deformations in the introduction

$$\begin{array}{ccc} \mathcal{X} & \longrightarrow & \mathcal{Y} \\ \alpha\downarrow & & \downarrow \\ \mathbf{A}^d & \longrightarrow & \mathbf{A}^d \end{array} \tag{7}$$

Note that $\mathcal{Y}$ has a $\mathbf{C}^*$-action with positive weights with a fixed point $0 \in Y$. Moreover, the diagram is $\mathbf{C}^*$-equivariant, and $\alpha : \mathcal{X} \to \mathbf{A}^d$ is a simultaneous resolution of $\mathcal{Y} \to \mathbf{A}^d$. Then we see that $\mathcal{X}$ is a C^∞-trivial fiber bundle over $\mathbf{A}^d$ by [S1, remark at the end of Section 4.2].† Let $\Omega^{\cdot}_{\mathcal{X}^{\mathrm{an}}/\mathbf{A}^d}$ be the relative complex-analytic de Rham complex. Let $\mathcal{K}$ be the subsheaf of $\Omega^2_{\mathcal{X}^{\mathrm{an}}/\mathbf{A}^d}$ which consists of d-closed relative 2-forms. By the natural map $\mathcal{K}[-2] \to \Omega^{\cdot}_{\mathcal{X}^{\mathrm{an}}/\mathbf{A}^d}$, we can define a sequence of maps:

†In [S1, 4.2], $\mathcal{Y}$ is assumed to be smooth. But the arguments can be applied to our case.

$$\alpha_*\mathcal{K} \to \mathbf{R}^2\alpha_*\Omega^{\cdot}_{\mathcal{X}^{\mathrm{an}}/\mathbf{A}^d} \cong R^2\alpha_*\alpha^{-1}\mathcal{O}^{\mathrm{an}}_{\mathbf{A}^d}.$$

Since $R^2\alpha_*\alpha^{-1}\mathcal{O}^{\mathrm{an}}_{\mathbf{A}^d} \cong R^2\alpha_*\mathbf{C} \otimes_{\mathbf{C}} \mathcal{O}^{\mathrm{an}}_{\mathbf{A}^d}$ (cf. [Lo, Lemma 8.2]), we have an isomorphism

$$R^2\alpha_*\alpha^{-1}\mathcal{O}^{\mathrm{an}}_{\mathbf{A}^d} \cong H^2(X,\mathbf{C}) \otimes_{\mathbf{C}} \mathcal{O}^{\mathrm{an}}_{\mathbf{A}^d}.$$

Since α is a Poisson deformation of X, $\mathcal{X}$ admits a relative symplectic 2-form $\sigma_{\mathcal{X}}$ such that $\sigma_{\mathcal{X}}|_X = \sigma$. Then $\sigma_{\mathcal{X}}$ gives a section s of the sheaf $H^2(X,\mathbf{C}) \otimes_{\mathbf{C}} \mathcal{O}^{\mathrm{an}}_{\mathbf{A}^d}$. Let

$$\mathrm{ev}_t : H^2(X,\mathbf{C}) \otimes_{\mathbf{C}} \mathcal{O}^{\mathrm{an}}_{\mathbf{A}^d} \to H^2(X,\mathbf{C})$$

be the evaluation map at $t \in \mathbf{A}^d$. We define a period map

$$p : \mathbf{A}^d \to H^2(X,\mathbf{C})$$

by $p(t) = \mathrm{ev}_t(s)$. By the construction, p is a holomorphic map. In [GK, Proposition 5.4], one can find another approach to the definition of the period map. The period map restricts to give a map

$$p_X : \mathrm{PDef}(X) \to H^2(X,\mathbf{C}).$$

We also call this map the period map for α. Since the tangential map $T_0\,\mathrm{PDef}(X) \to H^2(X,\mathbf{C})$ of p_X is an isomorphism (see [Na5, Corollary 10]), p_X is an open immersion.

(ii) Since S_B is an ADE singularity, it is isomorphic to the hypersurface defined by a weighted homogeneous polynomial $f(x,y,z)$. Then S_B has a $\mathbf{C}^*$-action with positive weights, and $\bar{\tau}_B := \mathrm{Res}(dx \wedge dy \wedge dz/f)$ is a generator of K_{S_B} with a positive weight. By the minimal resolution $\tilde{S}_B \to S_B$, $\bar{\tau}_B$ is pulled back to a symplectic 2-form τ_B on $\tilde{S}_B$. On the other hand, (B,b) is isomorphic to $(\mathbf{C}^{2n-2},0)$, and it admits a canonical symplectic 2-form $\tau_{\mathbf{C}^{2n-2}} := ds_1 \wedge dt_1 + \cdots + ds_{n-1} \wedge dt_{n-1}$ with the standard coordinates $(s_1,\dots,s_{n-1},t_1,\dots,t_{n-1})$ of $\mathbf{C}^{2n-2}$. By a generalization of Darboux's theorem (see [Na6, Lemma 1.3]), one can see that $(T_B,\bar{\sigma}_B)$ is equivalent to $(T_B,\bar{\tau}_B + \tau_{\mathbf{C}^{2n-2}})$ as a symplectic variety. Therefore, $(\tilde{T}_B,\sigma_B)$ is equivalent to $(\tilde{T}_B,\tau_B + \tau_{\mathbf{C}^{2n-2}})$. Let $\mathfrak{g}$ be the complex simple Lie algebra of the same type as S_B, and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. We denote by W the Weyl group of $\mathfrak{g}$. By using a special transversal slice of $\mathfrak{g}$, one can construct the universal Poisson deformation

$$\mathcal{S}_B \to \mathfrak{h}/W$$

of S_B and the universal Poisson deformation

$$\tilde{\mathcal{S}}_B \to \mathfrak{h}$$

of $\tilde{S}_B$ (see [Na6, Proposition 3.1(1)]). Moreover, the composite

$$\mathcal{S}_B \times (\mathbf{C}^{2n-2},0) \xrightarrow{p_1} \mathcal{S}_B \to \mathfrak{h}/W$$

and the composite

$$\tilde{\mathcal{S}}_B \times (\mathbf{C}^{2n-2},0) \xrightarrow{p_1} \tilde{\mathcal{S}}_B \to \mathfrak{h}$$

are, respectively, universal Poisson deformations of T_B and $\tilde{T}_B$ (see [Na6, Proposition 3.1(2)]). Note that T_B has a $\mathbf{C}^*$-action with positive weights. Since $\tilde{S}_B$ is the minimal resolution of S_B, this $\mathbf{C}^*$-action extends uniquely to that on $\tilde{T}_B$. We now have a $\mathbf{C}^*$-equivariant commutative diagram

$$\begin{array}{ccc} \tilde{\mathcal{S}}_B \times (\mathbf{C}^{2n-2}, 0) & \longrightarrow & \mathcal{S}_B \times (\mathbf{C}^{2n-2}, 0) \\ \downarrow & & \downarrow \\ \mathfrak{h} & \longrightarrow & \mathfrak{h}/W \end{array} \tag{8}$$

As is seen in (i), one can define a period map $\mathfrak{h} \to H^2(\tilde{T}_B, \mathbf{C})$. The Kuranishi space $\mathrm{PDef}(\tilde{T}_B)$ for the Poisson deformation of $\tilde{T}_B$ is an open neighborhood of $0 \in \mathfrak{h}$, and the period map above restricts to a map

$$p_B : \mathrm{PDef}(\tilde{T}_B) \to H^2(\tilde{T}_B, \mathbf{C}) \cong H^2(\tilde{S}_B, \mathbf{C}).$$

Since the tangential map $T_0 \mathrm{PDef}(\tilde{T}_B) \to H^2(\tilde{T}_B, \mathbf{C})$ of p_B is an isomorphism (see [Na5, Corollary 10]), p_B is an open immersion. Since $\mathrm{PDef}(X)$ (resp., $\mathrm{PDef}(\tilde{T}_B)$) can be regarded as open subsets of $H^2(X, \mathbf{C})$ (resp., $H^2(\tilde{T}_B, \mathbf{C})$) by the period maps, one can define a holomorphic map

$$\varphi_B : \mathrm{PDef}(X) \to \mathrm{PDef}(\tilde{T}_B)$$

so that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{PDef}(X) & \xrightarrow{p_X} & H^2(X, \mathbf{C}) \\ {\scriptstyle \varphi_B}\downarrow & & \downarrow \\ \mathrm{PDef}(\tilde{T}_B) & \xrightarrow{p_B} & H^2(\tilde{S}_B, \mathbf{C}) \end{array} \tag{9}$$

Since a Poisson deformation of X over a local Artinian base restricts to give a Poisson deformation of $\tilde{T}_B$ over the same base, we have a natural morphism of functors $\mathrm{PD}_X \to \mathrm{PD}_{\tilde{T}_B}$. By the (formal) universality of $\mathrm{PDef}(\tilde{T}_B)$, a formal map

$$\hat{\varphi}_B : \widehat{\mathrm{PDef}(X)} \to \widehat{\mathrm{PDef}(\tilde{T}_B)}$$

is uniquely determined. This is nothing but the formal completion of φ_B.

Step 3: Description of $\mathrm{Im}(\varphi_B)$. The Weyl group W (of the root system Φ associated with $\tilde{S}_B$) acts on $H^2(\tilde{S}_B, \mathbf{C})$. The period map $p : \mathfrak{h} \to H^2(\tilde{S}_B, \mathbf{C})$ is a W-equivariant linear map by [Ya] (cf. [Na6, proof of Proposition 3.2]). The commutative diagram

$$\begin{array}{ccc} \mathfrak{h} & \xrightarrow{p} & H^2(\tilde{S}_B, \mathbf{C}) \\ \downarrow & & \downarrow \\ \mathfrak{h}/W & \longrightarrow & H^2(\tilde{S}_B, \mathbf{C})/W \end{array} \tag{10}$$

induces a commutative diagram

$$
(11)\qquad
\begin{array}{ccc}
\mathrm{PDef}(\tilde{T}_B) & \xrightarrow{p_B} & H^2(\tilde{S}_B, \mathbf{C}) \\
{\scriptstyle f_B}\downarrow & & {\scriptstyle q}\downarrow \\
\mathrm{PDef}(T_B) & \longrightarrow & H^2(\tilde{S}_B, \mathbf{C})/W
\end{array}
$$

Let $E_i(B)$ $(1 \le i \le \bar{r})$ be the irreducible exceptional divisors of $\pi : X \to Y$ lying over B, and let $e_i(B) \in H^2(X, \mathbf{Z})$ be the cohomology class determined by $E_i(B)$. Even if $E_i(B)$ is irreducible, $E_i(B) \cap \tilde{S}_B$ might be reducible. Denote by r_B the restriction map $H^2(X, \mathbf{Z}) \to H^2(\tilde{S}_B, \mathbf{Z})$. Then $\mathrm{Im}(r_B \otimes \mathbf{C})$ is the $\mathbf{C}$-vector subspace generated by $r_B(e_i(B))$, $(1 \le i \le \bar{r})$. Let r be the number of (-2)-curves in $\tilde{S_B}$. As explained at the beginning of this section, when $\bar{r} < r$ we have a nontrivial graph automorphism τ of the Dynkin diagram. We then define $\Gamma_B := \langle \tau \rangle$. When $\bar{r} = r$, we just put $\Gamma_B = id$. Then $H^2(\tilde{S}_B, \mathbf{C})^{\Gamma_B} = \mathrm{Im}(r_B \otimes \mathbf{C})$. We put $\tilde{V}_B := \mathrm{Im}(r_B \otimes \mathbf{C})$. Let W' be the subgroup of W consisting of the elements which preserve $\tilde{V}_B$ as a set, and let $W_B := \{w \in W; \tau w \tau^{-1} = w\}$. This W_B is nothing but W^1 in [Ca1, Chapter 13]. It is obvious that $W_B \subset W'$.

LEMMA 1.2

The group W_B coincides with W'.

Proof

For simplicity we put $V := H^2(\tilde{S}_B, \mathbf{R})$ and $V^\tau := H^2(\tilde{S}_B, \mathbf{R})^{\Gamma_B}$. Let $(V^\tau)^\perp$ be the orthogonal complement of V^τ with respect to the inner product. Assume that $\tau^2 = 1$. Assume that $g \in W$ preserves V^τ as a set. Since g is an isometry of V, it acts on $(V^\tau)^\perp$. Since τ acts on $(V^\tau)^\perp$ by -1, we see that g commutes with τ. This means that $g \in W_B$. We next treat the case where τ has order 3 (i.e., the D_4 case). By the first part of [Ca1, Chapter 13], W is normalized by τ; in other words, $\tau W \tau^{-1} = W$ in the automorphism group of the root system. Assume that W_B does not coincide with W'. Then there is an element $w' \in W'$ such that $w := \tau w' \tau^{-1}$ does not equal w'. Note that $w' w^{-1} \neq 1$ and $w' w^{-1}$ acts trivially on V^τ. We show that such an element does not exist. Let $V = (\mathbf{R}^4, (\ ,\ ))$ be a 4-dimensional real vector space with a positive definite symmetric form. Let $e_1, \dots, e_4$ be an orthogonal basis such that $(e_i, e_j) = 0$ if $i \neq j$ and $(e_i, e_i) = 1$. One can choose simple roots for D_4 in such a way that $C_1 := e_1 - e_2$, $C_2 := e_2 - e_3$, $C_3 := e_3 - e_4$, and $C_4 := e_3 + e_4$. Define τ by $\tau(C_1) = c_3$, $\tau(C_3) = C_4$, and $\tau(C_4) = C_1$. Then V^τ is a 2-dimensional vector space spanned by $e_2 - e_3$ and $e_1 - e_2 + 2e_3$. Every element w of the Weyl group has the form

$$w(e_i) = (-1)^{\epsilon_i} e_{\sigma(i)}$$

with a permutation σ of 1, 2, 3, and 4. Here each ϵ_i is zero or one and $\Sigma \epsilon_i$ is even. It is easily checked that if w acts on V^τ trivially, then $w = id$. $\square$

The natural map $\tilde{V}_B \to q(\tilde{V}_B)$ factors through $\tilde{V}_B / W_B$. By Lemma 1.2, the map $\tilde{V}_B / W_B \to q(\tilde{V}_B)$ is the normalization map. Since W_B is generated by reflections (cf. [Ca1, Chapter 13]), $\tilde{V}_B/W_B$ is smooth. Put $V_B := \tilde{V}_B \cap \mathrm{PDef}(\tilde{T}_B)$. Then there is a commutative diagram

$$\begin{array}{ccc} V_B & \longrightarrow & \mathrm{PDef}(\tilde{T}_B) \\ \downarrow & & \downarrow{\scriptstyle f_B} \\ f_B(V_B) & \longrightarrow & \mathrm{PDef}(T_B) \end{array} \tag{12}$$

We prove that the image of the map $\mathrm{PDef}(X) \to \mathrm{PDef}(\tilde{T}_B)$ coincides with V_B. Since $\tilde{V}_B = \mathrm{Im}(r_B \otimes \mathbf{C})$, the image is contained in V_B by the commutative diagram of the period maps p_X and p_B. The tangent space $T_{([X,\sigma])}\mathrm{PDef}(X)$ (resp., $T_{([\tilde{T}_B,\sigma_B])}\mathrm{PDef}(\tilde{T}_B)$) is identified with $H^2(X,\mathbf{C})$ (resp., $H^2(\tilde{S}_B,\mathbf{C})$). Moreover, the tangential map

$$T_{([X,\sigma])}\mathrm{PDef}(X) \to T_{([\tilde{T}_B,\sigma_B])}\mathrm{PDef}(\tilde{T}_B)$$

is identified with the map

$$r_B \otimes \mathbf{C} : H^2(X,\mathbf{C}) \to H^2(\tilde{S}_B,\mathbf{C}).$$

This means that the image of the map $\mathrm{PDef}(X) \to \mathrm{PDef}(\tilde{T}_B)$ coincides with V_B.

Step 4: Proof that ι is an isomorphism.

(i) There is a commutative diagram of functors

$$\begin{array}{ccc} \mathrm{PD}_X & \longrightarrow & \mathrm{PD}_{\tilde{T}_B} \\ \downarrow & & \downarrow \\ \mathrm{PD}_Y & \longrightarrow & \mathrm{PD}_{T_B} \end{array} \tag{13}$$

Correspondingly we have a commutative diagram of formal spaces

$$\begin{array}{ccc} \widehat{\mathrm{PDef}(X)} & \xrightarrow{\hat{\varphi}_B} & \widehat{\mathrm{PDef}(\tilde{T}_B)} \\ {\scriptstyle \hat{f}}\downarrow & & \downarrow{\scriptstyle \hat{f}_B} \\ \widehat{\mathrm{PDef}(Y)} & \xrightarrow{\hat{\phi}_B} & \widehat{\mathrm{PDef}(T_B)} \end{array} \tag{14}$$

We have seen in Step 2 that the formal maps $\hat{f}$, $\hat{f}_B$, and $\hat{\varphi_B}$ are completions of the holomorphic maps f, f_B, and φ_B. Let us prove that $\hat{\phi}_B$ is also the completion of a holomorphic map $\phi_B : \mathrm{PDef}(Y) \to \mathrm{PDef}(T_B)$. In fact, there is a commutative diagram of local rings

$$\begin{array}{ccc} \mathcal{O}_{\mathrm{PDef}(T_B),0} & \xrightarrow{\hat{\phi}_B^*|_{\mathrm{PDef}(T_B),0}} & \hat{\mathcal{O}}_{\mathrm{PDef}(Y),0} \\ \downarrow & & \downarrow \\ \mathcal{O}_{\mathrm{PDef}(\tilde{T}_B),0} & \xrightarrow{\varphi_B^*} & \hat{\mathcal{O}}_{\mathrm{PDef}(X),0} \end{array} \tag{15}$$

Since $\mathrm{Im}(\varphi_B^*) \subset \mathcal{O}_{\mathrm{PDef}(X),0}$, we see that

$$\hat{\phi}_B^*(\mathcal{O}_{\mathrm{PDef}(T_B),0}) \subset \mathcal{O}_{\mathrm{PDef}(X),0} \cap \hat{\mathcal{O}}_{\mathrm{PDef}(Y),0}.$$

On the right-hand side, we take the intersection in $\hat{\mathcal{O}}_{\mathrm{PDef}(X),0}$. We prove that

$$\mathcal{O}_{\mathrm{PDef}(X),0} \cap \hat{\mathcal{O}}_{\mathrm{PDef}(Y),0} = \mathcal{O}_{\mathrm{PDef}(Y),0}.$$

For simplicity, we put $A = \mathcal{O}_{\mathrm{PDef}(Y),0}$ and $B = \mathcal{O}_{\mathrm{PDef}(X),0}$. Let m be the maximal ideal of A. Note that B is a finite A-module. Assume that $g \in \hat{A} \cap B$. For $n > 0$, we can write $g = g_n + h_n$ with $g_n \in A$ and $h_n \in m^n\hat{A}$. Since $h_n = g - g_n \in B$, we have

$$h_n \in m^n\hat{A} \cap B \subset m^n\hat{B} \cap B = m^nB.$$

In other words, $g \in A + m^nB$. Since n is arbitrary, we have

$$g \in \bigcap_{n>0} (A + m^nB) = A.$$

This shows that $\hat{\phi}_B$ induces a homomorphism $\mathcal{O}_{\mathrm{PDef}(T_B),0} \to \mathcal{O}_{\mathrm{PDef}(Y),0}$ and hence a holomorphic map

$$\phi_B : \mathrm{PDef}(Y) \to \mathrm{PDef}(T_B).$$

As a consequence we have a commutative diagram

$$\begin{array}{ccc} \mathrm{PDef}(X) & \xrightarrow{\varphi_B} & \mathrm{PDef}(\tilde{T}_B) \\ {\scriptstyle f}\downarrow & & {\scriptstyle f_B}\downarrow \\ \mathrm{PDef}(Y) & \xrightarrow{\phi_B} & \mathrm{PDef}(T_B) \end{array} \tag{16}$$

(ii) We briefly recall some results proved in [Na6]. Put $U := Y \setminus \Sigma_0$ and $\tilde{U} := \pi^{-1}(U)$. There are natural morphisms of functors $\mathrm{PD}_Y \to \mathrm{PD}_U$ and $\mathrm{PD}_X \to \mathrm{PD}_{\tilde{U}}$. By [Na6, Lemma 5.3], these are both isomorphisms. Denote by $\mathbf{PT}^1_U$ (resp., $\mathbf{PT}^1_{\tilde{U}}$) the tangent space of PD_U (resp., $\mathrm{PD}_{\tilde{U}}$). We have an isomorphism $\mathbf{PT}^1_{\tilde{U}} \cong H^2(\tilde{U}, \mathbf{C})$ (cf. [Na6, Theorem 5.1, Proof(i)]). In [Na6] we constructed a local system $\mathcal{H}$ of $\mathbf{C}$-modules on $\Sigma - \Sigma_0$. The local system $\mathcal{H}$ is the subsheaf of $\underline{\mathrm{Ext}}^1(\Omega^1_U, \mathcal{O}_U)$ which consists of local sections coming from the Poisson deformations (see [Na6, Sections 1.4, 1.5]). We have an exact sequence (cf. [Na6, Proposition 1.11])

$$0 \to H^2(U, \mathbf{C}) \to \mathbf{PT}^1_U \to H^0(\Sigma - \Sigma_0, \mathcal{H}).$$

Here the first term $H^2(U, \mathbf{C})$ is the space of locally trivial Poisson deformations of U. There is a commutative diagram of exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^2(U, \mathbf{C}) & \longrightarrow & H^2(\tilde{U}, \mathbf{C}) & \longrightarrow & H^0(U, R^2(\pi_{\tilde{U}})_*\mathbf{C}) \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H^2(U, \mathbf{C}) & \longrightarrow & \mathbf{PT}^1_U & \longrightarrow & H^0(\Sigma - \Sigma_0, \mathcal{H}) \end{array} \tag{17}$$

Theorem 5.1 of [Na6] claims that PD_U and $\mathrm{PD}_{\tilde{U}}$ are both unobstructed and have the same dimension. In the course of its proof, we also prove that two maps

$$H^2(\tilde{U}, \mathbf{C}) \to H^0\big(U, R^2(\pi_{\tilde{U}})_*\mathbf{C}\big)$$

and

$$\mathbf{PT}^1_U \to H^0(\Sigma - \Sigma_0, \mathcal{H})$$

are both surjective. Suppose that the exceptional locus of $\pi|_{\tilde{U}} : \tilde{U} \to U$ has exactly m irreducible components. Then we have

$$\dim \mathrm{Im}\big[H^2(\tilde{U}, \mathbf{C}) \to H^0\big(U, R^2(\pi_{\tilde{U}})_* \mathbf{C}\big)\big] = m$$

and

$$\dim \mathrm{Im}[\mathbf{PT}^1_U \to H^0(\Sigma - \Sigma_0, \mathcal{H})] = m.$$

There is a natural injection $H^0(U, R^2(\pi_{\tilde{U}})_* \mathbf{C}) \to \prod H^2(\tilde{S}_B, \mathbf{C})$, and the image of the composed map $H^2(\tilde{U}, \mathbf{C}) \to H^0(U, R^2(\pi_{\tilde{U}})_* \mathbf{C}) \to \prod H^2(\tilde{S}_B, \mathbf{C})$ has dimension m. Since $H^2(X, \mathbf{C}) \cong H^2(\tilde{U}, \mathbf{C})$ (cf. [Na6, Lemma 5.3]),

$$\dim \mathrm{Im}\Big[H^2(X, \mathbf{C}) \to \prod H^2(\tilde{S}_B, \mathbf{C})\Big] = m.$$

On the other hand, the left-hand side equals $\Sigma \dim V_B$ by the argument in Step 3. Hence we have $m = \Sigma \dim V_B$. Denote by $T^1_{S_B}$ the tangent space of $\mathrm{Def}(S_B)$ at the origin. The stalk $\mathcal{H}_b$ of $\mathcal{H}$ at $b \in B$ is isomorphic to $T^1_{S_B}$ (cf. [Na6, Sections 1.3, 1.5, 3.1]). Since $\mathcal{H}$ is a local system, there is a natural injection $H^0(\Sigma - \Sigma_0, \mathcal{H}) \to \prod_{B \in \mathcal{B}} T^1_{S_B}$. The image of the composed map $\mathbf{PT}^1_U \to H^0(\Sigma - \Sigma_0, \mathcal{H}) \to \prod_{B \in \mathcal{B}} T^1_{S_B}$ has dimension m. Note that $T^1_{S_B}$ is identified with the tangent space $T_0 \,\mathrm{PDef}(T_B)$ of $\mathrm{PDef}(T_B)$ (see [Na6, Proposition 3.1]). The map $\mathbf{PT}^1_U \to \prod T^1_{S_B}$ coincides with $T_0 \,\mathrm{PDef}(Y) \to \prod T_0 \,\mathrm{PDef}(T_B)$. Hence we have

$$\dim \mathrm{Im}\Big[T_0 \,\mathrm{PDef}(Y) \to \prod T_0 \,\mathrm{PDef}(T_B)\Big] = m.$$

On the other hand, the image of the map

$$\mathrm{PDef}(Y) \to \prod \mathrm{PDef}(T_B)$$

is $\prod f_B(V_B)$. Since $\mathrm{PDef}(Y)$ is smooth and $m = \Sigma \dim f_B(V_B)$, we see that $f_B(V_B)$ is smooth. In particular, the map $V_B \to f_B(V_B)$ is a finite Galois cover with Galois group W_B.

(iii) Let us consider the commutative diagram

$$\begin{array}{ccc} \mathrm{PDef}(X) & \xrightarrow{\prod \varphi_B} & \prod_{B \in \mathcal{B}} \mathrm{PDef}(\tilde{T}_B) \\ {\scriptstyle f}\big\downarrow & & {\scriptstyle \prod f_B}\big\downarrow \\ \mathrm{PDef}(Y) & \xrightarrow{\prod \phi_B} & \prod_{B \in \mathcal{B}} \mathrm{PDef}(T_B) \end{array} \tag{18}$$

By the argument in Step 3, the horizontal map at the bottom factorizes as $\mathrm{PDef}(Y) \to \prod_{B \in \mathcal{B}} f_B(V_B) \to \prod \mathrm{PDef}(T_B)$ and the following diagram commutes:

$$\begin{array}{ccc} \mathrm{PDef}(X) & \longrightarrow & \prod_{B \in \mathcal{B}} V_B \\ \big\downarrow & & \big\downarrow \\ \mathrm{PDef}(Y) & \longrightarrow & \prod_{B \in \mathcal{B}} f_B(V_B) \end{array} \tag{19}$$

The commutative diagram above induces a map

$$\iota: \ \mathrm{PDef}(X) \to \mathrm{PDef}(Y) \times_{\prod f_B(V_B)} \prod V_B.$$

First, we prove that $\mathrm{PDef}(Y) \times_{\prod f_B(V_B)} \prod V_B$ is smooth. Since $m = \Sigma \dim f_B(V_B)$ and each $f_B(V_B)$ is smooth, the map $\mathrm{PDef}(Y) \to \prod f_B(V_B)$ is a smooth map. Therefore, $\mathrm{PDef}(Y) \times_{\prod f_B(V_B)} \prod V_B \to \prod V_B$ is also a smooth map, and $\mathrm{PDef}(Y) \times_{\prod f_B(V_B)} \prod V_B$ is smooth. Finally, we prove that the map ι is an isomorphism. The tangent space T of $\mathrm{PDef}(Y) \times_{\prod f_B(V_B)} \prod V_B$ at the origin $\{0\} \times \prod\{0\}$ is isomorphic to $\mathbf{PT}^1_U \times_{\prod T_0(f_B(V_B))} \prod T_0 V_B$. Since $T_0(V_B) \to T_0(f_B(V_B))$ is the zero map, it is isomorphic to $H^2(U, \mathbf{C}) \oplus \prod V_B$. The map $d\iota: T_0 \mathrm{PDef}(X) \ (\cong H^2(\tilde{U}, \mathbf{C})) \to T$ is injective. In fact, if $v \in H^2(\tilde{U}, \mathbf{C})$ is mapped to zero by this map, then v must be sent to zero by the map $H^2(\tilde{U}, \mathbf{C}) \to T_0 V_B$ for each B. In other words, v is sent to zero by the composed map $H^2(\tilde{U}, \mathbf{C}) \to H^0(U, R^2(\pi_{\tilde{U}})_* \mathbf{C}) \to \prod H^2(\tilde{S}_B, \mathbf{C})$. Since the second map $H^0(U, R^2(\pi_{\tilde{U}})_* \mathbf{C}) \to \prod H^2(\tilde{S}_B, \mathbf{C})$ is an injection, v is already sent to zero by the first map $H^2(\tilde{U}, \mathbf{C}) \to H^0(U, R^2(\pi_{\tilde{U}})_* \mathbf{C})$. By the commutative diagram above, we see that $v \in H^2(U, \mathbf{C})$. On the other hand, v must be sent to zero by the map $H^2(\tilde{U}, \mathbf{C}) \to \mathbf{PT}^1_U$. The restriction of this map to $H^2(U, \mathbf{C})$ is an injection; hence, $v = 0$. Since both $T_0 \mathrm{PDef}(X)$ and T have the same dimension $h^2(U, \mathbf{C}) + m$, $d\iota$ is an isomorphism. Note that $\mathrm{PDef}(X)$ and $\mathrm{PDef}(Y) \times_{\prod f_B(V_B)} \prod V_B$ are both smooth; hence, ι is an isomorphism. (This is the end of the proof of Theorem 1.1) $\square$

REMARK

For $g \in W_B$, denote by $g_B: \ \mathrm{PDef}(\tilde{T}_B) \to \mathrm{PDef}(\tilde{T}_B)$ the automorphism induced by g. By Theorem 1.1, g also induces an automorphism $g_X: \mathrm{PDef}(X) \to \mathrm{PDef}(X)$. By pulling back the universal family $\tilde{\mathcal{T}}_B \to \mathrm{PDef}(\tilde{T}_B)$ by $g_B: \mathrm{PDef}(\tilde{T}_B) \to \mathrm{PDef}(\tilde{T}_B)$, we have a new family $\tilde{\mathcal{T}}'_B \to \mathrm{PDef}(\tilde{T}_B)$. Since G acts trivially on the universal family $\mathcal{T}_B \to \mathrm{PDef}(T_B)$, we have a diagram of the birational maps

$$\tilde{\mathcal{T}}_B \to \mathcal{T}_B \times_{\mathrm{PDef}(T_B)} \mathrm{PDef}(\tilde{T}_B) \leftarrow \tilde{\mathcal{T}}'_B.$$

If $g \neq 1$, the birational map $\tilde{\mathcal{T}}_B - - \to \tilde{\mathcal{T}}'_B$ is not regular. Similarly, by pulling back the universal family $\mathcal{X} \to \mathrm{PDef}(X)$ by g_X, we get a diagram of the birational maps

$$\mathcal{X} \to \mathcal{Y} \times_{\mathrm{PDef}(Y)} \mathrm{PDef}(X) \leftarrow \mathcal{X}'.$$

If $g \neq 1$, the birational map $\mathcal{X} - - \to \mathcal{X}'$ is not regular; moreover, some π-exceptional divisor $E \subset X$ lying over B is contained in its indeterminacy locus.

1.3.

The proof of Theorem 1.1 also works for the compact case. More exactly, let Y be a projective symplectic variety, and let $\pi: X \to Y$ be a crepant (projective) resolution. We assume that $h^0(X, \Omega^2_X) = 1$ and $h^1(X, \mathcal{O}_X) = 0$. By [Na1], the Kuranishi spaces $\mathrm{Def}(X)$ and $\mathrm{Def}(Y)$ are both nonsingular, and the induced map $\bar{f}: \ \mathrm{Def}(X) \to \mathrm{Def}(Y)$ is a finite surjective map. Markman [Ma] proved that $\bar{f}$ is actually a Galois covering whose Galois group coincides with $\prod W_B$. Let $\mathrm{PDef}(X)$ be the Kuranishi space for the Poisson deformations (symplectic deformations) of (X,σ). Let $\mu : \mathcal{X} \to \mathrm{Def}(X)$ be the Kuranishi family in the usual sense. Then $\mathcal{V} := \mu_*\Omega^2_{\mathcal{X}/\mathrm{Def}(X)}$ is a line bundle on $\mathrm{Def}(X)$, and $\mathcal{V}^* := \mathcal{V} - \{0-\text{section}\}$ is a $\mathbf{C}^*$-bundle over $\mathrm{Def}(X)$. The Kuranishi space $\mathrm{PDef}(X)$ for the Poisson deformation of (X,σ) is defined as an open neighborhood of $(X,\sigma) \in \mathcal{V}^*$. In particular, $\mathrm{PDef}(X)$ is a smooth variety. One can define a period map

$$p_X : \mathrm{PDef}(X) \to H^2(X,\mathbf{C})$$

by $p_X(X_t,\sigma_t) := [\sigma_t] \in H^2(X,\mathbf{C})$. Let $Q \subset H^2(X,\mathbf{C})$ be the hypersurface defined by $q = 0$ with the Beauville-Bogomolov form q (see [Be]). Let $\bar{Q} \subset \mathbf{P}(H^2(X,\mathbf{C}))$ be the projective hypersurface defined by $q = 0$. There is a commutative diagram

$$\begin{tikzcd} \mathrm{PDef}(X) \arrow[r, "p_X"] \arrow[d] & Q - \{0\} \arrow[d] \\ \mathrm{Def}(X) \arrow[r, "\bar{p}_X"] & \bar{Q} \end{tikzcd} \tag{20}$$

where $\bar{p}_X$ is the usual period map (see [Be]). The fibers of both vertical maps are $\mathbf{C}^*$, and p_X maps the fibers isomorphically. Since $\bar{p}_X$ is an open immersion by the local Torelli theorem, p_X is also an open immersion by the commutative diagram. As in the proof of Theorem 1.1, Step 2, we have a commutative diagram of period maps

$$\begin{tikzcd} \mathrm{PDef}(X) \arrow[r] \arrow[d] & \mathrm{PDef}(\tilde{T}_B) \arrow[d] \\ H^2(X,\mathbf{C}) \arrow[r] & H^2(\tilde{S}_B,\mathbf{C}) \end{tikzcd} \tag{21}$$

Note that $[\sigma] \in H^2(X,\mathbf{C})$ is not zero, but $[\sigma_B] = 0$ in $H^2(\tilde{S}_B,\mathbf{C})$ (cf. [Ka, Corollary 2.8]). Let PD_Y be the Poisson deformation functor of $(Y,\bar{\sigma})$ (see [Na6]). As in [Na6], we prove that PD_Y is unobstructed. Let $U := Y - \Sigma_0$, and put $\tilde{U} := \pi^{-1}(U)$. The key commutative diagram in the compact case is

$$\begin{tikzcd} 0 \arrow[r] & \mathbf{H}^2(U,\tilde{\Omega}^{\geq 1}_{\tilde{U}}) \arrow[r] \arrow[d] & \mathbf{H}^2(\tilde{U},\Omega^{\geq 1}_{\tilde{U}}) \arrow[r] \arrow[d] & H^0(U,R^2(\pi_{\tilde{U}})_*\mathbf{C}) \\ 0 \arrow[r] & \mathbf{H}^2(U,\tilde{\Omega}^{\geq 1}_{\tilde{U}}) \arrow[r] & \mathbf{PT}^1_U \arrow[r] & H^0(\Sigma - \Sigma_0,\mathcal{H}) \end{tikzcd} \tag{22}$$

The exact sequence on the first row comes from the Leray spectral sequence

$$E_2^{p,q} := H^p\big(U,R^q(\pi_{\tilde{U}})_*\Omega^{\geq 1}_{\tilde{U}}\big) \Rightarrow \mathbf{H}^{p+q}(\tilde{U},\Omega^{\geq 1}_{\tilde{U}}).$$

In the exact sequence on the second row, $\mathbf{H}^2(U,\tilde{\Omega}^{\geq 1}_{\tilde{U}})$ is the space of the first-order Poisson deformations of U which are locally trivial as flat deformations (cf. the proof of [Na6, Lemma 1.9]), and $\mathbf{PT}^1_U$ is the tangent space of PD_U. Note that $\mathbf{H}^2(\tilde{U},\Omega^{\geq 1}_{\tilde{U}})$ is the tangent space of $\mathrm{PD}_{\tilde{U}}$ (cf. [Na5, Propositions 8, 9]). Since $H^2(X,\mathbf{C}) \cong H^2(\tilde{U},\mathbf{C})$ (cf. the proof of [Na3, Proposition 2]), $H^2(\tilde{U})$ has a pure Hodge structure of weight 2. By the distinguished triangle

$$\Omega^{\geq 1}_{\tilde{U}} \to \Omega^{\cdot}_{\tilde{U}} \to \mathcal{O}_{\tilde{U}} \xrightarrow{[1]} \Omega^{\geq 1}_{\tilde{U}}[1]$$

and by the fact that $H^1(\tilde{U}, \mathcal{O}_{\tilde{U}}) = 0$, we have an exact sequence

$$0 \to \mathbf{H}^2(\tilde{U}, \Omega^{\geq 1}_{\tilde{U}}) \to H^2(\tilde{U}, \mathbf{C}) \to H^2(\tilde{U}, \mathcal{O}_{\tilde{U}}).$$

By the proof of [Na3, Proposition 2], $\mathrm{Gr}^0_F(H^2(\tilde{U})) = H^2(\tilde{U}, \mathcal{O}_{\tilde{U}})$. Therefore, $\mathbf{H}^2(\tilde{U}, \Omega^{\geq 1}_{\tilde{U}}) = F^1(H^2(\tilde{U}))$. Since $F^1(H^2(X)) = H^{2,0}(X) \oplus H^{1,1}(X)$, we see that the map

$$\mathbf{H}^2(\tilde{U}, \Omega^{\geq 1}_{\tilde{U}}) \to H^0\big(U, R^2(\pi_{\tilde{U}})_* \mathbf{C}\big)$$

is identified with the map

$$H^{2,0}(X) \oplus H^{1,1}(X) \subset H^2(X, \mathbf{C}) \to H^0\big(U, R^2(\pi_{\tilde{U}})_* \mathbf{C}\big).$$

Since U has only quotient singularities, U is $\mathbf{Q}$-factorial. Then by [KM, Proposition 12.1.6],

$$\mathrm{Im}\big[H^2(X, \mathbf{C}) \to H^0\big(U, R^2(\pi|_{\tilde{U}})_* \mathbf{C}\big)\big] = \mathrm{Im}\big[\Sigma_{B,i} \mathbf{C}[E_i(B)] \to H^0\big(U, R^2(\pi|_{\tilde{U}})_* \mathbf{C}\big)\big].$$

Since the map $H^0(U, R^2(\pi|_{\tilde{U}})_* \mathbf{C}) \to \prod H^2(\tilde{S}_B, \mathbf{C})$ is an injection, we see that

$$\begin{aligned}\dim \mathrm{Im}\big[H^2(X, \mathbf{C}) \to H^0\big(U, R^2(\pi|_{\tilde{U}})_* \mathbf{C}\big)\big] &= \dim \mathrm{Im}\Big[H^2(X, \mathbf{C}) \to \prod H^2(\tilde{S}_B, \mathbf{C})\Big] \\ &= m,\end{aligned}$$

where $m = \Sigma \dim V_B$. By the map $H^2(X, \mathbf{C}) \to H^0(Y, R^2\pi_* \mathbf{C})$, $H^{0,2}(X)$ is sent to zero. Therefore,

$$\dim \mathrm{Im}\big[\mathbf{H}^2(\tilde{U}, \Omega^{\geq 1}_{\tilde{U}}) \to H^0\big(U, R^2(\pi_{\tilde{U}})_* \mathbf{C}\big)\big] = m.$$

By [Na6, Proposition 4.2], $\dim H^0(\Sigma - \Sigma_0, \mathcal{H}) = m$. Then the same argument as in [Na1, Theorem 4.1] can be applied to our case to prove that PD_Y is unobstructed. Since $H^0(Y, \Theta_Y) = 0$, PD_Y is prorepresented by a complete regular local ring R over $\mathbf{C}$. We denote by $\widehat{\mathrm{PDef}(Y)}$ the formal scheme† defined by R. Since $R^1\pi_* \mathcal{O}_X = 0$ and $\pi_* \mathcal{O}_X = \mathcal{O}_Y$, the crepant resolution $\pi : X \to Y$ induces a morphism of functors $\mathrm{PD}_X \to \mathrm{PD}_Y$ (cf. [Na6, Theorem 5.1, proof (i)]). By the formal universality of $\widehat{\mathrm{PDef}(Y)}$, we have a formal map $\widehat{\mathrm{PDef}(X)} \to \widehat{\mathrm{PDef}(Y)}$. Note that $\dim \widehat{\mathrm{PDef}(X)} = \dim \widehat{\mathrm{PDef}(Y)}$, and as in [Na1, Lemma 4.2], the formal map is finite. There is a commutative diagram

$$\begin{array}{ccc} \widehat{\mathrm{PDef}(X)} & \longrightarrow & \widehat{\mathrm{PDef}(Y)} \\ \downarrow & & \downarrow \\ \widehat{\mathrm{Def}(X)} & \longrightarrow & \widehat{\mathrm{Def}(Y)} \end{array} \tag{23}$$

†The author has not yet constructed the Kuranishi space for PD_Y as a complex space. Let $\beta : \mathcal{Y} \to \mathrm{Def}(Y)$ be the universal family, and let $\mathcal{Y}^0 \subset \mathcal{Y}$ be the locus where β is smooth. We put $\beta^0 := \beta|_{\mathcal{Y}^0}$. Then $\bar{\mathcal{V}} := (\beta^0)_* \Omega^2_{\mathcal{Y}^0/\mathrm{Def}(Y)}$ seems very likely to be a line bundle on $\mathrm{Def}(Y)$ (see [Na7]). Then the Kuranishi space for PD_Y would be realized as an open subset of $\bar{\mathcal{V}} - \{0 - \mathrm{section}\}$.

The fibers of the maps $\widehat{\mathrm{PDef}(X)} \to \widehat{\mathrm{Def}(X)}$ and $\widehat{\mathrm{PDef}(Y)} \to \widehat{\mathrm{Def}(Y)}$ both have dimension 1, and they correspond to the Poisson deformations of X and Y, where the underlying flat deformations are fixed and the symplectic structures only vary. Therefore,

$$\widehat{\mathrm{PDef}(X)} \cong \widehat{\mathrm{PDef}(Y)} \times_{\widehat{\mathrm{Def}(Y)}} \widehat{\mathrm{Def}(X)}.$$

To prove Markman's result, we have to prove only that the map $\widehat{\mathrm{PDef}(X)} \to \widehat{\mathrm{PDef}(Y)}$ is a finite Galois cover with Galois group $\prod_{B\in\mathcal{B}} W_B$. The rest of the argument is similar to Theorem 1.1. Another approach avoiding formal schemes is the following. We first remark that Poisson deformations of $\tilde{T}_B$ and T_B are consequently determined only by the underlying flat deformations (cf. proof of Theorem 1.1, Step 2(ii) and [Na6, Proposition 3.1]). In particular, the map $\varphi_B : \mathrm{PDef}(X) \to \mathrm{PDef}(\tilde{T}_B)$ factorizes as

$$\mathrm{PDef}(X) \to \mathrm{Def}(X) \to \mathrm{PDef}(\tilde{T}_B).$$

Let V_B $(\subset \mathrm{PDef}(\tilde{T}_B))$ be the image of the second map. By the same reasoning as in Theorem 1.1, we then have a commutative diagram

$$\begin{array}{ccc} \mathrm{Def}(X) & \longrightarrow & \prod_{B\in\mathcal{B}} V_B \\ \downarrow & & \downarrow \\ \mathrm{Def}(Y) & \longrightarrow & \prod_{B\in\mathcal{B}} f_B(V_B) \end{array} \tag{24}$$

The induced map

$$\mathrm{Def}(X) \cong \mathrm{Def}(Y) \times_{\prod_{B\in\mathcal{B}} f_B(V_B)} \prod_{B\in\mathcal{B}} V_B$$

turns out to be an isomorphism.

2. Poisson deformations associated with nilpotent orbits

Let $\mathfrak{g}$ be a complex simple Lie algebra. We fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and let $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{g}$ such that $\mathfrak{h} \subset \mathfrak{p}$. Denote by $r(\mathfrak{p})$ (resp., $n(\mathfrak{p})$) the solvable radical (resp., nilpotent radical) of $\mathfrak{p}$. Define $\mathfrak{k}(\mathfrak{p}) := \mathfrak{h} \cap r(\mathfrak{p})$, and let $l(\mathfrak{p})$ be the Levi subalgebra of $\mathfrak{p}$ which contains $\mathfrak{h}$. Let G be the adjoint group of $\mathfrak{g}$, and let P (resp., L) be the closed subgroup of G corresponding to $\mathfrak{p}$ (resp., $l(\mathfrak{p})$). The cotangent bundle $T^*(G/P)$ of G/P is isomorphic to the vector bundle $G \times^P n(\mathfrak{p})$ over G/P. The Springer map $s : G \times^P n(\mathfrak{p}) \to \mathfrak{g}$ is defined by $s([g,x]) := \mathrm{Ad}_g(x)$ for $[g,x] \in G \times^P n(\mathfrak{p})$. The image $\mathrm{Im}(s)$ is the closure $\bar{O}$ of a nilpotent orbit O. The Springer map is a generically finite, surjective, projective morphism. Let $G \times^P n(\mathfrak{p}) \to \tilde{O} \to \bar{O}$ be the Stein factorization of s. Let W be the Weyl group of G. If we fix a Borel subalgebra $\mathfrak{b}$ such that $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{p}$, then $\mathfrak{p}$ is determined by a choice of a subset J of the set of simple roots (see [Na4, (P1)]). Then $W(L)$ is generated by reflections in elements of J; hence, $W(L)$ is a subgroup of W. Define $W' := N_W(L)/W(L)$, where $N_W(L)$ is the normalizer group of L in W, and $W(L)$ is the Weyl group of L. $N_W(L)$ acts on $\mathfrak{k}(\mathfrak{p})$, where $W(L)$ acts trivially on $\mathfrak{k}(\mathfrak{p})$. Therefore, W' acts effectively on $\mathfrak{k}(\mathfrak{p})$. Let us construct the Brieskorn-Slodowy diagram. There is a direct sum decomposition

$$r(\mathfrak{p}) = \mathfrak{k}(\mathfrak{p}) \oplus n(\mathfrak{p}) \quad (x \to x_1 + x_2),$$

where $n(\mathfrak{p})$ is the nil-radical of $\mathfrak{p}$. We have a well-defined map

$$G \times^P r(\mathfrak{p}) \to \mathfrak{k}(\mathfrak{p})$$

by sending $[g, x] \in G \times^P r(\mathfrak{p})$ to $x_1 \in \mathfrak{k}(\mathfrak{p})$ ([S2, Section 4.3]). On the other hand, define a map $G \times^P r(\mathfrak{p}) \to G \cdot r(\mathfrak{p})$ by $[g, x] \to \mathrm{Ad}_g(x)$. By the adjoint quotient map $\mathfrak{g} \to \mathfrak{h}/W$, we have a map from $G \cdot r(\mathfrak{p})$ to $\mathfrak{h}/W$. These maps form a commutative diagram (see [S2, Section 4.3])

$$\begin{array}{ccc} G \times^P r(\mathfrak{p}) & \longrightarrow & G \cdot r(\mathfrak{p}) \\ {\scriptstyle f}\downarrow & & \downarrow \\ \mathfrak{k}(\mathfrak{p}) & \longrightarrow & \mathfrak{h}/W \end{array} \tag{25}$$

One can find an instructive example of the diagram in [Na2, Example 7.10].

Let $\widetilde{G \cdot r(\mathfrak{p})}$ be the normalization of $G \cdot r(\mathfrak{p})$. Here the set $G \cdot r(\mathfrak{p})$ is irreducible since it is the image of the smooth variety $G \times^P r(\mathfrak{p})$; we regard $G \cdot r(\mathfrak{p})$ as a variety with the reduced structure. The normalization of the image of the map $G \cdot r(\mathfrak{p}) \to \mathfrak{h}/W$ coincides with $\mathfrak{k}(\mathfrak{p})/W'$. Then the map $\widetilde{G \cdot r(\mathfrak{p})} \to \mathfrak{h}/W$ factors through $\mathfrak{k}(\mathfrak{p})/W'$. By [Na4, Lemma 1.1], we already know that $G \cdot r(\mathfrak{p}) \times_{\mathfrak{h}/W} \mathfrak{k}(\mathfrak{p})$ is irreducible. Let $G \cdot r(\widetilde{\mathfrak{p}) \times_{\mathfrak{h}/W}} \mathfrak{k}(\mathfrak{p})$ be the normalization of the variety $(G \cdot r(\mathfrak{p}) \times_{\mathfrak{h}/W} \mathfrak{k}(\mathfrak{p}))_{\mathrm{red}}$. Then $\widetilde{G \cdot r(\mathfrak{p})}$ is the quotient variety of $G \cdot r(\widetilde{\mathfrak{p}) \times_{\mathfrak{h}/W}} \mathfrak{k}(\mathfrak{p})$ by W'. The variety $G \cdot r(\widetilde{\mathfrak{p}) \times_{\mathfrak{h}/W}} \mathfrak{k}(\mathfrak{p})$ has a resolution $G \times^P r(\mathfrak{p})$ whose canonical line bundle is trivial (see [Na4, Lemma 1.2]). In particular, $G \cdot r(\widetilde{\mathfrak{p}) \times_{\mathfrak{h}/W}} \mathfrak{k}(\mathfrak{p})$ has only rational singularities (cf. [Na4, Lemma 1.2]). Hence its quotient variety $\widetilde{G \cdot r(\mathfrak{p})}$ also has rational singularities. In particular, $\widetilde{G \cdot r(\mathfrak{p})}$ is Cohen-Macaulay.

LEMMA 2.1
The central fiber F of $G \cdot r(\widetilde{\mathfrak{p}) \times_{\mathfrak{h}/W}} \mathfrak{k}(\mathfrak{p}) \to \mathfrak{k}(\mathfrak{p})$ is isomorphic to $\tilde{O}$.

Proof
Since $G \cdot r(\widetilde{\mathfrak{p}) \times_{\mathfrak{h}/W}} \mathfrak{k}(\mathfrak{p})$ is Cohen-Macaulay and $\mathfrak{k}(\mathfrak{p})$ is smooth, the central fiber F is also Cohen-Macaulay. On the other hand, let us consider the birational map $G \times^P r(\mathfrak{p}) \to G \cdot r(\widetilde{\mathfrak{p}) \times_{\mathfrak{h}/W}} \mathfrak{k}(\mathfrak{p})$ and take their central fibers to get a map $T^*(G/P) \to F_{\mathrm{red}}$ with connected fibers. Since the Springer map is generically finite, this map is birational by Zariski's main theorem. Moreover, it is an isomorphism outside a certain codimension 2 subset Z of F_{red}. Take a point $x \in F_{\mathrm{red}} - Z$. Then we have a surjection

$$\mathcal{O}_{F,x} \to \mathcal{O}_{F_{\mathrm{red}},x} \cong \mathcal{O}_{T^*(G/P),x}.$$

By the lemma of Nakayama, this implies that $\mathcal{O}_{G\cdot \widetilde{r(\mathfrak{p})\times_{\mathfrak{h}/W}}\mathfrak{k}(\mathfrak{p}),x} \cong \mathcal{O}_{G\times^P r(\mathfrak{p}),x}$. Therefore, F is reduced at x, and moreover, F is smooth at x. Since F is Cohen-Macaulay and regular in codimension one, F is normal. This means that $F = \tilde{O}$. $\square$

In the remainder, we always assume the following.

ASSUMPTION

The Springer map $s : T^*(G/P) \to \bar{O}$ is birational.

We often write $X_{\mathfrak{p}}$ for $G \times^P r(\mathfrak{p})$ and $Y_{\mathfrak{k}(\mathfrak{p})}$ for $G \cdot \widetilde{r(\mathfrak{p}) \times_{\mathfrak{h}/W}} \mathfrak{k}(\mathfrak{p})$. Denote by $\mu_{\mathfrak{p}}$ the birational map from $X_{\mathfrak{p}}$ to $Y_{\mathfrak{k}(\mathfrak{p})}$. The map $\mu_{\mathfrak{p}}$ is a crepant resolution of $Y_{\mathfrak{k}(\mathfrak{p})}$, which is an isomorphism in codimension one (see [Na4, Theorem 1.3]). Let $X_{\mathfrak{p},0}$ (resp., $Y_{\mathfrak{k}(\mathfrak{p}),0}$) be the central fiber of $X_{\mathfrak{p}} \to \mathfrak{k}(\mathfrak{p})$ (resp., $Y_{\mathfrak{k}(\mathfrak{p})} \to \mathfrak{k}(\mathfrak{p})$). Note that $X_{\mathfrak{p},0} = T^*(G/P)$; and $Y_{\mathfrak{k}(\mathfrak{p}),0} = \tilde{O}$ by Lemma 2.1. The birational map $\mu_{\mathfrak{p},0} : X_{\mathfrak{p},0} \to Y_{\mathfrak{k}(\mathfrak{p}),0}$ coincides with the Stein factorization of the Springer map $s : T^*(G/P) \to \bar{O}$.

We briefly review [Na4, Section 1(P2), (P3)]. Let $\mathcal{S}(l(\mathfrak{p}))$ be the set of parabolic subalgebras $\mathfrak{p}'$ which contain $l(\mathfrak{p})$ as Levi subalgebras. Then every crepant resolution of $Y_{\mathfrak{k}(\mathfrak{p})}$ is isomorphic to $\mu_{\mathfrak{p}'} : X_{\mathfrak{p}'} \to Y_{\mathfrak{k}(\mathfrak{p})}$ with $\mathfrak{p}' \in \mathcal{S}(l(\mathfrak{p}))$ (see [Na4, Theorem 1.3]). Let $M(L) := \mathrm{Hom}_{\mathrm{alg.gp.}}(L, \mathbf{C}^*)$. The second cohomology group $H^2(X_{\mathfrak{p}'}, \mathbf{R})$ is isomorphic to $M(L) \otimes \mathbf{R}$. By this isomorphism, the nef cone $\overline{\mathrm{Amp}}(\mu_{\mathfrak{p}'})$ is regarded as a cone in $M(L) \otimes \mathbf{R}$. The cohomology group $H^2(X_{\mathfrak{p}',0}, \mathbf{R})$ is also isomorphic to $M(L) \otimes \mathbf{R}$. By this isomorphism, the nef cone $\overline{\mathrm{Amp}}(\mu_{\mathfrak{p}',0})$ is regarded as a cone in $M(L) \otimes \mathbf{R}$. Note that $\overline{\mathrm{Amp}}(\mu_{\mathfrak{p}'}) = \overline{\mathrm{Amp}}(\mu_{\mathfrak{p}',0})$. One has the following (see [Na4, Remark 1.6]):

$$M(L) \otimes \mathbf{R} = \bigcup_{\mathfrak{p}' \in \mathcal{S}(l(\mathfrak{p}))} \overline{\mathrm{Amp}}(\mu_{\mathfrak{p}'}).$$

We say that two nef cones $\overline{\mathrm{Amp}}(\mu_{\mathfrak{p}'})$ and $\overline{\mathrm{Amp}}(\mu_{\mathfrak{p}''})$ are adjacent to each other if they share a common codimension one face. In this case, we also say that $\mathfrak{p}'$ and $\mathfrak{p}''$ are adjacent to each other. If $\mathfrak{p}'$ and $\mathfrak{p}''$ are adjacent to each other, $\mathfrak{p}'$ and $\mathfrak{p}''$ are related by an operation called the *twist*. There are two kinds of twists, a twist of the first kind and a twist of the second kind. If $\mathfrak{p}'$ and $\mathfrak{p}''$ are adjacent to each other, $X_{\mathfrak{p}'}$ and $X_{\mathfrak{p}''}$ are connected by a flop (cf. [Na4, Section 1]). If the corresponding twist is of the first kind (resp., second kind), we say that the flop is of the first kind (resp., second kind). Let $\mathcal{S}^1(l(\mathfrak{p}))$ be the set of parabolic subalgebras in $\mathcal{S}(l(\mathfrak{p}))$ which can be obtained from $\mathfrak{p}$ by a finite succession of twists of the first kind. Then $\bigcup_{\mathfrak{p}' \in \mathcal{S}^1(l(\mathfrak{p}))} \overline{\mathrm{Amp}}(\mu_{\mathfrak{p}'})$ coincides with the movable cone $\overline{\mathrm{Mov}}(\mu_{\mathfrak{p},0})$. This movable cone $\overline{\mathrm{Mov}}(\mu_{\mathfrak{p},0})$ is a fundamental domain of $M(L) \otimes \mathbf{R}$ by the action of W'. In particular,

$$M(L) \otimes \mathbf{R} = \bigcup_{w \in W'} w(\overline{\mathrm{Mov}}(\mu_{\mathfrak{p},0})).$$

Moreover, $w(\overline{\mathrm{Mov}}(\mu_{\mathfrak{p},0}))$ coincides with the movable cone $\overline{\mathrm{Mov}}(\mu_{w(\mathfrak{p}),0})$. One has

$$\overline{\mathrm{Mov}}(\mu_{w(\mathfrak{p}),0}) = \bigcup_{\mathfrak{p}' \in \mathcal{S}^1(l(\mathfrak{p}))} \overline{\mathrm{Amp}}(\mu_{w(\mathfrak{p}'),0}).$$

By [Ho], W' is almost a reflection group. But W' turns out to be a reflection group under Assumption.

LEMMA 2.2

The group W' is generated by reflections of $\mathfrak{k}(\mathfrak{p})$. In particular, $\mathfrak{k}(p)/W'$ is smooth.

Proof

Assume that $\mathfrak{p}$ and $\mathfrak{p}'$ are adjacent to each other and that they are related by a second twist. Then $\mathfrak{p}' = w(\mathfrak{p})$ for some $w \in W'$. Let $\phi_w : X_{\mathfrak{p}} \to X_{w(\mathfrak{p})}$ be the isomorphism defined by $[g, x] \to [gw^{-1}, \mathrm{Ad}_w(x)]$ for $[g, x] \in G \times^P r(\mathfrak{p})$. Let $\bar{\phi}_w : Y_{\mathfrak{k}(\mathfrak{p})} \to Y_{\mathfrak{k}(\mathfrak{p})}$ be the automorphism induced by the map $id \times w : G \cdot r(\mathfrak{p}) \times_{\mathfrak{h}/W} \mathfrak{k}(\mathfrak{p}) \to G \cdot r(\mathfrak{p}) \times_{\mathfrak{h}/W} \mathfrak{k}(\mathfrak{p})$. Then we have a commutative diagram

$$\begin{array}{ccc} X_{\mathfrak{p}} & \xrightarrow{\phi_w} & X_{w(\mathfrak{p})} \\ \downarrow & & \downarrow \\ Y_{\mathfrak{k}(\mathfrak{p})} & \xrightarrow{\bar{\phi}_w} & Y_{\mathfrak{k}(\mathfrak{p})} \end{array} \tag{26}$$

The composite $X_{\mathfrak{p}} \to X_{w(\mathfrak{p})} - - \to X_{\mathfrak{p}}$ induces an automorphism of $H^2(X_{\mathfrak{p}}, \mathbf{R})$. We call this automorphism φ_w. In this way, W' acts on $H^2(X_{\mathfrak{p}}, \mathbf{R})$. Note that its dual action coincides with the natural action of W' on $\mathfrak{k}(\mathfrak{p})$ by [Na4, Lemma 2.1]. We prove that φ_w is a reflection, that is, an involution that fixes all points in a certain hyperplane. The flop $X_{w(\mathfrak{p})} - - \to X_{\mathfrak{p}}$ can be expressed more exactly by the diagram

$$X_{w(\mathfrak{p})} \to G \times^{\bar{P}} \widetilde{\bar{P} \cdot r(\mathfrak{p})} \times_{\mathfrak{k}(l(\bar{\mathfrak{p}}) \cap \mathfrak{p})/W''} \mathfrak{k}\big(l(\bar{\mathfrak{p}}) \cap \mathfrak{p}\big) \longleftarrow X_{\mathfrak{p}}.$$

Here W'' is the subgroup of the Weyl group $W(l(\bar{\mathfrak{p}}))$ which stabilizes $\mathfrak{k}(l(\bar{\mathfrak{p}}) \cap \mathfrak{p})$ as a set. The element w is contained in W''; hence, it acts on $G \times^{\bar{P}} \widetilde{\bar{P} \cdot r(\mathfrak{p})} \times_{\mathfrak{k}(l(\bar{\mathfrak{p}}) \cap \mathfrak{p})/W''} \mathfrak{k}(l(\bar{\mathfrak{p}}) \cap \mathfrak{p})$ by $id \times w$. Put

$$Z_{\bar{\mathfrak{p}}} := G \times^{\bar{P}} \widetilde{\bar{P} \cdot r(\mathfrak{p})} \times_{\mathfrak{k}(l(\bar{\mathfrak{p}}) \cap \mathfrak{p})/W''} \mathfrak{k}\big(l(\bar{\mathfrak{p}}) \cap \mathfrak{p}\big)$$

for short.

Then we have a commutative diagram

$$\begin{array}{ccc} X_{\mathfrak{p}} \overset{\phi_w}{\to} & X_{w(\mathfrak{p})} & - - \to X_{\mathfrak{p}} \\ \downarrow & \downarrow & \\ Z_{\bar{\mathfrak{p}}} \overset{id \times w}{\to} & Z_{\bar{\mathfrak{p}}} & \\ \downarrow & \downarrow & \\ Y_{\mathfrak{k}(\mathfrak{p})} \to & Y_{\mathfrak{k}(\mathfrak{p})} & \end{array}$$

The automorphism $Z_{\bar{\mathfrak{p}}} \stackrel{id\times w}{\to} Z_{\bar{\mathfrak{p}}}$ induces the identity map on $H^2(Z_{\bar{\mathfrak{p}}}, \mathbf{R})$. Since the image of the map $H^2(Z_{\bar{\mathfrak{p}}}, \mathbf{R}) \to H^2(X_{\mathfrak{p}}, \mathbf{R})$ has codimension one, the automorphism φ_w of $H^2(X_{\mathfrak{p}}, \mathbf{R})$ is a reflection. By [Na4, Proposition 2.3],

$$H^2(X_{\mathfrak{p}}, \mathbf{R}) = \bigcup_{w \in W'} w\big(\overline{\mathrm{Mov}}(\mu_{\mathfrak{p},0})\big).$$

Note that $w(\overline{\mathrm{Mov}}(\mu_{\mathfrak{p},0})) = \overline{\mathrm{Mov}}(\mu_{w(\mathfrak{p}),0})$. If $w(\overline{\mathrm{Mov}}(\mu_{\mathfrak{p},0}))$ and $w'(\overline{\mathrm{Mov}}(\mu_{\mathfrak{p},0}))$ are adjacent to each other, then $w^{-1}w'$ is a reflection. For any $w \in W'$, one can connect $w(\overline{\mathrm{Mov}}(\mu_{\mathfrak{p},0}))$ and $\overline{\mathrm{Mov}}(\mu_{\mathfrak{p},0})$ by a finite sequence of movable cones $w_1(\overline{\mathrm{Mov}}(\mu_{\mathfrak{p},0}))$, $w_2(\overline{\mathrm{Mov}}(\mu_{\mathfrak{p},0})), \ldots, w_n(\overline{\mathrm{Mov}}(\mu_{\mathfrak{p},0}))$ with $w_1 = 1$ and $w_n = w$ in such a way that $w_i(\overline{\mathrm{Mov}}(\mu_{\mathfrak{p},0}))$ and $w_{i+1}(\overline{\mathrm{Mov}}(\mu_{\mathfrak{p},0}))$ are adjacent for all i. Then w can be represented as a product of reflections: $w = (w_n \cdot w_{n-1}^{-1}) \cdots (w_2 \cdot w_1^{-1})$. $\square$

REMARK

When the Springer map $s : T^*(G/P) \to \bar{O}$ is not birational, Lemma 2.2 does not hold (see [Na4, Example 1.9, Remark 2.4]).

COROLLARY 2.3

$\widetilde{G \cdot r(\mathfrak{p})}$ is flat over $\mathfrak{k}(\mathfrak{p})/W'$.

Proof

First, note that $\widetilde{G \cdot r(\mathfrak{p})}$ is Cohen-Macaulay. Since every fiber of the map $\widetilde{G \cdot r(\mathfrak{p})} \to \mathfrak{k}(\mathfrak{p})/W'$ has the dimension equal to $\dim \widetilde{G \cdot r(\mathfrak{p})} - \dim \mathfrak{k}(\mathfrak{p})/W'$ and $\mathfrak{k}(\mathfrak{p})/W'$ is smooth, the map $\widetilde{G \cdot r(\mathfrak{p})} \to \mathfrak{k}(\mathfrak{p})/W'$ is flat. $\square$

LEMMA 2.4

$\widetilde{G \cdot r(\mathfrak{p})} \times_{\mathfrak{k}(\mathfrak{p})/W'} \mathfrak{k}(\mathfrak{p})$ is a variety.

Proof

Let B be the affine ring of $\widetilde{G \cdot r(\mathfrak{p})}$, and let A (resp., A') be the affine ring of $\mathfrak{k}(\mathfrak{p})/W'$ (resp., $\mathfrak{k}(\mathfrak{p})$). Denote by L the quotient field of B and by K (resp., K') the quotient field of A (resp., A'). Since B is flat over A, $B \otimes_A A' \subset B \otimes_A K' = B \otimes_A K \otimes_K K'$. Since $B \otimes_A K$ is the localization of B by the multiplicative set $S := A - \{0\}$, it is naturally contained in L. Therefore, $B \otimes_A K \otimes_K K' \subset L \otimes_K K'$. Since K' is a separable extension of K, $L \otimes_K K'$ is reduced. Since $G \cdot r(\mathfrak{p}) \times_{\mathfrak{h}/W} \mathfrak{k}(\mathfrak{p})$ is irreducible, we see that $L \otimes_K K'$ is an integral domain. Finally, we conclude that $B \otimes_A A'$ is an integral domain because $B \otimes_A A' \subset L \otimes_K K'$. $\square$

LEMMA 2.5

We have

$$\widetilde{G \cdot r(\mathfrak{p})} \times_{\mathfrak{k}(\mathfrak{p})/W'} \mathfrak{k}(\mathfrak{p}) \cong G \cdot \widetilde{r(\mathfrak{p}) \times_{\mathfrak{h}/W}} \mathfrak{k}(\mathfrak{p}).$$

The map $\widetilde{G\cdot r(\mathfrak{p})} \to \mathfrak{k}(\mathfrak{p})/W'$ is flat, and its central fiber coincides with $\tilde{O}$.

Proof
Let A, A', and B be the same as in the proof of Lemma 2.4. Consider the map $G\cdot r(\mathfrak{p}) \to \mathfrak{h}/W$. If $\bar{B}$ and C are affine rings of $G\cdot r(\mathfrak{p})$ and $\mathfrak{h}/W$, respectively, then $\bar{B}$ is a C-algebra. The origin $0\in\mathfrak{h}/W$ corresponds to a maximal ideal $\bar{m}$ of C. Let B' be the affine ring of $\widetilde{G\cdot r(\mathfrak{p})\times_{\mathfrak{h}/W}} \mathfrak{k}(\mathfrak{p})$. By Lemma 2.4, we already know that $B\otimes_A A'$ is an integral domain. We have an injection $B\otimes_A A' \to B'$. Let m' (resp., m) be the maximal ideal of A' (resp., A) corresponding to the origin $0\in\mathfrak{k}(\mathfrak{p})$ (resp., $0\in\mathfrak{k}(\mathfrak{p})/W'$). By the base change property, $B\otimes_A (A'/m') = B/mB$. Then the injection above induces a homomorphism $B/mB \to B'/m'B'$. Moreover, there is a map $\bar{B}/\bar{m}\bar{B} \to B/mB$. By the definition, $\mathrm{Spec}(\bar{B}/\bar{m}\bar{B})_{\mathrm{red}} = \bar{O}$. In our case, $\mathrm{Spec}(B'/m'B') = \tilde{O}$ is the normalization of $\bar{O}$. Note that the normalization map $\tilde{O}\to\bar{O}$ is an isomorphism in codimension one. Therefore, the cokernel of the map $\bar{B}/\bar{m}\bar{B} \to B'/m'B'$ has the support with codimension ≥ 2 in $\tilde{O}$. The cokernel Q of the map $B/mB \to B'/m'B'$ also has the support with codimension ≥ 2 in $\tilde{O}$. Take a point $\mathfrak{q}\in\tilde{O} - \mathrm{Supp}(Q)$. Then, by the lemma of Nakayama, we have an isomorphism $(B\otimes_A A')_{\mathfrak{q}} \cong B'_{\mathfrak{q}}$. In particular, $\mathrm{Spec}(B/mB)$ and $\mathrm{Spec}(B'/mB')$ are isomorphic in codimension one; hence, we see that $\mathrm{Spec}(B/mB)$ is regular in codimension one. On the other hand, by Corollary 2.3, B is Cohen-Macaulay and is flat over A. Therefore, $\mathrm{Spec}(B/mB)$ is Cohen-Macaulay. This means that $\mathrm{Spec}(B/mB)$ is normal; hence, $\mathrm{Spec}(B/mB) = \tilde{O}$. □

REMARK
When the Springer map $s: T^*(G/P)\to\bar{O}$ is not birational, the central fiber of the map $G\cdot r(\mathfrak{p})\to\mathfrak{h}/W$ is everywhere nonreduced.

PROPOSITION 2.6
Two flat morphisms

$$G\times^P r(\mathfrak{p}) \to \mathfrak{k}(\mathfrak{p})$$

and

$$\widetilde{G\cdot r(\mathfrak{p})} \to \mathfrak{k}(\mathfrak{p})/W'$$

are, respectively, Poisson deformations of $T^*(G/P)$ and $\tilde{O}$.

Proof
The smooth variety $G\times^P r(\mathfrak{p})$ over $\mathfrak{k}(\mathfrak{p})$ admits a G-invariant relative symplectic 2-form ω (see [CG, Proposition 1.4.14]).† Let ω_t be the restriction of ω to the

†In [CG], coajoint orbits of $\mathfrak{g}^*$ are treated. But the Killing form of $\mathfrak{g}$ identifies the coadjoint orbits with adjoint orbits. The variety $G\times_P(\lambda+\mathfrak{p}^\perp)$ is identified with a fiber of $G\times^P r(\mathfrak{p})\to \mathfrak{k}(\mathfrak{p})$ by the Killing form.

fiber $G \times^P (t + n(\mathfrak{p}))$ over $t \in \mathfrak{k}(\mathfrak{p})$. There is a G-equivariant map

$$\mu_t : G \times^P \big(t + n(\mathfrak{p})\big) \to \mathfrak{g}$$

defined by $\mu_t([g, t+x]) = \mathrm{Ad}_g(t+x)$. The image $\mathrm{Im}(\mu_t)$ coincides with the closure $\bar{O}_t$ of an adjoint orbit O_t. By [CG, Proposition 1.4.14(2)], ω_t is the pullback of the Kostant-Kirillov form on O_t by μ_t. Denote by $(\widetilde{G \cdot r(\mathfrak{p}) \times_{\mathfrak{h}/W}} \mathfrak{k}(\mathfrak{p}))_{\mathrm{reg}}$ the smooth part of $\widetilde{G \cdot r(\mathfrak{p}) \times_{\mathfrak{h}/W}} \mathfrak{k}(\mathfrak{p})$. There is a crepant resolution

$$G \times^P r(\mathfrak{p}) \to \widetilde{G \cdot r(\mathfrak{p}) \times_{\mathfrak{h}/W}} \mathfrak{k}(\mathfrak{p})$$

which does not change the smooth locus. Then ω determines a relative symplectic 2-form $\bar{\omega}$ of $(\widetilde{G \cdot r(\mathfrak{p}) \times_{\mathfrak{h}/W}} \mathfrak{k}(\mathfrak{p}))_{\mathrm{reg}} \to \mathfrak{k}(\mathfrak{p})$. Note that

$$\widetilde{G \cdot r(\mathfrak{p}) \times_{\mathfrak{h}/W}} \mathfrak{k}(\mathfrak{p}) \cong \widetilde{G \cdot r(\mathfrak{p})} \times_{\mathfrak{k}(\mathfrak{p})/W'} \mathfrak{k}(\mathfrak{p})$$

by Lemma 2.5. Since W' acts on $\mathfrak{k}(\mathfrak{p})$, it acts on $\widetilde{G \cdot r(\mathfrak{p})} \times_{\mathfrak{k}(\mathfrak{p})/W'} \mathfrak{k}(\mathfrak{p})$ in a natural manner. Let us check that $\bar{\omega}$ is W'-invariant. Take a general point $t \in \mathfrak{k}(\mathfrak{p})$ (or more precisely, take $t \in \mathfrak{k}(\mathfrak{p})^{\mathrm{reg}}$ in the notation of [Na4, Section 1, (P1)]). Then the fiber of the map

$$\bar{f} : \widetilde{G \cdot r(\mathfrak{p})} \times_{\mathfrak{k}(\mathfrak{p})/W'} \mathfrak{k}(\mathfrak{p}) \to \mathfrak{k}(\mathfrak{p})$$

over t is the semisimple (adjoint) orbit $G \cdot t$ (cf. [Na4, Lemma 1.1]). Then $\bar{\omega}_t$ coincides with the Kostant-Kirillov form on $G \cdot t$. If $w \in W'$, then $w(t) \in \mathfrak{k}(\mathfrak{p})^{\mathrm{reg}}$ and the fiber $\bar{f}^{-1}(w(t))$ is $G \cdot w(t)$. Moreover, $\bar{\omega}_{w(t)}$ coincides with the Kostant-Kirillov form on $G \cdot w(t)$. Since $G \cdot t = G \cdot w(t)$, we see that $\bar{\omega}_t$ and $\bar{\omega}_{w(t)}$ coincide. This argument shows that $\bar{\omega}$ is W'-invariant on a Zariski open subset of $(\widetilde{G \cdot r(\mathfrak{p})} \times_{\mathfrak{k}(\mathfrak{p})/W'} \mathfrak{k}(\mathfrak{p}))_{\mathrm{reg}}$. Therefore, $\bar{\omega}$ is W'-invariant. The relative 2-form $\bar{\omega}$ descends to a relative symplectic 2-form of $(\widetilde{G \cdot r(\mathfrak{p})})_{\mathrm{reg}} \to \mathfrak{k}(\mathfrak{p})/W'$, which determines a Poisson structure of $(\widetilde{G \cdot r(\mathfrak{p})})_{\mathrm{reg}}$ over $\mathfrak{k}(\mathfrak{p})/W'$. This Poisson structure uniquely extends to that of $\widetilde{G \cdot r(\mathfrak{p})}$. $\square$

PROPOSITION 2.7

The Poisson deformation $f : G \times^P r(\mathfrak{p}) \to \mathfrak{k}(\mathfrak{p})$ *of* $T^*(G/P)$ *is universal at* $0 \in \mathfrak{k}(\mathfrak{p})$.

Proof

For $\lambda \in \mathfrak{k}(\mathfrak{p})$, the vector space $\mathbf{C}\lambda \oplus n(\mathfrak{p})$ becomes a P-module by the adjoint action. Thus one can define a vector bundle $G \times^P (\mathbf{C}\lambda \oplus n(\mathfrak{p}))$ over G/P. This vector bundle fits into the exact sequence

$$0 \to G \times^P n(\mathfrak{p}) \to G \times^P \big(\mathbf{C}\lambda \oplus n(\mathfrak{p})\big) \to G/P \times \mathbf{C}\lambda \to 0.$$

Since $G \times^P n(\mathfrak{p}) \cong T^*(G/P)$, this extension gives an element $e(\lambda) \in H^1(G/P, \Omega^1_{G/P})$. Assume that $b_2(G/P) = n$, or equivalently, assume that $\dim \mathfrak{k}(\mathfrak{p}) = n$. Then one can find n maximal parabolic subgroups Q_i $(1 \le i \le n)$ such that $P \subset Q_i$ and each projection map $\pi_i : G/P \to G/Q_i$ determines an extremal ray of the nef cone $\overline{\mathrm{Amp}}(G/P)$, which is a simplicial polyhedral cone of dimension n. Note that each Q_i is the maximal parabolic subgroup associated to a vertex in the set of marked vertices corresponding to P (cf. [Na4, Section 1(P3)]). Let $\lambda_i \in \mathfrak{k}(\mathfrak{q}_i) \subset \mathfrak{k}(\mathfrak{p})$ be a nonzero element. Note that, since $\mathfrak{k}(\mathfrak{q}_i)$ is one-dimensional, λ_i is unique up to constant. Then one has an exact sequence of vector bundles on G/Q_i:

$$0 \to G \times^{Q_i} n(\mathfrak{q}_i) \to G \times^{Q_i} \big(\mathbf{C}\lambda \oplus n(\mathfrak{q}_i)\big) \to G/Q_i \times \mathbf{C}\lambda_i \to 0.$$

Let $f_i : G \times^{Q_i} (\mathbf{C}\lambda_i \oplus n(\mathfrak{q}_i)) \to \mathfrak{k}(\mathfrak{q}_i)$ be the map defined by $f_i([g, t\lambda_i + x]) := t\lambda_i$ with $x \in n(\mathfrak{q}_i)$. The fiber $f_i^{-1}(0) = T^*(G/Q_i)$ is not an affine variety, but $f_i^{-1}(\lambda_i)$ is an affine variety. Thus the exact sequence above does not split, and its extension class $e(\lambda_i) \in H^1(G/Q_i, \Omega^1_{G/Q_i})$ is not zero. The exact sequence is pulled back to the exact sequence of vector bundles on G/P:

$$0 \to G \times^P n(\mathfrak{q}_i) \to G \times^P \big(\mathbf{C}\lambda \oplus n(q_i)\big) \to G/P \times \mathbf{C}\lambda_i \to 0.$$

By the natural injection $G \times^P n(\mathfrak{q}_i) \to G \times^P n(\mathfrak{p})$, one obtains an exact sequence

$$0 \to G \times^P n(\mathfrak{p}) \to G \times^P \big(\mathbf{C}\lambda_i \oplus n(\mathfrak{p})\big) \to G/P \times \mathbf{C}\lambda_i \to 0.$$

The extension class of this exact sequence is the image of $e(\lambda_i) \in H^1(G/Q_i, \Omega^1_{G/Q_i})$ by the map $H^1(G/Q_i, \Omega^1_{G/Q_i}) \to H^1(G/P, \Omega^1_{G/P})$. Note that $\{e(\lambda_i)\}$ is a basis of $H^1(G/P, \Omega^1_{G/P})$. Therefore, the exact sequence

$$0 \to G \times^P n(\mathfrak{p}) \to G \times^P r(\mathfrak{p}) \to G/P \times \mathfrak{k}(\mathfrak{p}) \to 0$$

is the universal extension of $G \times^P n(\mathfrak{p})$ by the trivial line bundle. Let

$$0 \to \Omega^1_{G/P} \to \mathcal{E} \to \mathcal{O}^n_{G/P} \to 0$$

be the corresponding exact sequence of the sheaves. Let $p : T^*(G/P)\ (= G \times^P n(\mathfrak{p})) \to G/P$ be the canonical projection. Then we have a commutative diagram of exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & p^*\Omega^1_{G/P} & \longrightarrow & p^*\mathcal{E} & \longrightarrow & p^*\mathcal{O}^n_{G/P} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Theta_{T^*(G/P)} & \longrightarrow & \Theta_{G\times^P r(\mathfrak{p})}|_{T^*(G/P)} & \longrightarrow & N_{T^*(G/P)/G\times^P r(\mathfrak{p})} & \longrightarrow & 0 \end{array} \tag{27}$$

Identify $\mathfrak{k}(\mathfrak{p})$ with its tangent space at zero. The Kodaira-Spencer map θ_f of f is given by the composite

$$\mathfrak{k}(\mathfrak{p}) \to H^0\big(T^*(G/P), N_{T^*(G/P)/G\times^P r(\mathfrak{p})}\big) \to H^1\big(T^*(G/P), \Theta_{T^*(G/P)}\big).$$

On the other hand, by the identification of $\mathfrak{k}(\mathfrak{p})$ with $H^0(G/P, \mathcal{O}^n_{G/P})$, one has a map

$$\mathfrak{k}(\mathfrak{p}) \cong H^0(G/P, \mathcal{O}^n_{G/P}) \to H^1(G/P, \Omega^1_{G/P}).$$

By the construction, the Kodaira-Spencer map is factored by this map:

$$\mathfrak{k}(\mathfrak{p}) \to H^1(G/P, \Omega^1_{G/P}) \to H^1\big(T^*(G/P), \Theta_{T^*(G/P)}\big).$$

The first map is an isomorphism by the definition of $\mathcal{E}$. The second map is an injection. In fact, let $S \subset T^*(G/P)$ be the zero section. Then $N_{S/T^*(G/P)} \cong \Omega^1_S$, and the composite $H^1(G/P, \Omega^1_{G/P}) \to H^1(T^*(G/P), \Theta_{T^*(G/P)}) \to H^1(S, \Omega^1_S)$ is an isomorphism. Therefore, the Kodaira-Spencer map θ_f is an injection. Since f is a Poisson deformation of $T^*(G/P)$, the Kodaira-Spencer map θ_f is factored by the *Poisson Kodaira-Spencer map* θ^P_f:

$$\mathfrak{k}(\mathfrak{p}) \xrightarrow{\theta^P_f} H^2\big(T^*(G/P), \mathbf{C}\big) \to H^1\big(T^*(G/P), \Omega^1_{T^*(G/P)}\big).$$

Hence θ^P_f is also injective. Since $\dim \mathfrak{k}(\mathfrak{p}) = h^2(T^*(G/P), \mathbf{C}) = n$, θ^P_f is actually an isomorphism. □

THEOREM 2.8

The Poisson deformation $\widetilde{G \cdot r(\mathfrak{p})} \to \mathfrak{k}(\mathfrak{p})/W'$ *is universal at* $0 \in \mathfrak{k}(\mathfrak{p})/W'$.

Proof

By Proposition 2.7, $G \times^P r(\mathfrak{p}) \to \mathfrak{k}(\mathfrak{p})$ is the universal Poisson deformation of $T^*(G/P)$ around $0 \in \mathfrak{k}(\mathfrak{p})$. Since $\tilde{O}$ has a $\mathbf{C}^*$-action with positive weight, the universal Poisson deformation of $\tilde{O}$ is algebraized to a $\mathbf{C}^*$-equivariant map $\mathcal{Y} \to \mathbf{A}^d$ with $\mathcal{Y}_0 = \tilde{O}$. There is a $\mathbf{C}^*$-equivariant commutative diagram

$$\begin{array}{ccc} G \times^P r(\mathfrak{p}) & \longrightarrow & \mathcal{Y} \\ \downarrow & & \downarrow \\ \mathfrak{k}(\mathfrak{p}) & \xrightarrow{\pi} & \mathbf{A}^d \end{array} \tag{28}$$

If $t \in \mathfrak{k}(\mathfrak{p})$ is general, then the induced map $G \times^P (t + n(\mathfrak{p})) \to \mathcal{Y}_{\pi(t)}$ is an isomorphism. By the main theorem (Theorem 1.1), $\mathfrak{k}(\mathfrak{p}) \to \mathbf{A}^d$ is a finite Galois map. Denote by H its Galois group. By Proposition 2.6, we have seen that $\widetilde{G \cdot r(\mathfrak{p})} \to \mathfrak{k}(\mathfrak{p})/W'$ is a Poisson deformation of $\tilde{O}$. By the (formal) universality of $\mathcal{Y} \to \mathbf{A}^d$ at zero, there is a formal map $\widehat{\mathfrak{k}(\mathfrak{p})/W'} \to \hat{\mathbf{A}}^d$. Since $\widetilde{G \cdot r(\mathfrak{p})}$ has a $\mathbf{C}^*$-action and the Poisson deformation $\widetilde{G \cdot r(\mathfrak{p})} \to \mathfrak{k}(\mathfrak{p})/W'$ is $\mathbf{C}^*$-equivariant, the formal map is also $\mathbf{C}^*$-equivariant. Then the formal map determines a map $\mathfrak{k}(\mathfrak{p})/W' \to \mathbf{A}^d$ (cf. [Na5, Section 7]). The map $\mathfrak{k}(\mathfrak{p}) \to \mathbf{A}^d$ in the commutative diagram factorizes as

$$\mathfrak{k}(\mathfrak{p}) \to \mathfrak{k}(\mathfrak{p})/W' \to \mathbf{A}^d.$$

In fact,

$$\operatorname{Spec}\Gamma\big(G \times^P r(\mathfrak{p}), \mathcal{O}_{G \times^P r(\mathfrak{p})}\big) \to \mathfrak{k}(\mathfrak{p})$$

is a Poisson deformation of $\tilde{O}$, and it coincides with the pullback of $\mathcal{Y} \to \mathbf{A}^d$ by π. But

$$\operatorname{Spec}\Gamma\big(G \times^P r(\mathfrak{p}), \mathcal{O}_{G \times^P r(\mathfrak{p})}\big) = \widetilde{G \cdot r(\mathfrak{p})} \times_{\mathfrak{k}(\mathfrak{p})/W'} \mathfrak{k}(\mathfrak{p})$$

by Lemma 2.5. As a result, we have a commutative diagram of Poisson deformations of $\tilde{O}$:

$$
\begin{array}{ccccc}
\operatorname{Spec}\Gamma(G\times^P r(\mathfrak{p}), \mathcal{O}_{G\times^P r(\mathfrak{p})}) & \longrightarrow & \widetilde{G\cdot r(\mathfrak{p})} & \longrightarrow & \mathcal{Y} \\
\downarrow & & \downarrow & & \downarrow \\
\mathfrak{k}(\mathfrak{p}) & \longrightarrow & \mathfrak{k}(\mathfrak{p})/W' & \longrightarrow & \mathbf{A}^d
\end{array}
\tag{29}
$$

We prove that $H = W'$. We only have to show that $H \subset W'$. An element $h \in H$ induces an automorphism of $\mathfrak{k}(\mathfrak{p})$ over $\mathbf{A}^d$. Denote this automorphism also by h. Let us consider the automorphism of $\mathcal{Y}\times_{\mathbf{A}^d}\mathfrak{k}(\mathfrak{p})$ defined by $id\times h$. It induces a birational automorphism $G\times^P r(\mathfrak{p}) --\to G\times^P r(\mathfrak{p})$ and a commutative diagram

$$
\begin{array}{ccc}
G\times^P r(\mathfrak{p}) & --\to & G\times^P r(\mathfrak{p}) \\
\downarrow & & \downarrow \\
\mathcal{Y}\times_{\mathbf{A}^d}\mathfrak{k}(\mathfrak{p}) & \to & \mathcal{Y}\times_{\mathbf{A}^d}\mathfrak{k}(\mathfrak{p})
\end{array}
$$

Let $\mathcal{X}$ be the fiber product of the diagram

$$\mathcal{Y}\times_{\mathbf{A}^d}\mathfrak{k}(\mathfrak{p}) \overset{id\times h}{\to} \mathcal{Y}\times_{\mathbf{A}^d}\mathfrak{k}(\mathfrak{p}) \leftarrow G\times^P r(\mathfrak{p}).$$

Then there is a birational map (over $\mathcal{Y}\times_{\mathbf{A}^d}\mathfrak{k}(\mathfrak{p})$)

$$G\times^P r(\mathfrak{p}) --\to \mathcal{X},$$

which is an isomorphism in codimension one. Since $\mathcal{Y}\times_{\mathbf{A}^d}\mathfrak{k}(\mathfrak{p})$ is affine, its normalization is isomorphic to $Y_{\mathfrak{k}(\mathfrak{p})}$. Denote by $\mu_{\mathfrak{p}}$ (resp., μ) the birational map from $G\times^P r(\mathfrak{p})$ (resp., $\mathcal{X}$) to $Y_{\mathfrak{k}(\mathfrak{p})}$. In $H^2(G\times^P r(\mathfrak{p}), \mathbf{R})$, the movable cone $\mathrm{Mov}(\mu_0)$ coincides with $w(\mathrm{Mov}(\mu_{\mathfrak{p},0}))$ for some $w\in W'$. Let us consider the automorphism of $\mathcal{Y}\times_{\mathbf{A}^d}\mathfrak{k}(\mathfrak{p})$ defined by $id\times h\cdot w^{-1}$. Let $\mathcal{X}'$ be the fiber product of the diagram

$$\mathcal{Y}\times_{\mathbf{A}^d}\mathfrak{k}(\mathfrak{p}) \overset{id\times h\cdot w^{-1}}{\to} \mathcal{Y}\times_{\mathbf{A}^d}\mathfrak{k}(\mathfrak{p}) \leftarrow G\times^P r(\mathfrak{p}).$$

As above, there is a birational map (over $\mathcal{Y}\times_{\mathbf{A}^d}\mathfrak{k}(\mathfrak{p})$; hence, over $Y_{\mathfrak{k}(\mathfrak{p})}$)

$$G\times^P r(\mathfrak{p}) --\to \mathcal{X}'.$$

Denote by μ' the birational map from $\mathcal{X}'$ to $Y_{\mathfrak{k}(\mathfrak{p})}$. Then by the construction, $\mathrm{Mov}(\mu'_0)$ coincides with $\mathrm{Mov}(\mu_{\mathfrak{p},0})$. It follows that $G\times^P r(\mathfrak{p})$ and $\mathcal{X}'$ are connected by a sequence of flops of the first kind:

$$G\times^P r(\mathfrak{p}) --\to G\times^{P_1} r(\mathfrak{p}_1) --\to \cdots --\to G\times^{P_k} r(\mathfrak{p}_k) = \mathcal{X}'.$$

Let $E\subset T^*(G/P)$ be an exceptional divisor of the Springer map $s: T^*(G/P)\to \tilde{O}$. At a general point of E, the flop $G\times^P r(\mathfrak{p}) --\to G\times^{P_1} r(\mathfrak{p}_1)$ is an isomorphism. Let E_1 be the proper transform of E by this flop. At a general point of E_1, the next flop $G\times^{P_1} r(\mathfrak{p}_1) --\to G\times^{P_2} r(\mathfrak{p}_2)$ is also an isomorphism. Similar things happen for all flops of the first kind. If $h\cdot w^{-1}\neq 1$, then, by the main theorem (Theorem 1.1), the indeterminacy locus of the birational map

$G \times^P r(\mathfrak{p}) - - \to \mathcal{X}'$ must contain at least one s-exceptional divisor E (cf. remark below Theorem 1.1). Therefore, $h \cdot w^{-1} = 1$. $\square$

EXAMPLE 2.9
The following are standard examples due to Slodowy [S2]. Let $\mathfrak{g}$ be of type B_m, C_m, F_4, or G_2. Let B be a Borel subgroup of G. Then the Springer map $s : T^*(G/B) \to \mathfrak{g}$ gives a crepant resolution of the nilpotent cone N; namely, $\mathrm{Im}(s) = N$. Note that N is the closure of the regular nilpotent orbit of $\mathfrak{g}$. In this case, $\Sigma - \Sigma_0$ has only one connected component, say, B. Then the surface S_B has a singularity of type A_{2m-1}, D_{m+1}, E_6, or D_4 according as $\mathfrak{g}$ is of type B_m, C_m, F_4, or G_2. Note that W' is the Weyl group $W(G)$ of G. By Theorems 2.8 and 1.1, we see that $W_B = W(G)$. This means that $\mathrm{Exc}(s)$ is a divisor of $T^*(G/B)$ with exactly m (resp., 4, 2) irreducible components when $\mathfrak{g}$ is of type B_m or C_m (resp., F_4, G_2).

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Department of Mathematics, Faculty of Science, Kyoto University, Japan;
namikawa@math.kyoto-u.ac.jp