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ON TWO-MOVE PREDICTION GAMES[1]

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The following class of games is considered: a sequence chooser produces an infinite sequence of 0's and 1's, and a predictor observes the sequence for a finite time, stopping when he pleases and choosing an action from a finite set. The predictor wins an amount depending only on the action chosen and on the first two unobserved terms of the sequence.

The value of such games is determined and the formula obtained is used to give a derivation of the well-known values of the 1–0 game and the two-move lag bomber-battleship game. Values for some generalizations of the latter game are given. Optimal strategies for the sequence chooser are discussed.

1. Introduction. In this paper, we present a method to solve two-move prediction games, that is games in which a predictor observes an infinite sequence of 0's and 1's for a finite length of time, chooses an action, and wins an amount depending only on the action chosen and on the first two unobserved terms of the sequence. In Section 2 we state a general formulation of prediction games and discuss the existence of a value and optimal strategies for the sequence chooser. The main theorem, which gives us a formula to determine the value of a two-move prediction game, is proved in Section 3. In Section 4, we indicate that Blackwell's 1–0 game ([1]), and the two-move lag bomber-battleship game ([2], [3], [4], [6]) fall in the general formulation of two-move prediction games. We illustrate how to solve these games by the method developed in Section 3. It is confirmed that $\frac{1}{4}$ is the value of 1–0 game, and the value of the two-move lag bomber-battleship game is $(-1 + 5^{\frac{1}{2}})/2$. The method can also be used to solve the generalized two-move lag bomber-battleship game where the inaccuracy of the bomber is considered. Some results are given in a remark. Optimal strategies for the sequence chooser are discussed.

2. Description of prediction games. Blackwell in [1] proposed a class of games in which a sequence chooser (player I) and a predictor (player II) are involved. The games can be described as follows: the sequence chooser chooses an infinite sequence of 0's and 1's, and the predictor is allowed to observe this sequence as long as he wishes. At some time the predictor chooses an action from a

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finite set of N elements. A general mathematical formulation of the games was originally given by Blackwell in [1]. We will use his definitions in this study.

Let $S =$ set of all finite sequences $s = (\varepsilon_1, \cdots, \varepsilon_n)$ of 0's and 1's, including the empty sequence θ.
Let $X =$ set of all infinite sequences $x = (\varepsilon_1, \varepsilon_2, \cdots)$ of 0's and 1's.

DEFINITION. $E = (e_1, \cdots, e_m)$ is a *partition* if E is a subset of S and every $x \in X$ has exactly one e_i as an initial segment.

Associated with a pair (E, A), with $E = (e_1, \cdots, e_m)$ a partition and $A = \|A(i, j)\|$ an $m \times N$ real-valued matrix, is a *prediction game*. In this game, a pure strategy for I is a sequence $x \in X$ and a pure strategy for II is a pair $y = (F, g)$, where $F = (f_1, \cdots, f_r)$ is a partition and g is a function which associates with each f_k, an integer $j = g(f_k)$, $1 \leqq j \leqq N$. Player II's pure strategies are denumerable and may be denoted by $y_1, y_2, \cdots$. The payoff to I is $M(x, y) = A(i, g(f_k))$ where i and k are the unique integers such that (f_k, e_i) is an initial segment of x.

A mixed strategy for I is a probability measure P on the σ-field generated by the cylinder sets of X. For every $s \in S$, denote $P\{x\colon s$ is the initial segment of $x\}$ by $P(s)$. A mixed strategy for II is a sequence $Q = (\lambda_1, \lambda_2, \cdots)$ where $\lambda_i \geqq 0$ and $\sum \lambda_i = 1$. The payoff to I when I uses P and II uses Q is:

$$M(P, Q) = \textstyle\sum_i \lambda_i M(P, y_i)$$

where

$$M(P, y) = \textstyle\sum_{k,i} P(f_k, e_i) A(i, g(f_k)) \, .$$

In [1], Blackwell proved that every prediction game satisfies the hypotheses of the Wald-Karlin Theorem ([5], [7]), hence has a value, and player I has an optimal strategy. Player II, in general, will not have an optimal strategy (see [1]). Optimal solutions for the predictor will consist of classes of ε-optimal strategies, which are often technically difficult to obtain.

3. Two-move prediction games. We now study a class of problems in which the predictor observes as many terms of an infinite sequence of 0's and 1's as he may please, after which he chooses an action from a finite set of N possible actions. He wins an amount depending only on the action chosen and on the first two unobserved terms of the sequence. Hence we can take a partition $E = (e_1, e_2, e_3, e_4) = ((0, 0), (0, 1), (1, 0), (1, 1))$ and $A = \|A(i, j)\|$ a $4 \times N$ real-valued matrix. We call this type of game a two-move prediction game.

For any 0–1 process $\alpha = \{\alpha_n\}_{n \geqq 1}$, let γ_α be any mapping from S into $[0, 1]$ such that

$$\gamma_\alpha(s) = \text{Prob}\,\{\alpha_{n+1} = 1 \mid (\alpha_1, \cdots, \alpha_n) = s\}$$

for all finite sequences $s = (\varepsilon_1, \cdots, \varepsilon_n)$ of 0's and 1's such that Prob $\{(\alpha_1, \cdots, \alpha_n) = s\} > 0$. Note that γ_α is not uniquely defined, but any two versions differ only at points $s = (\varepsilon_1, \cdots, \varepsilon_n)$ with Prob $\{(\alpha_1, \cdots, \alpha_n) = s\} = 0$. Next define

$$q_\alpha(s) = (\gamma_\alpha(s), \gamma_\alpha(s, 0), \gamma_\alpha(s, 1)) \quad \text{for } s \in S$$

and

$$Q_\alpha = \text{range of } q_\alpha(s) \text{ as } s \text{ varies over all finite sequences.}$$

We emphasize here that Q_α, too, is not uniquely defined, but depends on the version of γ_α selected. In the following, unless stated otherwise, when we say that Q_α has a certain property, we mean there is at least one version of Q_α with that property.

DEFINITION. A subset T of the unit cube is *attainable* if there is an $\alpha = \{\alpha_n\}_{n \geqq 1}$ with $Q_\alpha \subset T$.

DEFINITION. A nonempty subset U of the unit cube is *regular* if $\pi_y(U) \subset \pi_x(U)$ and $\pi_z(U) \subset \pi_x(U)$ where π_x, π_y and π_z denote projections on the x, y, z axes respectively.

LEMMA 1. (i) *Each version of every Q_α is regular.*
(ii) *Every regular U contains a Q_α.*

PROOF. (i) For any $q_\alpha(s) \in Q_\alpha$, $q_\alpha(s) = (\gamma_\alpha(s), \gamma_\alpha(s, 0), \gamma_\alpha(s, 1))$. Then $\gamma_\alpha(s, 0)$, $\gamma_\alpha(s, 1)$ are the first components of $q_\alpha(s, 0)$ and $q_\alpha(s, 1)$ respectively (i.e. $\gamma_\alpha(s, 0) \in \pi_x(Q_\alpha)$ and $\gamma_\alpha(s, 1) \in \pi_x(Q_x)$). Therefore Q_α is regular.

(ii) Let U be regular. We want to find an $\alpha = \{\alpha_n\}_{n \geqq 1}$ with $Q_\alpha \subset U$. For this, note that since the distribution of α can be defined in terms of γ_α, it is sufficient to find a function γ from S into $[0, 1]$ satisfying $(\gamma(s), \gamma(s, 0), \gamma(s, 1)) \in U$ for all $s \in S$.

Take any $(x_0, y_0, z_0) \in U$, and set $\gamma(\theta) = x_0$, $\gamma(0) = y_0$, $\gamma(1) = z_0$. Having defined $\gamma(s)$ for s at level less than or equal to n, take any s^* at level $n - 1$. Let $x^* = \gamma(s^*)$, $y^* = \gamma(s^*, 0)$, $z^* = \gamma(s^*, 1)$. Since $(x^*, y^*, z^*) \in U$ and U is regular, there are points (y^*, a, b) and (z^*, c, d) in U. Set $\gamma(s^*, 0, 0) = a$, $\gamma(s^*, 0, 1) = b$, $\gamma(s^*, 1, 0) = c$, and $\gamma(s^*, 1, 1) = d$. The induction defines γ. Consequently we have defined an $\alpha = \{\alpha_n\}$ with $Q_\alpha \subset U$.

COROLLARY 1.1. *T is attainable if and only if it contains a regular subset.*

PROOF. Immediate from Lemma 1.

LEMMA 2. *T is attainable if and only if there exists a nonempty subset D of* $[0, 1]$ *such that*

$$\pi_x(T \cap D^3) = D .$$

PROOF. If there exists a nonempty subset D of $[0, 1]$ with $\pi_x(T \cap D^3) = D$, then $T \cap D^3$ is a regular subset of T. By Corollary 1.1, T is attainable.

If T is attainable, and $U \subset T$ is regular, define $D = \pi_x(U)$.

Since U is regular, we have $U \subset D^3$ and

$$\pi_x(T \cap D^3) \supseteqq \pi_x(U \cap D^3) = D .$$

But $\pi_x(T \cap D^3) \subseteqq \pi_x(D^3) = D$, hence we have

$$\pi_x(T \cap D^3) = D .$$

COROLLARY 2.1. *If T is closed and $\pi_x(T \cap D^3) = D$, then $\pi_x(T \cap \bar{D}^3) = \bar{D}$ where $\bar{D}$ is the closure of D.*

PROOF. T is closed, $\bar{D}^3$ is closed, so $T \cap \bar{D}^3$ is a closed subset of the unit cube. Therefore $\pi_x(T \cap \bar{D}^3)$ is closed and contains $\pi_x(T \cap D^3)$, i.e. $\pi_x(T \cap \bar{D}^3) \supseteqq \bar{D}$. But $\pi_x(T \cap \bar{D}^3) \subseteqq \pi_x(\bar{D}^3) = \bar{D}$, hence $\pi_x(T \cap \bar{D}^3) = \bar{D}$.

COROLLARY 2.2. *If T is convex and $\pi_x(T \cap D^3) = D$, then $\pi_x(T \cap H^3) = H$, where H is the convex hull of D.*

PROOF. T is convex, H^3 is convex, thus $T \cap H^3$ is convex. Therefore $\pi_x(T \cap H^3)$ is convex and contains $\pi_x(T \cap D^3)$, i.e. $\pi_x(T \cap H^3) \supseteqq H$. But $\pi_x(T \cap H^3) \subseteqq \pi_x(H^3) = H$, hence we have $\pi_x(T \cap H^3) = H$.

COROLLARY 2.3. *If T is closed and convex, and $\pi_x(T \cap D^3) = D$, then $\pi_x(T \cap F^3) = F$, where F is closed and convex.*

Equivalently, a closed convex set T is attainable if and only if there are two numbers a, b such that for every $x \in [a, b]$, there are $y, z \in [a, b]$, such that $(x, y, z) \in T$.

PROOF. Take $F = [a, b]$, with $a = \inf\{x : x \in D\}$ and $b = \sup\{x : x \in D\}$. The result follows from Corollaries 2.1 and 2.2.

THEOREM. *Let (E, A) be associated to a two-move prediction game with $E = ((0, 0), (0, 1), (1, 0), (1, 1))$ and $A = \|A(i,j)\|$ a $4 \times N$ real valued matrix. Let*

$$\phi(x, y, z) = \min_{1 \leqq j \leqq N} [(1-x)(1-y), (1-x)y, x(1-z), xz] \begin{bmatrix} A(1,j) \\ A(2,j) \\ A(3,j) \\ A(4,j) \end{bmatrix}$$

$$T_V = \{(x, y, z) : \phi(x, y, z) \geqq V\}$$

$$V_0 = \sup\{V : T_V \textit{ is attainable}\}.$$

Then (i) *T_{V_0} is attainable.*

(ii) *V_0 is the value of the game, and if α is such that $Q_\alpha \subset T_{V_0}$, the distribution of α is an optimal strategy for player* I.

(iii) $$V_0 = \max_{D \subset [0,1] D \text{ closed}} \min_{x \in D} \max_{y,z \in D} \phi(x, y, z) .$$

PROOF. (i) Let $\{V_m\}_{m \geqq 1}$ be such that $V_m \uparrow V_0$ and T_{V_m} is attainable for each m. Then by the definition of attainability, there exists a 0–1 process $\alpha^{(m)} = \{\alpha_n^{(m)}\}_{n \geqq 1}$ with $Q_{\alpha^{(m)}} \subset T_{V_m}$ for each m. We can now apply a diagonal argument to find a subsequence $\{\alpha^{(m')}\}$ of $\{\alpha^{(m)}\}$ and a 0–1 process $\alpha = \{\alpha_n\}$ such that for all n and all choices of finite 0–1 sequences s of length n,

(a) $\lim_{m' \to \infty} \text{Prob}\{(\alpha_1^{(m')}, \alpha_2^{(m')}, \cdots, \alpha_n^{(m')}) = s\} = \text{Prob}\{(\alpha_1, \cdots, \alpha_n) = s\}$ and
(b) $\gamma_{\alpha^{(m')}}(s)$ converges to a limit $\gamma(s)$ if $\text{Prob}\{(\alpha_1, \cdots, \alpha_n) = s\} = 0$.

We will now show that $Q_\alpha \subset T_{V_0}$. Since (a) above implies $\gamma_{\alpha(m')}(s) \to \gamma_\alpha(s)$ if $\text{Prob}\{(\alpha_1, \cdots, \alpha_n) = s\} > 0$, and since $\gamma_\alpha(s)$ may be defined to be $\gamma(s)$ of (b) if $\text{Prob}\{(\alpha_1, \cdots, \alpha_n) = s\} = 0$, we conclude that $\gamma_{\alpha(m')}(s) \to \gamma_\alpha(s)$ for all $s \in S$. Therefore

$$q_\alpha(s) = \lim_{m' \to \infty} (\gamma_{\alpha(m')}(s), \gamma_{\alpha(m')}(s, 0), \gamma_{\alpha(m')}(s, 1)) = \lim_{m' \to \infty} q_{\alpha(m')}(s)$$

for all $s \in S$. But $Q_{\alpha(m')} \subset T_{V_{m'}}$ means $q_{\alpha(m')}(s) \in T_{V_{m'}}$ for all $s \in S$, so that $\phi(q_{\alpha(m')}(s)) \geqq V_{m'}$ for all $s \in S$. Since A is a real-valued matrix, ϕ is continuous on the unit cube. Therefore for all $s \in S$

$$\phi(q_\alpha(s)) = \lim_{m' \to \infty} \phi(q_{\alpha(m')}(s)) \geqq \limsup_{m' \to \infty} V_{m'} = V_0 .$$

We conclude that $q_\alpha(s) \in T_{V_0}$ for all $s \in S$, i.e. $Q_\alpha \subset T_{V_0}$.

(ii) By (i), T_{V_0} is attainable and there exists a 0–1 process $\alpha = \{\alpha_n\}_{n \geqq 1}$ with $Q_\alpha \subset T_{V_0}$, i.e. for every $s \in S$, we have $\phi(q_\alpha(s)) \geqq V_0$. Let P^0 be the distribution of α, and recall that for $s = (\varepsilon_1, \cdots, \varepsilon_n)$, we have set $P^0(s) = \text{Prob}\{\alpha_1 = \varepsilon_1, \cdots, \alpha_n = \varepsilon_n\}$. For each $s \in S$, define

$$\begin{aligned} P_s^0(e_1) &= (1 - \gamma_\alpha(s))(1 - \gamma_\alpha(s, 0)) \\ P_s^0(e_2) &= (1 - \gamma_\alpha(s))\gamma_\alpha(s, 0) \\ P_s^0(e_3) &= \gamma_\alpha(s)(1 - \gamma_\alpha(s, 1)) \\ P_s^0(e_4) &= \gamma_\alpha(s)\gamma_\alpha(s, 1) . \end{aligned}$$

Clearly, $P^0(s, e_i) = P^0(s)P_s^0(e_i)$. Since $\phi(q_\alpha(s)) \geqq V_0$, we have

$$\textstyle\sum_{i=1}^4 P_s^0(e_i)A(i, j) \geqq V_0 \qquad \text{for all} \quad j \quad \text{and } s.$$

For any y, therefore,

$$\begin{aligned} M(P^0, y) &= \textstyle\sum_{k,i} P^0(f_k, e_i)A(i, g(f_k)) \\ &= \textstyle\sum_{k,i} P^0(f_k)[\sum_i P_{f_k}^0(e_i)A(i, g(f_k))] \geqq V_0 , \end{aligned}$$

which implies

$$\sup_P M(P, Q) \geqq M(P^0, Q) = \textstyle\sum_i \lambda_i M(P^0, y) \geqq V_0 \qquad \text{for all} \quad Q .$$

So if V^* is the value of the game, then $V^* = \inf_Q \sup_P M(P, Q) \geqq V_0$. Now let P^* be an optimal strategy for player I, and suppose $V > V_0$. Then the corresponding T_V is not attainable, so that there exist s^* and j^* such that $P^*(s^*) > 0$ and $\sum_{i=1}^4 P_{s^*}^*(e_i)A(i, j^*) < V$, where $P_{s^*}^*$ is the distribution of the conditional P^*-process starting from s^* and satisfies $P_{s^*}^*(t) = P^*(s^*, t)/P^*(s^*)$ for all $t \in S$. Hence $\inf_Q M(P_{s^*}^*, Q) < V$. But $P_{s^*}^*$ is an optimal strategy since P^* is (By Theorem 1 of [1]), so that $V^* \leqq \inf_Q M(P_{s^*}^*, Q) < V$. We conclude that $V^* = V_0$ and V_0 is the value of the game.

That P^0 is an optimal strategy follows from the previous discussion that $M(P^0, Q) \geqq V_0$ for all Q.

(iii) By Lemma 1 and Lemma 2, T_V is attainable if and only if there exists a subset D of $[0, 1]$ such that for every x in D, there exist y and z in D such that

(x, y, z) is in T_V. So

$$\begin{aligned} V_0 &= \sup\{V\colon T_V \text{ is attainable}\} \\ &= \sup\{V\colon \exists D \subset [0, 1] \quad \text{with} \quad \inf_{x \in D} \sup_{y,z \in D} \phi(x, y, z) \geqq V\} \\ &= \sup_{D \subset [0,1]} \inf_{x \in D} \sup_{y,z \in D} \phi(x, y, z) . \end{aligned}$$

Since the T_V's are closed (because ϕ is continuous on the unit cube), Corollary 2.1 implies

$$V_0 = \sup_{D \subset [0,1], D \text{ closed}} \inf_{x \in D} \sup_{y,z \in D} \phi(x, y, z) .$$

Again using the continuity of ϕ, we may replace the inner sup and inf by max and min, obtaining

$$V_0 = \sup_{D \subset [0,1], D \text{ closed}} \min_{x \in D} \max_{y,z \in D} \phi(x, y, z) .$$

But T_{V_0} is attainable, so that Lemma 1 and Lemma 2 imply that there exists D_0 closed with $\min_{x \in D_0} \max_{y,z \in D_0} \phi(x, y, z) \geqq V_0$, i.e. the supremum is attained, and

$$V_0 = \max_{D \subset [0,1], D \text{ closed}} \min_{x \in D} \max_{y,z \in D} \phi(x, y, z) .$$

4. Applications. We now solve the following games to see how the method developed in the previous section can be applied.

4.1. 1–0 *Game*. The game was first studied by Blackwell [1], in which player I writes an infinite sequence of 1's and 0's and player II is allowed to observe the successive sequence of 1's and 0's until he decides to stop (unknown to I). If the next two digits that I writes are 1, 0 in that order, then II loses, otherwise II wins. Let the payoff be 1 if II loses, 0 if II wins. This makes the expected payoff equal to the probability of losing for player II.

This game falls in the class of two-move prediction games with

$$A = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \text{and} \qquad \phi(x, y, z) = x(1 - z) .$$

By our theorem and Corollary 2.3, the value of the game is:

$$\begin{aligned} V_0 &= \max_{[a,b] \subset [0,1]} \min_{x \in [a,b]} \max_{y,z \in [a,b]} x(1 - z) \\ &= \max_{[a,b] \subset [0,1]} a(1 - a) . \end{aligned}$$

Since $a(1 - a)$ attains its maximum $\frac{1}{4}$ on $[0, 1]$ at $a = \frac{1}{2}$, we have $V_0 = \frac{1}{4}$. The set $T_0 = \{(x, y, z)\colon x(1 - z) \geqq \frac{1}{4}\}$ is attainable with $(\{\frac{1}{2}\}^3)$ as a regular subset. An optimal strategy for player I is to produce an independent identically distributed sequence by tossing a fair coin.

4.2. *Bomber-battleship game*. This problem was first formulated by R. Isaacs, and consists of a ship attempting to maneuver so as to minimize the probability of its being destroyed by a bomber flying overhead. It has received considerable attention since then by Isaacs and Karlin [3], Isaacs [4], Dubins [2] and Karlin [5], with different approaches. The game can be described as follows: at each unit of time the battleship can choose to move one unit to the right (southeast direction) or to the left (southwest direction), and he must move. The bomber watches the ship as long as he pleases, and then drops a bomb which takes two units of time to reach the plane of the ship after being released. It is assumed that the bomber has only one bomb and there are no aiming or ballistic errors. Also we assume that hitting the ship is equivalent to destroying it with probability one. (See Figure 1.)

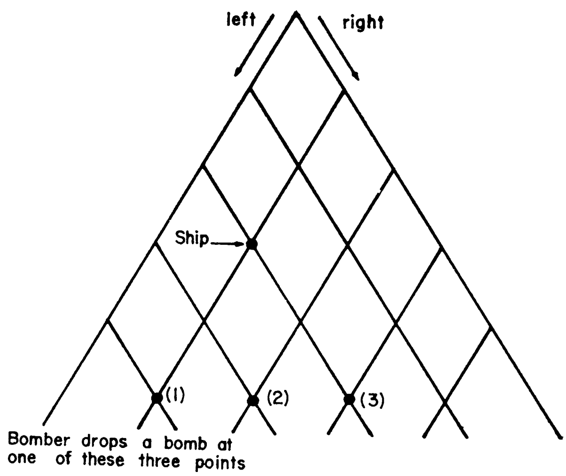

FIG. 1.

Let the payoff be 0 if the bomber destroys the battleship, 1 otherwise. This makes the expected payoff equal to the probability of the battleship's survival.

This game can be thought of as a two-move prediction game with

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

and

$$\phi(x, y, z) = \min\,[(1-(1-x)(1-y)), (1-y(1-x)-x(1-z)), (1-xz)]\,.$$

The following lemma will simplify things and be useful.

LEMMA 3. *Let* $T_V = \{(x, y, z) : \phi(x, y, z) \geqq V\}$. *If* $\phi(x, y, z) = \phi(1-x, y_0, z_0)$ *where* y_0, z_0 *each is one of the variables* $1-x$, $1-y$, $1-z$, *and if there exists a subset* D *of* $[0, 1]$ *with* $\pi_x(T_V \cap D^3) = D$, *then* $\pi_x(T_V \cap D^{*3}) = D^*$, *where* $D^* = D \cup \{(1-x) : x \in D\}$. (That is, when $\phi(x, y, z) = \phi(1-x, y_0, z_0)$, if we can find a regular subset D^3, we can find a regular subset D^{*3} where D^* is symmetric about $\frac{1}{2}$.)

PROOF. Denote $D_0 = \{(1 - x) : x \in D\}$, so that $D^* = D \cup D_0$. Since $\pi_x(T \cap D^3) = D$, we have for every $x \in D$ the existence of $y, z \in D$ such that $\phi(x, y, z) \geqq V$.

Now for every x in D_0, we have $(1 - x)$ in D, which implies there exist y and z in D such that $\phi(1 - x, y, z) \geqq V$. But $\phi(1 - x, y, z) = \phi(x, y_0, z_0)$, where each y_0 and z_0 is one of the variables $1 - x$, $1 - y$, $1 - z$. We conclude that $\phi(x, y_0, z_0) \geqq V$ and $y_0, z_0 \in D_0 \subset D^*$.

Consequently for every x in D^*, there exist y and z in D^* such that $\phi(x, y, z) \geqq V$, i.e. $\pi_x(T_V \cap D^{*3}) = D^*$.

We now return to the bomber and battleship game.

$$\begin{aligned}\phi(x, y, z) &= \min\,[(1 - (1 - x)(1 - y)), (1 - y(1 - x) - x(1 - y)), (1 - xz)] \\ &= \phi(1 - x, 1 - z, 1 - y)\,.\end{aligned}$$

By our Theorem and Lemma 3, the value of the game is:

$$\begin{aligned}V_0 &= \max_{D^* \subset [0,1]} \min_{x \in D^*} \max_{y,z \in D^*} \min\,[(1 - (1 - x)(1 - y)), \\ &\qquad (1 - y(1 - x) - x(1 - z)), (1 - zx)] \\ &= 1 - \min_{D^* \subset [0,1]} \max_{x \in D^*} \min_{y,z \in D^*} \max\,[(1 - x)(1 - y), \\ &\qquad y(1 - x) + x(1 - z), xz]\end{aligned}$$

where D^* denotes a closed set which is symmetric about $\frac{1}{2}$.

To find V_0, we first obtain an upper bound for V_0 and show next that this upper bound can be achieved, so that in fact it is V_0.

For any closed subset D^* of $[0, 1]$ which is symmetric about $\frac{1}{2}$, let us denote $\min \{x : x \in D^*\}$ by a and $\max \{x : x \in D^*\}$ by $1 - a$.

We have

$$\begin{aligned}&\max_{x \in D^*} \min_{y,z \in D^*} \max\,[(1 - x)(1 - y), y(1 - x) + x(1 - z), xz] \\ &\qquad \geqq \max_{x \in D^*} \min_{y,z \in D^*} \max\,[(1 - x)(1 - y), x(1 - z) + y(1 - x)] \\ (1) \qquad &\qquad \geqq \min_{y,z \in D^*} \max\,[(1 - a)(1 - y), a(1 - z) + y(1 - a)] \\ &\qquad = \min_{y \in D^*} \max\,[(1 - a)(1 - y), a[1 - (1 - a)] + y(1 - a)] \\ &\qquad \geqq \frac{1 - a + a^2}{2}\end{aligned}$$

and

$$\begin{aligned}&\max_{x \in D^*} \min_{y,z \in D^*} \max\,[(1 - x)(1 - y), y(1 - x) + x(1 - z), xz] \\ &\qquad \geqq \max_{x \in D^*} \min_{y,z \in D^*} [x(1 - z) + y(1 - x)] \\ (2) \qquad &\qquad \geqq \min_{y,z \in D^*} [a(1 - z) + y(1 - a)] \\ &\qquad = a[1 - (1 - a)] + a(1 - a) \\ &\qquad = a\,.\end{aligned}$$

Inequalities (1) and (2) together imply that

$$\max_{x \in D^*} \min_{y,z \in D^*} \phi(x, y, z) \geqq \max\left[a, \frac{1 - a + a^2}{2}\right].$$

We have assumed $a = \min \{x : x \in D^*\}$, so that $a \leqq 1 - a$ and $0 \leqq a \leqq \frac{1}{2}$.

The function $(1 - a + a^2)/2$ is a decreasing function of a on $[0, \frac{1}{2}]$; therefore $\max[a, (1 - a + a^2)/2]$ attains its minimum $(3 - 5^{\frac{1}{2}})/2$ at $a = (3 - 5^{\frac{1}{2}})/2$. That gives us

$$1 - V_0 \geqq (3 - 5^{\frac{1}{2}})/2 .$$

Consequently we have $V_0 \leqq (-1 + 5^{\frac{1}{2}})/2$.

This upper bound can be achieved, i.e. the set $T = \{(x, y, z) : \phi(x, y, z) \geqq (-1 + 5^{\frac{1}{2}})/2\}$ is attainable, with $\{(3 - 5^{\frac{1}{2}})/2, (-1 + 5^{\frac{1}{2}})/2\}^3$ as a regular subset of T. We conclude that $(-1 + 5^{\frac{1}{2}})/2$ is the value of the game. Optimal strategies for the battleship exist. We can describe them as follows: The battleship takes its initial move to the right with probability between $(3 - 5^{\frac{1}{2}})/2$ and $(-1 + 5^{\frac{1}{2}})/2$, continuing at each stage in the direction of its previous move with probability $(-1 + 5^{\frac{1}{2}})/2$. The bomber cannot destroy it with probability greater than $(3 - 5^{\frac{1}{2}})/2$.

REMARK. 1. Our method can also solve the more general form of the bomber-battleship game in which the inaccuracy of the bomber is considered.

Case 1. The bomber can destroy the battleship with probability $p (0 < p < 1)$ when he hits the middle position and the ship is in the middle, (position (2) in the Figure), and with probability one when he hits the ship and the ship is in one of the other two positions. This game is a two-move prediction game with

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1-p & 1 \\ 1 & 1-p & 1 \\ 1 & 1 & 0 \end{bmatrix} .$$

The value of the game can be obtained by the same method as in application 4.2. We have

$$V_0 = 1 - \frac{p[(2 + p) - (p^2 + 4p)^{\frac{1}{2}}]}{2} .$$

An optimal strategy for the battleship can be described as follows: The battleship takes its initial move to the right with probability $[(2 + p) - (p^2 + 4p)^{\frac{1}{2}}]/2$, continuing at each stage in the direction of its previous move with probability $[-p + (p^2 + 4p)^{\frac{1}{2}}]/2$. The bomber cannot destroy it with probability greater than $p[(2 + p) - (p^2 + 4p)^{\frac{1}{2}}]/2$.

Case 2. The bomber can destroy the ship with probability one when he hits the ship and the ship is either at the left or in the middle (positions (1) and (2) in the Figure) and with probability $p (0 < p < 1)$ when he hits the ship and the ship is at the right (position (3) in the Figure). This game is a two-move prediction game with

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1-p \end{bmatrix}$$

and $\phi(x, y, z) = 1 - \max[(1-x)(1-y), x(1-z) + y(1-x), pxz]$. By our theorem, the value of the game is

$$V_0 = 1 - \min_{D \subset [0,1], D \text{ closed}} \max_{x \in D} \min_{y,z \in D} \max[(1-x)(1-y), x(1-z) + y(1-x), pxz].$$

We will give an outline for computing V_0:
Let D be any closed subset of $[0, 1]$. Define

$$f(D) = \max_{x \in D} \min_{y,z \in D} \max[(1-x)(1-y), x(1-z) + y(1-x), pxz].$$

Let $\mathscr{C} = \{D : D \subset [0, 1], D \text{ closed}, \min\{x : x \in D\} = a \text{ and } \max\{x : x \in D\} = (p + 1 - 2pa)/(2p + a - pa) \text{ for some } a \in [0, (-2p + (3p^2 + 1)^{\frac{1}{2}})/(1 - p)]\}$. We first check that for every closed subset D of $[0, 1]$, there exists a set $\tilde{D} \in \mathscr{C}$ such that $f(D) \geqq f(\tilde{D})$. Consequently we have

$$V_0 = 1 - \min_{D \subset [0,1], D \text{ closed}} f(D) = 1 - \min_{D \in C} f(D).$$

Next, using calculus, we obtain for every $D \in \mathscr{C}$,

$$f(D) \geqq \frac{p(1 - a^* + a^{*2})}{2p + a^* - a^*p},$$

where a^* is a root of the equation

$$f(a) = a^3(1 - p) - a^2(2 - 3p) + a(1 - 4p) + p = 0$$

and

$$0 \leqq a^* \leqq (-2 + (3p^2 + 1)^{\frac{1}{2}})/(1 - p);$$

i.e.

$$\min_{D \in C} f(D) \geqq \frac{p(1 - a^* + a^{*2})}{2p + a^* - a^*p}.$$

Consequently, we have an upper bound for V_0:

$$V_0 \leqq \frac{p + a^* - a^{*2}p}{2p + a^* - a^*p}.$$

This upper bound of V_0 is in fact achieved. The set

$$T = \{(x, y, z) : \phi(x, y, z) \geqq (p + a^* - a^{*2})/(2p + a^* - a^*p)\}$$

is attainable with $\{a^*, b^*, c^*\}^3$ as a regular subset of T, where $b^* = (p + 1 - 2pa^*)/(2p + a^* - a^*p)$ and $c^* = (1 - a^* + a^{*2})/(p + 1 - 2pa^*)$. (One can check that the points (a^*, a^*, b^*), (b^*, a^*, c^*) and (c^*, a^*, b^*) are in T.) Therefore we conclude that

$$V_0 = \frac{p + a^* - a^{*2}p}{2p + a^* - a^*p}.$$

An optimal strategy for the battleship can be described as follows: He takes his initial move to the right with probability a^*. Then if $s = (\varepsilon_1, \cdots, \varepsilon_n)$, where the ε_i's are 0's and 1's, is the result of first n moves, he moves to the right at the next move with probability

$$\begin{aligned} \gamma(s) &= a^* && \text{if } K = n \\ &= b^* && \text{if } n - K \text{ is odd} \\ &= c^* && \text{if } n - K \text{ is even}, \end{aligned}$$

where

$$\begin{aligned} K &= \max\{i : \varepsilon_i = 0,\ 1 \leqq i \leqq n\} \\ &= 0 \quad \text{if } \varepsilon_1 = \cdots = \varepsilon_n = 1\,. \end{aligned}$$

The bomber cannot destroy him with probability more than $p(1 - a^* + a^{*2}) \div (2p + a^* - a^*p)$.

REMARK 2. When player I uses mixed strategies, that is, he chooses a path with probability distribution involved, it is a stochastic process. Let $\alpha_1, \alpha_2, \cdots$ be a stochastic process corresponding to a good strategy of player I. Let D^3 be a regular subset of T_{V_0} where V_0 is the value of the game. Then if D consists of only one point, $\alpha_1, \alpha_2, \cdots$ can be taken as an independent identically distributed sequence. The 1–0 game is an example. If D consists of two points, then $\alpha_1, \alpha_2, \cdots$ can be taken as a Markov chain. An optimal strategy in the original bomber and battleship game, for example, is a two-state Markov chain with starting distribution Prob $(\alpha_1 = 1)$ in the interval $[(3 - 5^{\frac{1}{2}})/2, (-1 + 5^{\frac{1}{2}})/2]$ and transition matrix

$$\begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} \quad 0 \quad & \quad 1 \quad \end{array} \\ \begin{bmatrix} \dfrac{-1 + 5^{\frac{1}{2}}}{2} & \dfrac{3 - 5^{\frac{1}{2}}}{2} \\ \dfrac{3 - 5^{\frac{1}{2}}}{2} & \dfrac{-1 + 5^{\frac{1}{2}}}{2} \end{bmatrix} \end{array}.$$

If D consists of a finite number of points, $\alpha_1, \alpha_2, \cdots$ often can be taken as a function of a finite state Markov chain. The generalized game in Case 2 of Remark 1, for example, is a function of a three-state Markov chain. We take $\alpha_n = f(X_n)$ where X_n is a three-state Markov chain with stationary transition probability matrix

$$\begin{array}{c} \\ \text{I} \\ \text{II} \\ \text{III} \end{array} \begin{array}{c} \begin{array}{ccc} \text{I} & \text{II} & \text{III} \end{array} \\ \begin{pmatrix} 1 - a^* & a^* & 0 \\ 1 - b^* & 0 & b^* \\ 1 - c^* & c^* & 0 \end{pmatrix} \end{array}$$

and

$$\begin{aligned} f(X_n) &= 1 && \text{if } X_n \text{ is in states II or III} \\ &= 0 && \text{if } X_n \text{ is in state I.} \end{aligned}$$

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