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UNIFORM LOCAL BEHAVIOR OF STABLE SUBORDINATORS[1]

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Appropriate constant multiples of the function $t^{1/\alpha}$ are "the" "maximal local lower envelope" and "the" "minimal local upper envelope" for the sample functions of a strictly stable subordinator of index α. The fact that the probability of extinction of a Galton-Watson process is less than one if the mean number of offspring is larger than one is used in the proofs.

1. Introduction. We shall study those increasing Lévy processes (processes with stationary, independent increments) $(t, \omega) \mapsto X(t, \omega) = X(t)$, $(t, \omega) \in [0, \infty] \times \Omega$, characterized by

$$\mathcal{E} e^{-\lambda X(t)} = e^{-tb\lambda^\alpha} \tag{1}$$

for some $\alpha \in (0, 1)$ and some $b > 0$. As is usual, we study a version such that $t \mapsto X(t)$ is right continuous and has left limits. The Lévy processes satisfying (1) are called strictly stable subordinators. The other stable subordinators (that is, increasing stable processes) are obtained by the addition of increasing deterministic linear functions to the sample functions of the strictly stable subordinators. These linear functions play a dominant role in the local behavior of the sample functions. Accordingly, we hereafter restrict our attention to strictly stable subordinators. Only the constants in the theorems in Section 4 depend on the value of the *scaling factor* b; for convenience we assume

$$\mathcal{E} e^{-\lambda X(t)} = e^{-t\Gamma(1-\alpha)\lambda^\alpha}; \tag{2}$$

α is called the *index* of the process.

A general problem is to study the behavior of

$$\frac{|X(s + t) - X(s)|}{h(|t|)} \tag{3}$$

as $t \to 0$ for various functions h. A probabilistic assertion that is true about (3) for some s will be true for any other one s. However, a statement that is true with probability one for some s will not necessarily be true with probability one for all s, since an uncountable number of null sets, one for each s, are involved.

The "uniform in s" theorems, Theorems 1–3 in Section 4, can not be obtained as corollaries of the known theorems for fixed s. See Section 6 of Fristedt (1974) for a survey of known theorems, including uniform theorems of a type different from those studied here, about the local behavior of subordinators. See Kahane (1974),

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Kahane (1976), Jain and Taylor (1973), and Orey and Taylor (1974) for some related results about Brownian motion.

Our goal is to prove Theorems 1–3 which, though stated in Section 4, can be read now. Section 3 contains the appropriate lemmas. Section 2 contains some known facts about stable subordinators.

I do not know whether a true statement is obtained by replacing "$t \to 0$" by "$t \downarrow 0$" in Theorem 3. The constant in Theorem 1 is no larger than the constant in Theorem 2. I do not know any other relationships among the constants in Theorems 1–3.

The methods in this paper depend heavily on the stability of the subordinators. I do not see how they can be used for general subordinators although it does seem likely that they can be adapted to apply to subordinators whose Lèvy measures have densities that vary regularly near 0.

2. Preliminary facts. The *scaling property* is that, for any $c > 0$, the process

$$t \mapsto c^{-1/\alpha}X(ct)$$

is identical in law to the process X.

A second important fact is that there exists a constant $c_1 > 0$ such that the density of $X(t)$ is asymptotic to $tc_1x^{-(1+\alpha)}$ as $x \to \infty$.

For each $(t, \omega) \in [0, \infty) \times \Omega$, let $J(t, \omega) = J(t) = X(t) - X(t-)$. For almost every ω, $J(t) = 0$ except for countably many values of t and

$$X(t) = \Sigma_{s \leqslant t}J(s).$$

Let $\{B_\beta\}$ denote a family of disjoint Borel subsets of $[0, \infty) \times (0, \infty)$. For each β let

$$V_\beta = \operatorname{card}\{t : (t, J(t)) \in B_\beta\}.$$

The family $\{V_\beta\}$ of random variables is an independent family; V_β is Poisson distributed with mean $(m \times \nu)(B_\beta)$ where m is Lebesgue measure and $\nu[x, \infty) = x^{-\alpha}$ for each $x > 0$ (A "Poisson random variable with infinite mean" equals ∞ almost surely). The case where each B_β is a rectangle is of particular interest. Also of interest is the consequence: the density of the random vector

$$(\max_{0 \leqslant t \leqslant 1}J(t), X(1) - \max_{0 \leqslant t \leqslant 1}J(t))$$

is

$$(x, y) \mapsto \alpha x^{-(1+\alpha)}e^{-x^{-\alpha}}f_x(y) \tag{4}$$

where the ordinary Fourier transform of the density f_x equals

$$\theta \mapsto \exp\left\{-\alpha\textstyle\int_0^x(1 - e^{i\theta z})z^{-(1+\alpha)}\,dz\right\}. \tag{5}$$

Notice that both the density of $X(1)$ and f_x are infinitely divisible densities determined by the Lévy measures

$$\alpha z^{-(1+\alpha)}\,dz$$

and

$$\begin{array}{ll} \alpha z^{-(1+\alpha)}\,dz & \text{if } z \leqslant x \\ 0\,dz & \text{if } z > x, \end{array}$$

respectively.

That the density f_x, rather than merely a distribution function, exists and is continuous follows from the fact that the characteristic function (5) is integrable. This integrability is a consequence of

$$\alpha \textstyle\int_0^x (1 - \cos \theta z) z^{-(1+\alpha)}\,dz \geqslant c_2 |\theta|^\alpha (1 \wedge |\theta x|^{2-\alpha}) \tag{6}$$

for an appropriate $c_2 > 0$, obtained by replacing x by $x \wedge |\theta|^{-1}$ and bounding $1 - \cos \theta z$ below by a multiple of $|\theta z|^2$. The continuity of $(x, y) \mapsto f_x(y)$ follows from the Fourier inversion formula and the explicit way that the characteristic functions (5) depend on x.

Throughout, $c_1, c_2, \cdots, c_{21}$ will denote appropriate positive, finite constants (depending on α); initial assertions involving them are assertions that positive constants exist making the assertions true.

3. Lemmas. The salient features of this paper are Lemmas 1, 2, 6, and 7. Theorems 1–3 in Section 4 are easy consequences of these lemmas. Were we able to replace "$t \to 0$" by "$t \downarrow 0$" in Lemma 7, there would be a "natural Theorem 4" appearing in Section 4. I do not know if such a modification of Lemma 7 is possible. The lack of left continuity of X implies that "$t \to 0$" can not be replaced by "$t \uparrow 0$".

Lemma 3, possibly of independent interest, and Lemma 4 are needed for the proof of Lemma 5, also possibly of independent interest, which is used in the proofs of Lemmas 6 and 7. I guess, but do not know how to prove, that the factor $\log y$ can be removed from Lemma 3.

LEMMA 1. $0 < \inf_{0 < s \leqslant 1} \limsup_{t \downarrow 0} (X(s + t) - X(s))/t^{1/\alpha}$ *with probability one.*

PROOF. Fix c such that $\exp(-1/4c^\alpha) = \frac{1}{3}$. For positive integers $M \leqslant N$ we define subsets $A_{M,N}$ of $\{0, 1\}^N : A_{M,M} = \{0, 1\}^M$ and, for $N > M$,

$$\begin{aligned} A_{M,N} = \big\{(a_1, \cdots, a_N) : (a_1, \cdots, a_{N-1}) \in A_{M,N-1} \quad &\text{and} \\ J(u) \notin \big[c2^{-(N-1)/\alpha}, c2^{-(N-2)/\alpha}\big) \quad &\text{for all} \\ u \in \big(2^{-N} + \textstyle\sum_{j=1}^N a_j 2^{-j}, 2^{-(N-1)} + \sum_{j=1}^N a_j 2^{-j}\big]\big\}. \end{aligned}$$

Since the events

$$\begin{aligned} \big\{\omega : J(u, \omega) \notin \big[c2^{-(n-1)/\alpha}, c2^{-(n-2)/\alpha}\big) \quad &\text{for all} \\ u \in \big(2^{-n} + \textstyle\sum_{j=1}^n a_j 2^{-j}, 2^{-(n-1)} + \sum_{j=1}^n a_j 2^{-j}\big]\big\} \end{aligned}$$

each have probability

$$\exp\big[2^{-n}(2^{n-1}c^{-\alpha} - 2^{n-2}c^{-\alpha})\big] = \tfrac{1}{3}$$

and are independent for various n and n-tuples $(a_1, \cdots, a_n)$,

$$\mathcal{P}(\{\omega : (a_1, \cdots, a_N) \in A_{M,N}(\omega)\}) = 3^{-(N-M)}.$$

Hence,

$$\mathcal{E}(\operatorname{card} A_{M,N}) = 2^N 3^{-(N-M)} \to 0 \quad \text{as} \quad N \to \infty.$$

From Fatou's lemma and the fact that card $A_{M,N}(\omega)$ is a nonnegative integral random variable we conclude that, for almost every ω and every M, there exists $N(\omega)$ such that $A_{M,N(\omega)}(\omega) = \varnothing$. Fix such an ω.

Let $s = \sum_{j=1}^{\infty} a_j 2^{-j}$, where $a_j = 0$ or $a_j = 1$, be an arbitrary member of $(0, 1]$. There exists $n \in (M, N(\omega)]$ such that, for some

$$u \in \left(2^{-n} + \textstyle\sum_{j=1}^{n} a_j 2^{-j}, 2^{-(n-1)} + \sum_{j=1}^{n} a_j 2^{-j}\right],$$

$$X(s + 2^{-(n-1)}) - X(s) \geqslant X\left(2^{-(n-1)} + \textstyle\sum_{j=1}^{n} a_j 2^{-j}\right) - X\left(2^{-n} + \sum_{j=1}^{n} a_j 2^{-j}\right)$$
$$\geqslant J(u) \geqslant c 2^{-(n-1)/\alpha}.$$

Let $M \to \infty$ to obtain an infinite sequence of n's for which this calculation is valid. Hence, with probability one,

$$\inf_{0<s\leqslant 1} \limsup_{t \downarrow 0} \frac{X(s+t) - X(s)}{t^{1/\alpha}} \geqslant c.$$

LEMMA 2. $0 < \sup_{0<s\leqslant 1} \liminf_{t\to 0} \dfrac{|X(s+t) - X(s)|}{|t|^{1/\alpha}}.$

PROOF. Since

$$\sup_{0<s\leqslant 1} \liminf_{t\to 0} \frac{|X(s+t) - X(s)|}{|t|^{1/\alpha}}$$

depends only on the small jumps, Kolmogorov's $0 - 1$ law can be used to show that it a.s. equals a constant belonging to $[0, \infty]$. (See Fristedt and Pruitt (1972) for details of such an argument.) We begin the proof that it is positive with positive probability by choosing c so that $\exp(-1/16c^{\alpha}) = \frac{1}{4}$.

For each nonnegative integer N we define a subset B_N of $\{0, 1\}^N$: $B_0 = \{$empty sequence$\}$ and

$$B_N = \Big\{(b_1, \cdots, b_N) : (b_1, \cdots, b_{N-1}) \in B_{N-1} \quad \text{and}$$
$$J(u) \in \left[c 2^{-(N-2)/\alpha}, c\left(\tfrac{4}{3}\right)^{1/\alpha} 2^{-(N-2)/\alpha}\right) \quad \text{for some}$$
$$u \in \left(-2^{-N} + \textstyle\sum_{j=1}^{N} b_j 2^{-j}, \sum_{j=1}^{N} b_j 2^{-j}\right] \quad \text{and}$$
$$J(v) \in \left[c\left(\tfrac{4}{3}\right)^{1/\alpha} 2^{-(N-2)/\alpha}, c 2^{-(N-3)/\alpha}\right) \quad \text{for some}$$
$$v \in \left(2^{-N} + \textstyle\sum_{j=1}^{N} b_j 2^{-j}, 2^{-(N-1)} + \sum_{j=1}^{N} b_j 2^{-j}\right]\Big\}.$$

For $N \geqslant 1$,

$$\mathcal{P}((b_1, \cdots, b_N) \in B_N | (b_1, \cdots, b_{N-1}) \in B_{N-1})$$
$$= \{1 - \exp[-2^{-N}(2^{N-2}c^{-\alpha} - 2^{N-2}c^{-\alpha}(\tfrac{3}{4}))]\}$$
$$\cdot\{1 - \exp[-2^{-N}(2^{N-2}c^{-\alpha}(\tfrac{3}{4}) - 2^{N-3}c^{-\alpha})]\}$$
$$= \{1 - \exp(-1/16c^{\alpha})\}^2 = \{1 - (\tfrac{1}{4})\}^2 = \tfrac{9}{16}.$$

If $(b_1, \cdots, b_{N-1}, b_N) \in B_N$, and therefore $(b_1, \cdots, b_{N-1}) \in B_{N-1}$, call $(b_1, \cdots, b_N)$ an offspring of $(b_1, \cdots, b_{N-1})$. Since there are two choices for b_N, the preceding calculation shows that the expected number of offspring is $2(\frac{9}{16}) = \frac{9}{8} > 1$. The fact that jumps of different sizes or of the same size and occurring in disjoint time sets are independent enables us to interpret B_N as the Nth generation of a Galton-Watson process. Since $\frac{9}{8} > 1$, with positive probability a sequence $(b_1, b_2, \cdots)$, depending on ω, exists such that $(b_1, \cdots, b_N) \in B_N$ for each N. Let $s = \sum_{j=1}^{\infty} b_j 2^{-j}$. For all t for which $|t| \leqslant s$ there exists an N such that $2^{-(N-1)} < |t| \leqslant 2^{-(N-2)}$ and a

$$u \in (-2^{-N} + \textstyle\sum_{j=1}^{N} b_j 2^{-j}, \sum_{j=1}^{N} b_j 2^{-j}] \cup (2^{-N} + \sum_{j=1}^{N} b_j 2^{-j}, 2^{-(N-1)} + \sum_{j=1}^{N} b_j 2^{-j}]$$

such that

$$|X(s + t) - X(s)| \geqslant J(u) \geqslant c2^{-(N-2)/\alpha} \geqslant c|t|^{1/\alpha}.$$

Hence,

$$\liminf_{t \to 0} \frac{|X(s + t) - X(s)|}{|t|^{1/\alpha}} \geqslant c.$$

The fact that $X(s_2) - X(s_1)$ is larger than $J(u)$ for every $u \in (s_1, s_2]$ was useful in the proofs of Lemmas 1 and 2, but adds difficulties to the proofs of Lemmas 6 and 7. We overcome these difficulties by proving Lemmas 3, 4 and 5.

LEMMA 3. *If $x \geqslant y \vee 1$, then*

$$f_x(y) \leqslant c_3(1 \vee \log y) y^{-(1+\alpha)},$$

where f_x is the density with the characteristic function (5).

PROOF. Let g denote the density of $X(1)$. We need only show that

$$|f_x - g|(y) \leqslant c_4(1 \vee \log y) y^{-(1+\alpha)}$$

for $x \geqslant y \vee 1$. By the Fourier inversion formula

$$2\pi|f_x - g|(y) = |\textstyle\int_{-\infty}^{\infty} e^{-iy\theta - \Gamma(1-\alpha)(-i\theta)^{\alpha}}\{\exp[\alpha \int_x^{\infty}(1 - e^{i\theta z}) z^{-(1+\alpha)}\, dz] - 1\}\, d\theta|.$$

Write $e^{-iy\theta}\, d\theta = (iy)^{-1} d[1 - e^{-iy\theta}]$ and integrate by parts to obtain the upper bound

$$y^{-1}\textstyle\int_0^{\infty} e^{-c_5\theta^{\alpha}}|1 - e^{-iy\theta}|\{c_6|\int_x^{\infty} - ie^{i\theta z} z^{-\alpha}\, dz| + c_7\theta^{\alpha-1}|\int_x^{\infty}(1 - e^{i\theta z}) z^{-(1+\alpha)}\, dz|\}\, d\theta. \tag{7}$$

To obtain the desired upper bound for (7) use:

$$|1 - e^{-i\theta y}| \leqslant (\theta y) \wedge 2,$$
$$|1 - e^{i\theta z}| \leqslant 2,$$
$$|\textstyle\int_x^\infty e^{-i\theta z} z^{-\alpha}\, dz| = \theta^{\alpha-1}|\int_{\theta x}^\infty e^{-iv} v^{-\alpha}\, dv| \leqslant c_8 \theta^{-1}(\theta^\alpha \wedge x^{-\alpha}),$$

and $x \geqslant y \vee 1$.

LEMMA 4. *If* $y \geqslant x \vee 1$, *then*

$$f_x(y) \leqslant c_9 y^{-2}\big[x^{-\frac{1}{2}} \vee (x^{1-\alpha} \log x)\big],$$

where f_x *is the density with the characteristic function* (5).

PROOF. Integrate the Fourier inversion formula by parts using

$$e^{-i\theta y}\, d\theta = (-iy)^{-1}\, de^{-i\theta y},$$

integrate by parts again using

$$e^{-i\theta y}\, d\theta = (iy)^{-1}\, d\big[1 - e^{-i\theta y}\big],$$

and use the inequality (6) to obtain

$$\begin{aligned}(8)\qquad f_x(y) \leqslant y^{-2}\textstyle\int_0^\infty |1 - e^{-i\theta y}| e^{-c_2\theta^\alpha[1 \wedge (\theta x)^{2-\alpha}]}\\ \cdot \big\{c_{10}|\textstyle\int_0^x e^{i\theta z} z^{-\alpha}\, dz|^2 + c_{11}|\int_0^x e^{i\theta z} z^{1-\alpha}\, dz|\big\}\, d\theta.\end{aligned}$$

To obtain the desired upper bound for the right side of (8) use: $y \geqslant x \vee 1$,

$$|1 - e^{-i\theta y}| \leqslant (\theta y) \wedge 2,$$
$$|\textstyle\int_0^x e^{i\theta z} z^{-\alpha}\, dz| = \theta^{\alpha-1}|\int_0^{\theta x} e^{iv} v^{-\alpha}\, dv| \leqslant c_{12}(x^{1-\alpha} \wedge \theta^{\alpha-1}),$$
$$|\textstyle\int_0^x e^{i\theta z} z^{1-\alpha}\, dz| = \theta^{\alpha-2}|\int_0^{\theta x} e^{iv} v^{1-\alpha}\, dv| \leqslant c_{13} x^{1-\alpha}(x \wedge \theta^{-1}),$$

and the substitution $w = \theta x^{(2-\alpha)/2}$ when integrating over $\theta \in [0, x^{-1}]$.

LEMMA 5. *For each* $\varepsilon > 0$ *there exists a* k *such that*

$$(9)\qquad \mathcal{P}(\sup\{J(t) : 0 < t \leqslant 1\} \leqslant z - k | X(1) = z) < \varepsilon$$

for almost every z.

PROOF. Using (4), we write the left side of (9) as

$$\frac{\alpha \int_0^{(z-k)\vee 0} x^{-(1+\alpha)} \exp(-x^{-\alpha}) f_x(z - x)\, dx}{g(z)},$$

where g denotes the density of $X(1)$. Accordingly, we need only show

$$(10)\qquad z^{\alpha+1} \textstyle\int_0^{z-k} x^{-(1+\alpha)} \exp(-x^{-\alpha}) f_x(z - x)\, dx \to 0$$

as $k \to \infty$, uniformly in $z > k$. We use $(z/2) \wedge (z - k)$ to divide the integral into two parts.

For one part we may assume that $z \geqslant 2k \geqslant 2e$ and we use Lemma 3:

$$\begin{aligned} z^{\alpha+1}\textstyle\int_{z/2}^{z-k} \cdots \, dx &\leqslant c_{14}\textstyle\int_{z/2}^{z-k} f_x(z-x)\, dx \\ &\leqslant c_{15}\textstyle\int_0^{z-k}[\log(z-x)](z-x)^{-(1+\alpha)}\, dx \\ &\leqslant c_{15}\textstyle\int_k^{\infty}(\log w)w^{-(1+\alpha)}\, dw \to 0 \end{aligned}$$

as $k \to \infty$.

For the other part of (10) we use Lemma 4:

$$\begin{aligned} &z^{\alpha+1}\textstyle\int_0^{z/2} \cdots \, dx \\ &\quad\leqslant c_{16} z^{\alpha-1}\textstyle\int_0^{z/2}\left[x^{-\frac{1}{2}} \vee (x^{1-\alpha}\log x)\right]x^{-(1+\alpha)}\exp(-x^{-\alpha})\, dx \\ &\quad\leqslant c_{17} z^{-[(1-\alpha)\wedge\alpha]}(1 \vee \log z) \to 0 \end{aligned}$$

as $k \to \infty$, uniformly in $z > k$.

Various theorems that, roughly speaking, say that for a subordinator X, $X(t)$ and $\sup\{J(s)\colon s \leqslant t\}$ behave similarly are in the literature. The preceding Lemma 5 is another such result.

LEMMA 6. $\inf_{0<s\leqslant 1}\limsup_{t\to 0}|X(s+t) - X(s)|/|t|^{1/\alpha} < \infty$.

PROOF. We see, as in the proof of Lemma 2, that we need only prove the apparently weaker statement obtained by replacing "with probability one" by "with positive probability" in Lemma 6. We may assume that, for every ω no two jumps have the same size.

Since the distribution of $\sup\{J(t)\colon 0 < t \leqslant 1\}$ is continuous, $h(x)$, for $x > 0$, can be defined so that

$$\mathcal{P}(\sup\{J(t)\colon 0 < t \leqslant 1\} > h(x)|X(1) = x) = \tfrac{3}{4}$$

for almost every x. According to Lemma 5,

$$x - h(x) \leqslant c_{18}. \tag{11}$$

For each nonnegative integer N we define a subset B_N of $\{1, 2\}^N$: $B_0 = \{$empty sequence$\}$ and

$$\begin{aligned} B_{N+1} = \Big\{(b_1, \cdots, b_{N+1}) : (b_1, \cdots, b_N) \in B_N \quad &\text{and the pair} \\ (u, z) \in \left(\textstyle\sum_{j=1}^N b_j 4^{-j},\, 4^{-N} + \sum_{j=1}^N b_j 4^{-j}\right] \times (0, \infty) \quad &\text{defined by} \\ J(u) = z = \sup\left\{J(t) : \textstyle\sum_{j=1}^N b_j 4^{-j} < t \leqslant 4^{-N} + \sum_{j=1}^N b_j 4^{-j}\right\} \quad &\text{satisfies} \\ u \notin \left(\textstyle\sum_{j=1}^{N+1} b_j 4^{-j},\, 4^{-(N+1)} + \sum_{j=1}^{N+1} b_j 4^{-j}\right] \quad &\text{and} \\ z > 4^{-N/\alpha} h\left(4^{N/\alpha}\left[X\left(4^{-N} + \textstyle\sum_{j=1}^N b_j 4^{-j}\right) - X\left(\sum_{j=1}^N b_j 4^{-j}\right)\right]\right)\Big\}. \end{aligned}$$

The random variables u and z, appearing in the definition of B_{N+1} are independent, u is distributed uniformly in

$$\left(\textstyle\sum_{j=1}^N b_j 4^{-j},\, 4^{-N} + \sum_{j=1}^N b_j 4^{-j}\right],$$

and the pair

$$(4^{N/\alpha}z,\ 4^{N/\alpha}\big[X\big(4^{-N} + \Sigma_{j=1}^{N} b_j 4^{-j}\big) - X\big(b_j 4^{-j}\big)\big]$$

has according to the scaling property, the same distribution as the pair

$$(\sup\{J(t)\colon 0 < t \leqslant 1\},\ X(1)).$$

Hence,

$$\mathcal{P}((b_1, \cdots, b_{N+1}) \in B_{N+1} | (b_1, \cdots, b_N) \in B_N) = \left(\tfrac{3}{4}\right)^2 = \tfrac{9}{16}$$

and

$$\mathcal{P}(b_1, \cdots, b_N, 1) \in B_{N+1} \quad \text{and} \quad (b_1, \cdots, b_N, 2) \in B_{N+1} | (b_1, \cdots, b_N) \in B_N) = \left(\tfrac{3}{4}\right)\left(\tfrac{1}{2}\right) = \tfrac{3}{8}.$$

Accordingly, we may regard B_N as the Nth generation of a Galton-Watson process; $(b_1, \cdots, b_N, b_{N+1})$ is an *offspring* of $(b_1, \cdots, b_N)$ provided $(b_1, \cdots, b_N, b_{N+1}) \in B_{N+1}$.

Since the mean number of offspring of any individual is $\left(\frac{9}{16}\right) + \left(\frac{9}{16}\right) = \frac{9}{8} > 1$, there exists, for each ω belonging to an event having positive probability, a sequence $(b_1, b_2, \cdots)$ such that $(b_1, \cdots, b_N) \in B_N$ for each N. For such an ω let $s = \Sigma_{j=1}^{\infty} b_j 4^{-j}$.

For $4^{-(N+2)} \leqslant |t| < 4^{-(N+1)}$ both s and $s + t$ are members of

$$\big[\Sigma_{j=1}^{N} b_j 4^{-j},\ 4^{-N} + \Sigma_{j=1}^{N} b_j 4^{-j}\big]$$

and, hence by (11),

$$\begin{aligned}|X(s + t) - X(s)| &\leqslant X\big(4^{-N} + \Sigma_{j=1}^{N} b_j 4^{-j}\big) - X\big(\Sigma_{j=1}^{N} b_j 4^{-j}\big)\\ &\leqslant 4^{-(N-1)/\alpha}\big[y - h(y)\big] \leqslant 4^{-(N-1)/\alpha} c_{18} \leqslant 64^{1/\alpha} c_{18} |t|^{1/\alpha}\end{aligned}$$

where

$$y = X\big(4^{-(N-1)} + \Sigma_{j=1}^{N-1} b_j 4^{-j}\big) - X\big(\Sigma_{j=1}^{N-1} b_j 4^{-j}\big).$$

Accordingly,

$$\inf_{0<s\leqslant 1} \limsup_{t\to 0} \frac{|X(s + t) - X(s)|}{|t|^{1/\alpha}} \leqslant 64^{1/\alpha} c_{18}.$$

LEMMA 7. *$\sup_{0<s\leqslant 1} \liminf_{t\to 0} |X(s + t) - X(s)| / |t|^{1/\alpha} < \infty$ with probability* 1.

PROOF. For each sequence $(a_1, \cdots, a_n)$ in $\{0, 1\}^n$ we define a random interval

$$I(a_1, \cdots, a_n) = \big[Y(a_1, \cdots, a_n), Z(a_1, \cdots, a_n)\big]$$

via:

I(empty sequence) $= [0, X(1)]$;

$Y(a_1, \cdots, a_n, 0) = Y(a_1, \cdots, a_n)$;

$Z(a_1, \cdots, a_n, 0) = \sup\{x \in \mathcal{R}X : x \leqslant (Y(a_1, \cdots, a_n) + Z(a_1, \cdots, a_n))/2\}$ ($\mathcal{R}$ denotes range);

$Y(a_1, \cdots, a_n, 1) = \inf\{x \in \mathcal{R}X : x \geqslant (Y(a_1, \cdots, a_n) + Z(a_1, \cdots, a_n))/2\}$;

$Z(a_1, \cdots, a_n, 1) = Z(a_1, \cdots, a_n)$.

Let $c = 12^{1/\alpha}k$ with k given by Lemma 5 for $\varepsilon = \frac{1}{5}$. For positive integers $M \leqslant N$ we define random subsets $A_{M,N}$ of $\{0, 1\}^N$: $A_{M,M} = \{0, 1\}^M$ and, for $N > M$,

$$A_{M,N} = \Big\{(a_1, \cdots, a_N) : (a_1, \cdots, a_{N-1}) \in A_{M,N-1} \quad \text{and} \quad Z(a_1, \cdots, a_N) - Y(a_1, \cdots, a_N) \geqslant c\big([X^{-1}(Z(a_1, \cdots, a_N)) - X^{-1}(Y(a_1, \cdots, a_N))]/2\big)^{1/\alpha}\Big\},$$

where $X^{-1}(x) = \inf\{t : X(t) \geqslant x\}$.

Our first objective is to show that, for $N \geqslant M + 2$,

$$\mathcal{P}((a_1, \cdots, a_{N-1}, a_N) \in A_{M,N} | (a_1, \cdots, a_{N-1}) \in A_{M,N-1}) \leqslant 3^{-1},$$

hence, that

$$\mathcal{P}((a_1, \cdots, a_N) \in A_{M,N}) \leqslant 3^{-(N-M-1)}$$

and, therefore,

$$\mathcal{E}(\operatorname{card} A_{M,N}) \leqslant 2^N 3^{-(N-M-1)} \to 0 \quad \text{as} \quad N \to \infty. \tag{12}$$

Fix $a_N \in \{0, 1\}$. Suppose that

$$\begin{aligned} &\max\{J(t) : X^{-1}(Y(a_1, \cdots, a_{N-1})) < t < X^{-1}(Z(a_1, \cdots, a_{N-1}))\} \\ &> Z(a_1, \cdots, a_{N-1}) - Y(a_1, \cdots, a_{N-1}) \\ &\quad - c12^{-1/\alpha}[X^{-1}(Z(a_1, \cdots, a_{N-1})) - X^{-1}(Y(a_1, \cdots, a_{N-1}))]^{1/\alpha} \end{aligned} \tag{13}$$

and that t_0 for which the maximum in the left side of (13) is achieved satisfies

$$\begin{aligned} &\frac{X^{-1}(Z(a_1, \cdots, a_{N-1})) - X^{-1}(Y(a_1, \cdots, a_{N-1}))}{6} \\ &\leqslant t_0 - X^{-1}(Y(a_1, \cdots, a_{N-1})) \quad \text{if} \quad a_N = 0 \\ &\leqslant X^{-1}(Z(a_1, \cdots, a_{N-1})) - t_0 \quad \text{if} \quad a_N = 1. \end{aligned} \tag{14}$$

The events described by (13) and (14) are conditionally independent given $A_{M,N-1}$. The probability of (14) given $A_{M,N-1}$ is $\frac{5}{6}$. That the probability of (13) given $A_{M,N-1}$ is greater than $\frac{4}{5}$ follows from the scaling property and a slight modification of Lemma 5. The modification is required because a possible jump at the time $X^{-1}(Z(a_1, \cdots, a_{N-1}))$ is being ignored. That ignoring is appropriate since $Z(a_1, \cdots, a_{N-1})$ is the left limit of the process at the time $X^{-1}(Z(a_1, \cdots, a_{N-1}))$. If $(a_1, \cdots, a_{N-1}) \in A_{M,N-1}$, $N \geqslant M + 2$, and (13) holds, then

$$\begin{aligned} &\max\{J(t) : X^{-1}(Y(a_1, \cdots, a_{N-1})) < t < X^{-1}(Z(a_1, \cdots, a_{N-1}))\} \\ &\qquad > [Z(a_1, \cdots, a_{N-1}) - Y(a_1, \cdots, a_{N-1})]/2 \end{aligned}$$

and, hence,

$$t_0 = X^{-1}(Z(a_1, \cdots, a_{N-1}, 0)) = X^{-1}(Y(a_1, \cdots, a_{N-1}, 1)).$$

Accordingly, if $(a_1, \cdots, a_{N-1}) \in A_{M,N-1}$, $N \geqslant M + 2$, and (13) and (14) hold, then

$$Z(a_1, \cdots, a_{N-1}, a_N) - Y(a_1, \cdots, a_{N-1}, a_N)$$
$$< c12^{-1/\alpha}\left[X^{-1}(Z(a_1, \cdots, a_{N-1})) - X^{-1}(Y(a_1, \cdots, a_{N-1}))\right]^{1/\alpha}$$
$$\leqslant c\left[\left[X^{-1}(Z(a_1, \cdots, a_{N-1}, a_N)) - X^{-1}(Y(a_1, \cdots, a_{N-1}, a_N))\right]/2\right]^{1/\alpha}$$

and, hence, $(a_1, \cdots, a_{N-1}, a_N) \notin A_{M,N}$. Therefore,

$$\mathcal{P}((a_1, \cdots, a_{N-1}, a_N) \in A_{M,N} | (a_1, \cdots, a_{N-1}) \in A_{M,N-1})$$
$$\leqslant 1 - \left(\tfrac{5}{6}\right)\left(\tfrac{4}{5}\right) = \tfrac{1}{3}, \qquad \text{as desired.}$$

From the now-proven (12), Fatou's lemma, and the fact that card $(A_{M,N})$ is integral valued, we obtain for each M and almost every ω the existence of $N(\omega)$ such that $A_{M,N(\omega)} = \varnothing$. We let $M \to \infty$ to obtain for almost every ω and every $s \in (0, 1)$ an infinite sequence of n's such that

$$Z(a_1, \cdots, a_n) - Y(a_1, \cdots, a_n)$$
$$< c\left(\left[X^{-1}(Z(a_1, \cdots, a_n)) - X^{-1}(Y(a_1, \cdots, a_n))\right]/2\right)^{1/\alpha}$$

and

$$s \in \left[X^{-1}(Y(a_1, \cdots, a_n)), X^{-1}(Z(a_1, \cdots, a_n))\right). \tag{15}$$

If s is at least as close to $X^{-1}(Y(a_1, \cdots, a_n))$ as it is to $X^{-1}(Z(a_1, \cdots, a_n))$, then

$$Z(a_1, \cdots, a_n) - X(s) \leqslant Z(a_1, \cdots, a_n) - Y(a_1, \cdots, a_n)$$
$$< c\left(X^{-1}(Z(a_1, \cdots, a_n)) - s\right)^{1/\alpha}.$$

For u less than but sufficiently close to $X^{-1}(Z(a_1, \cdots, a_n))$,

$$X(u) - X(s) \leqslant Z(a_1, \cdots, a_n) - X(s) < c(u - s)^{1/\alpha}.$$

If s is closer to $X^{-1}(Z(a_1, \cdots, a_n))$ than it is to $X^{-1}(Y(a_1, \cdots, a_n))$, then

$$X(s) - X\left(X^{-1}(Y(a_1, \cdots, a_n))\right) = X(s) - Y(a_1, \cdots, a_n)$$
$$\leqslant Z(a_1, \cdots, a_n) - Y(a_1, \cdots, a_n) < c\left(s - X^{-1}(Y(a_1, \cdots, a_n))\right)^{1/\alpha}.$$

If $s = 1$, the interval in (15) may be taken to be closed. The proof of Lemma 7 is complete.

4. Theorems. In each of Theorems 1–3, X is a strictly stable subordinator of index $\alpha \in (0, 1)$. The constants depend on the scaling factor as well as on the index α but not upon I, an arbitrary subinterval of $(0, \infty)$ having positive length.

THEOREM 1. *With probability* 1,

$$\inf_{s \in I} \limsup_{t \downarrow 0} \frac{X(s+t) - X(s)}{t^{1/\alpha}} = c_{19} \in (0, \infty).$$

THEOREM 2. *With probability* 1,

$$\inf_{s\in I}\limsup_{t\to 0}\frac{|X(s+t)-X(s)|}{|t|^{1/\alpha}} = c_{20} \in (0, \infty).$$

THEOREM 3. *With probability* 1,

$$\sup_{s\in I}\liminf_{t\to 0}\frac{|X(s+t)-X(s)|}{|t|^{1/\alpha}} = c_{21} \in (0, \infty).$$

PROOF. The presence of a general I, rather than $(0, 1]$, causes no problems. That the quantities of interest lie in $(0, \infty)$ follows from Lemmas 1 and 6 in the case of Theorems 1 and 2 and Lemmas 2 and 7 in the case of Theorem 3. That they are not random is a consequence of Kolmogorov's $0 - 1$ law (see the beginning of the proof of Lemma 2).

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