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# THE MULTITYPE GALTON-WATSON PROCESS WITH IMMIGRATION[1]

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A multitype branching process with immigration is considered. A necessary and sufficient condition is found for when the process has a limiting stationary distribution. As a corollary to this result we are able to give necessary and sufficient conditions for when the extinction time for an ordinary multitype process has finite expectation.

**1. Introduction.** The theory of a single type Galton Watson process with immigration has been treated by many authors. Heathcote [4], [5] and Seneta [10] among others have found under certain assumptions, necessary and sufficient conditions for the process to possess a stationary limiting distribution. The most general result of this nature appears in a paper by Foster and Williamson [2]. It is the purpose of this note to extend the results of Heathcote and Seneta to a multitype process. In Section 2, we present a necessary and sufficient condition for the existence of a limiting distribution in the multitype case. As a corollary of this result, we give a necessary and sufficient condition for when the extinction time of a multitype process has finite expectation. This result is a direct generalization of a theorem of Seneta [9].

We now introduce our notation and formally define the process of interest. For any square matrix $A$ we write as usual, $A > 0$ if all elements of $A$ are positive, $A \geqq 0$ if all elements of $A$ are nonnegative. We call $X$ the set of all $p$ tuples $i = (i_1, \cdots, i_p)'$ where each element $i_v$ is some nonnegative integer. $e_\alpha$ denotes the vector with $i_v = \delta(\alpha, v)$, $v = 1, \cdots, p$ where $\delta(\cdot, \cdot)$ is the Kronecker delta. The $p$-dimensional unit cube of points $s = (s_1, \cdots, s_p)'$, $0 \leqq s_v \leqq 1$, $v = 1, \cdots, p$, is denoted by $C$. For a given $s \in C$ and $i \in X$, we put $s^i = \prod_{v=1}^{p} s_v^{i_v}$. We write $Ds$ for the vector whose components are all equal to $s$, $0 \leqq s \leqq 1$. For ease of notation we write $\mathbf{0}$ for $D0$ and $\mathbf{1}$ for $D1$.

For our process we consider a Markov chain $\{Z_n\}_{n\geqq 0}$ defined on $A$ consisting of a $p$-type Galton Watson process $\{Y_n = (Y_{n1}, \cdots, Y_{np})\}_{n\geqq 0}$ (the offspring distribution) augmented by an independent random $p$-dimensional immigration component at each generation. For the offspring process, we assume given, an offspring vector of $p$-dimensional probability generating functions (pgf)

$$F(s) = F_1(s) = (F_{11}(s), \cdots, F_{1p}(s))$$

with

$$F_{1j}(s) = \sum_{i \in X} f_j(i) s^i \qquad 1 \leqq j \leqq p\,.$$

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The $f_j(i)$ are interpreted as

$$f_j(i) = P\{Y_1 = i \mid Y_0 = e_j\}. \qquad i \in X,\ 1 \leqq j \leqq p.$$

It is well known ([3] page 36) that the pgf vector of the $n$th generation of an ordinary $p$-type branching process is the $n$th functional iterate of $F(s)$, $F_n(s) = (F_{n1}(s), F_{n2}(s), \cdots, F_{np}(s))$. It is consistent with this to define $F_0(s) = s$. For the immigration component, we are given the pgf.

$$B(s) = \sum_{i \in X} b(i)s^i \qquad s \in C.$$

It is assumed that $B(\mathbf{0}) < 1$, i.e. that some immigration occurs with positive probability. Define:

$$m_{ij} = \frac{\partial F_{1i}}{\partial s_j}(\mathbf{1}) = E\{Y_{1j} \mid Y_0 = e_i\}.$$

Let $M$ be the matrix $(m_{ij})_{i,j=1}^p$. It is obvious that $M \geqq 0$. Throughout this paper, we make the following assumption.

Assumption 1. *The elements of $M$ are all finite and $M$ is positive-regular, i.e. there exists an integer $n_0$ such that $M^{n_0} > 0$.*

It is proven in [7] that if $M$ is positive-regular then $M$ has an eigenvalue $\rho$, which is real, positive, simple and greater in modulus that any other eigenvalue. Let $u$ and $v$ be the right and left eigenvectors of $\rho$ respectively, normalized so that $(u, \mathbf{1}) = 1$ and $(v, \mathbf{1}) = 1$. It is also proven in [7] that $u > 0$ and $v > 0$.

It has been shown in [3] that the value of $\rho$ determines the asymptotic behavior of the $p$-type Galton Watson process $\{Y_n\}$. As is usual, we will call the $\{Z_n\}$ process supercritical, critical or subcritical, depending on whether $\rho > 1 = 1$ or $< 1$. When $\rho = 1$, we will always rule out the case $F_1(s) = Ms$, $s \in C$. Define:

$$P_n(i, s) = \sum_{j \in X} s^j P\{Z_n = j \mid Z_0 = e_i\} \qquad s \in C. \tag{1.1}$$

It has been shown in [8] that

$$P_n(i, s) = F_{ni}(s) \cdot \prod_{r=0}^{n-1} B(F_r(s)). \tag{1.2}$$

Our main result gives necessary and sufficient conditions for when $\lim_{n\to\infty} P_n(i, s)$ is nontrivial.

Theorem 1. *A necessary and sufficient condition for the $\{Z_n\}$ process to satisfy for each $j \in X$*

$$\lim_{n\to\infty} P\{Z_n = j \mid Z_0 = e_i\} = \prod(j)$$

*independent of $i \in X$ where $\sum_{j \in X} \prod(j) = 1$ is*

$$\int_0^1 \frac{1 - B(Ds)}{(v, F_1(Ds)) - s}\, ds < \infty. \tag{1.3}$$

If $p = 1$, condition (1.3) reduces exactly to the condition given in [2], [9]. Condition (1.3) is the same as for the univariant Galton Watson process with immigration having offspring pgf $(v, F_1(Ds))$ and immigration pgf $B(Ds)$.

We can obtain two interesting corollaries of Theorem 1.

COROLLARY 1. *Assume* $\rho < 1$. *Then the result of Theorem* 1 *is valid* iff

$$\sum_{j \in X-\{0\}} b(j) \log |j| < \infty \quad (|j| = \sum_{i=1}^{p} j_i) .$$

Corollary 1 is a result proven by Quine [8]. Our second Corollary deals with the $p$-type Galton Watson process $\{Y_n\}$. Define:

$$T = \inf\{k : Y_k = \mathbf{0}\}$$

i.e., $T$ is the time to extinction. Then:

COROLLARY 2.

$$E_i(T) = E\{T \mid Y_0 = e_i\} < \infty \qquad \textit{for some} \quad i \quad \text{iff} \quad \int_0^1 \frac{1-s}{(v, F_1(Ds)) - s}\, ds < \infty ,$$

*in which case* $E_i(T) < \infty$ *for all* $i$.

Corollary 2 generalizes a result of Seneta [9] to $p$-dimensions.

**2. Proofs of results.** Without loss of generality, we can assume that $\rho \leqq 1$. If $\rho > 1$, then it is a simple matter to show that $Z_n \to \infty$ w.p.1. To prove Theorem 1, we must show that $P(s) = \lim_{n\to\infty} P_n(i, s)$ is nonzero. Since we are assuming that $\rho \leqq 1$, $\lim_{n\to\infty} F_{ni}(s) = 1$, [5]. Therefore, from (1.2)

$$P(s) = \lim_{n\to\infty} P_n(i, s) = \lim_{n\to\infty} [\prod_{r=0}^{n-1} B(F_r(s))] . \tag{2.1}$$

(2.1) shows that $P(s)$ is indeed independent of the choice of $Z_0$. Since $0 \leqq B(F_r(s)) \leqq 1$ for all $r \geqq 0$, a necessary and sufficient condition for

$$\lim_{n\to\infty} [\prod_{r=0}^{n-1} B(F_r(s))] > 0$$

is $\sum_{r=0}^{\infty} 1 - B(F_r(s)) < \infty$. This last sum converges if and only if

$$\sum_{r=0}^{\infty} 1 - B(F_r(\mathbf{0})) < \infty . \tag{2.2}$$

This is a consequence of the fact that there exists an integer $l = l(s)$ such that $s \leqq F_{li}(\mathbf{0})$, $i \leqq i \leqq p$ and hence $F_r(\mathbf{0}) \leqq F_r(s) \leqq F_{r+l}(\mathbf{0})$, $r \geqq 1$.

The next series of lemmas will prove that (2.2) holds iff (1.3) is valid. Define:

$$h(s) = \frac{1 - B(Ds)}{(v, F_1(Ds)) - s} \qquad s \in [0, 1) .$$

LEMMA 2.1. $h(s)$ *is a positive increasing function for* $s \in [0, 1)$.

PROOF. Since $(v, \mathbf{1}) = 1$, $(v, F_1(Ds))$ is a legitimate 1-dimensional pgf. Furthermore,

$$\begin{aligned} \frac{d}{ds}(v, F_1(Ds))\Big|_{s=1-} &= (vM, \mathbf{1}) \\ &= \rho(v, \mathbf{1}) = \rho . \end{aligned}$$

Since $\rho$ is assumed to be less than or equal to one, we can appeal to the argument given in [2] to prove the result. □

Define:

$$g(s) = (v, F_1(s) - s) \qquad s \in C.$$

LEMMA 2.2. *$g(s)$ is decreasing in $s$, ($s \in C$) i.e. if $s_1 \leqq s_2$ then $g(s_1) \geqq g(s_2)$.*

PROOF. Joffe and Spitzer [6] showed that if Assumption 1 holds then there exists a matrix $E(s)$ such that

$$\mathbf{1} - F_1(s) = (M - E(s))(\mathbf{1} - s) \qquad s \in C. \tag{2.3}$$

The matrix $E(s)$ has the properties that

(a) $0 < E(s) \leqq M$ if $s > 0$, and
(b) $E(s)$ is "monotone", i.e. $t \geqq s \Rightarrow E(t) \leqq E(s)$.

Therefore:

$$\begin{aligned} g(s) &= (v, F_1(s) - s) = (v, \mathbf{1} - s) - (v, \mathbf{1} - F_1(s)) \\ &= (v, \mathbf{1} - s) - (v, (M - E(s))(\mathbf{1} - s)) \\ &= (1 - \rho)(v, \mathbf{1} - s) + (v, E(s)(\mathbf{1} - s)). \end{aligned}$$

However, both of the final terms on the right are decreasing in $s$. □

Define:

$$\alpha_n = \min_{1 \leqq i \leqq p} F_{ni}(\mathbf{0}) \qquad \text{and} \qquad \beta_n = \max_{1 \leqq i \leqq p} F_{ni}(\mathbf{0}).$$

LEMMA 2.3. *There exist constants $N_0$, $K_1$ and $K_2$ such that for all $n > N_0$*

$$\int_{\alpha_n}^{\alpha_{n+1}} h(s)\, ds \leqq K_1(1 - B(F_{n+1}(\mathbf{0}))) \tag{2.4}$$

*and*

$$\int_{\beta_n}^{\beta_{n+1}} h(s)\, ds \geqq K_2(1 - B(F_n(\mathbf{0}))). \tag{2.5}$$

PROOF. It follows from the properties of $F_n(\mathbf{0})$ that the $\{\alpha_n\}$ and $\{\beta_n\}$ are increasing sequences of real numbers. Also, since $\rho \leqq 1$, $\lim_{n\to\infty} F_n(\mathbf{0}) = \mathbf{1}$, [6]. Therefore,

$$\lim_{n\to\infty} \alpha_n \uparrow 1 \qquad \text{and} \qquad \lim_{n\to\infty} \beta_n \uparrow 1.$$

We will first establish (2.4).

By Lemma 2.1

$$\int_{\alpha_n}^{\alpha_{n+1}} h(s)\, ds \leqq h(\alpha_{n+1})(\alpha_{n+1} - \alpha_n). \tag{2.6}$$

By Lemma 2.2

$$\begin{aligned} (v, F_1(D\alpha_{n+1}) - D\alpha_{n+1}) &\geqq (v, F_1(F_{n+1}(\mathbf{0})) - F_{n+1}(\mathbf{0})) \\ &= (v, F_{n+2}(\mathbf{0}) - F_{n+1}(\mathbf{0})). \end{aligned} \tag{2.7}$$

Also:

$$\frac{1 - B(D\alpha_{n+1})}{1 - \alpha_{n+1}} \leqq \frac{1 - B(D\beta_{n+1})}{1 - \beta_{n+1}} \leqq \frac{1 - B(F_{n+1}(\mathbf{0}))}{1 - \beta_{n+1}}. \tag{2.8}$$

From (2.6), (2.7) and (2.8), we obtain

$$\int_{\alpha_n}^{\alpha_{n+1}} h(s)\, ds \leqq [1 - B(F_{n+1}(\mathbf{0}))]\left[\frac{1 - \alpha_{n+1}}{1 - \beta_{n+1}}\right]\left[\frac{\alpha_{n+1} - \alpha_n}{(v, F_{n+2}(\mathbf{0}) - F_{n+1}(\mathbf{0}))}\right].$$

To prove (2.4) we need to show that for some $N_0$.

$$\sup_{n \geqq N_0}\left[\frac{1-\alpha_n}{1-\beta_n}\right] < \infty \tag{2.9}$$

and

$$\sup_{n \geqq N_0}\left[\frac{\alpha_{n+1}-\alpha_n}{(v, F_{n+2}(\mathbf{0}) - F_{n+1}(\mathbf{0}))}\right] < \infty\,. \tag{2.10}$$

Since $\lim_{n\to\infty} F_n(\mathbf{0}) = \mathbf{1}$, there exists an integer $n_1$ such that $F_{n_1}(\mathbf{0}) > \mathbf{0}$. Set $N_0 = n_1 + n_0$ and let $n > N_0$. If we apply (2.3) to the pgf $F_{n_0}(s)$ we obtain the existence of a matrix $E_{n_0}(s)$ such that

$$\mathbf{1} - F_{n_0}(s) = (M_{n_0} - E_{n_0}(s))(\mathbf{1} - s) \qquad s \in C \tag{2.11}$$

where

$$M_{n_0} = \left(\frac{\partial F_{n_0 i}}{\partial s_j}(\mathbf{1})\right)_{1 \leqq i,j \leqq p}$$

Put $s = F_{n-n_0}(\mathbf{0})$. It follows from the properties of $E_{n_0}(s)$ that

$$M_{n_0} - E_{n_0}(F_{n_1}(\mathbf{0})) \leqq M_{n_0} - E_{n_0}(F_{n-n_0}(\mathbf{0})) \leqq M_{n_0} \qquad n \geqq N_0\,. \tag{2.12}$$

Due to Assumptions 1, our choice of $n_0$ and $n_1$, and (2.12) we can assert that there exists positive constants $L_1$ and $L_2$ such that

$$L_1 \leqq M_{n_0} - E_{n_0}(F_{n-n_0}(\mathbf{0})) \leqq L_2 \qquad n \geqq N_0\,.$$

Using (2.11) it is not difficult to show that

$$\frac{1-\alpha_n}{1-\beta_n} \leqq \frac{L_2}{L_1} < \infty \qquad n \geqq N_0\,.$$

This proves (2.9). To prove (2.10) we argue as follows. Assume $\alpha_n = F_{ni}(\mathbf{0})$ where $i$ depends on $n$. Then

$$\gamma_n = \frac{\alpha_{n+1}-\alpha_n}{(v, F_{n+2}(\mathbf{0}) - F_{n+1}(\mathbf{0}))} \leqq \frac{F_{n+1i}(\mathbf{0}) - F_{ni}(\mathbf{0})}{v_i[F_{n+2i}(\mathbf{0}) - F_{n+1i}(\mathbf{0})]} \qquad n \geqq N_0\,.$$

The Mean Value Theorem for $p$-dimensions ([1], page 117) gives for each $1 \leqq i \leqq p$

$$F_{n+1i}(\mathbf{0}) - F_{ni}(\mathbf{0}) = \textstyle\sum_{j=1}^{p} \dfrac{\partial F_{n_0 i}}{\partial s_j}(S_n)(F_{n-n_0+1j}(\mathbf{0}) - F_{n-n_0 j}(\mathbf{0}))$$

and

$$F_{n+2i}(\mathbf{0}) - F_{n+1i}(\mathbf{0}) = \textstyle\sum_{j=1}^{p} \dfrac{\partial F_{n_0+1i}}{\partial s_j}(T_n)(F_{n-n_0+1j}(\mathbf{0}) - F_{n-n_0 j}(\mathbf{0})) \qquad n \geqq N_0$$

where

$$F_{n-n_0}(\mathbf{0}) \leqq S_n,\, T_n \leqq F_{n-n_0+1}(\mathbf{0})\,.$$

It follows by monotonicity that

$$\frac{\partial F_{n_0 i}}{\partial s_j}(S_n) \leqq \max_{1 \leqq i,j \leqq p} \frac{\partial F_{n_0 i}}{\partial s_j}(\mathbf{1}) = L_4$$

and

$$\frac{\partial F_{n_0+1i}}{\partial s_j}(T_n) \geqq \min_{1\leqq i,j\leqq p} \frac{\partial F_{n_0+1i}}{\partial s_j}(F_{n_1}(\mathbf{0})) = L_3 \qquad n \geqq N_0 .$$

Due to Assumption 1 and our choices of $n_0$ and $n_1$, $0 < L_3 < L_4 < \infty$. Thus

$$\gamma_n \leqq \frac{L_4}{L_3} \frac{1}{\min_{1\leqq i\leqq p} v_i} < \infty \qquad n \geqq N_0 .$$

This proves (2.10). The proof of (2.5) is done in a similar way and the details will be omitted. □

We are now in a position to prove Theorem 1. Since the $\{\alpha_n\}$ and the $\{\beta_n\}$ are increasing,

$$\int_0^1 h(s)\,ds < \infty \Leftrightarrow \sum_{n=1}^\infty \int_{\alpha_n}^{\alpha_n+1} h(s)\,ds < \infty$$
$$\Leftrightarrow \sum_{n=1}^\infty \int_{\beta_n}^{\beta_n+1} h(s)\,ds < \infty .$$

However, by Lemma 2.3

$$\sum_{n=1}^\infty \int_{\beta_n}^{\beta_n+1} h(s)\,ds < \infty \Rightarrow \sum_{n=1}^\infty \{1 - B(F_n(\mathbf{0}))\} < \infty$$

and

$$\sum_{n=1}^\infty \{1 - B(F_n(\mathbf{0}))\} < \infty \Rightarrow \sum_{n=1}^\infty \int_{\alpha_n}^{\alpha_n+1} h(s)\,ds < \infty .$$

All that remains to be proven is that $\sum_{j\in X} \prod(j) = 1$ or equivalently $\lim_{s\to 1-} P(s) = 1$. To do this, one can proceed exactly as in [8]. The details are omitted.

We now turn to the proofs of Corollary 1 and Corollary 2. Consider first Corollary 1. It is easy to show that

$$\lim_{s\to 1} \frac{\dfrac{1 - B(Ds)}{(v, F_1(Ds)) - s}}{\dfrac{1 - B(Ds)}{1 - s}} = (1 - \rho)^{-1} .$$

Therefore, if $\rho < 1$,

$$\int_0^1 h(s)\,ds < \infty \qquad \text{iff} \quad \int_0^1 \frac{1 - B(Ds)}{1 - s}\,ds < \infty .$$

But

$$\int_0^1 \frac{1 - B(Ds)}{1 - s}\,ds = \sum_{j\in X} h(j) \int_0^1 \frac{1 - s^{|j|}}{1 - s}\,ds$$
$$= \sum_{j\in X} h(j)\Big(\sum_{i=1}^{|j|} 1/i\Big) .$$

Therefore:

$$\int_0^1 \frac{1 - B(Ds)}{1 - s}\,ds < \infty \qquad \text{iff} \quad \sum_{j\in X-\{0\}} b(j) \log|j| < \infty .$$

This proves Corollary 1. To prove Corollary 2, we observe that

$$E_i(T) = \sum_{n=1}^\infty P\{T \geqq n \mid Y_0 = e_i\}$$
$$= \sum_{n=1}^\infty [1 - F_{ni}(\mathbf{0})] .$$

Choose $B(s) = s_i$. Then,

$$\sum_{n=1}^\infty 1 - B(F_n(\mathbf{0})) = \sum_{n=1}^\infty 1 - F_{ni}(\mathbf{0}) .$$

So by Theorem 1,

$$\sum_{n=1}^{\infty} 1 - F_{ni}(0) < \infty \qquad \text{iff} \quad \int_0^1 \frac{1-s}{(v, F_1(Ds)) - s}\, ds \, .$$

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