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PROBABILITY BOUNDS FOR FIRST EXITS THROUGH MOVING BOUNDARIES

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Let $S_1, S_2, \cdots$ be partial sums of independent and identically distributed random variables and let $f(n)$ and $g(n)$ be increasing positive sequences. Nearly sharp bounds are presented for the probabilities $P\{S_i \geqq g(i), i = 1, \cdots, n\}$ and $P\{-f(i) \leqq S_i \leqq f(i), i = 1, \cdots, n\}$ under conditions on f and g. The most difficult results are the lower bounds in the normal case. Results are obtained by an embedding method which approximates Brownian motion by sums of independent random variables taking on only two or three values.

1. Introduction. This paper is concerned with obtaining sharp bounds on first passage probabilities of the following form: let $S_1, S_2, \cdots$ be the partial sums of an i.i.d. sequence of random variables and let $f(n)$ and $g(n)$ be increasing (smooth) positive sequences. Then define

$$q_n = P\{-f(i) \leqq S_i \leqq f(i); i = 1, \cdots, n\}$$
$$r_n = P\{S_i \geqq g(i); i = 1, \cdots, n\}$$

where $f(n)$ and $g(n)$ are regularly varying functions with index $\alpha \in [0, \frac{1}{2}]$ and $\beta \in [\frac{1}{2}, 1)$ respectively. Section 2 obtains lower bounds for q_n and r_n for Brownian motion (from which results for sums of normal random variables follow directly). Using an embedding technique, lower bounds are given for q_n of the form $p_1 \exp\{-c_1 \sum_{i=1}^{n} f^{-2}(i)\}$, and for r_n of the form $p_2 \exp\{-c_2 \sum_{i=1}^{n} (g(i) - g(i-1))^2\}$, where p_1 and p_2 are constants depending on f and g and c_1 and c_2 are absolute constants. For $\alpha \neq \frac{1}{2}$ and $\beta \neq \frac{1}{2}$, the bounds have exponents $n^{1-2\alpha}$ and $n^{2\beta-1}$ respectively.

Using more elementary techniques, Section 3 presents upper bounds of the same form for q_n and r_n. Section 4 discusses how results can be extended to sums of nonnormal random variables. In particular, results of Lai [6] discussed below show that bounds for q_n hold whenever the second moment is finite (in the i.i.d. case, and less generally in the independent case). Using the recent invariance principle of Komlós, Major and Tusnády [5], bounds on r_n are extended to the case when the distribution has a finite moment generating function.

It is important to note that results giving bounds on q_n have been independently obtained by Lai [6]. His results are much better in the sense that he is able to eliminate a technical assumption required here (equations (2.1) and (2.16)) and,

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as noted above, his results hold for sums of independent random variables (satisfying an appropriate moment condition). However, the results here may have some independent interest, since the proof here is completely different and it does have some minor advantages: (1) it applies to Brownian motion for a lower bound, (2) the expression for the bound is much simpler (although the bounds appear to agree in general), and (3) values (not the best possible) for the absolute constants c_1 and c_2 are given.

I would also like to note that this research was originally suggested by Professor Wijsman ([9], Section 3.3) who was interested in a sequential analysis problem with boundaries $|S_n^2/n - b \log n| \leqq c$. In particular, he wanted to know if $p(\tau \geqq n) \leqq pe^{-cn}$ (for some p and c) where τ is the first time the inequality does not hold. It is possible to use the method of proof for Theorems 2.1 and 2.2 to show that there are constants p and c such that $P\{\tau \geqq n\} \geqq pe^{-cn^\varepsilon}$ for any $\varepsilon > 0$. However, the method does not seem to give reasonable bounds for the more general probability $P\{g(i) - f(i) \leqq S_i \leqq g(i) + f(i), i = 1, \cdots, n\}$. In a recent preliminary announcement [7], Lai and Wijsman have announced that they now have results which lead to bounds on this more general probability.

2. Lower bounds.

THEOREM 2.1. *Let $X(t)$ be a Brownian motion and let $f(t) = L_0(t)t^\alpha$ be a regularly varying nondecreasing function such that* $\inf\{f(t): t \geqq 0\} > 1$ *and also*

$$\text{(2.1)} \qquad \text{either} \quad 0 < \alpha < \tfrac{1}{2}, \quad \text{or} \quad \alpha = \tfrac{1}{2} \quad \text{and} \quad L_0(t) \leqq \frac{1}{R(t)(\log t)}$$

$$\text{where} \quad \frac{R(t)}{(\log\log t)^{\frac{1}{2}}} \to +\infty \quad \text{as} \quad t \to \infty .$$

Then for any $\varepsilon > 0$ (small enough so that $f^2(t) \geqq 1 + \frac{1}{2}\varepsilon$ for all $t \geqq 0$) there is $p > 0$ such that

$$\text{(2.2)} \qquad P\{-f(t) \leqq X(t) \leqq f(t), 0 \leqq t \leqq m\} \geqq p \exp\left\{-\sum_{i=1}^{n(m)} \frac{(1+\varepsilon)c_0}{f^2(i)}\right\}$$

where $c_0 = (\frac{3}{2})^{2\alpha}$ and $n(m) = [\frac{3}{2}(1+\varepsilon)m]$.

PROOF. For $\varepsilon > 0$, define

$$\text{(2.3)} \qquad h(t) = (1 + \tfrac{1}{2}\varepsilon)^{-\frac{1}{2}} f(\tfrac{2}{3}t) ,$$

then $h(n) = (\frac{2}{3})^\alpha(1 + \frac{1}{2}\varepsilon)^{-\frac{1}{2}} L_0(\frac{2}{3}n)n^\alpha = L(n)n^\alpha$ where $L(n)$ is slowly varying and (for ε small enough) $h(n) > 1$, $n = 1, 2, \cdots$. Also note that h satisfies the condition (2.1) above. Let $Y_1, Y_2, \cdots$ be independent,

$$\begin{aligned} Y_n &= h(n) && \text{with prob.} \quad 1/(2h^2(n)) \\ &= 0 && \text{with prob.} \quad 1 - 1/h^2(n) \\ &= -h(n) && \text{with prob.} \quad 1/(2h^2(n)) \end{aligned}$$

(note: $EY_n = 0$, $\operatorname{Var} Y_n = 1$). Let $X(t)$ be a Brownian motion process and define

$T_1, T_2, \cdots$ such that

$$X(T_n) \sim \sum_{i=1}^{n} Y_i \qquad n = 1, 2, \cdots . \tag{2.4}$$

In particular, let $T_n = \sum_{i=1}^{n} W_i$ where $W_i = U_i + V_i$ with (U_i, V_i) independent for different i and for each i

$$\begin{aligned} U_i &\sim \inf\left\{t: |X(t)| > \frac{1}{h(i)}\right\} \\ V_i &\sim \inf\{t: X(t) \text{ is not between } 0 \text{ and } (\operatorname{sgn} X(U_i))h(i)\} . \end{aligned} \tag{2.5}$$

Then it is easy to see that (2.4) holds.

It must first be shown that there is a monotonic sequence $l(k)$ such that

$$P\{l(k) \leqq T_k, k = 1, 2, \cdots \mid X(T_k) = 0, k = 1, 2, \cdots\} = p_0 > 0 . \tag{2.6}$$

First note that (since $h(i) > 1$) conditional on $X(T_k) = 0$ for $k = 1, 2, \cdots$, U_i and V_i are independent, U_i is distributed as the time to leave $\pm 1/h(i)$ and V_i is distributed as the time to leave $[0, h(i)]$ given that the exit is through 0. From Breiman ([2], page 289) it then follows that the moment generating function for W_n (conditional on $X(T_k) = 0$, $k = 1, 2, \cdots$) is

$$M_n(s) = \frac{h^2(n)}{h^2(n) - 1} \frac{\sin (2s)^{\frac{1}{2}}(h(n) - 1/h(n))}{\sin ((2s)^{\frac{1}{2}}h(n))} \cdot \frac{2 \sin ((2s)^{\frac{1}{2}}/h(n))}{\sin (2(2s)^{\frac{1}{2}}/h(n))} . \tag{2.7}$$

A direct computation (using $W_n = U_n + V_n$) shows that

$$\begin{aligned} E^*W_n &= \tfrac{2}{3}(1 + 1/h^2(n)) , \\ \operatorname{Var}^* W_n &= \tfrac{2}{45}(4h^2(n) - 6 + 4/h^2(n) + 14/h^4(n)) \end{aligned} \tag{2.8}$$

(where the asterisk denotes the above conditioning).

Now let $\tilde{W}_1, \tilde{W}_2, \cdots$ be independent with m.g.f. $\tilde{M}_1, \tilde{M}_2, \cdots$ where

$$\tilde{M}_n(s) = M_n(s)e^{-sE^*W_n} . \tag{2.9}$$

We first check the condition for the result of Feller [3].

$$\begin{aligned} \tilde{M}_n(s) &\geqq E^* \frac{e^{\tilde{W}_n s}}{\tilde{W}_n^{\,2}} \tilde{W}_n^{\,2} I\left(\tilde{W}_n^{\,2} \geqq \frac{v_n}{(\log_2 v_n)^3}\right) \\ &\geqq \exp\left\{\frac{(v_n)^{\frac{1}{2}} s}{(\log_2 v_n)^{\frac{3}{2}}}\right\} \frac{(\log_2 v_n)^3}{v_n} E^* W_n^{\,2} I\left(W_n^{\,2} \geqq \frac{v_n}{(\log_2 v_n)^3}\right) \\ &\qquad \text{for} \quad \frac{(v_n)^{\frac{1}{2}}}{(\log_2 v_n)^{\frac{3}{2}}} \geqq \frac{2}{s} \end{aligned} \tag{2.10}$$

where $\log_2$ represents the iterated logarithm. Here

$$v_n = \sum_{i=1}^{n} \operatorname{Var}^* (W_i) = c_n L^2(n) \cdot n^{2\alpha+1} \tag{2.11}$$

for appropriate c_n such that $c_n \to c$ as $n \to \infty$ (this uses the definition of $h(n)$, (2.8) and Feller [4], page 273). Let $s(n) = a/h^2(n)$ for some $a < \pi^2/8$. Then a direct calculation shows $\tilde{M}_n(s_n) \leqq B$ (for B independent of n). Now,

$$\frac{(v_n)^{\frac{1}{2}}}{(\log_2 v_n)^{\frac{3}{2}}} = \frac{(c_n)^{\frac{1}{2}} L(n) n^{\alpha+\frac{1}{2}}}{(\log_2 v_n)^{\frac{3}{2}}} \quad \text{and} \quad \frac{2}{s(n)} = \frac{2L^2(n) n^{2\alpha}}{a} .$$

Therefore, since h satisfies condition (2.1),

$$\frac{(v_k)^{\frac{1}{2}}}{(\log_2 v_k)^{\frac{3}{2}}} \geqq \frac{2}{s(k)}\,.$$

Hence, applying (2.10),

$$\frac{(\log_2 v_n)^3}{v_n} E^* \tilde{W}_n^{\,2} I\left(\tilde{W}_n^{\,2} \geqq \frac{v_n}{(\log_2 v_n)^3}\right) \leqq B \exp[(-a/h^2(n))(v_n)^{\frac{1}{2}}/(\log_2 v_n)^{\frac{3}{2}}]$$
$$= B \exp[-a(c_n)^{\frac{1}{2}}(n^{\frac{1}{2}-\alpha}/L(n)(\log_2 v_n)^{\frac{3}{2}})]\,.$$

This sums as long as h satisfies condition (2.1). Therefore, the result of Feller [3] shows that

(2.12) $$P^*\{|T_n - d_n| > ((2+\varepsilon)v_n \log_2 v_n)^{\frac{1}{2}} \text{ i.o.}\} = 0$$

for any $\varepsilon > 0$ and where $d_n = \frac{2}{3}n + \frac{2}{3}\sum_{i=1}^n 1/h^2(i)$.

Now, define a monotonic regularly varying sequence $l(t)$ with $l(0) = 0$ such that for n large enough,

(2.13) $$l(n) = d_n - ((2+\varepsilon)v_n \log_2 v_n)^{\frac{1}{2}}$$

(note: l can be chosen monotonic by (2.11) for $\alpha \leqq \frac{1}{2}$). Using a standard argument, it follows from (2.12) that

(2.14) $$P\{l(n) \leqq T_n, n = 1, 2, \cdots \mid X(T_k) = 0, k = 1, 2, \cdots\} = p_0 > 0\,.$$

We now complete the proof. If $X(T_k) = 0$ for $k = 1, 2, \cdots, n$ then $|X(t)| \leqq h(k)$ for $T_{k-1} \leqq t \leqq T_k$ ($k = 1, 2, \cdots, n$). If also $l(k) \leqq T_k$ for $k = 0, 1, 2, \cdots, n$, then

$$-h(l^{-1}(t)+1) \leqq X(t) \leqq h(l^{-1}(t)+1) \qquad \text{for} \quad 0 \leqq t \leqq [l(n)]\,.$$

Therefore, using (2.14),

$$\begin{aligned}
&P\{-h(l^{-1}(t)+1) \leqq X(t) \leqq h(l^{-1}(t)+1), 0 \leqq t \leqq [l(n)]\} \\
&\quad = P\{-h(l^{-1}(t)+1) \leqq X(t) \leqq h(l^{-1}(t)+1), \\
&\qquad\qquad 0 \leqq t \leqq [l(n)] \mid X(T_k) = 0, k = 1, \cdots, n\} \\
&\qquad \times P\{X(T_k) = 0, k = 1, 2, \cdots, n\} \\
&\quad \geqq P\{l(k) \leqq T_k, k = 1, \cdots, n \mid X(T_k) = 0, k = 1, 2, \cdots\} \textstyle\prod_{i=1}^n (1 - 1/h^2(i)) \\
&\quad \geqq p_0 \textstyle\prod_{i=1}^n (1 - 1/h^2(i))\,.
\end{aligned}$$

Now, using (2.13) and the definition of d_n, $l(t)/t \to \frac{2}{3}$ as $t \to \infty$. Since $l(t)$ goes to $+\infty$, it follows that $l^{-1}(s)/s \to \frac{3}{2}$ as $s \to \infty$. Since h varies regularly, $h(ct)/t \to c^\alpha$ uniformly for $c_0 \leqq c \leqq c_1$ (with $c_0 > 0$, $c_1 < +\infty$) (see Feller [4], page 274). Therefore,

$$\lim_{t\to\infty} \frac{h(l^{-1}(t)+1)}{h(\frac{3}{2}t)} = 1\,.$$

Therefore, using (2.3), it follows that for any $\varepsilon > 0$ (small enough so that

$1 + \frac{1}{2}\varepsilon < f^2(\frac{2}{3}i)$ for all i) there is p_1, p_1' and i_1 such that

$$P\{-f(t) \leqq X(t) \leqq f(t), 0 \leqq t \leqq [l(n)]\} \geqq p_1 \prod_{i=1}^{n} \left(1 - \frac{1 + \frac{1}{2}\varepsilon}{f^2(\frac{2}{3}i)}\right)$$
$$\geqq p_1' \prod_{i=i_1}^{n} \left(1 - \frac{(1 + \frac{2}{3}\varepsilon)c_0}{f^2(i)}\right).$$

Letting $m = [l(n)]$, $n \leqq [\frac{3}{2}(1 + \varepsilon)m]$ for m large enough; and since f varies regularly, Theorem 2.1 follows using the inequality $1 - x \geqq \exp[-x(1 + \varepsilon)/(1 + \frac{2}{3}\varepsilon)]$ for x small enough. □

REMARK. If $0 < \alpha < \frac{1}{2}$, one can show (using Feller [4], page 273) that

$$\text{(2.15)} \qquad \exp\{-\textstyle\sum_{i=1}^{n(m)} (1 + \varepsilon)c_0/f^2(i)\}$$
$$\geqq \exp\{-(1 + \varepsilon)c_0 c^*(m)[\tfrac{3}{2}(1 + \varepsilon)m]^{1-2\alpha}/L_0^2(m)\}$$

where $c^*(n) \to (1 - 2\alpha)^{-1}$. Thus by changing ε, the lower bound could be written as $p \exp\{-3(1 + \varepsilon)m^{1-2\alpha}/2(1 - 2\alpha)L_0^2(m)\}$. If $\alpha = \frac{1}{2}$, the result in Feller is inapplicable; but if $L_0(t)$ is smooth enough, we may often obtain

$$\textstyle\sum_{i=1}^{n} 1/f^2(i) = \sum_{i=1}^{n} \dfrac{1}{iL_0^2(i)} \geqq c_1 \log n/L^2(n).$$

THEOREM 2.2. *Let $X(t)$ be a Brownian motion and let $g(t)$ be a monotonic increasing function with $g(0) < 0$ and such that the function $b_0(t) = g(\frac{2}{3}t) - g(\frac{2}{3}(t - 1))$ (for $t \geqq 1$) is regularly varying of the form $b_0(t) = L_0(t)t^{\beta-1}$ (so that from Feller [4, page 273], $g(t)$ is regularly varying with index β). Further suppose,* $\sup\{b_0(t) : t \geqq 0\} < 1$ *and also*

$$\text{(2.16)} \qquad \textit{either} \quad \tfrac{1}{2} < \beta < 1, \quad \textit{or} \quad \beta = \tfrac{1}{2} \quad \textit{and} \quad L_0(t) \geqq (\log t)R(t)$$
$$\textit{where} \quad \frac{R(t)}{(\log\log t)^{\frac{3}{2}}} \to +\infty.$$

Then for $\varepsilon > 0$ (small enough so that $(1 + \varepsilon)b_0^2(t) < 1$ for all t) there is $p > 0$ such that with $n(m) = [\frac{3}{2}(1 + \varepsilon)m]$,

$$\text{(2.17)} \qquad P\{X(t) \geqq g(t), 0 \leqq t \leqq m\} \geqq p \exp\{-\textstyle\sum_{i=1}^{n(m)} (1 + \varepsilon)b_0^2(i)\}.$$

PROOF. The proof proceeds exactly as the proof of Theorem 2.1, although here it is the differences $\{h(n) - h(n - 1)\}$ which play the fundamental role. As in (2.3), for $\varepsilon > 0$ define $h(t) = g(\frac{2}{3}t)(1 + \varepsilon)^{\frac{1}{2}}$. Define

$$\text{(2.18)} \qquad b(n) = h(n) - h(n - 1) \qquad n = 1, 2, \cdots.$$

Then $b(n) = (1 + \varepsilon)^{\frac{1}{2}}(g(\frac{2}{3}n) - g(\frac{2}{3}(n - 1)))$; and, hence, $b(n) < 1$ for $n = 1, 2, \cdots$ and also $b(n)$ is regularly varying, say

$$\text{(2.19)} \qquad b(n) = L(n)n^{\beta-1}.$$

Furthermore, $L(n)$ satisfies (2.16); and, by Feller [4, page 273] $h(n)$ is also regularly varying with index β. Now define $Y_1, Y_2, \cdots$ independent with

$$\begin{aligned} Y_n &= b(n) && \text{with prob.} \quad 1/(1 + b^2(n)) \\ &= -1/b(n) && \text{with prob.} \quad b^2(n)/(1 + b^2(n)) \,. \end{aligned}$$

Again $EY_n = 0$, Var $Y_n = 1$, and $\sum_{i=1}^n Y_i \sim X(T_n)$ where $T_n = \sum_{i=1}^n W_i$ and W_i are independent and distributed as the time for $X(t)$ to leave the interval $[b(n), -1/b(n)]$. Conditional on $X(T_i) = h(i)$, $i = 1, 2, \cdots, W_n$ has moment generating function $M_n(s) = (1 + b^2(n)) \sin((2s)^{\frac{1}{2}}/b(n))/\sin((2s)^{\frac{1}{2}}(b(n) + 1/b(n)))$ (at least for s small enough; see Breiman [2], page 289). Here, $E^*(W_n) = \frac{2}{3} + \frac{1}{3}b^2(n)$ and Var* $(W_n) = \frac{2}{45}(4/b^2(n) + 6 + 4b^2(n) + b^4(n))$. Again letting $\tilde{W}_1, \tilde{W}_2, \cdots$ be independent with moment generating function $\tilde{M}_n(s) = M_n(s)e^{-sE^*W_n}$, (2.10) can be obtained. Here, using (2.19) and Feller [4], page 273, we have

$$v_n \equiv \sum_{i=1}^n \mathrm{Var}^* (W_i) = c_n n^{3-2\beta}/L^2(n)$$

for appropriate c_n such that $c_n \to c$ as $n \to \infty$. As before, let $s(n) = ab^2(n)$ where $a < \pi^2/8$. Then $\tilde{M}_n(s_n) \leqq B$ (for B independent of n) and $(v_n)^{\frac{1}{2}}/(\log_2 v_n)^{\frac{3}{2}} \geqq 2/s(n)$.

Once again (2.10) can be applied to obtain an upper bound which will sum since $L(n)$ satisfies (2.16). Thus, Feller [3] can be applied to obtain (conditional on $X(T_i) = h(i)$)

$$P\{T_n \geqq d_n + ((2 + \varepsilon)v_n \log_2 v_n)^{\frac{1}{2}} \text{ i.o.}\} = 0$$

where $d_n = \frac{2}{3}n + \frac{1}{3}\sum_{i=1}^n b^2(n)$. Defining $u(n) = d_n + ((2 + \varepsilon)v_n \log_2 v_n)^{\frac{1}{2}}$, u is monotonic, $u(n)/n \to \frac{2}{3}$ as $n \to \infty$, and $P\{T_n \leqq u(n), n = 1, 2, \cdots \mid X(T_i) = h(i), i = 1, 2, \cdots\} = p_0 > 0$. Now if $T_i \leqq u(i)$ for $i = 1, 2, \cdots, n$ and $X(T_k) = h(k)$, then $X(t) \geqq h(k) - 1/b(k + 1) \geqq h(u^{-1}(t) - 1) - 1/b(u^{-1}(t))$ for $T_{k-1} \leqq t \leqq T_k$ $(k = 1, 2, \cdots, n)$. Furthermore, $h(\frac{3}{2}t)^{-1}(h(u^{-1}(t) - 1) - 1/b(u^{-1}(t))) \to 1$ as $t \to \infty$; and, therefore, there is p such that

$$\begin{aligned} P\{X(t) \geqq g(t), 0 \leqq t \leqq [u(n)]\} &\geqq pP\{X(T_i) = h(i), i = 1, \cdots, n\} \\ &= p \textstyle\prod_{i=1}^n (1/(1 + b_2(i))) \\ &= p \textstyle\prod_{i=1}^n (1/(1 + (1 + \varepsilon)b_0^2(i))) \,. \end{aligned}$$

Again the result follows using standard inequalities (since $b_0(i) \to 0$ as $i \to \infty$). $\square$

REMARK. Here, if $\frac{1}{2} < \beta < 1$, one will be able to obtain the bound $\exp\{-cm^{2\beta-1}L_0^2(m)\}$ where $c = (\frac{3}{2})^{2\beta-1}(1 + \varepsilon)/(2\beta - 1)$; and similar bounds can generally be obtained if $\beta = \frac{1}{2}$ and $L_0(t)$ is smooth enough.

3. Upper bounds. An upper bound for q_n can be easily derived from bounds on the probability, $P\{|S_i| \leqq B, i = 1, 2, \cdots, n\}$, where $S_1, S_2, \cdots$ are partial sums of i.i.d. $\mathscr{N}(0, 1)$ random variables. The following lemma shows that this bound is the same as the bound for Brownian motion except for the multiplicative constant and the factor $(1 - \varepsilon)$ in the exponent. It would be interesting to see if the factor of $(1 - \varepsilon)$ could be eliminated (by allowing a multiplicative constant) but I conjecture that this cannot be done.

LEMMA. *With $S_1, S_2, \cdots$ as above, for any $\varepsilon > 0$ there are constants K and B_0 such that for $B \geqq B_0$ and $n \geqq K$,*

$$P\{|S_i| \leqq B;\ i = 1, \cdots, n\} \leqq \exp\left\{-\frac{\pi^2}{8}\frac{n(1-\varepsilon)}{B^2}\right\}. \tag{3.1}$$

PROOF. Let $S_i = X(i)$, $i = 1, 2, \cdots$ where $X(t)$ is Brownian motion, let $N = \min(i: |S_i| > B + A)$, and let $T = \inf\{t: |X(t)| > B + A\}$ (where $A > 0$ will be chosen later). Let $M = [T+1]$; with probability $\Phi(A/(M-T)^{\frac{1}{2}})$, $|X(M)| > B$ and $N \leqq T + 1$. Otherwise, N equals M plus the (conditionally independent) time, N', for a sequence of partial sums to leave the interval $[-B, B]$ given that the sequence starts at a (random) position $X(M) \in [-B, B]$. Let N^* be distributed as N but be independent of $X(t)$. From Anderson's theorem (see [1], page 173) it follows that N' (conditionally on $X(M)$) is stochastically smaller than N^* (and, thus, $E[\exp(tN') \mid X(M)] \leqq E\exp(tN^*)$ for $t \geqq 0$). Therefore, for $t \geqq 0$,

$$\begin{aligned} Ee^{tN} &\leqq Q(A)Ee^{t(T+1)} + (1 - Q(A))E[Ee^{t(T+1+N')} \mid X(M)] \\ &\leqq Q(A)e^tEe^{tT} + (1 - Q(A))e^tEe^{tT}Ee^{tN^*}, \end{aligned} \tag{3.2}$$

where $Q(A) = E\Phi(A/(M-T)^{\frac{1}{2}})$. Now, from Breiman [2], page 289,

$$Ee^{tT} = \frac{2\sin(2t)^{\frac{1}{2}}(B+A)}{\sin 2(2t)^{\frac{1}{2}}(B+A)}.$$

Let $t = \pi^2(1-\varepsilon/2)/(8(A+B)^2)$. Then $E\exp(tT) \leqq 2/\sin\pi(1-\varepsilon/2)$, and A can be chosen so that $(1-Q(A))e^tEe^{tT} \leqq \frac{1}{2}$. Thus, using a Chebyshev-type inequality and (3.2),

$$\begin{aligned} P\{|S_i| \leqq B;\ i = 1, \cdots, n\} &\leqq e^{-tn}Ee^{tN} \\ &\leqq e^{-tn}\frac{Q(A)e^tEe^{tT}}{1 - (1-Q(A))e^tEe^{tT}} \\ &\leqq c_0\exp\left\{-\frac{\pi^2}{8}\frac{(1-\varepsilon/2)n}{(A+B)^2}\right\} \end{aligned}$$

for some c_0 (depending on ε); and the result follows. □

THEOREM 3.1. *Let $S_1, S_2, \cdots$ be partial sums of* i.i.d. $\mathscr{N}(0, 1)$ *random variables, and let $f(i)$ be positive and increasing in i. Let $m = [\log_2 n]$* (log base 2). *Then for any $\varepsilon > 0$ there is $p > 0$ such that*

$$P\{|S_i| \leqq f(i);\ i = 1, \cdots, n\} \leqq p\exp\left\{-c_0(1-\varepsilon)\textstyle\sum_{i=1}^{2(m+1)}\frac{1}{f^2(i)}\right\} \tag{3.3}$$

where $c_0 = \pi^2/16$.

PROOF. For $k = 1, 2, \cdots, m$, define

$$B_k = \{|S_i| \leqq f(2^k);\ i = 2^{k-1}+1, \cdots, 2^k\}.$$

Then,

$$\begin{aligned} P\{|S_i| \leqq f(i);\ i = 1, \cdots, n\} &\leqq P(\textstyle\bigcap_{k=1}^m B_k) \\ &= \textstyle\prod_{k=1}^m P\{B_k \mid B_1, B_2, \cdots, B_{k-1}\}. \end{aligned}$$

Now with $j(k) = 2^{k-1}$,

$$
\begin{aligned}
P(\bigcap_{j=1}^{k} B_m) &= EP\{\bigcap_{j=1}^{k} B_j \mid S_{j(k)}\} \\
&= EP\{|S_i| \leqq f(i), i = 1, \cdots, j(k) - 1 \mid S_{j(k)}\} \\
&\quad \times P\{B_k \mid S_{j(k)}\} I\{S_{j(k)} \geqq f(j(k))\} .
\end{aligned}
$$

By Anderson's theorem (see [1]), $P\{B_k \mid S_{j(k)} = x\} \leqq P\{B_k \mid S_{j(k)} = 0\}$; and for k large enough this can be bounded by (3.1). Therefore, for $k \geqq K_0$,

$$P\{B_k \mid B_1, B_2, \cdots, B_{k-1}\} \leqq \exp\{-2c_0(1 - \varepsilon)2^{k-1}/f^2(2^k)\} .$$

Thus,

$$
\begin{aligned}
P\{|S_i| \leqq f(i); i = 1, \cdots, n\} &\leqq \prod_{k=1}^{K_0} P\{B_k \mid B_1, \cdots, B_{k-1}\} \\
&\quad \times \exp\{-2c_0(1 - \varepsilon) \textstyle\sum_{k=k_0+1}^{m} 2^{k-1}/f^2(2^k)\} \\
&\leqq p \exp\left\{-c_0(1 - \varepsilon) \sum_{i=1}^{2^{(m+1)}} \frac{1}{f^2(i)}\right\} . \qquad \square
\end{aligned}
$$

REMARK. If f is regularly varying with index $0 < \alpha < \frac{1}{2}$, (3.3) will yield the same bound as the lower bound (2.15) up to the constant c_0 in the exponent. I conjecture that the actual rate is of the form $\exp\{-(\pi^2/8)(1/1 - 2\alpha)(1 + o(1))n^{1-2\alpha}/L^2(n)\}$, but I cannot prove this.

THEOREM 3.2. *Let $S_1, S_2, \cdots$ be partial sums of* i.i.d. $\mathscr{N}(0, 1)$ *random variables, let $g(i)$ be increasing in i and assume the differences $b(i) = g(i) - g(i - 1)$ are non-increasing. If $1 \leqq m \leqq n$ and $g(m)/m - b(m + 1) \geqq 0$, then*

$$P\{S_i \geqq g(i); i = m, \cdots, n\} \leqq \exp\left\{-\frac{1}{2} \frac{g^2(m)}{m} - \tfrac{1}{2} \textstyle\sum_{i=m}^{n} b^2(i)\right\} . \tag{3.4}$$

PROOF. Using the joint moment generating function for $(S_m, \cdots, S_n)$, a Chebyshev-type inequality shows that for any $(t_m, \cdots, t_n)$ with all $t_i \geqq 0$,

$$
\begin{aligned}
P\{S_i \geqq g(i); i = m, \cdots, n\} &\leqq \exp\{-\textstyle\sum_{i=m}^{n} t_i g(i)\} M_{S_m,\cdots,S_n}(t_m, \cdots, t_n) \\
&= \exp\{-\textstyle\sum_{i=m}^{n} t_i g(i) + \frac{1}{2}\sigma^2\}
\end{aligned}
$$

where

$$\sigma^2 = \operatorname{Var}(\textstyle\sum_{i=m}^{n} t_i S_i) = \sum_{j=1}^{n} (\sum_{i=j\vee m}^{n} t_i)^2 .$$

Thus, the minimizing value, t^*, of the quadratic form, $q(t)$, in the exponent satisfies for $k = m, \cdots, n$

$$0 = -g(k) + \textstyle\sum_{j=1}^{k} (\sum_{i=j\vee m}^{n} t_i^*) .$$

Therefore, $g(m) = m \sum_{i=m}^{n} t_i^*$ and for $k = m + 1, \cdots, n$,

$$b(k) = g(k) - g(k - 1) = \textstyle\sum_{i=k}^{n} t_i^* .$$

Hence, $g(m) = m(t_m^* + b(m + 1))$; and $t_m^* = g(m)/m - b(m + 1)$,

$$
\begin{aligned}
&t_k^* = b(k) - b(k + 1) , \qquad \text{for} \quad k = m + 1, \cdots, n - 1 , \\
&t_n^* = b(n) .
\end{aligned}
$$

Since $b(k)$ is positive and nondecreasing, $t_k^* \geqq 0$ for $k = m, \cdots, n$. Furthermore, since q is a quadratic form, the quadratic term at t^* is minus the linear; and

$$q(t^*) = -\tfrac{1}{2} \sum_{j=1}^n (\sum_{i=j\vee m}^n t_i^*)^2 = -\frac{1}{2}\frac{g^2(m)}{m} + \sum_{j=m+1}^n b^2(j) . \qquad \square$$

4. Extensions to sums of nonnormal variables. The results of earlier sections can all be extended to sums of i.i.d. random variables with more general distributions. For example, as was noted in Section 1, Lai [6] has proven versions of Theorems 2.1 and 3.1 which hold for sums of i.i.d. random variables with finite second moment (and, in fact, for sums of independent random variables satisfying a slightly stronger condition). However, as the following argument shows, the result of Theorem 3.2 (and, hence, a result analogous to Lai's) can not hold in such generality:

PROPOSITION. *Let $S_1, S_2, \cdots$ be partial sums of a sequence of* i.i.d. *random variables with finite second moment and a* cdf *satisfying* $1 - F(x) \geqq c_1/x^k$. *Let $\{g(n)\}$ be any sequence with $g(n) \leqq c_2 n$, $n = 1, 2, \cdots$. Then there is a constant b such that*

$$P\{S_i \geqq g(i), i = 1, 2, \cdots, n\} \geqq b/n^{k+\frac{1}{2}} , \qquad n = 1, 2, \cdots . \tag{4.1}$$

PROOF. From Spitzer [8], Theorem 3.5, there is a constant c_3 such that

$$P\{S_i \geqq 0, i = 1, \cdots, n-1\} \geqq c_3/n^{\frac{1}{2}} , \qquad n = 1, 2, \cdots .$$

Therefore,

$$\begin{aligned} P\{S_i \geqq g(i), i = 1, \cdots, n\} &\geqq P\{X_1 \geqq g(n); S_i - X_1 \geqq 0, i = 2, \cdots, n\} \\ &\geqq P(X_1 \geqq c_2 n) \cdot c_3/n^{\frac{1}{2}} \\ &\geqq c_1 c_3/[(c_2 n)^k \cdot n^{\frac{1}{2}}] = b/n^{k+\frac{1}{2}} . \qquad \square \end{aligned}$$

However, if the underlying distribution has a finite moment generating function in a neighborhood of the origin, the results of Komlós, Major and Tusnády [5] (hereafter abbreviated KMT) can be used to extend the results of Theorems 2.2 and 3.2.

THEOREM 4.1. *Let $S_1, S_2, \cdots$ be partial sums of* i.i.d. *standardized random variables, X_i, satisfying $Ee^{tX_i} < +\infty$ for $|t| < t_0$; and suppose $g(t)$ satisfies the hypotheses of Theorem* 2.2. *Let d be such that*

$$g(j) \geqq g(j+N) - d - g(N) , \qquad j = 1, 2, \cdots; N = 1, 2, \cdots \tag{4.2}$$

(*the existence of d uses the fact that $g(n) - g(n-1)$ is eventually decreasing*). *Let $a(n)$ be the exponent in the lower bound of equation* (2.17) *defined using* $2(g(t) + d)$ *to replace $g(t)$. Then for any $\varepsilon > 0$, there is a positive constant p^* such that*

$$P\{S_i \geqq g(i), i = 1, \cdots, n\} \geqq p^* \exp(-(1+\varepsilon)a(n)) . \tag{4.3}$$

PROOF. For integers $N_1 < N_2 < n$,

$$
\begin{aligned}
&P\{S_i \geqq g(i),\, i = 1, \cdots, n\} \\
&\qquad \geqq P\{S_i \geqq 3g(i),\, i = 1, \cdots, N_1;\; S_i \geqq 2g(i),\, i = N_1 + 1, \cdots, N_2; \\
&\qquad\qquad S_i \geqq g(i),\, i = N_2 + 1, \cdots, n\} \\
&\qquad \geqq P\{S_i \geqq 3g(i),\, i = 1, \cdots, N_1; \\
&\qquad\qquad S_i - S_{N_1} \geqq 2g(i) - 3g(N_1),\, i = N_1 + 1, \cdots, N_2; \\
&\qquad\qquad S_i - S_{N_2} \geqq g(i) - 2g(N_2),\, i = N_2 + 1, \cdots, n\} \\
&\qquad = p_1 p_2 p_3 ,
\end{aligned}
\tag{4.4}
$$

where p_1, p_2, and p_3 are the probabilities of the three events described in the previous line.

Let $\delta > 0$ be such that $P(X_1 \geqq \delta) = \rho > 0$ and let k_0 be such that $g(n) - g(n-1) < \delta/3$ for $n \geqq k_0$. Then

$$
\begin{aligned}
p_1 &\geqq P\{S_i \geqq 3g(i),\, i = 1, \cdots, k_0; \\
&\qquad S_i \geqq 3g(k_0) + \delta(i - k_0),\, i = k_0 + 1, \cdots, N_1\} \\
&\geqq P\{S_i \geqq 3g(i),\, i = 1, \cdots, k_0\} \\
&\qquad \times P\{S_i - S_{k_0} \geqq \delta(i - k_0),\, i = k_0 + 1, \cdots, N_1\} \\
&\geqq p_1' P\{X_i \geqq \delta,\, i = k_0 + 1, \cdots, N_1\} \\
&= p_1' e^{-N_1(-\log \rho)} .
\end{aligned}
\tag{4.5}
$$

For p_2, use Theorem 1 of KMT [5] to construct new random variables, $\{\tilde{S}_i\}$; with the same distribution as $\{S_{i+N_1} - S_{N_1}\}$, and partial sums $\{T_i\}$ of i.i.d. $\mathscr{N}(0, 1)$ random variables satisfying the conclusion of the theorem. Then

$$
\begin{aligned}
p_2 &= P\{\tilde{S}_i \geqq 2g(i) - 3g(N_1),\, i = N_1 + 1, N_1 + 2, \cdots, N_2\} \\
&\geqq P\{T_i \geqq 2g(i) + 2d;\; |T_i - \tilde{S}_i| \leqq g(N_1),\, i = 1, \cdots, N_2 - N_1\} \\
&\geqq P\{T_i \geqq 2g(i) + 2d,\, i = 1, \cdots, N_2\} - P\{\sup_{1<i<N_2} |T_i - \tilde{S}_i| > g(N_1)\} \\
&\geqq p e^{-a(N_2)} - K e^{-\lambda(g(N_1) - C \log N_2)} ,
\end{aligned}
\tag{4.6}
$$

where (4.2) is used to obtain the first inequality, p comes from Theorem 2.2 and K, λ, and C come from KMT. The same inequality holds for p_3 with N_1 replaced by N_2 and N_2 by $N_3 = n$. Now, if N_1 and N_2 are chosen (depending on n) so that (for $i = 1, 2$)

$$
\begin{aligned}
&N_1 < \frac{\varepsilon a(n)}{-\log(\rho)} ; \qquad a(N_{i+1}) \ll g(N_i) ; \\
&\log N_{i+1} \ll g(N_i) ; \qquad a(N_2) \ll a(n) ;
\end{aligned}
\tag{4.7}
$$

then, considering the dominating terms and combining (4.6) (for p_2 and p_3), (4.5) and (4.4) yield a bound $p_0 \exp\{-\varepsilon a(n) - a(N_2) - a(n)\}$ for n large enough. The result (4.3) follows directly from this; so it remains to check (4.7).

First consider the case where $a(n) \leqq n^{\delta}$ for some $\delta < \frac{1}{4}$ for n large enough. Note that $g(n) \gg n^{\frac{1}{2}}$, $a(n)$ is increasing and $a(n) \gg \log n$ (this only requires that

$L_0(t)$ in Theorem 2.2 tend to $+\infty$). Define $N_1 = ca(n)$ where $c < -\varepsilon/\log\rho$ and $N_2 = a^{-1}((ca(n))^{\frac{1}{2}})$. Now $N_1^{\frac{1}{2}} = (ca(n))^{\frac{1}{2}} = a(N_2)$; so $g(N_1) \gg a(N_2)$. Since $a(n) \gg \log n$, for n large enough

$$\begin{aligned} a(e^{(ca(n))^{\frac{1}{2}}}) \geqq (ca(n))^{\frac{1}{2}} &\Rightarrow e^{(ca(n))^{\frac{1}{2}}} \geqq a^{-1}((ca(n))^{\frac{1}{2}}) \\ &\Rightarrow N_1^{\frac{1}{2}} = (ca(n))^{\frac{1}{2}} \geqq \log N_2 \\ &\Rightarrow g(N_1) \gg \log N_2 . \end{aligned}$$

Now, if we define $N_3 = a^{-1}((N_2)^{\frac{1}{2}})$, then as above, $a(N_3) \ll g(N_2)$ and $\log N_3 \ll g(N_2)$, so it remains to show that $N_3 \geqq n$ (at least for n large). But if $\delta < \frac{1}{4}$, for n large enough,

$$(a^2(n))^{2\delta} \leqq a^{4\delta}(n) \leqq ca(n) .$$

But if $a(n) \leqq n^\delta$, $(a^2(n))^{2\delta} \geqq a^2(a^2(n))$. Therefore, $a^2(a^2(n)) \ll ca(n)$ or $a^2(n) \leqq a^{-1}((ca(n))^{\frac{1}{2}}) = N_2$ or $n \leqq a^{-1}((N_2)^{\frac{1}{2}}) = N_3$.

Lastly if there is no $\delta < \frac{1}{4}$ such that $a(n) \leqq n^\delta$, then the conditions on g imply that $g(n) = n^\beta L_1(n)$ for some β (satisfying $2\beta - 1 \geqq \frac{1}{4}$) and L_1 slowly varying; and also $a(n) = n^{2\beta-1}L_2(n)$. Here define $N_1 = n^{2\beta-1-\varepsilon_1}$ and $N_2 = n^{\beta-\varepsilon_2}$ where ε_1 and ε_2 are such that both exponents are positive and $(\beta-1)^2 > \varepsilon_2\beta$ and $\varepsilon_1\beta < (2\beta-1)\varepsilon_2$. So $N_1 \leqq a(n)$; and since $a(N_2) = n^{(2\beta-1)(\beta-\varepsilon_2)}L_2(N_2)$ and $g(N_1) = n^{\beta(2\beta-1-\varepsilon_1)}L_1(N_1)$, $a(N_2) \ll g(N_1)$. Trivially, $g(N_1) \gg \log N_2$. Similarly, $g(N_2) \gg a(n)$ and $g(N_2) \gg \log n$; so (4.7) holds in general and the proof is complete. $\square$

THEOREM 4.2. *Let $S_1, S_2, \cdots$ be partial sums of* i.i.d. *standardized random variables with finite moment generating function in a neighborhood of the origin, and let K, λ, and C be the constants given by Theorem* 1 *of* KMT [5]. *Let $g(n)$ be a positive increasing sequence, $b(n) = g(n) - g(n-1)$ (with $b(1) = g(1)$), and $a(n) = \sum_{i=1}^n b^2(i)$. Suppose $g(n)$ is such that*

(1) $\dfrac{\log n}{a(n)} \to 0$ *as* $n \to \infty$, *and*

(2) *for each n there is m_n such that*

  (i) $\dfrac{a(m_n)}{a(n)} \to 0$ *as* $n \to \infty$, *and*

  (ii) *if* $c_n \equiv \dfrac{1}{m_n}\left(g(m_n) - \dfrac{1}{\lambda}a(n)\right) - b(m_n+1)$, $c_n \geqq 0$.

Then for any $\varepsilon > 0$ there is N such that for $n \geqq N$

$$P\{S_i \geqq g(i),\ i = 1, 2, \cdots, n\} \leqq \exp\{-\tfrac{1}{2}(1-\varepsilon)\textstyle\sum_{i=1}^n b^2(i)\} .$$

REMARK. If $b(n)$ are regularly varying with index $\beta - 1$ for $\beta < 1$, then condition (2)(ii) can be satisfied by choosing $m_n = n^d$ for some $d < 1$.

PROOF. Let $T_1, T_2, \cdots$ be i.i.d. $\mathcal{N}(0, 1)$ random variables given by KMT [5].

Then

$$
\begin{aligned}
&P\{S_i \geqq g(i),\, i = 1, \cdots, n\} \\
&\quad \leqq P\{S_i \geqq g(i),\, i = m_n, \cdots, n\} \\
&\quad \leqq P\left\{S_i \geqq g(i),\, i = m_n, \cdots, n;\, |S_i - T_i| \leqq \frac{1}{\lambda}\, a(n),\, i = 1, \cdots, n\right\} \\
&\qquad + P\left\{\sup\nolimits_{1 \leqq i \leqq n} |S_i - T_i| > \frac{1}{\lambda}\, a(n)\right\} \\
&\quad \leqq P\left\{T_i \geqq g(i) - \frac{1}{\lambda}\, a(n),\, i = m_n, \cdots, n\right\} + K \exp\{-a(n) + C \log n\} \\
&\quad \leqq \exp\{-\tfrac{1}{2} \textstyle\sum_{i=m_n+1}^{n} b^2(i)\} + K \exp\{-a(n) + C \log n\} \\
&\quad \leqq \exp\left\{-\tfrac{1}{2} a(n) \left(1 - \frac{a(m_n)}{a(n)}\right)\right\} [1 + K \exp\{-\tfrac{1}{2} a(n) + C \log n\}]
\end{aligned}
$$

where the next to last line uses Theorem 3.2 (which required $c_n \geqq 0$). The result follows directly using conditions (1) and (2). □

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