The Annals of Probability
1979, Vol. 7, No. 3, 515–525

TIME-DEPENDENT FUNCTIONS OF BROWNIAN MOTION THAT ARE MARKOVIAN[1]

BY ALBERT T. WANG

University of Tennessee

The class of continuous functions $f(t, x)$ for which $f(t, X(t))$ are Markov processes is explicitly determined, where $X(t)$ is a Brownian motion on the real line. This extends a result by Walsh.

1. Introduction. Let $X(t)$ be a one-dimensional Brownian motion and $f(t, x)$ be a continuous function from $[0, \infty) \times R$ to R; then which properties of f make $f(t, X(t))$ a (homogeneous or nonhomogeneous) Markov process? When $f(t, x) = h(x)$ for all t, similar problems were studied by Rosenblatt [6]; Dynkin [3], page 325; and Walsh [7].

Rosenblatt [6] studied functions $h(X_n)$ of a stationary Markov chain X_n. Under the hypothesis that the transition probabilities of X_n are dominated by a sigma finite measure, he obtained some necessary and sufficient conditions on h for $h(X_n)$ to be Markov. For a stationary Markov process $Z(t)$, Dynkin ([3], page 325) has a sufficient condition on function h to make $h(Z(t))$ Markov. In the case of a one-dimensional Brownian motion $X(t)$, the following explicit criterion of h was obtained by Walsh [7] for $h(X(t))$ to be a Markov process.

PROPOSITION 1. *Let* $l_0(x) = 0$, $l_1(x) = x$, $l_2(x) = |x|$, *and* $l_3(x) = d(x, E)$, *where* $E = \{$*all even integers*$\}$ *and* $d(x, E) = \inf \{|x - y| | y \in E\}$. *Then* $h(X(t))$ *is a Markov process if and only if for some* i, $0 \leqslant i \leqslant 3$

$$h(x) = g \circ l_i(ax + b), \tag{1}$$

where a and b are constants, g is continuous and strictly monotone.

The purpose of this paper is to obtain a result similar to that of Walsh for time-dependent functions of Brownian motion. To state our problem more precisely, let $(X(t), \mathfrak{F}_t^s, P_{s,x})$, $t \geqslant 0$, be a standard Brownian motion on the real line $(R, \mathfrak{B})$ with transition functions

$$p(s, x, t, B) = P_{s,x}(X(t) \in B) = \bar{p}(t - s, x, B), \qquad \text{where} \quad 0 \leqslant s \leqslant t, B \in \mathfrak{B},$$

and $\mathfrak{F}_t^s$ is the completed sigma field generated by $\{X(u), s \leqslant u \leqslant t\}$. Let $Y(t) = f(t, x(t))$ and $\mathcal{G}_t^s$ be the completed sigma field generated by $\{Y(u), s \leqslant u \leqslant t\}$. For convenience, we define $\mathfrak{F}^s$ to be the sigma field generated by $\{X(u), u \geqslant s\}$ and we define $\mathcal{G}^s$ similarly. We shall obtain a necessary and sufficient condition on f under

Received August 24, 1977; revised April 19, 1978.

[1]Research sponsored by University of Tennessee faculty research funds and the National Science Foundation Grant MCS77-02110.

AMS 1970 *subject classifications.* Primary 60J25, 60J65.

Key words and phrases. Brownian motion, Markov processes.

which the process $Y(t)$ satisfies both (2) and (3) given in the following:

$$P_{s,x}[Y(t) \in B|\mathcal{F}_u^s] = P_{s,x}[Y(t) \in B|Y(u)] = q(u, Y(u), t, B), \tag{2}$$

$$P_{s,x}[Y(t) \in B|\mathcal{G}_u^s] = P_{s,x}[Y(t) \in B|Y(u)] = q(u, Y(u), t, B), \tag{3}$$

where $0 \leqslant s \leqslant u \leqslant t$, $B \in \mathfrak{B}$, $x \in R$. In both (2) and (3), $q(u, y, t, B)$ are Markov transition functions defined for $\{(u, y)|0 \leqslant u < \infty, y \in f(u, R)\}$, $u \leqslant t$, $B \in \mathfrak{B}$.

Throughout this paper, unless stated otherwise, f denotes a continuous function from $[0, \infty) \times R$ *to* R. We shall use $f_t^{-1}(B)$ to denote $f(t, \cdot)^{-1}(B)$. We put

$$S = \{f|f(s, x) = f(s, x') \text{ implies } p(s, x, t, f_t^{-1}(B)) = p(s, x', t, f_t^{-1}(B)), \text{ for all } t \geqslant s \geqslant 0, B \in \mathfrak{B}\}. \tag{4}$$

$$T = \{f|f \text{ satisfies formula (2)}\}, \tag{5}$$

$$U = \{f|f \text{ satisfies formula (3)}\}. \tag{6}$$

In Lemma 1, we obtain $S = T = U$. This is an extension of some results obtained in [6] and [3]. The main result of this paper is given in the following:

THEOREM. *$f \in U$ if and only if f is in one of the following four forms.*

(i) *$f(t, \cdot) = C(t)$ for all t;*

(ii) *there exists $t_0 > 0$ (t_0 may be ∞) such that $f(t, \cdot)$ are strictly increasing (decreasing) for all $t < t_0$, and $f(t, \cdot) = C(t)$ for all $t \geqslant t_0$;*

(iii) *there exists $t_0 > 0$ such that for $t < t_0$, $f(t, x) = g(t, |ax + b|)$, where g is a continuous function such that $g(t, \cdot)$ is strictly monotone for each t, and $f(t, \cdot) = C(t)$ for $t \geqslant t_0$;*

(iv) *there exists $t_0 > 0$ such that for $t < t_0$, $f(t, x) = g(t, l_3(ax + b))$, where a, b are constants and g is as given in* (iii), *and $f(t, \cdot) = C(t)$ for $t \geqslant t_0$.*

2. Some preliminary results. We shall show that $S = T = U$. Before we do so, we state without proof the following proposition which is needed later on.

PROPOSITION 2. *Let g and h be continuous functions from R to R. Suppose g is constant on sets $\{x|h(x) = \lambda\}$, $\lambda \in R$; then there exists a Borel measurable function v from R to R such that $g = v \circ h$.*

The following lemma is an extension of a result in [6] and [3].

LEMMA 1. $S = T = U$.

PROOF. Clearly, $T \subseteq U$. We only need to prove $S \subseteq T$ and $U \subseteq S$. For $0 \leqslant s \leqslant u \leqslant t$,

$$\begin{aligned} P_{s,x}[Y(t) \in B|\mathcal{F}_u^s] &= P_{s,x}[X(t) \in f_t^{-1}(B)|\mathcal{F}_u^s] \\ &= P_{s,x}[X(t) \in f_t^{-1}(B)|X(u)] \\ &= p(u, X(u), t, f_t^{-1}(B)). \end{aligned} \tag{7}$$

Assume $f \in S$. Then $p(u, x, t, f_t^{-1}(B)) = p(u, x', t, f_t^{-1}(B))$ whenever $f(u, x) = f(u, x')$. Now let u, t, and B be fixed; we put $g(x) = p(u, x, t, f_t^{-1}(B))$ and $h(x) = f(u, x)$. By Proposition 2, we know that there exists a Borel measurable function ν such that $p(u, x, t, f_t^{-1}(B)) = \nu[f(u, x)]$. Hence $p(u, X(u), t, f_t^{-1}(B)) = \nu[Y(u)]$ is $Y(u)$ measurable and

$$(8) \qquad P_{s,x}[Y(t) \in B \mid \mathcal{F}_u] = p(u, X(u), t, f_t^{-1}(B)) = \nu[Y(u)] = P_{s,x}[Y(t) \in B \mid Y(u)].$$

The first equality of (2) is obtained. Now we define

$$(9) \qquad q(u, y, t, B) = P_{u,x}[X(t) \in f_t^{-1}(B)],$$

when $f(u, x) = y$. By (4), $q(u, y, t, B)$ are well defined for $\{(u, y) | 0 \leqslant u < \infty, y \in f(u, R)\}$, $t \geqslant u$. Combining (7), (8), and (9), we get

$$(10) \qquad \begin{aligned} P_{s,x}[Y(t) \in B \mid Y(u)] &= p(u, X(u), t, f_t^{-1}(B)) \\ &= P_{u,X(u)}[X(t) \in f_t^{-1}(B)] \\ &= q(u, Y(u), t, B). \end{aligned}$$

That is, $f \in T$.

Now assume $f \in U$. When $s = u$ and $f(s, x) = y$, we use (3) to obtain

$$(11) \qquad \begin{aligned} P_{s,x}[Y(t) \in B] &= P_{s,x}[Y(t) \in B \mid Y(s)] \\ &= q(s, y, t, B). \end{aligned}$$

Then, for $f(s, x) = f(s, x')$

$$(12) \qquad \begin{aligned} p(s, x, t, f_t^{-1}(B)) &= P_{s,x}[Y(t) \in B] = q(s, y, t, B) \\ &= p(s, x', t, f_t^{-1}(B)). \end{aligned}$$

The technique used in proving the following lemma is very similar to that given in [2], Lemma 5A. This lemma is an extension of [7], Lemma 1.

LEMMA 2. *Let $f \in S$. Then any proper local maximum* (resp. *minimum*) *of $f(t, \cdot)$ is also a global maximum* (resp. *minimum*).

PROOF. We will give the argument only for the case of maximum. Assume $t > 0$. Let $f(t, x_0) = b$ be a proper local maximum of $f(t, \cdot)$, but not an absolute maximum. Then there exist x_1, x_2, x_3 such that $x_1 < x_0 < x_2$, $x_3 \notin [x_1, x_2]$ and that

$$(13) \qquad \max\{f(t, x_1), f(t, x_2)\} < b, \qquad \max_{x_1 \leqslant x \leqslant x_2}\{f(t, x)\} = b,$$

$$(14) \qquad f(t, x_3) > b \text{ (we may assume } x_3 > x_2).$$

By the continuity of f, there exists $\varepsilon > 0$, $\alpha(s) \in (x_1, x_2)$ such that for all $s \in (t - \varepsilon, t]$, (15) and (16) hold.

$$(15) \qquad \max_{x_1 \leqslant x \leqslant x_2}\{f(s, x)\} = f(s, \alpha(s)) > \max\{f(s, x_1), f(s, x_2)\}.$$

$$(16) \qquad \max_{x_1 \leqslant x \leqslant x_2}\{f(s, x)\} = f(s, \alpha(s)) < f(s, x_3).$$

For $s \in (t-\varepsilon, t]$, we define

$$\beta(s) = \sup\{x \leqslant x_3 | f(s, x) = f(s, \alpha(s))\}. \tag{17}$$

Since f is a continuous function, we can take $\alpha(s)$, $\beta(s)$ to be measurable. Hence we can find $\{s_n\}_1^\infty \subseteq (t-\varepsilon, t]$ such that $\{s_n\}$ increases to $s_0 \in (t-\varepsilon, t]$, and that

$$\lim_{n\to\infty} \beta(s_n) = \beta(s_0), \qquad \lim_{n\to\infty} \alpha(s_n) = \alpha(s_0). \tag{18}$$

Given any positive $\varepsilon < \frac{1}{4}$, by the fact that $f(s, x) > f(s, \beta(s))$ for all $x \in (\beta(s), x_3]$, there exists N_0 such that for all $n \geqslant N_0$,

$$p\big(s_n, \beta(s_n), s_0, f_{s_0}^{-1}\{[f(s_0, \beta(s_0)), \infty)\}\big) > \tfrac{1}{2} - \varepsilon. \tag{19}$$

On the other hand, by (15) we can find N_1 such that for all $n \geqslant N_1$

$$p\big(s_n, \alpha(s_n), s_0, f_{s_0}^{-1}\{[f(s_0, \alpha(s_0)), \infty)\}\big) < \tfrac{1}{2} - \varepsilon. \tag{20}$$

But $f(s_n, \alpha(s_n)) = f(s_n, \beta(s_n))$, $n = 0, 1, 2 \cdots$, (19) and (20) contradict the definition of S.

The case for which $t = 0$ can be easily obtained by a continuity argument.

Lemma 3. *Let $f \in S$. If $f(t, \cdot)$ has an improper local maximum (or improper local minimum) then $f(u, \cdot) = C(u)$ for all $u \geqslant t$, where $C(u)$ is a continuous function of u only.*

Proof. Suppose $f(t, x_0) = b$ is an improper local maximum. Then there exists a sequence $\{x_n\}_1^\infty$ of distinct terms such that x_n converges to x_0 and that $f(t, x_n) = b$ for all n. Hence for all fixed $u > t$, $B \in \mathfrak{B}$, $p(t, x_n, u, f_u^{-1}(B))$ is a constant. Since

$$p(t, x, u, f_u^{-1}(B)) = \int_{f_u^{-1}(B)} [2\pi(u-t)]^{-\frac{1}{2}} \exp[-(x-y)^2/2(u-t)]\, dy, \tag{21}$$

it is a real analytic function of x when u and B are fixed. Now $p(t, \cdot, u, f_u^{-1}(B))$ is a real analytic function which equals to b at $\{x_n\}_{n=0}^\infty$; it must be a constant function. Differentiating both sides of (21) with respect to x we get for all $x \in R$

$$\int_{f_u^{-1}(B)} (y-x)[2\pi(u-t)^3]^{-\frac{1}{2}} \exp[-(x-y)^2/2(u-t)]\, dy = 0. \tag{22}$$

This is possible only when $f_u^{-1}(B)$ or its complement is of Lebesgue measure zero. Let B be any open interval. Then $f_u^{-1}(B)$ is an open set. Hence $f_u^{-1}(B)$ is either R or the empty set. That is, $f(u, \cdot) = C(u)$ for all $u > t$. It follows by the continuity of f that $f(t, \cdot) = C(t)$.

Remark 1. Let $f \in S$. Let $f(t, \cdot)$ be a nondegenerate function. Then Lemma 3 implies that $f(s, \cdot)$ are nondegenerate for all $0 \leqslant s \leqslant t$. Furthermore, by using the arguments of Lemma 3 we can prove that $\{x | f(t, x) = b\}$ does not have a limit point for any $b \in R$.

The following lemma is simply a restatement of definition (4). We present it as a lemma because of its importance.

Lemma 4. *Let $f \in S$ and let $f(s, x) = f(s, x')$. Then $P_{s,x}$ is equal to $P_{s,x'}$ on $\mathcal{G}^s$.*

Let us now stop for a moment to see one of the implications of our main theorem. Let $f \in S$ and $f(t, \cdot)$ be nondegenerate and not strictly monotone. Then our theorem implies that $f(t, \cdot)$ should be of the form as given in either (iii) or (iv) in the statement of the main theorem. We shall prove this is indeed the case. We need more notations to proceed. Define

$$m(t) = \inf_x\{f(t, x)\}, \qquad M(t) = \sup_x\{f(t, x)\}, \tag{23}$$

$$\begin{aligned} r_c(t) &= m(t) + c \qquad \text{if } m(t) > -\infty, \\ &= M(t) - c \qquad \text{if } m(t) = -\infty, \end{aligned} \tag{24}$$

$$\begin{aligned} \Lambda_c(t) &= \{x | f(t, x) = r_c(t)\}, \\ \Lambda_m(t) &= \{x | f(t, x) = m(t)\}, \\ \Lambda_M(t) &= \{x | f(t, x) = M(t)\}. \end{aligned} \tag{25}$$

LEMMA 5. *Let $f \in S$ and $f(t, \cdot)$ be nondegenerate and not strictly monotone. Then f has the following properties*:

(i) *Any local extremum of $f(s, \cdot)$, $0 \leqslant s \leqslant t$, is a global extremum of $f(s, \cdot)$. Conversely, any global extremum of $f(s, \cdot)$, $0 \leqslant s \leqslant t$, is a proper local extremum of $f(s, \cdot)$.*

(ii) $\Lambda_M(t) \cup \Lambda_m(t) \neq \varnothing$, $\Lambda_M(t) \cap \Lambda_m(t) = \varnothing$.

(iii) *For each s, $0 \leqslant s \leqslant t$, $\Lambda_M(s)$ and $\Lambda_m(s)$ contain only isolated points. Let x_1 and x_2 be two adjacent extrema of $f(s, \cdot)$ (when there is no extremum adjacent in either the left or the right, one or both of x_1 and x_2 can be taken from $\{-\infty, \infty\}$). Then $f(s, \cdot)$ is strictly monotone in the open interval spanned by x_1 and x_2.*

(iv) *For each $x \in \Lambda_M(t)$ (or $\Lambda_m(t)$, $\Lambda_c(t)$), $t > 0$, and any given $\varepsilon > 0$, one can find $\delta > 0$ and a unique x_s for each $s \in (t - \delta, t]$ such that $x_s \in \Lambda_M(s)$ (or $\Lambda_m(s)$, $\Lambda_c(s)$) and $|x_s - x| < \varepsilon$.*

PROOF.

(i) Since $f \in S$ and $f(t, \cdot)$ is nondegenerate, by Lemma 3 $f(s, \cdot)$, $0 \leqslant s \leqslant t$, cannot have improper local extremum. Hence any local extremum of $f(s, \cdot)$, $0 \leqslant s \leqslant t$, must be a proper local extremum. But proper local extrema of $f(s, \cdot)$ are global extrema by Lemma 2. The converse can be shown by similar arguments.

(ii) By our hypothesis $f(t, \cdot)$ is nondegenerate and not strictly monotone. If $f(t, \cdot)$ is monotone but not strictly monotone, then for some $b\{x | f(t, x) = b\}$ has a limit point. Then it follows from Remark 1 that $f(t, \cdot)$ is degenerate, a contradiction to our hypothesis. Now we know $f(t, \cdot)$ is not monotone, hence by the continuity of $f(t, \cdot)$ it has a local extremum. By using (i) one obtains $\Lambda_M(t) \cup \Lambda_m(t) \neq \varnothing$. It is also true that $\Lambda_M(t) \cap \Lambda_m(t) = \varnothing$, because $f(t, \cdot)$ is nondegenerate.

(iii) If either $\Lambda_M(s)$ or $\Lambda_m(s)$, $0 \leqslant s \leqslant t$, contains an accumulation point, then by Lemma 3 $f(u, \cdot)$, $u \geqslant s$, are degenerate, hence contradicting our hypothesis. Now assume that x_1 and x_2, $x_1 < x_2$, are two adjacent extrema of $f(s, \cdot)$, $0 \leqslant s \leqslant t$. If $f(s, \cdot)$ is not monotone in (x_1, x_2), then it has a local extremum in (x_1, x_2) and hence a global extremum (by (i)) in (x_1, x_2). This contradicts our assumption that x_1 and x_2 are adjacent. If $f(s, \cdot)$ is monotone but not strictly monotone in (x_1, x_2), then for some $b\{x | f(s, x) = b\}$ has a limit point. This would imply that $f(u, \cdot)$, $u \geqslant s$, are degenerate and is a contradiction to our hypothesis.

(iv) First, we prove the existence of δ and x_s in the case when $x \in \Lambda_M(t)$. By (i) x is a proper local maximum of $f(t, \cdot)$. Given $\varepsilon > 0$, by the continuity of f one can find $\delta > 0$ and an x_s for each $s \in (t - \delta, t]$ such that x_s is a local maximum of $f(s, \cdot)$ and that $|x_s - x| < \varepsilon$. By (i) again $x_s \in \Lambda_M(s)$. Similarly, one can obtain the existence of δ and x_s in the case when $x \in \Lambda_m(t)$. Note that in the proof presented above we only used the property (i) and the continuity of f.

Now assume $x \in \Lambda_c(t)$. For simplicity we will also assume $m(t) > -\infty$. The case when $m(t) = -\infty$ can be treated similarly. Since $m(t) > -\infty$, there exists $\delta_0 > 0$ such that $m(s) > -\infty$ for all $s \in [t - \delta_0, t]$. Now for $s \in [t - \delta_0, t]$, we define a continuous function

$$
\begin{aligned}
(27)\qquad \nu_c(s, z) &= [f(s, z) - m(s)]/[r_c(s) - m(s)] \text{ if } f(s, z) \leqslant r_c(s), \\
&= 1 - [f(s, z) - m(s)]/[r_c(s) - m(s)] \text{ if } f(s, z) > r_c(s).
\end{aligned}
$$

Since $f(s, \cdot)$, $0 \leqslant s \leqslant t$, are piecewise strictly monotone (see (iii)), the local extrema of $\nu_c(s, \cdot)$ are also proper for $s \in [t - \delta_0, t]$. It is easy to see that x is a proper local maximum of $\nu_c(s, \cdot)$ if and only if $x \in \Lambda_c(s)$. Then our arguments used in the case when $x \in \Lambda_M(t)$ can be applied here.

Now we start to prove the uniqueness. We will only prove the case when $x \in \Lambda_M(t)$; the other cases can be proved similarly. Given arbitrary $\varepsilon > 0$, assume that for each $\delta > 0$ we can find s, $s \in (t - \delta, t]$, and two distinct points x_s, x_s' in $\Lambda_M(s)$ such that $|x - x_s| < \varepsilon$ and $|x - x_s'| < \varepsilon$. Then one can find a point x_s'' which is located in between x_s and x_s' and belongs to $\Lambda_m(s)$. This implies that x is a limit point of $\{\Lambda_m(s) | s \in (t - \delta, t)\}$ for any $\delta > 0$. It follows then that $f(t, \cdot)$ is degenerate, a contradiction to our hypothesis.

Let f satisfy the conditions of Lemma 5. Then by 5 (ii) $\Lambda_M(t) \cup \Lambda_m(t) \neq \varnothing$. Let $\Lambda_M(t) \neq \varnothing$. For $x \in \Lambda_M(t)$, we define

$$
\begin{aligned}
(28)\qquad L(t, x) &= \sup \{z | z < x, z \in \Lambda_m(t)\}, \\
R(t, x) &= \inf \{z | z > x, z \in \Lambda_m(t)\}.
\end{aligned}
$$

Clearly, we have $L(t, x) < x < R(t, x)$.

LEMMA 6. *Let $f \in S$ and $f(t, \cdot)$ be nondegenerate and not strictly monotone. Let $\Lambda_M(t) \neq \varnothing$ and $x \in \Lambda_M(t)$. Then $f(t, \cdot)$, when restricted to $(L(t, x), R(t, x))$, is symmetric to x.*

PROOF. Assume $t > 0$. From Lemma 5(iii), $f(t, \cdot)$ is strictly increasing in $(L(t, x), x)$ and is strictly decreasing in $(x, R(t, x))$. Suppose that $f(t, \cdot)$, when restricted to $(L(t, x), R(t, x))$, is not symmetric to x; then one can find two distinct points x' and x'' in $(L(t, x), R(t, x))$ such that $f(t, x') = f(t, x'') = r_b(t)$ for some b but $|x'' - x| \neq |x' - x|$. Without loss of generality, we can assume that $x'' - x > x - x' > 0$. By using Lemma 5(iv) we can find $\delta > 0$ and a unique $\gamma(s)$ for each $s \in (t - \delta, t]$ such that $\gamma(s) \in \Lambda_M(s)$ and $\lim_{s \to t} \gamma(s) = \gamma(t) = x$. Further, for each $s \in (t - \delta, t]$ we define

$$\alpha(s) = \sup\{z | f(s, z) = r_b(s), z < \gamma(s)\}, \tag{29}$$
$$\beta(s) = \inf\{z | f(s, z) = r_b(s), z > \gamma(s)\}.$$

Clearly, $\alpha(t) = x'$, $\beta(t) = x''$, $f(s, \alpha(s)) = f(s, \beta(s)) = r_b(s)$, $f(s, \gamma(s)) = M(s)$. Because $x'' - x > x - x'$, we know $\beta(s) - \gamma(s) > \gamma(s) - \alpha(s)$ when s is very close to t. Indeed, there exists $\delta' > 0$ such that

$$\begin{aligned}(30)\ \inf\{\beta(s) - \gamma(s') | t - \delta' &\leqslant \min(s, s') \\ &\leqslant \max(s, s') \leqslant t\} > \sup\{\gamma(s) - \alpha(s') | t - \delta' \\ &\leqslant \min(s, s') \leqslant \max(s, s') \leqslant t\}.\end{aligned}$$

Pick an $s \in (t - \delta', t]$. By Lemma 5(iii), $f(s, \cdot)$ is strictly increasing in $(\alpha(s), \gamma(s))$ and strictly decreasing in $(\gamma(s), \beta(s))$. Further, we know $f(s, \alpha(s)) = f(s, \beta(s))$. Hence

$$\{y = f(s, z) | \alpha(s) \leqslant z \leqslant \gamma(s)\} = \{y = f(s, z) | \gamma(s) \leqslant z \leqslant \beta(s)\}. \tag{31}$$

Now let $T = \inf\{u | Y(u) = M(u) \text{ or } Y(u) = r_b(u)\}$. We will restrict our attention to T under the measure $P_{s,x}$. Note that $P_{s,x}$ is a measure defined on $\mathcal{F}^s$. Then the set $\{Y(s) = x, T < t\}$ is $\mathcal{G}_t^s$ measurable under $P_{s,x}$. We put

$$A = \sup\{P_{s,z}[Y(t) \in [r_b(t), M(t)], t \leqslant T] | \gamma(s) \leqslant z \leqslant \beta(s)\}, \tag{32}$$

$$B = \sup\{P_{s,z}[Y(t) \in [r_b(t), M(t)], t \leqslant T] | \alpha(s) \leqslant z \leqslant \gamma(s)\}. \tag{33}$$

From Lemma 4 and (31), it follows that $A = B$. On the other hand, for $z \in [\gamma(s), \beta(s)]$

$$\begin{aligned}(34)\quad \{X(s) = z,\ Y(t) \in [r_b(t), M(t)], t \leqslant T\} \\ = \{X(s) = z, \gamma(u) < X(u) < \beta(u) \text{ for } s \leqslant u < t, \\ X(t) \in [\gamma(t), \beta(t)]\}\end{aligned}$$

and for $z \in [\alpha(s), \gamma(s)]$

$$\begin{aligned}(35)\quad \{X(s) = z,\ Y(t) \in [r_b(t), M(t)], t \leqslant T\} \\ = \{X(s) = z, \alpha(u) < X(u) < \gamma(u) \text{ for } s \leqslant u < t, \\ X(t) \in [\alpha(t), \gamma(t)]\}.\end{aligned}$$

Knowing (30) and the properties of $P_{s,x}$ we can see that $A > B$. Hence $f(t, \cdot)$ must be symmetric about x.

Now let $t = 0$. By our assumption $\Lambda_M(0) \neq \varnothing$. By Lemma 5(i), we know that $x \in \Lambda_M(0)$ is also a proper local maximum of $f(0, \cdot)$. By the continuity of f it follows that there exists $\delta > 0$ such that for all $s \in [0, \delta]$, $\Lambda_M(s)$ are nonempty and $f(s, \cdot)$ are nondegenerate and not strictly monotone. Hence $f(s, \cdot)$, $s \in (0, \delta]$, are symmetric to its absolute maximum when they are restricted to the proper intervals. Then by a continuity argument, one can prove Lemma 6 for the case when $t = 0$.

REMARK 2. In Lemma 6 if one replaces the condition $\Lambda_M(t) \neq \varnothing$ by $\Lambda_m(t) \neq \varnothing$ and assumes that $x \in \Lambda_m(t)$, then $f(t, \cdot)$ is symmetric to x in an interval defined similar to $(L(t, x), R(t, x))$.

REMARK 3. Let t be positive and $f(t, \cdot)$ be nondegenerate and not strictly monotone. From Lemma 5(ii) $\Lambda_M(t) \cup \Lambda_m(t) \neq \varnothing$. Further, Lemma 5(i) implies that the global extrema of $f(t, \cdot)$ are proper local extrema. By the continuity of f, it follows that $f(s)$, $s \in [t - \delta, t + \delta]$, are nondegenerate and not strictly monotone for some $\delta > 0$. Hence $f(s, \cdot)$, $t - \delta \leqslant s \leqslant t + \delta$, also have the kind of symmetric property given in Lemma 6 and Remark 2.

REMARK 4. Let $f \in S$ and $f(t, \cdot)$ be nondegenerate and not strictly monotone. Hence $\Lambda_M(t) \cup \Lambda_m(t) \neq \varnothing$ and $\Lambda_M(t) \cap \Lambda_m(t) = \varnothing$. Then $f(t, \cdot)$ either attains one of its absolute extrema or it attains both of them. If it attains only one of its absolute extrema, then by Lemma 5(iii) it can only attain this extrema at one point, say at point x. Then $f(t, \cdot)$ is symmetric to x by either Lemma 6 or Remark 2. Similarly one can see that if $f(t, \cdot)$ attains both of its extrema, then $f(t, \cdot)$ is a periodic function.

Now we define $\psi(t)$ to be the period of $f(t, \cdot)$ if it is a periodic function and we put $\psi(t)$ to be ∞ when it attains only one of its absolute extrema. Note that the extended valued function $\psi(t)$ is undefined when $f(t, \cdot)$ is degenerate or strictly monotone.

LEMMA 7. *Let $f \in S$ and $f(t, \cdot)$ be nondegenerate and not strictly monotone. Let $x \in \Lambda_M(t) \neq \varnothing$ (or $x \in \Lambda_m(t) \neq \varnothing$). Then $x \in \Lambda_M(s)$ (or $x \in \Lambda_m(s)$) for all $s \in [0, t]$.*

PROOF. Let $\alpha = \inf \{u | x \in \Lambda_M(v) \text{ for all } v \in [u, t]\}$. Clearly, $0 \leqslant \alpha \leqslant t$. By Lemma 3 $f(\alpha, \cdot)$ cannot be degenerate. By the definition of α and the continuity of f, $f(\alpha, \cdot)$ cannot be strictly monotone either. Hence $f(\alpha, \cdot)$ satisfies the conditions of Lemma 5. Further, it is easy to see that $x \in \Lambda_M(\alpha)$. If $\alpha = 0$, then there is nothing more to be shown. Assume $\alpha > 0$. Given a positive number $\varepsilon < \psi(\alpha)/4$, by Lemma 5(iv) there exists $\delta > 0$ such that for each $u \in (\alpha - \delta, \alpha]$ one can find a unique $x_u \in \Lambda_M(u)$ satisfying $|x_u - x| < \varepsilon$. By the definition of α, one can find $\beta \in (\alpha - \delta, \alpha]$ such that $x_\beta \neq x$. Without loss of generality, we can assume $x_\beta < x$. By Lemma 6 $f(\beta, \cdot)$ is symmetric to x_β, when it is restricted on a proper region. Hence one can find $x' < x_\beta < x$ such that $f(\beta, x') = f(\beta, x)$. Then

$p(\beta, x, \alpha, f_\alpha^{-1}(B)) = p(\beta, x', \alpha, f_\alpha^{-1}(B))$ for all $B \in \mathfrak{B}$. Now we choose a number $a \in (m(\alpha), M(\alpha))$ and put $A = [a, M(\alpha)]$. Clearly,

$$(36)\quad p\big(\beta, z, \alpha, f_\alpha^{-1}(A)\big) = \int_{f_\alpha^{-1}(A)} \big[2\pi(\alpha - \beta)\big]^{-\frac{1}{2}} \exp\big[(z - y)^2/2(\alpha - \beta)\big]\, dy.$$

By Remark 4, $f_\alpha^{-1}(A)$ is a set which is symmetric about point x. Note that

$$(37)\qquad x' - x = (x' - x_\beta) + (x_\beta - x) < \psi(\beta)/2 + \psi(\alpha)/4 < \psi(\alpha).$$

The last inequality (37) is true when we choose β very close to α. Hence $x' \notin \Lambda_M(\alpha)$. Calculating the first and the second derivative of $p(\beta, \cdot, \alpha, f_\alpha^{-1}(A))$ by differentiating under the integral sign of (36), one can easily see that $z = x$ is a proper local maximum of $p(\beta, z, \alpha, f_\alpha^{-1}(A))$. Indeed $p(\beta, z, \alpha, f_\alpha^{-1}(A))$ attains absolute maximum at point z if and only if $z \in \Lambda_M(\alpha)$. Hence $p(\beta, x', \alpha, f_\alpha^{-1}(A)) < p(\beta, x, \alpha, f_\alpha^{-1}(A))$. This contradicts our assumption that $f \in S$ and $f(\beta, x) = f(\beta, x')$.

LEMMA 8. *Let $f \in S$ and $f(t, \cdot)$ be nondegenerate and not strictly monotone. Then $\Lambda_M(t) = \Lambda_M(0)$ and $\Lambda_m(t) = \Lambda_m(0)$.*

PROOF. We will only prove that $\Lambda_M(t) = \Lambda_M(0)$. By Lemma 7, $\Lambda_M(t) \subseteq \Lambda_M(0)$. We only need to show $\Lambda_M(0) \subseteq \Lambda_M(t)$. Let $x \in \Lambda_M(0)$. Define $\alpha = \sup\{u | x \in \Lambda_M(v)$ for all $v \in [0, u]\}$. Clearly, $\alpha \geqslant 0$. We want to show $\alpha > t$. By the definition of α and the continuity of f, one sees that $f(\alpha, \cdot)$ cannot be strictly monotone and that $x \in \Lambda_M(\alpha)$. Assume $\alpha < t$. Then by Lemma 3 $f(\alpha, \cdot)$ cannot be degenerate. By going through a similar argument as given in Remark 3, we know there exists $\delta_0 > 0$ such that for all $s \in [\alpha, \alpha + \delta_0]$, $f(s, \cdot)$ are nondegenerate and not strictly monotone. Given $\varepsilon > 0$, by an argument similar to Lemma 5(iv) one can find $\delta_1 > 0$ and a unique x_s for each $s \in [\alpha, \alpha + \delta_1)$ such that $x_s \in \Lambda_M(s)$ and $|x_s - x| < \varepsilon$. Put $\delta = \min(\delta_0, \delta_1)$ and applying Lemma 7 to $f(s, \cdot)$ for each $s \in [\alpha, \alpha + \delta)$, one obtains $x_s = x$ for all $s \in [\alpha, \alpha + \delta]$, a contradiction to the definition of α. Hence $\alpha > t$.

3. Proof of the main theorem. Let $f \in S$. By Remark 4 $f(0, \cdot)$ can only be of one of the following four forms:

(38) $f(0, \cdot) = C(0)$ is degenerate;

(39) $f(0, \cdot)$ is strictly increasing (decreasing);

(40) $f(0, \cdot)$ attains only one of its absolute extrema and does so only at one point x, and $f(0, \cdot)$ is strictly monotone in $(-\infty, x)$ and (x, ∞) (see Lemma 5(iii)). Further, $f(0, \cdot)$ is symmetric to x;

(41) $f(0, \cdot)$ is a periodic function which satisfies the symmetric properties given in Lemma 6 and Remark 2. Further, $f(0, \cdot)$ is piecewise strictly monotone (Lemma 5(iii)).

THEOREM A. *Let $f \in S$. Let $f(0, \cdot) = C(0)$. Then $f(u, \cdot) = C(u)$ for all u.*

PROOF. This is a consequence of Lemma 3.

THEOREM B. *Let $f \in S$. Let $f(0, \cdot)$ be strictly increasing (decreasing). Then there exists $t_0 > 0$ (t_0 may be ∞) such that $f(t, \cdot)$ are strictly increasing (decreasing) for all $t < t_0$, and $f(t, \cdot) = C(t)$ for all $t \geqslant t_0$.*

PROOF. Let $t_0 = \sup \{u | f(v, \cdot)$ are increasing for all $v \in [0, u]\}$. If $t_0 = \infty$, there is nothing more to be proved. We assume $t_0 < \infty$. Given $\varepsilon > 0$ there exists t_1, $t_1 \in [t_0, t_0 + \varepsilon)$, such that $f(t_1, \cdot)$ is not strictly increasing. If $f(t_1, \cdot)$ is nondegenerate and not strictly monotone, then by Lemma 5(ii) $\Lambda_M(t_1) \cup \Lambda_m(t_1) \neq \varnothing$. Using Lemma 8, we have $\Lambda_M(0) \cup \Lambda_M(0) \neq \varnothing$, a contradiction to the hypothesis that $f(0, \cdot)$ is strictly increasing. Hence $f(t_1, \cdot)$ can only be degenerate or strictly decreasing. One of the following two cases has to happen. Either there exists a sequence t_n decreasing to t such that $f(t_n, \cdot)$ are strictly decreasing for all n or $f(t, \cdot)$, $t_0 < t \leqslant t_0 + \varepsilon$, are degenerate for a certain $\varepsilon > 0$. Both cases will lead to $f(t_0, \cdot) = C(t_0)$. Then by Lemma 3, $f(t, \cdot) = C(t)$ for all $t \geqslant t_0$.

THEOREM C. *Let $f \in S$ and $f(0, \cdot)$ be as given in* (40). *Then there exists $t_0 > 0$ such that for $t < t_0$, $f(t, x) = g(t, |ax + b|)$, where g is a continuous function such that $f(t, \cdot)$ is strictly monotone for each t, a and b are two constants, and $f(t, \cdot) = C(t)$ for $t \geqslant t_0$.*

PROOF. Without loss of generality, we assume $\Lambda_M(0) = x$ and $\Lambda_m(0) = \varnothing$. We put $t_0 = \sup \{t | f(s, \cdot)$ is nondegenerate and not strictly monotone for $s \in [0, t]\}$. By Lemma 8, we know $\Lambda_M(s) = x$, $\Lambda_m(s) = \varnothing$ for all $s \in [0, t_0)$. Clearly, $f(t_0, \cdot)$ cannot be strictly monotone. If $f(t_0, \cdot)$ is also nondegenerate, then by the arguments of Remark 3 we would have $f(t_0, \cdot)$ nondegenerate and not strictly monotone for $t_0 + \delta > t > t_0$, $\delta > 0$. Hence $f(t_0, \cdot)$ must be degenerate. Then $f(t, \cdot)$, $t \geqslant t_0$, are degenerate by Lemma 3. Now we know for all $s \in [0, t_0)$, $f(s, \cdot)$ are symmetric to x and are strictly increasing in $(-\infty, x)$ and strictly decreasing in (x, ∞). The existence of function g and constants a, b as given in the theorem is not hard to prove and is left for the reader.

THEOREM D. *Let $f \in S$ and $f(0, \cdot)$ be as given in* (41). *Then $f(t, x)$ is of the form as given in* (iv) *of the main theorem.*

We omit the proof of Theorem D, because it is similar to the proof of Theorem C. Now the necessity of our main theorem is obtained. The sufficiency is easy and hence omitted.

Acknowledgments. The author wishes to thank the referee for his helpful comments. Thanks also go to Professors Steven Orey and Bert Fristedt for their clarifying remarks; and to Professor Naresh Jain for his helpful comments, criticisms, and corrections.

REFERENCES

[1] DOOB, J. L. (1955). A probability approach to the heat equation. *Trans. Amer. Math. Soc.* **80** 216–280.

[2] Dudley, R. M. (1971). Non-linear equivalence transformations of Brownian motion. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **20** 249–258.

[3] Dynkin, E. B. (1965). *Markov Processes* I (English transl.). Springer-Verlag, Berlin.

[4] Freedman, D. (1971). *Brownian Motion and Diffusion*. Holden-Day, San Francisco.

[5] Gihman, I. I. and Skorohod, A. V. (1975). *The Theory of Stochastic Processes* II (English transl.). Springer-Verlag, New York.

[6] Rosenblatt M. (1959). Functions of a Markov process that are Markovian. *J. Math. Mech.* **8** 585–596.

[7] Walsh, J. B. (1975). Functions of Brownian motion. *Proc. Amer. Math. Soc.* **49** 227–231.

Department of Mathematics
University of Southern California
Los Angeles, California 90007