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ON ONE-DIMENSIONAL DIFFUSIONS WITH TIME PARAMETER SET $(-\infty, \infty)$

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Let p_t, $t \geqq 0$ be the probability transition semigroup for a continuous one-dimensional diffusion. We examine continuous Markov processes ξ_s, defined for all $-\infty < s < \infty$, which are governed by p_t. We determine necessary and sufficient conditions for the set of such processes governed by p_t to be nontrivial, and give an example where these conditions are satisfied.

0. Introduction. In [5] and [6] Dynkin uses his work on excessive functions and measures to develop a theory for processes ξ_s ($s \in T$, an interval of real numbers) which are Markovian between random birth time α and death time β. We propose to study a subclass of these processes, obtaining results similar to those in [4]. We consider the case $T = \mathbb{R}$, $\alpha = -\infty$, $\beta = +\infty$, with ξ_s taking values in an interval I. Thus we are looking at one-dimensional diffusions ξ_s, defined for all $-\infty < s < \infty$, which did not "begin" at any instant in time, but have instead been forever evolving according to some fixed (time homogeneous) transition mechanism.

Section 1 presents the necessary definitions and framework. Section 2 deals with the extreme points of the set of measures corresponding to the processes ξ_s; it presents a theorem of Dynkin's, a characterization of the extreme measures, and some results on the boundary theory involved. In Section 3 we give a necessary and sufficient condition for a transition semigroup to have associated nontrivial processes ξ_s. Finally, a manageable sufficiency condition is given and examples provided.

1. Notation and definitions. Let I be an interval of real numbers, $\mathscr{B}$ the Borel subsets of I. For convenience we assume $r_1 < 0 < r_2$, r_1, r_2 are the endpoints of I (which may or may not belong to I). Let $p_t = \{p_t(x, B) : t \geqq 0, x \in I, B \in \mathscr{B}\}$ be the probability transition semigroup of a diffusion $\mathbf{P} = \{P_x\}_{x \in I}$ on I as defined in [7]. X_t ($t \geqq 0$) will denote the process, P_x is the p.m. (probability measure) on the path space satisfying $P_x[X_0 = x] = 1$, and E_x is the corresponding expectation operator. Throughout this paper p_t is assumed to be *regular*, i.e., if $r_1 < x < r_2$, $y \in I$, then $P_x[\tau_y < \infty] > 0$, where τ_y is the usual first hitting time of y.

p_t has an associated speed measure m and scale function S, as described in [7]. We mention here that S is a strictly increasing, continuous function defined on

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I such that for all $x \in [a, b] \subset I$,

$$P_x[\tau_a < \tau_b] = \frac{S(b) - S(x)}{S(b) - S(a)}\,. \tag{1.1}$$

If $r_i \notin I$, let $S(r_i) = \lim_{x \to r_i} S(x)$.

Let $C_b(I)$ be the bounded continuous functions defined on I, and $C_c(I)$ those members of $C_b(I)$ with compact support. If $f \in C_b(I)$, the map

$$(x, t) \to E_x[f(X_t)] \tag{1.2}$$

is jointly continuous.

p_t is now regarded as fixed, and we wish to investigate continuous, strong Markov processes ξ_s, $s \in \mathbb{R}$, which have transition probabilities p_t. To this end we let Ω be the set of continuous functions from $\mathbb{R}$ to I, ξ_s the coordinate map, $\xi_s(\omega) = \omega(s)$, $\mathscr{F} = \sigma[\xi_s, s \in \mathbb{R}]$, $\mathscr{F}_s = \sigma[\xi_u, u \leqq s]$, and $\mathscr{F}_{-\infty} = \bigcap_s \mathscr{F}_s$.

DEFINITION 1.1. $\mathscr{M}(p_t)$ is the set of all p.m.'s $\mathscr{P}$ on $(\Omega, \mathscr{F})$ such that for all $s \in \mathbb{R}$, $t \geqq 0$, $B \in \mathscr{B}$,

$$\mathscr{P}[\xi_{s+t} \in B \mid \mathscr{F}_s] = p_t(\xi_s, B) \quad \mathscr{P}\text{-a.e.} \tag{1.3}$$

It is not difficult to show that (1.3) is equivalent to

$$\begin{aligned}\mathscr{P}[\xi_{s_1} &\in B_1, \cdots, \xi_{s_n} \in B_n] \\ &= \int_{B_1} \cdots \int_{B_n} \mathscr{P}[\xi_{s_1} \in dx_1] p_{s_2 - s_1}(x_1, dx_2) \cdots p_{s_n - s_{n-1}}(x_{n-1}, dx_n)\end{aligned}$$

for $s_1 < \cdots < s_n$, $B_i \in \mathscr{B}$. If we set $\mu_s(B) = \mathscr{P}[\xi_s \in B]$, and write $\nu p_t(B) = \int \nu(dx) p_t(x, B)$, then

$$\mu_s p_t = \mu_{s+t}\,, \qquad s \in \mathbb{R},\ t \geqq 0\,. \tag{1.4}$$

Conversely, if μ_s is any family of p.m.'s on $(I, \mathscr{B})$ satisfying (1.4), then μ_s and p_t define an element $\mathscr{P} \in \mathscr{M}(p_t)$. $\{\mu_s\}_{s \in \mathbb{R}}$ is an entrance law for p_t.

$\mathscr{P}$ will be called trivial if $\mathscr{P}[\xi_s \in B]$ is independent of s for all $B \in \mathscr{B}$, otherwise $\mathscr{P}$ is nontrivial. If p_t has an invariant p.m. ν, $\nu p_t = \nu$, then $\mathscr{M}(p_t)$ contains the trivial $\mathscr{P}$. $\mathscr{M}(p_t)$ will be called nontrivial if it contains a nontrivial element. An example presented in Section 3 shows that $\mathscr{M}(p_t)$ can indeed be nontrivial. We regard $\mathscr{M}(p_t)$ as the proper realization of the intuitive description given in the introduction, and will now proceed to study $\mathscr{M}(p_t)$.

2. The extreme points. Let $\mathscr{M}(p_t)$ have the topological σ-algebra determined by weak convergence (see [1]). $\mathscr{M}(p_t)$ is convex, and $\mathscr{M}_e(p_t)$ will be the set of extreme points. The following, due to Dynkin, is a special case of Theorem 2.1 in [5].

THEOREM 2.1. *$\mathscr{M}_e(p_t)$ is a measurable subset of $\mathscr{M}(p_t)$. Each $\mathscr{P} \in \mathscr{M}(p_t)$ is uniquely represented in the form*

$$\mathscr{P} = \int_{\mathscr{M}_e(p_t)} \mathscr{P}'\, d\lambda(\mathscr{P}') \tag{2.1}$$

for some p.m. λ *on* $\mathscr{M}_e(p_t)$. *Furthermore,* $\mathscr{P} \in \mathscr{M}_e(p_t)$ iff $\mathscr{F}_{-\infty}$ *is* $\mathscr{P}$*-trivial,* i.e., $\mathscr{P}(A) = 0$ *or* 1 *for all* $A \in \mathscr{F}_{-\infty}$.

It will be convenient in what follows to work with *vague convergence* ($\to_v$) of sub-p.m.'s as defined in [2] (i.e., $\nu_n \to_v \nu$ if $\int f\,d\nu_n \to \int f\,d\nu$ for all $f \in C_c$). The next result gives more information about $\mathscr{M}_e(p_t)$.

PROPOSITION 2.2. (i) *If $\mathscr{P} \in \mathscr{M}_e(p_t)$, then there is a continuous function z: $\mathbb{R} \to I$ such that for all $s \in \mathbb{R}$,*

$$p_{s+t}(z(t), dy) \to_v \mathscr{P}[\xi_s \in dy] \tag{2.2}$$

as $t \to +\infty$. If $\mathscr{P}$ is nontrivial, then $z(t) \to r$, where $r = r_1$ or $r = r_2$, and $r \notin I$, $|S(r)| = \infty$.

(ii) *If $\mathscr{P} \in \mathscr{M}_e(p_t)$ is nontrivial, $[a, b] \subset I$, then $\mathscr{P}[a \leqq \xi_s \leqq b] \to 0$ as $s \to -\infty$.*

Before turning to the proof we list a few preliminary results. As shown in [7] (Section 2.7) and [8] (Theorems IV.4, IV.6, IV.7), for each $x \in I$ there is a sub-p.m. π_x

$$p_t(x, dy) \to_v \pi_x(dy) \tag{2.3}$$

as $t \to \infty$. π_x may or may not depend on x, and $\pi_x(I) = 0$ is possible.

LEMMA 2.3. (i) *Assume $x(n) \to x \in I$, $t(n) \to \infty$. Then $p_{t(n)}(x(n), dy) \to_v \pi_x(dy)$ as $n \to \infty$.*

(ii) *Assume $r \notin I$, $|S(r)| = \infty$, $r = r_1$ or $r = r_2$. Then for all $x \in I$, $p_t(x, dy) \to_v \pi(dy)$ independently of x, where $\pi(I) = 0$ or $\pi(I) = 1$.*

(iii) *As in* (ii), *assume also that K is a compact continuity set for π, $[a, b] \subset I$. Then* $\sup_{a \leqq x \leqq b} |p_t(x, K) - \pi(K)| \to 0$.

PROOF. We omit the details. Only equations (1.1) and (2.3) above, and Corollary 2.18 in [7] are used. □

LEMMA 2.4. *Fix $s \in \mathbb{R}$, $f \in C_c(I)$, $\mathscr{P} \in \mathscr{M}(p_t)$, define Z_u by*

$$Z_u = \mathscr{E}[f(\xi_s) \mid \mathscr{F}_u], \qquad u \leqq s.$$

Then $(Z_u, \mathscr{F}_u)_{u \leqq s}$ is a martingale with continuous sample paths, and

$$Z_u = \int_I p_{s-u}(\xi_u, dy) f(y). \tag{2.4}$$

PROOF. $\mathscr{E}$ denotes expectation wrt $\mathscr{P}$. The first assertion follows from the definition of Z_u. Equation (2.4) holds because of equation (1.3). Finally equations (1.2) and (2.4) and the fact that ξ_u is continuous imply that Z_u is continuous. □

The martingale argument which follows was used by Dynkin in [6] and [7] in a more general setting to derive his integral representation theorem. In addition, equation (2.2) falls easily from his construction.

PROOF OF PROPOSITION 2.2. (i) Define Z_u as above. By the martingale convergence theorem there is an $\mathscr{F}_{-\infty}$-measurable random variable $Z_{-\infty}$ such that $Z_u \to Z_{-\infty}$ $\mathscr{P}$-a.e. as $u \to -\infty$. By Dynkin's theorem $\mathscr{F}_{-\infty}$ is $\mathscr{P}$-trivial, which means $Z_{-\infty}$ is a constant ($\mathscr{P}$-a.e.) and

$$Z_{-\infty} = \mathscr{E}[Z_{-\infty}] = \int_I \mathscr{P}[\xi_s \in dy] f(y).$$

Let Q be the rationals, D a countable dense (in the topology of uniform convergence) subset of $C_c(I)$. Then there is a subset $\Omega_0 \subset \Omega$, $\mathscr{P}(\Omega_0) = 1$, such that

$$\int_I p_{s-u}(\xi_u, dy) f(y) \to \int_I \mathscr{P}[\xi_s \in dy] f(y) \tag{2.5}$$

as $u \to -\infty$ for all $\omega \in \Omega_0$, $s \in \mathbb{R}$, $f \in D$. Standard arguments imply the truth of (2.5) for all $\omega \in \Omega_0$, $s \in \mathbb{R}$, $f \in C_c(I)$. Since $\mathscr{P}(\Omega_0) = 1$, $\Omega_0 \neq \emptyset$. Choose $\bar{\omega} \in \Omega_0$, set $z(t) = \bar{\omega}(-t)$, and (2.2) follows.

Assume $\mathscr{P}$ is nontrivial, $t(n) \to \infty$, $z(n) = z(t(n)) \to r$. If $r \in I$, then by (i) of Lemma 2.3,

$$p_{s+t(n)}(z(n), dy) \to_v \pi_r(dy) .$$

The limit π_r (even if it is a p.m.) is independent of s, and so (2.2) implies $\mathscr{P}$ is trivial. Hence $r \notin I$. Assume $r = r_2$, $|S(r_2)| < \infty$, $[a, b] \subset I$. Eventually $z(n) > b$, and then

$$\begin{aligned} p_{s+t(n)}(z(n), [a, b]) &\leqq P_{z(n)}[\tau_b < \infty] \\ &= \frac{S(r_2) - S(z(n))}{S(r_2) - S(b)} \\ &\to 0 \end{aligned}$$

as $n \to \infty$. Hence $\mathscr{P}[\xi_s \in I] = 0$, which is impossible, and $S(r_2) = +\infty$. Finally, since $z(t)$ is continuous, $z(t)$ must tend to one of r_1 or r_2.

(ii) Follow the proof of Proposition 2.2 in [4]. The missing technical details are provided by (i) of this proposition and Lemma 2.3. □

Note that a *necessary* condition for $\mathscr{M}(p_t)$ to be nontrivial is that one of the endpoints r *not* belong to I and $|S(r)| = \infty$. This condition is far from being sufficient. (2.2) describes $\mathscr{M}(p_t)$ (via 2.1) completely, but it is difficult to apply unless the form of p_t is explicitly known.

3. $\mathscr{M}(p_t)$ and entries from the boundary.

We introduce the following definition as a refinement of the idea implicit in (2.2).

DEFINITION 3.1. A sequence $z(n)$, $n = 1, 2, \cdots$ of elements of I is an *entry from* ∂I (the boundary of I) for p_t if there are constants $c(n) \to \infty$ and a p.m. Φ on $\mathbb{R}$ such that

$$P_{z(n)}[\tau_0 - c(n) \in du] \to_v \Phi(du) \quad \text{as} \quad n \to \infty . \tag{3.1}$$

Roughly speaking, $z(n)$ tends to some point in a manner which captures the distribution of the (infinite) time it takes to reach 0 from this point. Further explanation of the terminology is provided by the next results.

THEOREM 3.2. *$\mathscr{M}(p_t)$ is nontrivial* iff *p_t has an entry from ∂I.*

COROLLARY. *Let p_t have speed m and scale S. If*

$$\int_0^{r_2} \int_y^{r_2} dm(u)\, dS(y) = +\infty , \qquad \int_0^{r_2} \int_y^{r_2} \left(\int_z^{r_2} dm(u)\right)^2 dS(z)\, dS(y) < \infty , \tag{3.2}$$

then $\mathscr{M}(p_t)$ is nontrivial.

LEMMA 3.3. *Assume $z(n)$ is an entry from ∂I for p_t.*

(i) *If $z(n')$ is a convergent subsequence of $z(n)$, then $z(n') \to$ an endpoint r, $r \notin I$, $|S(r)| = +\infty$.*

(ii) *Assume $z(n) \to r$. Then there is a family of* p.m.'s $\{\Phi_x\}_{x \in I}$ *on $\mathbb{R}$ such that $P_{z(n)}[\tau_x - c(n) \in du] \to_v \Phi_x(du)$.*

PROOF. Assume $z(n') \uparrow r \in I$. Let u be a continuity point of $d\Phi$, so

$$\begin{aligned}\Phi((-\infty, u]) &= \lim_{n'\to\infty} P_{z(n')}[\tau_0 \leqq u + c(n')] \\ &\geqq \lim_{n'\to\infty} P_r[\tau_0 \leqq u + c(n')] \\ &= P_r[\tau_0 < \infty]\,.\end{aligned}$$

Let $u \downarrow -\infty$ to obtain $P_r[\tau_0 < \infty] = 0$, which is impossible (by regularity) unless r is an endpoint of I. Similar arguments, using the properties of S and the fact that Φ is a p.m., establish $r \notin I$ and $|S(r)| = \infty$.

(ii) The details, similar to those in the proof of Theorem 4.1 in [4], are omitted. □

PROOF OF THEOREM 3.2. The plan is to imitate the proof of Theorems 2.3 and 2.4 in [4] whenever possible. Assume $z(n)$ is an entry from ∂I, $z(n) \uparrow r_2$. Fix $s \in \mathbb{R}$, $f \in C_c(I)$, $b = \sup\{y : f(y) \neq 0\}$. Then $b < r_2$, $z(n) > b$ eventually, and

$$\begin{aligned}\textstyle\int_I p_{s+c(n)}(z(n), dy)f(y) &= E_{z(n)}[f(X_{s+c(n)});\ \tau_b \leqq s + c(n)] \\ &= \textstyle\int_{-\infty}^{\infty} E_b[f(X_{s-u})]P_{z(n)}[\tau_b - c(n) \in du] \\ &\to \textstyle\int_{-\infty}^{\infty} E_b[f(X_{s-u})]\Phi_b(du)\end{aligned}$$

by (3.1), the fact that Φ_b is a p.m., and $u \to E_b[f(X_u)]$ is a bounded continuous function on $\mathbb{R}$. Here we have set $E_b[f(X_u)] = 0$ for $u < 0$. Hence there is a sub-p.m. μ_s on $(I, \mathscr{B})$ such that $p_{s+c(n)}(z(n), dy) \to_v \mu_s(dy)$. Now the argument in [4] shows $\mu_s(I) = 1$, $\mu_s p_t = \mu_{s+t}$, and hence μ_s defines an element $\mathscr{P} \in \mathscr{M}(p_t)$. To see that it is nontrivial let $s \to -\infty$ in the last expression above.

Conversely, if $\mathscr{M}(p_t)$ is nontrivial, Dynkin's theorem guarantees the existence of a nontrivial $\mathscr{P} \in \mathscr{M}_e(p_t)$. Let $\mathscr{P}$ have the representation in (2.2). The argument in [4] shows the family $P_{z(t)}[\tau_0 - t \in du]$ is tight, and hence there is a vague limit Φ which is a p.m.; say $t(n) \to \infty$, $P_{z(t_n)}[\tau_0 - t(n) \in du] \to_v \Phi(du)$. The technicalities are overcome by Lemmas 2.3 and 3.3. □

PROOF OF COROLLARY. The assumptions imply $S(r_2) = \infty$, $r_2 \notin I$, hence $P_b[\tau_a < \infty] = 1$ for $b > a \in I$. In fact, as shown in [8] (or [7])

$$E_b[\tau_a] = \textstyle\int_a^b \int_y^{r_2} dm(u)\, dS(y)\,, \qquad E_b[\tau_a{}^2] = 2 \int_a^b \int_y^{r_2} E_u[\tau_a]\, dm(u)\, dS(y)\,.$$

From these formulas one derives

$$\operatorname{Var}_b[\tau_a] = E_b[\tau_a{}^2] - (E_b[\tau_a])^2 = \textstyle\int_a^b \int_y^{r_2} (\int_z^{r_2} dm(u))^2\, dS(z)\, dS(y)\,.$$

Hence (3.2) implies

$$\lim_{b\to r_2} E_b[\tau_0] = +\infty\,, \qquad \lim_{b\to r_2} \operatorname{Var}_b[\tau_0] = \sigma^2 < \infty\,.$$

Since the time from b to 0 is equal in distribution to the sum of (any number of) independent random variables, there must be a p.m. Φ on $\mathbb{R}$ such that

$$P_b[\tau_0 - E_b[\tau_0] \in du] \to_v \Phi(du)\,, \quad b \to r_2 \tag{3.3}$$

(see Theorem 4.2 in [4]). So p_t has an entry from ∂I, and $\mathscr{M}(p_t)$ is nontrivial. □

We note that (3.3) is apparently stronger than (3.1). This is not so, provided we replace $E_b[\tau_0]$ with constants $c(b)$, $c(b) \to \infty$ as $b \to r_2$. Using concentration functions one can "interpolate" between the $z(n)$, as shown in Theorem 3.2.1 in [3].

EXAMPLES. (a) Brownian motion, $I = \mathbb{R}$. $\mathscr{M}(p_t) = \varnothing$, even though each of the boundaries $r = \pm\infty$ do not belong to I and $|S(\pm\infty)| = \infty$. Since p_t does not have an invariant probability measure $\mathscr{M}(p_t)$ cannot contain the trivial element. Now use Theorem 3.2 and the well-known formula for $P_x[\tau_0 \leqq t]$ to show $\mathscr{M}(p_t) = \varnothing$.

(b) Brownian motion on [0, 1] with reflecting barriers. Here $\mathscr{M}(p_t) = \{$the trivial element$\}$. $\mathscr{M}(p_t)$ cannot be nontrivial by Proposition 2.2, but the diffusion is ergodic, so $\mathscr{M}(p_t) \neq \varnothing$.

(c) Ornstein–Uhlenbeck process. For a, b, δ positive constants, consider the diffusion defined on $I = \mathbb{R}$ by

$$\lim_{t\downarrow 0} \frac{E_x[f(X_t)] - f(x)}{t} = \tfrac{1}{2}af''(x) - b(\operatorname{sgn} x)|x|^\delta f'(x)\,,$$

or equivalently,

$$m(B) = \int_B a^{-1} \exp\left[\frac{-2b|y|^{1+\delta}}{a(1+\delta)}\right] dy$$

$$S(x) = \int_0^x \exp\left[\frac{2b|y|^{1+\delta}}{a(1+\delta)}\right] dy\,.$$

If $\delta = 1$ we obtain the Ornstein–Uhlenbeck process. Computations show using (3.2) that $\mathscr{M}(p_t)$ is nontrivial for $\frac{1}{3} < \delta \leqq 1$. If $\delta = 1$, using (2.2) and the explicit form of p_t, it can be shown that

$$\mathscr{M}_e(p_t) = \Big\{\mathscr{P}_\alpha : \alpha \in \mathbb{R},\ \text{ and for }\ s \in \mathbb{R},\, B \in \mathscr{B},$$
$$\mathscr{P}_\alpha[\xi_s \in B] = (\pi a/b)^{-\frac{1}{2}} \int_B \exp\left[\frac{(y - \alpha e^{-2bs})^2}{a/b}\right] dy\Big\}\,.$$

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