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PRECISION BOUNDS FOR THE RELATIVE ERROR IN THE APPROXIMATION OF $E|S_n|$ AND EXTENSIONS[1]

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Let S_n denote the nth partial sum of i.i.d. nonconstant mean zero random variables. Given an approximation $K(n)$ of $E|S_n|$, tight bounds are obtained for the ratio $E|S_n|/K(n)$. These bounds are best possible as n tends to infinity. Implications of this result relate to the law of the iterated logarithm for mean zero variables, Chebyshev's inequality and Markov's inequality. Asymptotically exact lower-bounds are obtained for expectations of functions of row-sums of triangular arrays of independent but not necessarily identically distributed random variables. Expectations of "Poissonized random sums" are also treated.

0. Introduction. Let $S_n = X_1 + \cdots + X_n$ be the nth partial sum of independent identically distributed (i.i.d.) nonconstant mean zero random variables. A function $K(\cdot)$ depending on 1-dimensional X-integrals was introduced in Klass [3] to approximate the n-dimensional integral $E|S_n|$. There it was shown that $E|S_n|/K(n) \leqslant 2$, a bound which is best possible asymptotically. A less precise lower-bound was derived in Klass [5]. A lower-bound is presented in Section 1 of this paper which is within a factor of $1 + O(n^{-\frac{1}{2}})$ of being best possible, and is therefore asymptotically exact. The proof relies on an integral representation of any nonnegative real number $|x|$. Due to a certain convexity property, this representation affords a sharp lower-bound of $E|S_{T_n}|$, which is then used to bound $E|S_n|$. (T_n is a Poisson random variable with parameter n, where T_n is independent of the X_j's.) Consequences of these results are discussed in Section 2. These relate to the law of the iterated logarithm for mean zero variables, generalization of Chebyshev's inequality, and a best possible improvement of Markov's inequality. A technique for extending theorems involving sums of random variables with finite variance to those with infinite variance is enunciated. Section 3 extends the method of lower-bounding $E|S_{T_n}|$ to expectations of functions of "Poissonized sums" of independent (but not necessarily i.i.d. or finite mean) random variables subject to a constraint. This result is applied in Section 4 to obtain asymptotic lower-bounds for situations involving triangular arrays. Section 5 contains some closing remarks together with a few conjectures.

1. Approximation of $E|S_n|$ and $E|S_{T_n}|$. Let $X_1, X_2, \cdots$ be a sequence of i.i.d. random variables. Let $S_n = X_1 + \cdots + X_n$. The formula to follow allows computation of $E|S_n|$ from the characteristic function of X_1. Von Bahr and Esseen [2]

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used a generalization of the formula to upper-bound $E|S_n|^\beta$ for $1 \leqslant \beta \leqslant 2$. We employ it to obtain a lower-bound. By change of variables, letting $y = tx$, it is clear that for any real number x,

$$(1.1) \qquad |x| = C_1 \int_0^\infty (1 - \cos tx) t^{-2}\, dt,$$

where

$$C_1 = \left(\int_0^\infty (1 - \cos t) t^{-2}\, dt\right)^{-1}.$$

Replace x by the random variable $S_n = X_1 + \cdots + X_n$ and take expectations. Using Fubini's theorem, as the nonnegative integrand allows,

$$(1.2) \qquad \begin{aligned} E|S_n| &= C_1 \int_0^\infty (1 - \operatorname{Re} Ee^{itS_n}) t^{-2}\, dt \\ &= C_1 \int_0^\infty \left(1 - \operatorname{Re}(Ee^{itX})^n\right) t^{-2}\, dt. \end{aligned}$$

This formula holds whether or not the integral is finite.

Closely related to the right-hand side of (1.2) is the integral $C_1 \int_0^\infty (1 - \operatorname{Re} \exp n(Ee^{itX} - 1)) t^{-2}\, dt$. A probabilistic relation links the two expressions. Note that for any random variable Y and any $\lambda > 0$,

$$(1.3) \qquad \exp \lambda(Ee^{itY} - 1) = E \exp it\Sigma_{j=1}^T Y_j,$$

where $T, Y_1, Y_2, \cdots$ are independent random variables such that T has Poisson distribution with parameter λ and the Y_j have the same distribution as Y. Therefore let T_n be independent of $X_1, X_2, \cdots$ and have Poisson distribution with parameter n. Then

$$(1.4) \qquad E|S_{T_n}| = C_1 \int_0^\infty (1 - \operatorname{Re} \exp n(Ee^{itX} - 1)) t^{-2}\, dt.$$

The method used in this paper enables one to lower-bound $E|S_{T_n}|$ directly. Therefore, to approximate $E|S_n|$, it is necessary to relate the magnitude of the expectation of $|S_{T_n}|$ to that of $|S_n|$.

PROPOSITION 1. *Let $T_n, X_1, X_2, X_3, \cdots$ be independent random variables, where the X_j's are identically distributed having expectation zero and T_n is Poisson with parameter n. Let $S_j = X_1 + \cdots + X_j$. Then*

$$(1.5) \qquad E|S_{T_n}| \geqslant E|S_n|\left(1 - e^{-n}\frac{n^n}{n!}\right)$$

and

$$(1.6) \qquad E|S_{T_n}| \leqslant E|S_n|\left(1 + e^{-n}\left(\frac{n^n}{n!} - 1\right)\right).$$

REMARK 2. It seems likely that $E|S_{T_n}| \leqslant E|S_n|$ for all $n \geqslant 1$ and all X-distributions. Such an inequality is easily proved if n is odd and the X_j's are symmetric. However, L. A. Shepp has pointed out that the inequality fails for $n = 2$ if $P(X = 1) = P(X = -1) = \frac{1}{2}$.

PROOF OF PROPOSITION 1. Since $EX_j = 0$, both $|S_1|, |S_2|, |S_3|, \cdots$ and $\cdots, |S_k|/k, |S_{k-1}|/k - 1, \cdots, |S_1|/1$ are submartingale sequences. Hence $E|S_1| \leqslant E|S_2| \leqslant \cdots$ and $\cdots \leqslant E|S_k|/k \leqslant E|S_{k-1}|/k - 1 \leqslant \cdots \leqslant E|S_1|/1$.

The expectation of $|S_{T_n}|$ is computable as

$$E|S_{T_n}| = \Sigma_{k=1}^{\infty} P(T_n = k) E|S_k|$$

$$= \Sigma_{k=1}^{n} e^{-n}\frac{n^k}{k!} E|S_k| + \Sigma_{k=n+1}^{\infty} e^{-n}\frac{n^k}{k!} E|S_k|.$$

To obtain the lower-bound, note that

$$\frac{n^k}{k!} E|S_k| \geqslant E|S_n|\frac{n^{k-1}}{(k-1)!} \qquad \text{for} \quad 1 \leqslant k \leqslant n$$

and

$$\frac{n^k}{k!} E|S_k| \geqslant E|S_n|\frac{n^k}{k!} \qquad \text{for} \quad k > n.$$

Inserting these smaller quantities, $E|S_{T_n}| \geqslant E|S_n|\Sigma_{j=0;\, j\neq n}^{\infty} e^{-n} n^j/j!$, which gives (1.5). The upper-bound is similar:

$$\frac{n^k}{k!} E|S_k| \leqslant E|S_n|\frac{n^k}{k!} \qquad \text{for} \quad 1 \leqslant k \leqslant n$$

and

$$\frac{n^k}{k!} E|S_k| \leqslant E|S_n|\frac{n^{k-1}}{(k-1)!} \qquad \text{for} \quad k > n.$$

Thus

$$E|S_{T_n}| \leqslant E|S_n|\left(\Sigma_{k=1}^{\infty} e^{-n}\frac{n^k}{k!} + \frac{e^{-n}n^n}{n!}\right),$$

which equals the right-hand side of (1.6). □

REMARK 3. Observe that $\lim_{n\to\infty} E|S_n|/E|S_{T_n}| = 1$ (provided $P(X_1 = 0) < 1$). Thus, for asymptotic purposes, any lower (upper) bound of $E|S_{T_n}|$ is essentially a lower (upper) bound of $E|S_n|$. This phenomenon is not unique to the function $f(x) = |x|$. In fact, given any nonnegative symmetric function $f(\cdot)$, nondecreasing on $[0, \infty)$ and satisfying

$$\text{(1.7)} \qquad \lim_{\alpha\searrow 1} \limsup_{x\to\infty} \frac{f(\alpha x)}{f(x)} = 1,$$

we have

$$\text{(1.8)} \qquad \lim_{n\to\infty} \frac{Ef(S_n)}{Ef(S_{T_n})} = 1,$$

provided that $0 < Ef(S_n) < \infty$ for some n. (Here we do not assume that X_1 has finite expectation but we do retain the other previous assumptions.)

The requirement that $f(x)$ not increase at exponential rate as $x \to \infty$ is crucial. Without it (1.8) is no longer valid. The proof of (1.8) is somewhat tedious and involved. The technique is based in part on the methods to be found in Klass [5].

Since the result itself is not central to the concerns of this paper, its proof will be omitted.

Lemma 4 (below) provides a technique for generating a sharp lower-bound of $E|S_{T_n}|$. An appropriate extension of this method will subsequently be used to derive approximations in more general contexts.

LEMMA 4. *Let X be a random variable such that*

$$nE(X^2 \wedge |X|) = 1. \tag{1.9}$$

Then

$$\int_0^\infty (1 - \operatorname{Re} \exp n(Ee^{itX} - 1))t^{-2}\,dt \geq \inf_{x>0} \int_0^\infty \left(1 - \exp\left(\frac{\cos tx - 1}{x^2 \wedge x}\right)\right)t^{-2}\,dt. \tag{1.10}$$

($a \wedge b$ denotes the minimum of a and b).

PROOF. Letting $d\nu(x) = n(x^2 \wedge x)\,dP(|X| \leq x)$, observe that $\nu(\cdot)$ is a probability measure on $(0, \infty)$.

$$\begin{aligned}
\operatorname{Re} \exp n(Ee^{itX} - 1) &\leq |\exp n(Ee^{itX} - 1)| \\
&= \exp nE(\cos tX - 1) \\
&= \exp \int_0^\infty \frac{\cos tx - 1}{x^2 \wedge x}\,d\nu(x) \\
&\leq \int_0^\infty \exp\left(\frac{\cos tx - 1}{x^2 \wedge x}\right) d\nu(x) \qquad \text{(Jensen's inequality)}.
\end{aligned}$$

Combine this inequality with an application of Fubini to obtain

$$\begin{aligned}
\int_0^\infty (1 - \operatorname{Re} \exp n(Ee^{itX} - 1))t^{-2}\,dt
&\geq \int_0^\infty \int_0^\infty \left(1 - \exp\left(\frac{\cos tx - 1}{x^2 \wedge x}\right)\right)t^{-2}\,d\nu(x)\,dt \\
&= \int_0^\infty \left(\int_0^\infty \left(1 - \exp\left(\frac{\cos tx - 1}{x^2 \wedge x}\right)\right)t^{-2}\,dt\,d\nu(x)\right. \\
&\geq \inf_{x>0} \int_0^\infty \left(1 - \exp\left(\frac{\cos tx - 1}{x^2 \wedge x}\right)\right)t^{-2}\,dt.
\end{aligned} \tag{1.11}$$

Equality in (1.10) is achieved by a symmetric random variable X such that $P(|X| = 1) = 1 - P(X = 0) = 1/n$, as may be deduced from Proposition 5 (below).

Let

$$g_\alpha(x) = \int_0^\infty \left(1 - \exp\left(\frac{\cos tx - 1}{x^\alpha}\right)\right)t^{-2}\,dt. \tag{1.12}$$

Then the right-hand side of (1.10) equals $\min\{\inf_{0<x<1} g_2(x), \inf_{x\geq 1} g_1(x)\}$.

We will show that

$$g_1(x) \nearrow \text{ on } (0, \infty) \tag{1.13}$$

and

$$g_2(x) \searrow \text{ on } (0, \infty). \tag{1.14}$$

Consequently,

PROPOSITION 5.

$$\inf_{x>0} \int_0^\infty \left(1 - \exp\left(\frac{\cos tx - 1}{x^2 \wedge x}\right)\right) t^{-2}\, dt = \int_0^\infty (1 - \exp(\cos t - 1)) t^{-2}\, dt. \tag{1.15}$$

Verification of (1.13) is not difficult: by change of variables to $y = tx$,

$$g_1(x) = x \int_0^\infty \left(1 - \exp\left(\frac{\cos y - 1}{x}\right)\right) y^{-2}\, dy. \tag{1.16}$$

A standard application of dominated convergence shows that $g_1(\cdot)$ is differentiable on $(0, \infty)$ and may be differentiated under the integral sign. Thus

$$g_1'(x) = \int_0^\infty \left(1 - \left(1 - \frac{\cos y - 1}{x}\right) \exp\left(\frac{\cos y - 1}{x}\right)\right) y^{-2}\, dy. \tag{1.17}$$

Since $e^u > 1 + u$ for $u > 0$, the integrand in (1.17) above is positive for $y \notin \{2k\pi : k = 0, 1, 2, \cdots\}$ and nonnegative otherwise. Hence $g_1'(x) > 0$ and therefore $g_1(\cdot)$ is strictly increasing.

A more subtle argument is required to prove that $g_2(\cdot)$ decreases.

PROPOSITION 6.

$$\int_0^\infty \left(1 - \exp\left(\frac{\cos tx - 1}{x^2}\right)\right) t^{-2}\, dt = \tfrac{1}{2} \int_0^\infty 1 - \exp\left(-\frac{2}{x^2 + v^2}\right) dv, \tag{1.18}$$

so that $g_2(x) \searrow$ *on* $(0, \infty)$.

PROOF. To utilize the periodicity of $\cos y$, change variables to $y = tx$. Thus

$$\begin{aligned}
\int_0^\infty \left(1 - \exp\left(\frac{\cos tx - 1}{x^2}\right)\right) t^{-2}\, dt &= x \int_0^\infty \left(1 - \exp\left(\frac{\cos y - 1}{x^2}\right)\right) y^{-2}\, dy \\
&= \frac{x}{2} \int_{-\infty}^\infty \left(1 - \exp\left(\frac{\cos y - 1}{x^2}\right)\right) y^{-2}\, dy \\
&= \frac{x}{2} \int_{-\pi}^\pi \left(1 - \exp\left(\frac{\cos y - 1}{x^2}\right)\right) \left(\Sigma_{k=-\infty}^\infty (y + 2\pi k)^{-2}\right) dy.
\end{aligned}$$

By a standard application of complex analysis (or a consultation of Abramowitz and Stegun [1], page 75, formula 4.3.92),

$$\Sigma_{k=-\infty}^\infty (y + 2\pi k)^{-2} = 2^{-1}(1 - \cos y)^{-1} \tag{1.19}$$

Therefore

$$g_2(x) = \frac{x}{4}\int_{-\pi}^{\pi} \frac{1 - \exp\left(\frac{\cos y - 1}{x^2}\right)}{1 - \cos y}\, dy$$

$$= \frac{x}{2}\int_0^{\pi} \frac{1 - \exp\left(\frac{\cos y - 1}{x^2}\right)}{1 - \cos y}\, dy.$$

We change variables twice more. First let $w = (1 - \cos y)^{-1}$. Then

$$dw = -(1 - \cos y)^{-2} \sin y\, dy$$

$$= -w^2\left(1 - (1 - w^{-1})^2\right)^{\frac{1}{2}}\, dy$$

$$= -w(2w - 1)^{\frac{1}{2}}\, dy$$

so that

$$g_2(x) = \frac{x}{2}\int_{\frac{1}{2}}^{\infty} \frac{1 - \exp(-w^{-1}x^{-2})}{(2w - 1)^{\frac{1}{2}}}\, dw.$$

For fixed $x > 0$ let $v = x(2w - 1)^{\frac{1}{2}}$. Then $v^2/x^2 = 2w - 1$ so that $dw = vx^{-2}\, dv$. Hence $g_2(x) = \frac{1}{2}\int_0^{\infty}(1 - \exp(-(2/(x^2 + v^2))))\, dv$.

To see that $g_2(\cdot)$ is strictly decreasing, observe that the integrand on the right-hand side of (1.18) is strictly decreasing as x increases. □

THEOREM 7. *Let $X, X_1, X_2, \cdots$ be* i.i.d. *nonconstant mean zero random variables. Let T_n be a random variable independent of the X_j's, which is Poisson with parameter n. Let $S_n = X_1 + \cdots + X_n$. For $y \geq 1$ let $K(y)$ be the unique positive real number satisfying*

$$yE\left((X/K(y))^2 \wedge |X/K(y)|\right) = 1. \tag{1.20}$$

Then

$$K(n)E|Y_1 - Y_2| \Big/ \left(1 + e^{-n}\left(\frac{n^n}{n!} - 1\right)\right) \leq E|S_n| \leq 2K(n) \tag{1.21}$$

and

$$K(n)E|Y_1 - Y_2| \leq E|S_{T_n}| \leq \left(1 + \left(1 + \frac{1}{n}\right)^{\frac{1}{2}}\right)K(n), \tag{1.22}$$

where Y_1 and Y_2 are independent Poisson random variables, each having parameter $\lambda = \frac{1}{2}$. The quantity $E|Y_1 - Y_2|$ equals $.673^+$.

PROOF. Due to the integral representation of $|X|$,

$$
\begin{aligned}
E|Y_1 - Y_2| &= C_1\int_0^\infty(1 - \operatorname{Re} E(e^{itY_1}e^{-itY_2}))t^{-2}\,dt \\
&= C_1\int_0^\infty(1 - \operatorname{Re} Ee^{itY_1}Ee^{-itY_2})t^{-2}\,dt \\
&= C_1\int_0^\infty\left(1 - \operatorname{Re}\exp\left(\left(\frac{e^{it}-1}{2}\right) + \left(\frac{e^{-it}-1}{2}\right)\right)\right)t^{-2}\,dt \\
&= C_1\int_0^\infty(1 - \exp(\cos t - 1))t^{-2}\,dt \\
&\leqslant C_1\int_0^\infty(1 - \operatorname{Re}\exp n(Ee^{itX/K(n)} - 1))t^{-2}\,dt \\
&\qquad\qquad\qquad\qquad \text{(by (1.15) and (1.10))} \\
&= E|S_{T_n}|/K(n) \qquad\qquad \text{(by (1.4)).}
\end{aligned}
$$

This establishes the left-hand side of (1.22). The lower-bound in (1.21) now follows by application of (1.6). The right-hand side of (1.21) was proved in Klass [3], Theorem 2.1. Essentially the same technique as used in that paper will be used to prove the right-hand side of (1.22). Fix $b > 0$. Let $S_n(b) = \Sigma_{j=1}^n X_j I(|X_j| \leqslant b)$ and $U_n(b) = \Sigma_{j=1}^n X_j I(|X_j| > b)$. Let τ be any nonnegative integer-valued random variable such that $E\tau^2 < \infty$ and $\{\tau = n\}$ is independent of the Borel field generated by $X_1, \cdots, X_n$. Note that $0 = ES_1 = EXI(|X| \leqslant b) + EXI(|X| > b)$ so that $(EXI(|X| \leqslant b))^2 = (EXI(|X| > b))^2 \leqslant (E|X|I(|X| > b))^2$.

$$
\begin{aligned}
E|S_\tau| &= E|S_\tau(b) + U_\tau(b)| \\
&\leqslant E|S_\tau(b)| + E|U_\tau(b)| \\
&\leqslant (ES_\tau^2(b))^{\frac{1}{2}} + E\Sigma_{j=1}^\tau |X_j| I(|X_j| > b).
\end{aligned}
$$

By Wald's equation, the second quantity equals $E\tau E|X|I(|X| > b)$. Using independence,

$$
\begin{aligned}
ES_\tau^2(b) &= \Sigma_{n=1}^\infty P(\tau = n)ES_n^2(b) \\
&= \Sigma_{n=1}^\infty P(\tau = n)\{n \operatorname{Var} XI(|X| \leqslant b) + (nEXI(|X| \leqslant b))^2\} \\
&\leqslant E\tau EX^2 I(|X| \leqslant b) + E\tau^2(E|X|I(|X| > b))^2.
\end{aligned}
$$

Let $\tau = T_n$. Then $E\tau = n$ and $E\tau^2 = \operatorname{Var} T_n + (ET_n)^2 = n + n^2$. Next, let $b = K(n)$. By construction,

$$
\begin{aligned}
K^2(n) &= nEX^2I(|X| \leqslant K(n)) + nK(n)E|X|I(|X| > K(n)) \\
&\equiv \lambda_n K^2(n) + (1 - \lambda_n)K^2(n).
\end{aligned}
$$

Combining these results, it is easily seen that for $\tau = T_n$ and $b = K(n)$,

$$ES_\tau^2(b) \leqslant K^2(n)\left(\lambda_n + (1 - \lambda_n)^2\left(1 + \frac{1}{n}\right)\right)$$

and

$$E\tau E|X|I(|X| > b) = (1 - \lambda_n)K(n).$$

Therefore,

$$E|S_{T_n}| \leqslant K(n)\left(1 - \lambda_n + \left(\lambda_n + (1 - \lambda_n)^2\left(1 + \frac{1}{n}\right)\right)^{\frac{1}{2}}\right)$$

$$\equiv K(n)g_n(\lambda_n) \leqslant K(n) \sup_{0 \leqslant \lambda \leqslant 1} g_n(\lambda).$$

Since $g_n'(\lambda) < 0$ for $0 \leqslant \lambda \leqslant 1$, $\sup_{0 \leqslant \lambda \leqslant 1} g_n(\lambda) = g_n(0) = 1 + (1 + \frac{1}{n})^{\frac{1}{2}}$. □

REMARK 8. In Klass [3] (Theorem 1.1) it was asserted without proof that $K(n) \leqslant 3E|S_n|$ and that $\limsup_{n\to\infty} K(n)/E|S_n| < 2.25$. Theorem 7 substantiates a stronger result and the bounds given are asymptotically exact. For example, let X have mean zero, be bounded below, and have positive tail satisfying $P(X > y) = 1/y \log^2 y$ for $y \geqslant e$. Then, letting $b_n = nE(|X| - b_n)^+$, $b_n \sim K(n)$ and so by Theorem 5 of Klass-Teicher [6], $\lim_{n\to\infty} E|S_n|/K(n) = 2$. Since $E|S_n| \sim E|S_{T_n}|$ we also have $\lim_{n\to\infty} E|S_{T_n}|/K(n) = 2$. The lower-bound is most easily obtained from triangular arrays. Simply let X_{nj} be i.i.d. *symmetric* random variables for $j = 1, 2, \cdots$, such that $P(|X_{nj}| = 1) = 1 - P(X_{nj} = 0) = 1/n$. Then $nE(X_{n1}^2 \wedge |X_{n1}|) = 1$. Furthermore, $S_{nT_n} = \sum_{j=1}^{T_n} X_{nj}$ has the same distribution as $Y_1 - Y_2$. Therefore $E|S_{nT_n}| = E|Y_1 - Y_2|$; whence the lower bound of (1.22) is achieved for each n. Recalling Proposition 1, $\lim_{n\to\infty} \inf_{\{X:\, EX=0\}} E|S_n|/K(n) = E|Y_1 - Y_2|$.

A single distribution can be constructed such that $\liminf_{n\to\infty} E|S_n|/K(n) = E|Y_1 - Y_2|$. Let $0 < a_1 < a_2 < \cdots$ be a sequence of reals such that $a_{n+1}/a_n \to \infty$. Assume that $\sum_{n=1}^{\infty} a_n^{-\frac{3}{2}} \leqslant 1$. Let X be a symmetric random variable taking values in the set $\{0, \pm a_1, \pm a_2, \cdots\}$ according to the probabilities $P(|X| = a_n) = a_n^{-\frac{3}{2}}$. Let j_n be the greatest integer not exceeding $a_n^{\frac{3}{2}}$. Then $j_n EX^2 I(|X| \leqslant a_n) + j_n a_n E|X| I(|X| > a_n) \sim a_n^2$ so that $K(j_n) \sim a_n$. $S_{j_n}/K(j_n)$ converges in distribution to $Y_1 - Y_2$, and, more to the point, $E|S_{j_n}/K(j_n)| \to E|Y_1 - Y_2|$.

By intermingling the distributions given in the first and third examples, one can construct a mean-zero X-distribution such that

$$E|Y_1 - Y_2| = \liminf_{n\to\infty} E|S_n|/K(n) \leqslant \limsup_{n\to\infty} E|S_n|/K(n) = 2.$$

The same inequalities hold if $E|S_n|$ is replaced by $E|S_{T_n}|$.

REMARK 9. To evaluate $E|Y_1 - Y_2|$ "directly," it is necessary to compute a double summation. There is an equivalent method which requires but a single summation. Let $W_1, W_2, \cdots$ be i.i.d. symmetric random variables such that $P(|W_j| = 1) = 1$. Note that

$$E\left|\sum_{j=1}^{2k-1} W_j\right| = E\left|\sum_{j=1}^{2k} W_j\right| = 2k4^{-k}\binom{2k}{k}.$$

Let T be independent of the W_j's and be Poisson (1). Then since $Y_1 - Y_2$ and

$\Sigma_{j=1}^{T} W_j$ have the same characteristic function,

$$E|Y_1 - Y_2| = E|\Sigma_{j=1}^{T} W_j| = e^{-1}\Sigma_{n=1}^{\infty}\frac{E|\Sigma_{j=1}^{n} W_j|}{n!}$$
$$= 2e^{-1}\Sigma_{k=1}^{\infty}\frac{k4^{-k}\binom{2k}{k}(1 + 1/2k)}{(2k-1)!}$$
$$= e^{-1}\Sigma_{k=0}^{\infty} 4^{-k}(k!)^{-2}(1 + 1/2(k+1)).$$

2. Consequences of Theorem 7. Theorem 7 has two corollaries which may be of interest. The first relates to the law of the iterated logarithm for sums of i.i.d. mean-zero variables.

COROLLARY 10. *Let $X_1, X_2, \cdots$ be* i.i.d. *nonconstant mean zero random variables. Let $S_n = X_1 + \cdots + X_n$ and let the constant C be determined by the relation*

$$\limsup_{n\to\infty} \frac{S_n}{(\log\log n) ES^{+}_{[n/\log\log n]}} = C \text{ a.s.} \tag{2.1}$$

Then $C < \infty$ iff *$\Sigma_n P(X \geqslant (\log\log n) ES^{+}_{[n/\log\log n]}) < \infty$. When $C < \infty$,*

$$1 \leqslant C \leqslant 3/E|Y_1 - Y_2| < 4.46. \tag{2.2}$$

PROOF. This follows from Klass [4] (Theorem 2.5), the fact that $E|S_n| = 2ES_n^+$, and the fact that $.673^+ = E|Y_1 - Y_2| \leqslant \liminf_{n\to\infty} E|S_n|/K(n) \leqslant \limsup_{n\to\infty} E|S_n|/K(n) \leqslant 2$ where $K(\cdot)$ is defined as in (1.20). ∎

The next corollary may be regarded either as a method of lower-bounding $E|S_n|$ in terms of the $1/n$th quantiles of the $|X|$-distribution or else as a method of upper-bounding a tail probability.

COROLLARY 11. *Let $X, X_1, X_2, \cdots$ be* i.i.d. *nonconstant mean-zero random variables. Let $S_n = X_1 + \cdots + X_n$. Then*

$$P(|X| \geqslant r_n E|S_n|) \leqslant 1/n, \tag{2.3}$$

where $r_n = (1 + e^{-n}((n^n/n!) - 1))/E|Y_1 - Y_2|$ and Y_1 and Y_2 are independent Poisson random variables, each having parameter $\lambda = \frac{1}{2}$.

PROOF. Since $r_n E|S_n| \geqslant K(n)$,

$$\begin{aligned} nP(|X| \geqslant r_n E|S_n|) &\leqslant nP(|X| \geqslant K(n)) \\ &\leqslant nE(|X|/K(n))I(|X| \geqslant K(n)) \\ &\qquad \text{(Markov's inequality)} \\ &\leqslant nE\big((X/K(n))^2 \wedge |X/K(n)|\big) \\ &= 1 \text{ (by construction of } K(n)). \end{aligned}$$

∎

REMARK 12. Corollary 11 generalizes Chebyshev's inequality to random variables with finite mean. Let $S_n = X_1 + \cdots + X_n$ be a sum of independent mean-zero random variables with finite variance. Consider any theorem about the behavior of S_n whose statement involves the quantity $(\operatorname{Var} S_n)^{\frac{1}{2}}$. How is $(\operatorname{Var} S_n)^{\frac{1}{2}}$ to be interpreted? Suppose it represents some constant times the median (or other quantile) of $|S_n|$. Because $\operatorname{med}|S_n|$ is *always* defined and finite, one expects that it is often feasible to extend such a theorem to random variables without finite variance. Both the proof and utilization of such a result would seem to require approximation of $\operatorname{med}|S_n|$. When this is difficult, $E|S_n|$ or, more generally, $f^{-1}(Ef(|S_n|))$ (some suitable function f) is often a convenient and adequate substitute. Previously, this idea was used (though not explicitly enunciated) on $\sigma(n \log\log n)^{\frac{1}{2}} = (\log\log n)(\operatorname{Var}(S_{n/\log\log n}))^{\frac{1}{2}}$ to suggest an appropriate generalization of the law of the iterated logarithm (see Klass [3] and [4]). We now show how application of this idea leads to the conjecture of Corollary 11.

Suppose that the X_j's are identically distributed with finite positive variance σ^2. Chebyshev's inequality gives

$$P\big(|X| \geqslant (\operatorname{Var} S_n)^{\frac{1}{2}}\big) = P\big(|X| \geqslant \sigma n^{\frac{1}{2}}\big) \leqslant EX^2/n\sigma^2 = 1/n. \tag{2.4}$$

Now when $\operatorname{Var} X = \infty$ this inequality lacks content. However, according to our "folk theorem," the extremes of the above inequality remain valid for sums of arbitrary i.i.d. mean-zero variates provided one replaces $(\operatorname{Var} S_n)^{\frac{1}{2}}$ by an appropriate multiple of $E|S_n|$. Corollary 11 proves just this fact; namely, $P(|X| \geqslant r_n E|S_n|) \leqslant 1/n$.

Another heuristic argument can be forwarded to motivate inequality (2.3): imagine a gambling situation. Suppose X_j denotes one's winnings during the jth repeated game (i.i.d. trials). Then $E|S_n| = E|S_n - 0|$ represents the expected amount one's fortune will change after n games. Inverting the statement, normally it takes about n games for one's fortune to change by amount $a = E|S_n|$. Therefore, the chance that one's fortune changes by at least some suitable multiple of $E|S_n|$ in a single trial (game) cannot greatly exceed $1/n$. Since $|X|$ represents the amount one's fortune changes in a single trial, an inequality of the form $P(|X| \geqslant r_n E|S_n|) \leqslant 1/n$ must hold for some suitable and uniformly bounded real number r_n.

REMARK 13. Corollary 11 is an improvement of Markov's inequality of best possible type. For example, let r be any positive constant less than $1/E|Y_1 - Y_2|$ and let X be distributed as in the third example of Remark 8. Then $\limsup_{n\to\infty} nP(|X| \geqslant rE|S_n|) = 1$. As another illustration of the sharpness of Corollary 11, let X be a stable random variable of index $1 < \alpha < 2$ (or else just in the domain of attraction of such a distribution). Then there exists an $\varepsilon > 0$ depending on X such that $\varepsilon \leqslant nP(|X| \geqslant r_n E|S_n|) \leqslant 1$ for all $n \geqslant 1$. By comparison, Markov's inequality gives $nP(|X| \geqslant r_n E|S_n|) \leqslant nE|X|/r_n E|S_n|$, which tends to infinity as $n \to \infty$. Corollary 11 gives a tighter bound than Markov's inequality only because it assumes more information. Markov's inequality bounds $P(|X| \geq t)$ for all $t > 0$. Corollary 11 bounds $P(|X| \geq t)$ subject to the constraint $t \geq r_n E|S_n|$.

3. Extension to functions of random sums. The lower-bound for $E|S_{T_n}|$ based on Lemma 4 can be generalized in two ways. We extend the result by admitting nonidentically distributed variates and by using functions $f(\cdot)$ other than $f(x) = |x|$.

Let $\mu(\cdot)$ be any positive σ-finite measure on $(0, \infty)$ satisfying

$$\int_0^\infty t^2 \wedge 1 \, d\mu(t) < \infty. \tag{3.1}$$

Define

$$\mathcal{F} = \{f(\cdot) : f(x) = \int_0^\infty (1 - \cos tx) \, d\mu(t) \text{ and } \mu(\cdot) \text{ satisfies (3.1)}\}. \tag{3.2}$$

Condition (3.1) ensures that every function $f \in \mathcal{F}$ is finite-valued. In addition, note that each $f \in \mathcal{F}$ is nonnegative, symmetric, and continuous, with $f(0) = 0$.

Many familiar functions belong to $\mathcal{F}$. For example, if $0 < \beta < 2$, $f_\beta(x) \equiv |x|^\beta \in \mathcal{F}$. To see this, merely confirm the integral representation given below by changing variables to $y = tx$.

$$|x|^\beta = C_\beta \int_0^\infty (1 - \cos tx) t^{-1-\beta} \, dt \qquad 0 < \beta < 2, \tag{3.3}$$

where $C_\beta = (\int_0^\infty (1 - \cos t) t^{-1-\beta} \, dt)^{-1}$. Contour integration shows that $C_\beta = 2\Gamma(\beta + 1)(\sin \pi\beta/2)/\pi$, where $\Gamma(\cdot)$ is the gamma function.

Some notation will be helpful. Given any random variable Y, let $\tilde{Y}$ denote $\Sigma_{j=1}^T Y_j$, where $Y, Y_1, Y_2, \cdots$ are i.i.d. and T is a Poisson random variable with parameter $\lambda = 1$, which is independent of the Y_j's.

THEOREM 14. *Let $h(\cdot)$ be a symmetric, nonnegative continuous function, zero only at zero. Let $X_1, X_2, \cdots, X_n$ be random variables such that*

$$\Sigma_{j=1}^n E h(X_j) = 1. \tag{3.4}$$

Suppose that $\tilde{X}_1, \tilde{X}_2, \cdots, \tilde{X}_n$ are independent. Let $f(x) = \int_0^\infty (1 - \cos tx) \, d\mu(t)$ belong to $\mathcal{F}$. Then

$$Ef(\Sigma_{j=1}^n \tilde{X}_j) \geq \inf_{x>0} \int_0^\infty \left(1 - \exp\left(\frac{\cos tx - 1}{h(x)}\right)\right) d\mu(t). \tag{3.5}$$

PROOF.

$$\begin{aligned} Ef(\Sigma_{j=1}^n \tilde{X}_j) &= \int_0^\infty \left(1 - \operatorname{Re} \Pi_{j=1}^n \exp E(e^{itX_j} - 1)\right) d\mu(t) \\ &\geq \int_0^\infty \left(1 - |\exp \Sigma_{j=1}^n E(e^{itX_j} - 1)|\right) d\mu(t) \\ &= \int_0^\infty \left(1 - \exp \Sigma_{j=1}^n E(\cos tX_j - 1)\right) d\mu(t) \\ &= \int_0^\infty \left(1 - \exp \int_0^\infty \frac{\cos tx - 1}{h(x)} \, d\nu(x)\right) d\mu(t), \end{aligned} \tag{3.6}$$

where $d\nu(x) = h(x)\Sigma_{j=1}^n \, dP(|X_j| \leq x)$. Since $h(x) \geq 0$, ν is a positive measure. Due to condition (3.4) and the fact that $h(0) = 0$, ν is a probability measure on $(0, \infty)$.

Arguing exactly as in Lemma 4,

$$\int_0^\infty\left(1 - \exp \int_0^\infty \frac{\cos tx - 1}{h(x)}\, d\nu(x)\right) d\mu(t) \geq \inf_{x>0} \int_0^\infty\left(1 - \exp\frac{\cos tx - 1}{h(x)}\right) d\mu(t).$$

$\square$

Remark 15. Typically, the function $f(\cdot)$ and random variables $X_1, \cdots, X_n$ are given. To apply Theorem 14 it is then necessary to construct a function $h(\cdot)$ satisfying (3.4). This can be done as follows. Assuming that $f(x)$ is strictly increasing on $[0, \infty)$, for each $x \neq 0$, $(x/b)^2 \wedge (f(x)/f(b))$ is strictly decreasing in $b > 0$. Owing to the continuity of $f(\cdot)$ and the fact that $f(0) = 0$, whenever $0 < \Sigma_{j=1}^n Ef(X_j) < \infty$ there exists a unique positive number K_n such that

$$\Sigma_{j=1}^n E\big((X_j/K_n)^2 \wedge (f(X_j)/f(K_n))\big) = 1. \tag{3.7}$$

Now a lower-bound for $Ef(\Sigma_{j=1}^n \tilde{X}_j)$ can be given; merely invoke Theorem 14, using $h(x) = (x/K_n)^2 \wedge (f(x)/f(K_n))$.

Remark 16. Theorem 14 can be improved somewhat if $f(x) = |x|^\beta$ $(0 < \beta < 2)$. Letting K_n satisfy (3.7) and scaling, Theorem 14 gives

$$E|\Sigma_{j=1}^n(\tilde{X}_j/K_n)|^\beta \geq C_\beta \inf_{x>0} \int_0^\infty\left(1 - \exp\left(\frac{\cos tx - 1}{x^2 \wedge x^\beta}\right)\right)t^{-1-\beta}\, dt.$$

Change variables to $y = tx$ and differentiate w.r.t. x to see that $\int_0^\infty(1 - \exp((\cos tx - 1)/x^\beta))t^{-1-\beta}\, dt$ increases as x increases on $(0, \infty)$. (The complete derivation parallels that of (1.13), which required use of (1.16) and (1.17).) Hence

$$E|\Sigma_{j=1}^n \tilde{X}_j|^\beta \geq C_\beta(K_n)^\beta \inf_{0<x\leq 1} \int_0^\infty\left(1 - \exp\left(\frac{\cos tx - 1}{x^2}\right)\right)t^{-1-\beta}\, dt. \tag{3.8}$$

It can be shown (see J. Reeds [8]) that for all $0 < \beta < 2$ the infimum in (3.8) occurs at $x = 1$, as was proved for $\beta = 1$. This leads to an extension of Theorem 7 via generalization of (1.6) of Proposition 1. Specifically, fix $1 \leq \beta < 2$ and let $X, X_1, X_2, \cdots$ be i.i.d. nonconstant zero-mean random variables such that $E|X|^\beta < \infty$. Define S_n, T_n, Y_1 and Y_2 as in Theorem 7. Note that $\{|S_k|^\beta\}_{k=1}^\infty$ and $\{|S_k/k|^\beta\}_{k=\infty}^1$ are both submartingales. Hence $E|S_k|^\beta \leq E|S_n|^\beta$ for $k \leq n$ and $E|S_k|^\beta \leq (k/n)^2 E|S_n|^\beta$ for $k > n$. Argue as in Proposition 1 to obtain $E|S_{T_n}|^\beta \leq E|S_n|^\beta(1 + e^{-n}((2n^n/n!) - 1 + \Sigma_{k=n}^\infty(n^{k-1}/k!)))$. Therefore

$$E|S_n|^\beta \geq (K_n)^\beta E|Y_1 - Y_2|^\beta \Big/ \left(1 + e^{-n}\left(\frac{2n^n}{n!} - 1 + \Sigma_{k=n}^\infty \frac{n^{k-1}}{k!}\right)\right). \tag{3.9}$$

Inequality (3.9) is within a factor of $1 + O(n^{-\frac{1}{2}})$ of being best possible, as consideration of the analogue of (1.5) used in conjunction with Stirling's approximation verifies.

4. Asymptotics. Theorem 14 can be applied to triangular arrays. Let

$$\mathfrak{F}_1 = \{f \in \mathfrak{F} : f(\cdot) \text{ is strictly increasing on } [0, \infty)\} \tag{4.1}$$

and

(4.2) $\mathfrak{F}_2 = \{f \in \mathfrak{F}_1 : \exists\, \alpha > 1 \text{ such that } f(2x) \geqslant \alpha f(x) \text{ for all } x \geqslant 0\}.$

Certain properties of the functions in these collections will be needed and so are recorded below.

PROPOSITION 17. *Let* $f(x) = \int_0^\infty (1 - \cos tx)\, d\mu(t)$. *Then*

(4.3) $$f(x) \geqslant x^2(1 - \cos 1)\int_0^{1/|x|} t^2\, d\mu(t) \qquad \text{if } f \in \mathfrak{F}$$

and

(4.4) $$f(x) \geqslant (1 - \sin 1)\int_{1/|x|}^\infty d\mu(t) \qquad \text{if } f \in \mathfrak{F}_1.$$

Finally, suppose $f \in \mathfrak{F}_2$. *Then for every* $\varepsilon > 0$ *there exists* $b \geqslant 1$ *such that*

(4.5) $$\int_{b/|x|}^\infty d\mu(t) \leqslant \varepsilon f(x) \qquad \text{for all } x \neq 0.$$

PROOF OF (4.3).

$$\begin{aligned} f(x) &\geqslant \int_0^{1/|x|}(1 - \cos tx)\, d\mu(t) \\ &\geqslant \inf_{|y| \leqslant 1} y^{-2}(1 - \cos y)\int_0^{1/|x|} t^2x^2\, d\mu(t) \\ &= x^2(1 - \cos 1)\int_0^{1/|x|} t^2\, d\mu(t). \end{aligned}$$

PROOF OF (4.4). We may assume $x > 0$.

$$\begin{aligned} f(x) &\geqslant (1/x)\int_0^x f(y)\, dy \\ &= \int_0^\infty \int_0^x \frac{1 - \cos ty}{x}\, dy\, d\mu(t) \qquad \text{(by Fubini)} \\ &= \int_0^\infty \left(1 - \frac{\sin tx}{tx}\right) d\mu(t) \\ &\geqslant \int_{1/x}^\infty \left(1 - \frac{\sin tx}{tx}\right) d\mu(t) \\ &\geqslant (1 - \sin 1)\int_{1/x}^\infty d\mu(t). \end{aligned}$$

PROOF OF (4.5). Select $\alpha > 1$ such that $f(2x) \geqslant \alpha f(x)$ for $x \neq 0$. Fix $\varepsilon > 0$. Choose $n \geqslant 1$ so that $\alpha^{-n} \leqslant \varepsilon(1 - \sin 1)$. Let $b = 2^n$. Clearly $f(x/b) \leqslant \alpha^{-n} f(x)$. Thus

$$\begin{aligned} \int_{b/|x|}^\infty d\mu(t) &\leqslant f(x/b)/(1 - \sin 1) \qquad \text{(by (4.4))} \\ &\leqslant \alpha^{-n} f(x)/(1 - \sin 1) \\ &\leqslant \varepsilon f(x). \end{aligned}$$ □

The main theorem of this section is based on an approximation lemma whose proof is facilitated by the next result.

PROPOSITION 18. *For each* $n \geqslant 1$ *let* $Y_{n1}, \cdots, Y_{nk_n}$ *be a sequence of independent random variables. Suppose that for each* $\varepsilon > 0$ *there exists* α_n *(which depends on* ε*) such that*

(4.6) $$\max_{1 \leqslant j \leqslant k_n} P(|Y_{nj}| \geqslant \varepsilon) \leqslant \alpha_n \to 0 \qquad \text{as } n \to \infty.$$

Then, given positive reals b_1, b_2 and b_3, there exists $\varepsilon_n = \varepsilon_n(b_1, b_2, b_3)$ such that for all $|u| \leqslant b_3$ and all $1 \leqslant j \leqslant k_n$,

$$(4.7)\qquad \big(E(\sin uY_{nj})I(|Y_{nj}| > b_1)\big)^2 \leqslant \varepsilon_n E(1 - \cos uY_{nj}),$$

$$(4.8)\qquad \big(E(1 - \cos uY_{nj})\big)^2 \leqslant \varepsilon_n E(1 - \cos uY_{nj}),$$

and

$$(4.9)\qquad \big(E \sin uY_{nj}I(|Y_{nj}| \leqslant b_2)\big)^2$$
$$\leqslant 2\big(uEY_{nj}I(|Y_{nj}| \leqslant b_2)\big)^2 + (\varepsilon_n/2)u^2EY_{nj}^2I(|Y_{nj}| \leqslant b_2),$$

where $\varepsilon_n \to 0$ as $n \to \infty$.

REMARK 19. Condition (4.6) is the well-known (Loeve [7], page 302) *uniformly asymptotically negligible* (u.a.n.) *condition*. It ensures that no single variable Y_{nj} will dominate the sum $S_n \equiv \Sigma_{j=1}^{k_n} Y_{nj}$ unless S_n itself is negligible.

PROOF. First we verify (4.7). By Cauchy-Schwarz

$$\begin{aligned}\big(E \sin uY_{nj}I(|Y_{nj}| > b_1)\big)^2 &\leqslant P(|Y_{nj}| > b_1)E \sin^2 uY_{nj}\\ &= P(|Y_{nj}| > b_1)E(1 - \cos uY_{nj})(1 + \cos uY_{nj})\\ &\leqslant 2\alpha_n E(1 - \cos uY_{nj}).\end{aligned}$$

According to Loeve ([7], page 302), the u.a.n. condition implies that

$$\lim_{n\to\infty} \sup_{|u|\leqslant b_3} \max_{1\leqslant j\leqslant k_n}\big(E(1 - \cos uY_{nj}) + u^2EY_{nj}^2I(|Y_{nj}| \leqslant b_2)\big) = 0.$$

This proves (4.8) and will be used in (4.9). Observe that for any twice continuously differentiable function $q(\cdot)$, $q(u) = q(0) + uq'(0) + \int_0^u\int_0^v q''(w)\,dw\,dv$. Therefore

$$\begin{aligned}\big(E \sin uY'_{nj}\big)^2 &\leqslant \Big(|uEY'_{nj}| + \textstyle\int_0^{|u|}\int_0^v E(Y'_{nj})^2\,dw\,dv\Big)^2\\ &\leqslant 2(uEY'_{nj})^2 + 2\Big(\frac{u^2}{2}E(Y'_{nj})^2\Big)^2\\ &\leqslant 2(uEY'_{nj})^2 + (\varepsilon_n/2)u^2E(Y'_{nj})^2\end{aligned}$$

for some appropriate ε_n which tends to zero as $n \to \infty$. □

Now for the lemma.

LEMMA 20. (*Asymptotic approximation lemma*). *Let $f(x) = \int_0^\infty(1 - \cos tx)\,d\mu(t)$ belong to $\mathcal{F}_1$. For each $n \geqslant 1$ let $Y_{n1}, \cdots, Y_{nk_n}$ be a sequence of independent random variables which satisfy the* u.a.n. *condition* (4.6). *Suppose further that for some constant $0 \leqslant c < \infty$ independent of n,*

$$(4.10)\qquad \Sigma_{j=1}^{k_n} EY_{nj}^2I(|Y_{nj}| \leqslant 1) \leqslant c$$

and that

$$\lim_{n\to\infty} \Sigma_{j=1}^{k_n}\big(EY_{nj}I(|Y_{nj}| \leqslant 1)\big)^2 = 0. \tag{4.11}$$

Then for any real numbers $b \geqslant 1$ *and* $K_n > 0$,

$$\liminf_{n\to\infty}(1/f(K_n))\textstyle\int_0^{b/K_n}\big(1 - \operatorname{Re}\Pi_{j=1}^{k_n}Ee^{itK_nY_{nj}}\big)\,d\mu(t) \tag{4.12}$$

$$\geqslant \liminf_{n\to\infty}(1/f(K_n))\textstyle\int_0^{b/K_n}\big(1 - \Pi_{j=1}^{k_n}\exp(E\cos tK_nY_{nj} - 1)\big)\,d\mu(t).$$

Moreover, if $f \in \mathfrak{F}_2$ *then*

$$\liminf_{n\to\infty} Ef\big(\Sigma_{j=1}^{k_n}K_nY_{nj}\big)/f(K_n) \tag{4.13}$$

$$\geqslant \liminf_{n\to\infty}(1/f(K_n))\textstyle\int_0^{\infty}\big(1 - \Pi_{j=1}^{k_n}\exp(E\cos tK_nY_{nj} - 1)\big)\,d\mu(t).$$

REMARK 21. Condition (4.11) may be thought of as an antidegeneracy condition. It ensures that whenever $\Sigma_{j=1}^{k_n}EY_{nj}^2I(|Y_{nj}| \leqslant 1)$ is not negligible, $\Sigma_{j=1}^{k_n}Y_{nj}I(|Y_{nj}| \leqslant 1)$ retains some randomness and does not degenerate about its expectation. The conclusions (4.12) and (4.13) remain valid if the two conditions (4.10) and (4.11) are replaced by the single condition: for every $\varepsilon > 0$ there exists $\alpha_n \to 0$ such that

$$\Sigma_{j=1}^{k_n}\big(EY_{nj}I(|Y_{nj}| \leqslant \varepsilon)\big)^2 \leqslant \alpha_n\Sigma_{j=1}^{k_n}EY_{nj}^2I(|Y_{nj}| \leqslant \varepsilon). \tag{4.14}$$

PROOF OF LEMMA 20. Assuming (4.12), the second assertion (4.13) follows directly from (4.5) and the fact that $Ef(\Sigma_{j=1}^{k_n}Y_{nj}) = \int_0^\infty(1 - \operatorname{Re}\Pi_{j=1}^{k_n}Ee^{itY_{nj}})\,d\mu(t)$. We therefore prove only (4.12). Write $Y_{nj} = Y_{nj}I(|Y_{nj}| \leqslant 1) + Y_{nj}I(|Y_{nj}| > 1) \equiv Y'_{nj} + Y''_{nj}$. Fix $b \geqslant 1$ and let $u = tK_n$. Due to Proposition 18, there exists $\varepsilon_n \to 0$ such that for all $|u| \leqslant b$,

$$4\Sigma_{j=1}^{k_n}(E\sin uY''_{nj})^2 \leqslant \varepsilon_n\Sigma_{j=1}^{k_n}E(1 - \cos uY_{nj}), \tag{4.15}$$

$$2\Sigma_{j=1}^{k_n}\big(E(1 - \cos uY_{nj})\big)^2 \leqslant \varepsilon_n\Sigma_{j=1}^{k_n}E(1 - \cos uY_{nj}), \tag{4.16}$$

and

$$\Sigma_{j=1}^{k_n}(E\sin uY'_{nj})^2 \leqslant 2\Sigma_{j=1}^{k_n}(uEY'_{nj})^2 + (\varepsilon_n/2)\Sigma_{j=1}^{k_n}u^2E(Y'_{nj})^2. \tag{4.17}$$

Furthermore, ε_n may be chosen so that

$$4\Sigma_{j=1}^{k_n}(EY'_{nj})^2 \leqslant \varepsilon_n. \tag{4.18}$$

Observe that for any complex number z sufficiently close to 1 ($|z - 1| \leqslant .5$ will do) there exists a complex number θ of modulus at most 1 such that $z = \exp(z - 1 + \theta(z - 1)^2)$. The u.a.n. condition (see Loève [7], page 302) ensures that $f_{nj}(u) \equiv E\exp iuY_{nj}$ converges to 1 uniformly in $|u| \leqslant b$ and $1 \leqslant j \leqslant k_n$ as $n \to \infty$. Writing $f_{nj}(u)$ in exponential form and suppressing the dependence of θ_{nj} on u, we lower-bound the integrand in (4.12) thusly:

$$\begin{aligned}
1 - \mathrm{Re}\, \Pi_{j=1}^{k_n} f_{nj}(u) &= 1 - \mathrm{Re}\, \Pi_{j=1}^{k_n} \exp\big(f_{nj}(u) - 1 + \theta_{nj}(f_{nj}(u) - 1)^2\big) \\
&= 1 - \mathrm{Re} \exp \Sigma_{j=1}^{k_n}\big(f_{nj}(u) - 1 + \theta_{nj}(f_{nj}(u) - 1)^2\big) \\
&\geqslant 1 - \exp\big(\Sigma_{j=1}^{k_n}(E \cos uY_{nj} - 1) + \Sigma_{j=1}^{k_n}(E(1 - \cos uY_{nj}))^2 \\
&\quad + \Sigma_{j=1}^{k_n}(|E \sin uY'_{nj}| + |E \sin uY''_{nj}|)^2\big) \\
&\geqslant 1 - \exp\big(\Sigma_{j=1}^{k_n}\big((E \cos uY_{nj} - 1) + \big(E(1 - \cos uY_{nj})\big)^2 \\
&\quad + 2(E \sin uY''_{nj})^2\big) \\
&\quad + 2\Sigma_{j=1}^{k_n}(E \sin uY'_{nj})^2\big) \\
&\geqslant 1 - \big(\exp(1 - \varepsilon_n)\Sigma_{j=1}^{k_n}(E \cos uY_{nj} - 1)\big) \exp \Sigma_{j=1}^{k_n}\big(4u^2(EY'_{nj})^2 \\
&\quad + \varepsilon_n u^2 E(Y'_{nj})^2\big) \qquad \text{(by (4.16), (4.15) and (4.17))} \\
&\geqslant 1 - \exp\big((1 - \varepsilon_n)\Sigma_{j=1}^{k_n}(E \cos uY_{nj} - 1)\big) \exp \varepsilon_n u^2(1 + c) \\
&\qquad \text{(using (4.18) and (4.10))} \\
&\geqslant 1 - \exp\big((1 - \varepsilon_n)\Sigma_{j=1}^{k_n}(E \cos uY_{nj} - 1)\big) - 2\varepsilon_n u^2(1 + c),
\end{aligned}$$

provided n is sufficiently large. Next, since

$$\lim_{n\to\infty} \sup_{a<0}\left|\frac{1 - \exp(a(1 - \varepsilon_n))}{1 - \exp a} - 1\right| = 0,$$

there exists $\delta_n \to 0$ such that for all $|u| \leqslant b$,

$$1 - \exp\big((1 - \varepsilon_n)\Sigma_{j=1}^{k_n}(E \cos uY_{nj} - 1)\big) \geqslant (1 - \delta_n)\big(1 - \exp \Sigma_{j=1}^{k_n}(E \cos uY_{nj} - 1)\big).$$

Therefore

$$\begin{aligned}
&\liminf_{n\to\infty} \\
&\frac{\int_0^{b/K_n}\big(1 - \mathrm{Re}\, \Pi_{j=1}^{k_n} f_{nj}(tK_n)\big) - (1 - \delta_n)\big(1 - \Pi_{j=1}^{k_n} \exp(E \cos tK_n Y_{nj} - 1)\big)}{f(K_n)}\, d\mu(t) \\
&\geqslant \liminf_{n\to\infty} -(2\varepsilon_n(1 + c)/f(K_n))\int_0^{b/K_n} t^2K_n^2\, d\mu(t) \\
&\geqslant -5b^2(1 + c) \limsup_{n\to\infty} \varepsilon_n f(K_n/b)/f(K_n) \\
&\qquad \text{(from (4.3))} \\
&= 0 \qquad \text{(since } f \text{ is nondecreasing).}
\end{aligned}$$

Inequality (4.12) is an immediate consequence. □

THEOREM 22. *Let $f(x) = \int_0^\infty(1 - \cos tx)\, d\mu(t)$ be any function in $\mathfrak{F}_2$ such that $f(x)x^{-2}$ is nonincreasing. For each $n \geqslant 1$, let $X_{n1}, \cdots, X_{nk_n}$ be a sequence of* *independent random variables such that* $0 < \Sigma_{j=1}^{k_n} Ef(X_{nj}) < \infty$. *Let* K_n *be the unique positive real number satisfying*

(4.19) $$\Sigma_{j=1}^{k_n} E\big((X_{nj}/K_n)^2 \wedge (f(X_{nj})/f(K_n))\big) = 1.$$

Suppose that

(4.20) $$\lim_{n\to\infty} \Sigma_{j=1}^{k_n}\big(E(X_{nj}/K_n)I(|X_{nj}| \leqslant K_n)\big)^2 = 0$$

and that, for every $\varepsilon > 0$,

(4.21) $$\lim_{n\to\infty} \max_{1\leqslant j\leqslant k_n} P(|X_{nj}| > \varepsilon K_n) = 0.$$

Then

(4.22) $$\liminf_{n\to\infty} Ef\big(\Sigma_{j=1}^{k_n} X_{nj}\big)/f(K_n)$$
$$\geqslant \liminf_{n\to\infty} \inf_{x>0}(1/f(K_n))\int_0^\infty\left(1 - \exp\frac{\cos tx - 1}{x^2K_n^{-2} \wedge f(x)/f(K_n)}\right) d\mu(t).$$

PROOF. Let $Y_{nj} = X_{nj}/K_n$. Clearly, Y_{nj} satisfies (4.6) and (4.11). Moreover, since $(x/K_n)^2 \leqslant f(x)/f(K_n)$ for $|x| \leqslant K_n$,

$$1 \geqslant \Sigma_{j=1}^{k_n} E\big((X_{nj}/K_n)^2 \wedge (f(X_{nj})/f(K_n))\big)I(|X_{nj}| \leqslant K_n) = \Sigma_{j=1}^{k_n} EY_{nj}^2 I(|Y_{nj}| \leqslant 1).$$

Hence (4.10) holds with $c = 1$. Therefore we may invoke the asymptotic approximation Lemma 20. To complete the theorem, apply Theorem 14 to the X_{nj}'s, using $h_n(x) = x^2K_n^{-2} \wedge f(x)/f(K_n)$. □

5. Concluding remarks and conjectures. The lower-bound in (4.22) is sharp. Furthermore, it is not difficult to show the existence of a finite, positive real number x_n which achieves the infimum over $x > 0$ of

$$g(x, K_n) \equiv \int_0^\infty\left(1 - \exp\frac{\cos txK_n - 1}{x^2 \wedge (f(xK_n)/f(K_n))}\right) d\mu(t).$$

It seems possible that x_n always equals 1.

Certain results for finite n are indicated. Henceforth suppose, in addition to (4.19), that $X_{n1}, \cdots, X_{nk_n}$ are i.i.d. Whenever X_{n1} is symmetric and k_n is either odd or sufficiently large, $Ef(\Sigma_{j=1}^{k_n} X_{nj}) \geqslant g(x_n, K_n)$. It seems natural to conjecture that this inequality holds for all $k_n \geqslant 1$ irrespective of whether X_{n1} is symmetric. Even if true, however, equality is not achieved for any finite k_n. To obtain the exact lower bound for finite n, the extremal X_{n1} distribution must be constructed. Going out on a limb, I conjecture that (at least if n is sufficiently large) it may be found among the symmetric distributions which assume at most three values $-xK_n$, 0, xK_n. An argument based on convexity (akin to the proof of Lemma 4) verifies the conjecture whenever X_{n1} is symmetric and k_n is even.

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