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MINIMAX ESTIMATION OF POWERS OF THE VARIANCE OF A NORMAL POPULATION UNDER SQUARED ERROR LOSS[1]

BY WILLIAM E. STRAWDERMAN

Rutgers University

The problem of estimating powers of the variance of a normal distribution is considered when loss is essentially squared error. A class of minimax estimators is found by extending the techniques of Stein. It is shown, at least for estimating the variance, that a subclass of the above consists of generalized Bayes estimators.

1. Introduction. This paper deals with the problem of estimating positive powers of the variance of a normal distribution when the loss is given by

$$L(\eta, \sigma^{2\alpha}) = \left(\frac{\eta}{\sigma^{2\alpha}} - 1\right)^2. \tag{1.1}$$

More specifically, we assume we are given a sample $X_1, \cdots, X_n$ of independent identically distributed normal random variables with unknown mean μ and unknown variance σ^2. Stein [4] showed that the usual best fully invariant estimator of σ^2 is inadmissible with respect to the loss (1.1). Brown [3] showed in a more general setting the best fully invariant estimator of the αth power of a scale parameter in the presence of an unknown location parameter is inadmissible for a large class of loss functions. In this paper we give a class of minimax estimators for any positive power of the variance for the model given above and show, at least for estimating the variance itself, that a subclass is generalized Bayes. Brewster and Zidek [2] have results that are similar in conclusion but different in technique to those presented here. They also have results on scale admissibility as well as on estimation of quantiles of normal distributions, which are not considered in this paper.

Section 2 presents a class of minimax estimators for powers of the variance using an extension of the methods of Stein. Section 3 presents a class of generalized Bayes estimators of powers of the variance. In Section 3 it is shown also that for the estimating the variance this class has non-empty intersection with the class of minimax estimators of Section 2. Section 4 presents some remarks primarily on extending the results of Sections 2 and 3 to the case of several unknown means. Some remarks are also given concerning the extending of Brown's [3] results to give smooth minimax estimators.

2. A class of minimax estimators of σ^2. In this section we produce a class of

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minimax estimators for any power of the variance when the loss in given by (1.1). Let $\bar{X} = \sum_{i=1}^{n} X_i$ and $S = \sum_{i=1}^{n} (X_i - \bar{X})^2$. We consider estimators of the form

$$\eta(\bar{X}, S) = \psi(S/(S + n\bar{X}^2))(S + n\bar{X}^2)^\alpha \,, \tag{2.1}$$

i.e. estimators based on the sufficient statistics which are orthogonally invariant and scale invariant. These are obvious extensions of the class of estimators considered by Stein [4] for the case of estimating the variance.

The risk of an estimator of the form (2.1) may be calculated as in Stein [4] by introducing an auxiliary variable L distributed as a Poisson variable with parameter $\lambda = \mu^2/2\sigma^2$ such that L is independent of S and that $n\bar{X}^2$ given S and L is central χ^2 with $1 + 2L$ degrees of freedom. A straightforward calculation leads to the following expression for the risk of (2.1):

$$E\left\{\frac{2^{2\alpha}\Gamma\left(\frac{2+2L}{2} + 2\alpha\right)}{\Gamma\left(\frac{n+2L}{2}\right)} \times \left[E\left(\left(\psi(S/S + n\bar{X}^2) - \frac{\Gamma\left(\frac{n+2L}{2} + \alpha\right)}{\Gamma\left(\frac{n+2L}{2} + 2\alpha\right)2^\alpha}\right)^2 \middle| L\right)\right] + C(L)\right\}, \tag{2.2}$$

where $C(L)$ is a function depending only on L, and whose value will play no role in the ensuing argument.

We now consider estimators of the form (2.1) with

$$\psi(U) = CU^\alpha(1 - \varepsilon(U)U^\delta) \,, \qquad 0 \leqq U \leqq 1 \,, \tag{2.3}$$

where

$$C = \Gamma\left(\frac{n-1}{2} + \alpha\right)\Big/\left(\Gamma\left(\frac{n-1}{2} + 2\alpha\right)2^\alpha\right)$$

(i.e. the multiplier for the best fully invariant estimator) and $\varepsilon(U)$ is non-decreasing. We will prove the following result.

THEOREM 1. *Estimators of the form* (2.1) *with* $\psi(U)$ *given by* (2.3) *are minimax for the loss function given by* (1.1) *provided* $\delta \geqq 0$ *and* $0 \leqq \varepsilon(U) \leqq D(\delta)$ *where*

$$\begin{aligned} D(\delta) = \min\Bigg\{2\alpha/(\alpha + \delta), \Bigg(2\Bigg[&\beta\left(\frac{n-1}{2} + 2\alpha + \delta, \tfrac{1}{2}\right) \\ &\times \beta\left(\frac{n-1}{2} + \alpha, \alpha\right) - \beta\left(\frac{n-1}{2} + \alpha + \delta, \tfrac{1}{2}\right)\beta\left(\frac{n}{2} + \alpha, \alpha\right)\Bigg]\Bigg) \\ &\div \left(\beta\left(\frac{n-1}{2} + \alpha, \alpha\right)\beta\left(\frac{n-1}{2} + 2\alpha + 2\delta, \tfrac{1}{2}\right)\right)\Bigg\} \,. \end{aligned} \tag{2.4}$$

We prove this result by showing that the risk of such an estimator is no larger than the risk of the best fully invariant estimator—i.e. that given by $\varepsilon(\cdot) = 0$, which is minimax. This will be accomplished if we show for each $L = 0, 1, \cdots$ that the "conditional risk given L" of the best fully invariant estimator is at least as great as that for the estimator given in the theorem. We will use the fact that conditional on L, $U = S/(S + n\bar{X}^2)$ has a Beta distribution with parameters $\frac{1}{2}(n-1)$ and $\frac{1}{2}(1+2L)$. Letting $A = \Gamma(\frac{1}{2}(n+2L)+\alpha)/\Gamma(\frac{1}{2}(n+2L)+2\alpha)2^\alpha$, it suffices to show for each L that

$$(2.5)\qquad \begin{aligned} 0 &\leqq E(CU^\alpha - A)^2 - E(CU^\alpha(1 - U^\delta\varepsilon(U)) - A)^2 \\ &= E\{C\varepsilon(U)[2CU^{2\alpha+\delta} - 2AU^{\alpha+\delta} - C\varepsilon(U)U^{2\alpha+2\delta}]\}\,. \end{aligned}$$

Since C and $\varepsilon(U)$ are all nonnegative, and $0 \leqq \varepsilon(U) \leqq D(\delta) = D$, the last expression in (2.5) is bounded below by

$$(2.6)\qquad E\{C\varepsilon(U)[2CU^{2\alpha+\delta} - 2AU^{\alpha+\delta} - CDU^{2\alpha+2\delta}]\}\,.$$

Also we note that the expression in brackets in (2.6) changes sign at most once from negative on the interval $[0, 1]$ provided that

$$(2.7)\qquad D \leqq 2\alpha/(\alpha+\delta)\,.$$

Hence if

$$U_0 = \sup\{U: 2CU^{2\alpha+\delta} - 2AU^{\alpha+\delta} - CDU^{2\alpha+2\delta} < 0\}\,,$$

we have that the expression in (2.6) is bounded below by

$$\begin{aligned} &C\varepsilon(U_0)E[2CU^{2\alpha+\delta} - 2AU^{\alpha+\delta} - CDU^{2\alpha+2\delta}] \\ &\quad = \frac{CD\varepsilon(U_0)}{\beta\left(\frac{n-1}{2}, \frac{1+2L}{2}\right)}\left[2C\beta\left(\frac{n-1}{2} + 2\alpha + \delta, \frac{1+2L}{2}\right)\right. \\ &\qquad - 2A\beta\left(\frac{n-1}{2} + \alpha + \delta, \frac{1+2L}{2}\right) \\ &\qquad \left. - CD\beta\left(\frac{n-1}{2} + 2\alpha + 2\delta, \frac{1+2L}{2}\right)\right]. \end{aligned}$$

This will be nonnegative whenever

$$(2.8)\qquad \begin{aligned} D &\leqq \frac{\left[2C\beta\left(\frac{n-1}{2} + 2\alpha + \delta, \frac{1+2L}{2}\right) - 2A\beta\left(\frac{n-1}{2} + \alpha + \delta, \frac{1+2L}{2}\right)\right]}{\left[C\beta\left(\frac{n-1}{2} + 2\alpha + 2\delta, \frac{1+2L}{2}\right)\right]} \\ &= D(\delta, L)\,. \end{aligned}$$

We show that $D(\delta, L)$ is nonnegative and non-decreasing in L for $\delta > 0$. This, together with (2.7) and the fact that $D(\delta, 0)$ is equal to last expression in (2.4), will complete the proof.

We show first that $D(\delta, 0) \geqq 0$. This will be so provided

$$\frac{\beta\left(\frac{n-1}{2}+\alpha, \alpha\right)}{\beta\left(\frac{n-1}{2}+\alpha+\delta, \alpha\right)} \geqq \frac{\beta\left(\frac{n}{2}+\alpha, \alpha\right)}{\beta\left(\frac{n}{2}+\alpha+\delta, \alpha\right)}, \tag{2.9}$$

which is equivalent to

$$\frac{1}{E(Y^\delta)} \geqq \frac{1}{E(X^\delta)} \tag{2.10}$$

where Y is a Beta random variable with parameters $\frac{1}{2}(n-1)+\alpha$ and α and X is Beta with parameters $\frac{1}{2}n+\alpha$ and α. But (2.10) follows from the fact that the distribution of X has a monotone increasing likelihood ratio with respect to that of Y and X therefore is stochastically larger than Y.

We now turn to proving $D(\delta, L)$ is monotone non-decreasing in L for fixed $\delta > 0$. Rewriting (2.8) we have that

$$\begin{aligned} D(\delta, L) = & \left[\frac{2\beta(\alpha, \delta)\beta\left(\frac{n-1}{2}+\alpha+\delta, \alpha+\delta\right)}{\beta\left(\frac{n-1}{2}+\alpha, \alpha\right)\beta\left(\frac{n-1}{2}+2\alpha+\delta, \delta\right)}\right] \\ & \times\left[\frac{\beta\left(\frac{n-1}{2}+\alpha, \alpha\right)}{\beta\left(\frac{n-1}{2}+\alpha+\delta, \alpha\right)} - \frac{\beta\left(\frac{n+2L}{2}+\alpha, \alpha\right)}{\beta\left(\frac{n+2L}{2}+\alpha+\delta, \alpha\right)}\right] \\ \equiv & [M(L)][Q-N(L)], \end{aligned}$$

where it is easily seen that $M(L)$ is non-decreasing in L and $N(L)$ is non-increasing in L. This implies that $D(\delta, L)$ is non-decreasing which completes the proof.

3. A class of generalized Bayes minimax estimators of the variance. In this section we find a class of generalized Bayes estimators of $\sigma^{2\alpha}$. We then show for the case $\alpha = 1$ a subclass of the above class consists of minimax estimators. The same type of result holds for integral α by the same type of proof but we have been unable to show the result for non-integral values of α. The class of generalized prior distributions is related to the priors used by Strawderman [5] in proving the existence for dimension greater than or equal to five of proper Bayes minimax estimators of the multivariate normal mean vector when the covariance matrix is given by $\sigma^2 I$, when σ^2 is unknown. The prior distributions used in this paper, however, are not integrable.

We consider the following class of generalized prior distributions: Reparametrize so that $\eta^2 = (\sigma^2)^{-1}$. Conditional on λ and η the distribution of θ is normal with mean 0 and variance $n^{-1}\eta^{-2}\lambda^{-1}(1-\lambda)$. The generalized density of (λ, η) is given by $g(\lambda, \eta) = \lambda^{-\alpha}\eta^{-1+\epsilon}$, $0 < \lambda < 1$, $\eta > 0$.

The generalized Bayes estimator of $\sigma^{2\alpha}$ with respect to the loss (1.1) is then given by

$$\eta(\bar{X}, S) = E(\eta^{2\alpha} \mid \bar{X}, S)/(\eta^{4\alpha} \mid \bar{X}, S)\,, \tag{3.1}$$

where, if we neglect factors which disappear by cancellation in (3.1), for $\gamma = 2\alpha, 4\alpha$:

$$E(\eta^\gamma \mid \bar{X}, S) \propto \int_0^1 d\lambda \int_0^\infty dy \int_{-\infty}^\infty d\theta \lambda^{\frac{1}{2}-a}(1-\lambda)^{-\frac{1}{2}}\eta^{n+\varepsilon+\gamma} \times \exp\left[\frac{-n\eta^2}{2(1-\lambda)}(\theta-(1-\lambda)\bar{X})^2 - \frac{\eta^2}{2}(S+n\lambda\bar{X}^2)\right].$$

By a straightforward calculation

$$E(\eta^\gamma \mid \bar{X}, S) \propto S^{-\frac{1}{2}\gamma}\Gamma\left(\frac{n+\varepsilon+\gamma}{2}\right)\int_U^1 dw(1-w)^{\frac{1}{2}-a}w^{\frac{1}{2}(n+\varepsilon+2\alpha+2a-5)}\,.$$

Thus

$$\eta(\bar{X}, S) = \frac{2^{-\alpha}S^\alpha\Gamma\left(\frac{n+\varepsilon+2\alpha}{2}\right)[\int_U^1 dw(1-w)^{\frac{1}{2}-a}w^{\frac{1}{2}(n+\varepsilon+2\alpha+2a-5)}]}{\Gamma\left(\frac{n+\varepsilon+4\alpha}{2}\right)[\int_U^1 dw(1-w)^{\frac{1}{2}-a}w^{\frac{1}{2}(n+\varepsilon+4\alpha+2a-5)}]}\,,$$

which is of the form (2.1) with

$$\begin{aligned}\phi(U) &= \frac{U^\alpha 2^{-\alpha}\Gamma\left(\frac{n+\varepsilon+2\alpha}{2}\right)[\int_U^1 dw(1-w)^{\frac{1}{2}-a}w^{\frac{1}{2}(n+\varepsilon+2\alpha+2a-5)}]}{\Gamma\left(\frac{n+\varepsilon+4\alpha}{2}\right)[\int_U^1 dw(1-w)^{\frac{1}{2}-a}w^{\frac{1}{2}(n+\varepsilon+4\alpha+2a-5)}]} \\ &= \left(U^\alpha 2^{-\alpha}\Gamma\left(\frac{n+\varepsilon+2\alpha}{2}\right)[\beta(\tfrac{1}{2}(n+\varepsilon+2\alpha+2a-3), \tfrac{3}{2}+a)\right. \\ &\qquad \left. - \beta_U(\tfrac{1}{2}(n+\varepsilon+2\alpha+2a-3), \tfrac{3}{2}+a)]\right) \\ &\qquad \div \left(\Gamma\left(\frac{n+\varepsilon+4\alpha}{2}\right)[\beta(\tfrac{1}{2}(n+\varepsilon+4\alpha+2a-3), \tfrac{3}{2}-a)\right. \\ &\qquad \left. - \beta_U(\tfrac{1}{2}(n+\varepsilon+4\alpha+2a-3), \tfrac{3}{2}-a)]\right)\end{aligned} \tag{3.2}$$

where $\beta_U(\cdot, \cdot)$ denotes the incomplete Beta function.

We now specialize to the case $\alpha = 1$. In this case (3.2) becomes

$$\begin{aligned}\phi(U) &= \frac{U[\beta(\frac{1}{2}(n+\varepsilon+2a-1), \frac{3}{2}-a) - \beta_U(\frac{1}{2}(n+\varepsilon+2\alpha-1), \frac{3}{2}-a)]}{n+\varepsilon+2[\beta(\frac{1}{2}(n+\varepsilon+2a+1), \frac{3}{2}-a) - \beta_U(\frac{1}{2}(n+\varepsilon+2a+1), \frac{3}{2}-a]} \\ &= \frac{U}{(n+\varepsilon+2)}\,\frac{\beta(\frac{1}{2}(n+\varepsilon+2a-1), \frac{3}{2}-a)}{\beta(\frac{1}{2}(n+\varepsilon+2a-1)+1, \frac{3}{2}-a)} \\ &\qquad \times \left\{\frac{[1-I_U(\frac{1}{2}(n+\varepsilon+2a-1), \frac{3}{2}-a)]}{[1-I_U(\frac{1}{2}(n+\varepsilon+2a-1)+1, \frac{3}{2}-a)]}\right\}\end{aligned}$$

where $I_U(a, b) = \beta_U(a, b)/\beta(a, b)$. We now use the fact that (see [1] page 944, equation 26.5.16) $I_U(a, b) = (a\beta(a, b))^{-1}U^a(1-U)^b + I_U(a+1, b)$ to obtain

$$(3.3)\qquad \psi(U) = \frac{U}{(n+\varepsilon+2a-1)}\{(1-(U^{\frac{1}{2}(n+\varepsilon+2a-1)}(1-U)^{\frac{3}{2}-a})) \\ \div([\tfrac{1}{2}(n+\varepsilon+2a-1)\beta(\tfrac{1}{2}(n+\varepsilon+2a-1), \tfrac{3}{2}-a) \\ \times(1-I_U(\tfrac{1}{2}(n+\varepsilon+2a+1), \tfrac{3}{2}-a))])\}\,.$$

If ε and a are chosen so that $\varepsilon + 2a - 1 = 1$ (of course $\frac{3}{2} - a > 0$, also) (3.3) is of the form (2.3) with $1 - D\,\varepsilon(U)U^\delta$ equal to the expression in braces.

We now demonstrate that a subclass of these estimators is minimax. We do so by showing that certain of them satisfy the conditions of Theorem 1 for $\delta = \frac{1}{2}(n-1)$. For this choice of δ

$$(3.4)\qquad \varepsilon(U) = [U(1-U)^{\frac{3}{2}-a}] \\ \div\{[\tfrac{1}{2}(n+1)\beta(\tfrac{1}{2}(n+1), \tfrac{3}{2}-a)][1-I_U(\tfrac{1}{2}(n+3), \tfrac{3}{2}-a)]\}\,.$$

$$(3.5)\qquad [\tfrac{1}{2}(n+1)\beta(\tfrac{1}{2}(n+1), \tfrac{3}{2}-a)/\beta(\tfrac{1}{2}(n+3), \tfrac{3}{2}-a)]\varepsilon(U) \\ = [U(1-U)^{\frac{3}{2}-a}]/\textstyle\int_U^1 dv\, v^{\frac{1}{2}(n+1)}(1-v)^{\frac{1}{2}-a}\,.$$

We first show $\varepsilon(U)$ is non-decreasing.

Letting $w = (1-v)/(1-U)$, the integral in (3.5) becomes

$$(1-U)^{\frac{3}{2}-a}\int_0^1 dw\, w^{\frac{1}{2}-a}[1-(1-U)w]^{(n+1)/2}\,.$$

Hence if $X = 1 - U$, $\varepsilon(U)$ will be non-decreasing if and only if $\bar{\varepsilon}(X)$ is non-decreasing where

$$(3.6)\qquad \bar{\varepsilon}(X) = \frac{\int_0^1 dw\, w^{\frac{1}{2}-a}[1-Xw]^{(n+1)/2}}{(1-X)}\,.$$

But the numerator of derivative of $\bar{\varepsilon}(X)$ is equal to

$$\int_0^1 dw\, w^{\frac{1}{2}-a}[1-Xw]^{(n+1)/2}[1-w]$$

which is negative. This completes the proof that $\varepsilon(U)$ is non-decreasing.

Our generalized Bayes estimator is therefore minimax provided

$$\varepsilon(1) = (3-2a)(n+4-2a)^{-1} \leqq D\left(\frac{n-1}{2}\right),$$

or equivalently, if

$$(3.7)\qquad \max\left\{\tfrac{1}{2}, \tfrac{1}{2}\left[3-(n+4)D\left(\frac{n-1}{2}\right)\right]\left[1-D\left(\frac{n-1}{2}\right)\right]\right\} \leqq a \leqq \tfrac{3}{2}\,.$$

The following table gives values of $D(\frac{1}{2}(n-1))$ and the lower bound a_L for a given by (3.7) for $n = 2(1)\,15$.

n	$D\left(\frac{n-1}{2}\right)$	a_L
2	.144	1.30
3	.145	1.15
4	.144	1.09
5	.133	1.04
6	.122	1.02
7	.112	.99
8	.104	.98
9	.096	.97
10	.090	.96
11	.084	.96
12	.079	.95
13	.074	.94
14	.070	.94
15	.066	.93

Hence we have the following result:

THEOREM 2. *For the case* $\alpha = 1$, *estimators of the form* (2.1) *with* $\phi(U)$ *given by* (3.5) *are generalized Bayes and minimax provided that* $\varepsilon + 2a - 1 = 1$ *and a satisfies* (3.7).

We remark that a similar result can be obtained for integral α by the same method. The result is almost certainly true for non-integral value of α but the author has had difficulty in proving that $\varepsilon(U)$ can be made monotone.

4. Remarks. In Sections 2 and 3 we have produced minimax and generalized Bayes minimax estimators for the case of n independent identically distributed $N(\mu, \sigma^2)$ random variables. Stein [4] treats the problem in a more general canonical form; namely $X_1, \cdots, X_n, Y_1, \cdots, Y_k$ are independent normal random variables with variances σ^2 and $EX_i = 0$, $EY_j = \theta_j$. The methods of Sections 2 and 3 apply to this more general setting as well. In particular, if $S = \sum_1^n X_i^2$, $T = \sum_1^k Y_j^2$, $U = S/(S/T)$, then $\eta(S, T) = \phi(S/(S+T))(S+T)^\alpha = CU^\alpha(1 - U^\delta\varepsilon(U))$ is minimax for the loss function (1.1) provided $0 \leqq \varepsilon(U) \leqq D(k, \delta)$ and $\varepsilon(U)$ is non-decreasing where

$$D^*(k, \delta) = \min\left\{ 2\alpha/(\alpha+\delta), \; \frac{2\left[\beta\left(\frac{n}{2}+\alpha, \alpha\right)\beta\left(\frac{n}{2}+2\alpha+\delta, \frac{k}{2}\right) - \beta\left(\frac{n}{2}+\alpha+\delta, \frac{k}{2}\right)\beta\left(\frac{n+k}{2}+\alpha, \alpha\right)\right]}{\beta\left(\frac{n}{2}+\alpha, \alpha\right)\beta\left(\frac{n}{2}+2\alpha+2\delta, \frac{k}{2}\right)} \right\}.$$

Furthermore, consider the following prior distribution: Conditional on (λ, η), the distribution of $\boldsymbol{\theta} = (\theta_1, \cdots, \theta_k)$ is $N_k(\mathbf{O}, \eta^{-2}\lambda^{-1}(1-\lambda)I_k)$. The (unconditional) generalized density of (λ, η) is then given by $g(\lambda, \eta) = \lambda^{-a}\eta^{-1+\varepsilon}$, $0 < \lambda < 1$, $\eta > 0$. The generalized Bayes estimator of $\sigma^{2\alpha}$ with respect to the above prior is given by

$$\frac{U^\alpha 2^{-\alpha}\Gamma\left(\frac{n+k+\varepsilon+2\alpha}{2}\right)}{\Gamma\left(\frac{n+k+\varepsilon+4\alpha}{2}\right)} \frac{\int_U^1 dw(1-w)^{\frac{1}{2}k-a}w^{\frac{1}{2}(n+\varepsilon+2\alpha+2a-4)}}{\int_U^1 dw(1-w)^{\frac{1}{2}k-a}w^{\frac{1}{2}(n+\varepsilon+4\alpha+2a-4)}}\,.$$

If we specialize to the case $\alpha = 1$, and require $\varepsilon + 2a = 2$, we find that the above generalized Bayes estimator is minimax provided that $[(k+2) - (n+k+4)D(k, \frac{1}{2}n)][1 - D(k, \frac{1}{2}n)]^{-1} \leqq a \leqq \frac{1}{2}(k+2)$. The proofs of all of the above statements are identical modulo change of notation, to the corresponding results of Section 2 and 3.

The results of Sections 2 and 3 constitute an extension of the results of Stein [4] in the sense that his estimator (6), i.e.

$$\min\left(\frac{1}{n+1}\sum_1^n (X_i - \bar{X})^2, \frac{1}{n+2}\sum_1^n X_i^2\right)$$

is of the form given by Theorem 1 with $\delta = n+1$.

Brown [3] also has results on the inadmissiblity of estimators of powers of a scale parameter in the presence of an unknown location parameter which can be extended in the normal case. We briefly indicate one such extension. Brown gives the following class of estimators which beat the usual estimator for $\sigma^{2\alpha}$.

$$\begin{aligned} \delta_K(\bar{X}, S) &= C_1(K)^{2\alpha} && \text{if } |\bar{X}/S| < K \\ &= C_2 S^{2\alpha} && \text{if } |\bar{X}/S| > K \end{aligned}$$

where C_2 is the multiplier for the usual best invariant estimator and $C_1(K)$ is given by

$$C_1(K) = \frac{[\int S^{2\alpha}[\Phi(n^{\frac{1}{2}}KS) - \Phi(-n^{\frac{1}{2}}KS)]f(S)\,dS]}{[\int S^{4\alpha}[\Phi(n^{\frac{1}{2}}KS) - \Phi(n^{\frac{1}{2}}KS)]f(S)\,dS]} \tag{4.1}$$

where $\Phi(\cdot)$ is the cdf of a standard normal distribution and $f(\cdot)$ is the density of a χ^2 variable with $n-1$ df. In fact it is clear from Brown that we may take $C_1(K)$ to be any value between $C_1'(K) = C_2 - 2(C_2 - C_1(K))$ and C_2 since the expression leading to (4.1) ((6.2) of Brown) is a quadratic with minimum at $C_1(K) < C_2$. It is also clear then that any estimator of the form $\delta_F(\bar{X}, S) = \int_0^\infty \delta_K(\bar{X}, S)\,dF(K)$ is minimax provided that $F(\cdot)$ is the cdf of a probability measure concentrated on $[0, \infty]$. For such an estimator we have

$$\begin{aligned} \delta_F(\bar{X}, S) &= \delta^{2\alpha}[\textstyle\int_0^\infty [C_1(K)I_{[0,K]}(\bar{X}/S) + C_2 I_{(K,\infty)}(\bar{X}/S)]\,dF(K) \\ &= S^{2\alpha}[\textstyle\int_{\bar{X}/S} C_1(K)\,dF(K) + aF(\bar{X}/S)] = S^{2\alpha}H(\bar{X}/S)\,. \end{aligned}$$

It is easy to see that $H(\cdot)$ is monotone non-decreasing and bounded above by C_2. In addition, if we require $H(\cdot)$ to be continuously differentiable, (which is a reasonable requirement if we are searching for generalized Bayes minimax estimators) it is easy to show that we must have

$$\int_0^\infty dKH'(K)/[C_2 - C_1'(K)] \leqq 1 . \tag{4.2}$$

It is interesting to note that if we take $H(K) = C_1(K)$ where $C_1(K)$ is given by (4.1) that the resulting estimator is identical to the estimator (3.3) with $a = 1$. (It is straightforward but tedious to show this.) However, the method of this section will not suffice to show minimaxity of this estimator, since for this choice of $H(\cdot)$ we have divergence of the integral in (4.2).

Brewster and Zidek [2] have extended the result of Brown [3] in a way that includes the above remarks, by partitioning the real line into K sets and applying Brown's technique inductively. Their procedure suffices to show minimaxity of the estimator in the previous paragraph.

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STATISTICS CENTER
RUTGERS UNIVERSITY
NEW BRUNSWICK, N. J. 08903