The Annals of Probability
1996, Vol. 24, No. 4, 1842–1870

A CENTRAL LIMIT THEOREM FOR REVERSIBLE EXCLUSION AND ZERO-RANGE PARTICLE SYSTEMS

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We give easily verifiable conditions under which a functional central limit theorem holds for additive functionals of symmetric simple exclusion and symmetric zero-range processes. Also a reversible exclusion model with speed change is considered. Let $\eta(t)$ be the configuration of the process at time t and let $f(\eta)$ be a function on the state space. The question is: For which functions f does $\lambda^{-1/2}\int_0^{\lambda t} f(\eta(s))\,ds$ converge to a Brownian motion? A general but often intractable answer is given by Kipnis and Varadhan. In this article we determine what conditions beyond a mean-zero condition on $f(\eta)$ are required for the diffusive limit above. Specifically, we characterize the H^{-1} space in an applicable way.

Our method of proof relies primarily on a sharp estimate on the "spectral gap" of the process and weak regularity properties for the invariant measures.

1. Introduction and results. One of the difficult problems in the recent study of interacting particle systems is the characterization of the motion of a specific, or tagged, particle. A suitable description of the tagged particle motion has been shown to imply various physical properties of the process [9]. The investigation of the motion of any single particle, however, is complicated by the fact that it depends on its environment, that is, the other particles. Therefore, by itself, the tagged particle motion is not Markovian. However, the interaction of the other particles is weak and it is usually expected that the tagged particle motion, appropriately scaled, will converge to a diffusion. How to prove this? The general method, outlined by Kipnis and Varadhan [6], is to evaluate the tagged particle motion as the sum of a martingale and an additive functional. The martingale, by standard limit theorems, converges. It is therefore enough to show that the scaled additive functional converges also. When such a functional central limit theorem is true for the general stationary reversible Markov process is determined in [6]. Although Kipnis and Varadhan give an abstract condition under which additive functionals converge, this condition is often too involved to verify for any particular case. In particular, if one desires to solve the tagged particle problem in an interacting particle system with drift, that is, when the process is nonreversible, then more tractable conditions are required.

Received May 1995; revised November 1995.

AMS 1991 *subject classifications*. Primary 60K35; secondary 60F05.

Key words and phrases. Simple exclusion process, zero-range process, invariance principle, central limit theorem.

The aim of this article is to provide such easy-to-verify conditions in the context of three types of conservative interacting particle systems, two versions of the exclusion process and the zero-range process. Although our motivation for the problem derives from tagged particle considerations, we note that our conditions represent simple criteria under which an invariance principle holds for these particle systems.

Before we proceed further, let us describe the result in [6]. Consider a Markov process $\eta(t)$ defined on the state space Σ and let it be reversible with respect to the probability distribution $\pi(d\eta)$. Suppose also that the process $\eta(t)$ is stationary and that π is the ergodic measure. Let $f(\eta) \in L_2(\Sigma, \pi)$ and normalize f so that $E^\pi[f] = 0$. Define L to be the infinitesimal generator of the process. Now specify the time integral

$$S(t) = \int_0^t f(\eta(s))\, ds.$$

It is proved in [6] that $\lambda^{-1/2} S(\lambda t)$ converges weakly, with respect to π, to a Brownian motion $B(\sigma^2(f)t)$ if the limiting variance is finite, $\sigma^2(f) < \infty$.

This condition is further analyzed by calculating the variance:

$$\begin{aligned}\sigma^2(f) &= \lim_{t \to \infty} \frac{1}{t} E^\pi\left[S(t)^2\right] \\ &= 2\int_0^\infty E^\pi[f(\eta(s)) f(\eta(0))]\, ds \\ &= 2E^\pi\left[(-L)^{-1} f, f\right] \\ &= 2E^\pi\left[(-L)^{-1/2} f, (-L)^{-1/2} f\right].\end{aligned}$$

Now it is clear that $\sigma^2(f) < \infty$ is equivalent to

$$f(x) \in \operatorname{Range}\left[(-L)^{1/2}\right], \tag{1}$$

which in turn is equivalent to $\|f\|_{-1} < \infty$, the H^{-1} bound on f:

$$|E^\pi[f, \phi]| \le c(f)\sqrt{E^\pi[(-L)\phi, \phi]} \tag{2}$$

for all locally supported ϕ; here, in fact, $\|f\|_{-1}$ is the smallest $c(f)$.

The above calculation also identifies the finite variance condition as an equivalent measure of the asymptotic independence of the state variables:

$$\int_0^\infty E^\pi[f(x(s)) f(x(0))]\, ds < \infty. \tag{3}$$

As terminology, we will call a function $f(x)\colon \Sigma \to R^1$ *admissible* for the generator L if $f(x)$ satisfies either $\sigma^2(f) < \infty$ or its equivalents (1)–(3). This characterization of f is naturally the best possible because $\sigma^2(f)$ is the limiting variance of $(1/\sqrt{t})S(t)$.

As mentioned before, it is difficult to verify these conditions for a particular $f(x)$, even if the function is locally supported. For certain conservative dynamics, we will determine in this article simpler admissibility conditions for locally supported functions. In other words, we will characterize the space H^{-1} in a more transparent way. Specifically, we investigate the symmetric simple exclusion and the symmetric zero-range processes. In the last section we show that our techniques also apply to a more general one-dimensional nearest-neighbor exclusion model whose invariant measure is a Gibbs distribution.

Conservative particle systems consist of an infinite number of particles moving on Z^d according to a Markovian law: particles are neither created nor destroyed. Hence the title "conservative." These systems were first introduced in 1970 by Spitzer [14].

Informally, both symmetric simple exclusion and symmetric zero-range processes are systems of particles performing random walks according to symmetric translation invariant finite range irreducible jump probabilities $p(i,j) = p(0, j-i) = p(j-i)$, where $i,\ j \in Z^d$. The difference occurs in the nature of their interactions. For the simple exclusion system, the particles interact only in that jumps to occupied sites are suppressed. Hence, the state space of the simple exclusion process may be realized as $\Sigma = \{0,1\}^{Z^d}$ and a configuration $\eta = \{\eta_i\colon i \in Z^d\}$ as a collection of occupation variables η_i, where $\eta_i = 0$ or 1 if site i is empty or full. However, for the zero-range systems, the interaction is given in terms of a rate function $c\colon N \to R^1$, $c(0) = 0$ and $c(i) > 0$ for $i \ge 1$ (here, N denotes the non-negative integers). If there are k particles at a vertex $i \in Z^d$, then one of them jumps to site j at rate $c(k)p(j-i)$; this happens independently at every site. A configuration η then is represented as a collection of occupation numbers η_i taking values in the state space $\Sigma = N^{Z^d}$. When the rate function $c(k) \equiv k$, it is not hard to see that the associated zero-range process is noninteractive.

The more general symmetric exclusion process, for convenience, will be defined on the space of charges $\Sigma = \{-1,1\}^{Z}$. Besides the basic exclusion, the charges interact in that different configurations possess different rates of change. This will be made clearer in the following.

Note that throughout this article, we consider only symmetric finite range irreducible $p(\cdot)$ (see Remarks 1.1 and 1.3).

The Markovian evolution of the simple exclusion is given precisely through the action of its symmetric infinitesimal generator L on test functions ϕ:

$$(L\phi)(\eta) = \sum_{i,j} \eta_i(1-\eta_j)\big(\phi(\eta^{ij}) - \phi(\eta)\big)p(j-i),$$

where η^{ij} is the "switched" configuration formed from η by exchanging the values of η_i and η_j.

The symmetric infinitesimal generator L for the zero-range process is given as:

$$(L\phi)(\eta) = \sum_{i,j} c(\eta_i)\big(\phi(\eta^{ij}) - \phi(\eta)\big)p(j-i),$$

where

$$\eta_k^{ij} = \begin{cases} \eta_i - 1, & \text{if } k = i, \\ \eta_j + 1, & \text{if } k = j, \\ \eta_k, & \text{if } k \neq i, j, \end{cases}$$

provided $\eta_i \geq 1$ and $i \neq j$; otherwise, $\eta^{ij} \equiv \eta$.

The simple exclusion process may be constructed now without further assumptions. However, in order for the zero-range processes to make sense, we must impose on the rate the following condition:

(LG) There exists a constant $a_1 < \infty$, where $|c(k+1) - c(k)| < a_1$ for all $k \geq 0$.

We refer to [10] and [1] for the details of these constructions.

We now specify the invariant measures for these processes. Because both systems are conservative, it is expected that these processes possess a family of mutually orthogonal extremal invariant states $\{P_\theta\}$, each concentrated on configurations of fixed density θ. In fact, this is the case and, furthermore, these measures are product measures. As the jump probabilities $p(\cdot)$ are symmetric, these measures are reversible. Expectation with respect to P_θ is denoted by E_θ. When the density θ is fixed, we will denote by $\|\cdot\|_p$ the $L^p(\Sigma, P_\theta)$ norm.

For the simple exclusion process, these extremal measures P_θ are Bernoulli product measures with density $\theta \in [0, 1]$ (see [10]).

In order to describe these measures for the zero-range processes, define the partition function $Z(\cdot)$ on R_+^1 (where R_+^1 are the positive real numbers) by

$$Z(\alpha) = \sum_{k \geq 0} \frac{\alpha^k}{c(1) \cdots c(k)}.$$

It is clear that $Z(\cdot)$ is an increasing function. Let α^* be the radius of convergence of Z:

$$\alpha^* = \sup\{\alpha; Z(\alpha) < \infty\}.$$

To avoid degeneracy, we will assume that $Z(\alpha)$ diverges as it approaches the boundary:

$$\lim_{\alpha \to \alpha^*} Z(\alpha) = \infty. \tag{4}$$

For $0 \leq \alpha < \alpha^*$, consider the translation invariant product measure $\overline{P}_\alpha$ defined on Σ with marginal μ_α:

$$\mu_\alpha\{\eta_i = k\} = \frac{1}{Z(\alpha)} \frac{\alpha^k}{c(1) \cdots c(k)} \tag{5}$$

for $k \geq 0$. In the literature, the parameter α is called the "fugacity" of the process. However, a more intuitive parametrization of the invariant measures is through the particle density of the system. Let $\theta(\alpha)$ be the density of particles for the measure $\bar{P}_\alpha$. That is,

$$\theta(\alpha) = \bar{E}_\alpha[\eta_0],$$

where $\bar{E}_\alpha$ refers to expectation with respect to $\bar{P}_\alpha$.

From (4) it follows that $\theta\colon [0, \alpha^*) \to R^1_+$ is a smooth strictly increasing bijection. We may write, therefore,

$$P_\theta = \bar{P}_{\alpha(\theta)}, \qquad \theta \geq 0.$$

Under this convention it follows then that

$$\alpha(\theta) = E_\theta\big[c(\eta_0)\big], \qquad \theta \geq 0.$$

It is shown in [1] that these measures are invariant for the zero-range process with rate $c(\cdot)$, and extremal when $c(\cdot)$ non-decreases (the "attractive" case) and $p(\cdot)$ corresponds to a null recurrent walk. So, under our assumptions on $p(\cdot)$, we have extremality of P_θ in $d = 1, 2$ for the attractive systems. This extremal property for P_θ is believed to be true, but not proven, in general.

We may now define the Dirichlet form $D_\theta(\phi) = -E_\theta[\phi(L\phi)]$ for both of these models associated with L under P_θ.

For simple exclusion,

$$D_\theta(\phi) = \sum_{i,j} E_\theta\Big[\eta_i(1-\eta_j)\big(\phi(\eta^{ij}) - \phi(\eta)\big)^2\Big]p(j-i).$$

For zero-range processes,

$$D_\theta(\phi) = \sum_{i,j} E_\theta\Big[c(\eta_i)\big(\phi(\eta^{ij}) - \phi(\eta)\big)^2\Big]p(j-i).$$

We now define a one-dimensional nearest-neighbor reversible generalized exclusion process. The "speed change" dynamics remarked on earlier is given implicitly by the Dirichlet form

$$D_0(\phi) = E_0\left[\sum_{|i-j|=1} \big(\phi(\eta^{ij}) - \phi(\eta)\big)^2\right],$$

where the expectation here is with respect to the invariant measure

$$P_0(\eta_{-k}, \dots, \eta_k) = \frac{\exp\big(\beta\sum_{i=-k}^{k-1}\eta_i\eta_{i+1}\big)}{Z_k(\beta)} \tag{6}$$

defined in terms of its finite-dimensional projections; $Z_k(\beta)$ is the normalization. We refer to [10] for the construction of this process; see also [15] for a discussion.

The Gibbs distribution P_0 is Markovian for all real β and therefore has unique extension to all of Z^1. Therefore, P_0 is also ergodic for the process (see [15]). When $\beta = 0$, the measure is the familiar Bernoulli product measure. By the usual transfer matrix methods, we may specify the transition probability matrix A:

$$A = \frac{1}{e^{\beta} + e^{-\beta}} \begin{pmatrix} e^{\beta} & e^{-\beta} \\ e^{-\beta} & e^{\beta} \end{pmatrix}.$$

The one coordinate marginal is the fair coin-tossing measure corresponding to zero density: $\mu\{\eta_i\} = P_0\{\eta_i\}$: $\mu\{\eta_i = -1\} = \mu\{\eta_i = 1\} = 1/2$.

Associated with P_0 are the modified measures P_θ^λ defined in terms of their finite-dimensional projections:

$$P_0^\lambda(\eta_{-k}, \eta_{-k+1}, \dots, \eta_k) = \frac{\exp\left(\lambda \sum_{i=-k}^{k} (\eta_i - \theta)\right)}{Z_k(\lambda)} P_\theta(\eta_{-k}, \eta_{-k+1}, \dots, \eta_k);$$

$Z_k(\lambda)$ again is the new normalization. The mean $m(\lambda)$ of these distributions is given by

$$m(\lambda) = \lim_{k \to \infty} \int \frac{1}{2k+1} \left(\sum_{i=-k}^{k} (\eta_i - \theta) \right) dP_0^\lambda.$$

Expectation with respect to P_0^λ will be denoted as E_0^λ.

We now define the notion of the finite volume "spectral gap" for these processes. Let $B_n^d \subset Z^d$ be a cube of width n. Let $P_{n,K}(\eta) = P_\theta(\eta \mid \sum_{i \in B_n^d} \eta_i = K)$ be the conditioned measure on the hyperplane corresponding to K particles. This measure is reversible and ergodic for the process localized on the hyperplane $\Sigma_{n,K} = \{\sum_{i \in B_n^d} \eta_i = K\}$. The dynamics for these processes are driven by L_n, the truncated generator corresponding to bonds only in B_n^d. Denote by $E_{n,K}[f]$ and $D_{n,K}(f)$ the expectation and the Dirichlet form of f with respect to $P_{n,K}$ and L_n. Note that these definitions are independent of θ.

Now, as these processes defined on $\Sigma_{n,K}$ are finite state irreducible Markov chains, the operator L_n exhibits a discrete spectrum and we may define the difference between 0 and the next largest eigenvalue as the "spectral gap" for these processes. This quantity is also understood in terms of the constant $W(n,K)$ appearing in Poincaré's inequality:

$$E_{n,K}[\phi - E_{n,K}[\phi]]^2 \le W(n,K) D_{n,K}(\phi).$$

This constant $W(n,K)$ is the reciprocal of the gap and necessarily depends on the infinitesimal rates of the processes.

Recall that $K = \sum_{i \in B_n^d} \eta_i$. The following condition on $W(n,K)$ will be useful in the statement of our results:

(A1) There is a constant $C(\theta) < \infty$ such that $E_\theta[W(n,K)^2] < C(\theta) n^4$.

REMARK 1.1. For exclusion processes with finite range jump probabilities p, it is known (see [11] and [13]) that $W(n,K) < Cn^2$, where C is a constant independent of n and K. In these cases, condition (A1) is trivially satisfied for any θ.

The situation is not as clear for zero-range processes. However, in a forthcoming paper [8], a similar bound, $W(n, K) < Cn^2$, is shown for those nearest-neighbor symmetric systems with rate function c satisfying (LG) and the following condition:

(M) There exists $k_0 \in N$ and $a_2 > 0$ such that $c(k) - c(j) \geq a_2$ for all $k \geq j + k_0$.

These conditions include the rate $c(k) = k$ corresponding to the independent random walk model. However, the important rate $c(k) = I(k \geq 1)$ [where $I(\cdot)$ is the indicator] does not satisfy (M). Yet, by a transform to the exclusion process [4], we may bound $W(n, K) < C(n + K)^2$ in dimension $d = 1$ and therefore condition (A1) holds also for this rate. In fact, it is expected that (A1) is only a technical condition and that all zero-range systems with rates $c(\cdot)$ which satisfy (LG) alone satisfy (A1).

We are now in a position to state the main result of this article.

THEOREM 1.1. *Let $\eta(t)$ correspond to the symmetric finite range simple exclusion process. Let also $\Lambda \subset Z^d$ be a finite set. Suppose that $f(\eta)$ is a local function supported in $\{\eta_i: i \in \Lambda\}$. Let now $\theta \in (0,1)$. Then, f is admissible for L acting on $L^2(\Sigma, P_\theta)$ if and only if*

$$\frac{d^n}{dy^n} E_y[f(\eta)]|_{y=\theta} = 0$$

for

$$n = \begin{cases} 0,1,2, & \text{in } d = 1, \\ 0,1, & \text{in } d = 2, \\ 0, & \text{in } d \geq 3, \end{cases}$$

THEOREM 1.2. *Let $\eta(t)$ correspond to a symmetric finite range zero-range process satisfying (A1). Let also $\Lambda \subset Z^d$ be a finite set. Suppose that $f(\eta)$ is a local function supported in $\{\eta_i: i \in \Lambda\}$. Suppose also that f obeys the growth condition $E_\theta[f^4] < \infty$ for some $\theta > 0$. Then f is admissible for L acting on $L^4(\Sigma, P_\theta)$ if and only if*

$$\frac{d^n}{dy^n} E_\theta[f(\eta)]|_{y=\theta} = 0$$

for

$$n = \begin{cases} 0,1,2, & \text{in } d = 1, \\ 0,1, & \text{in } d = 2, \\ 0, & \text{in } d \geq 3. \end{cases}$$

THEOREM 1.3. *Let $\eta(t)$ correspond to the one-dimensional symmetric nearest neighbor generalized exclusion process with Gibbs invariant measure P_0. Let also $\Lambda = \{1, 2, \dots, |\Lambda|\} \subset Z^1$ be a finite set. Suppose that $f(\eta)$ is a local* *function supported in* $\{\eta_i\colon i \in \Lambda\}$. *Then* f *is admissible for* L *acting on* $L^2(\Sigma, P_0)$ *if and only if*

$$\frac{d^n}{dy^n} \int f(\eta)\, dP_0^{\lambda(y)} \mid_{y=\theta} = 0 \quad \text{for } n = 0, 1, 2,$$

where $\lambda(y)$ *is chosen so that* $m(\lambda(y)) = y$.

REMARK 1.2. It should be noted that we expect Theorem 1.3 to extend in a similar way to higher dimensions as in Theorems 1.1 and 1.2; the problem lies in applying a computationally difficult cluster expansion to estimate the characteristic function of the invariant measures. In one dimension, such estimates are more manageable, due to the Markovian nature of these measures.

REMARK 1.3. The finite range assumption, besides being useful in satisfying (A1), is also exploited to prove the "necessity" parts of the claims.

The following remarks indicate a few basic equivalences which lead to a corollary.

REMARK 1.4. We note that the conditions

$$\frac{d^n}{dy^n} E_y[f(\eta)]\big|_{y=\theta} = 0 \quad \text{for } n = 0, 1, 2, \dots \tag{7}$$

or for the Gibbs state P_0,

$$\frac{d^n}{dy^n} \int f(\eta)\, dP_0^{\lambda(y)}\bigg|_{y=\theta} = 0 \quad \text{for } n = 0, 1, 2, \dots,$$

may be recast as

$$E_\theta\left[f(\eta)\left(\sum_{i\in Z^d} (\eta_i - \theta)^n \right)\right] = 0 \quad \text{for } n = 0, 1, 2, \dots . \tag{8}$$

In fact, for models with product invariant measures, such as the simple exclusion and zero-range processes, these last conditions become

$$E_\theta f(\eta)\left(\sum_{i\in\Lambda} (\eta_i - \theta)\right)^n = 0 \quad \text{for } n = 0, 1, 2, \dots . \tag{9}$$

REMARK 1.5. As a point of clarification for the proofs of Theorems 1.1 and 1.2, we note that any local function f satisfies $\|f\|_p < \infty$ for any $p \ge 0$ when the underlying probability structure is a Bernoulli-type measure, that is, when the site number is bounded.

REMARK 1.6. A refinement of Theorem 1.2 holds when the zero-range rate satisfies the assumptions (LG) and (M). In this case, because $\|W(n, K)\|_\infty <$

Cn^2, we may replace the growth condition $\|f\|_4 < \infty$ with the less restrictive, more natural condition $\|f\|_2 < \infty$.

These remarks and the previous discussion of extremality of P_θ and the Kipnis–Varadhan theorem imply:

COROLLARY 1.1. *Let $f(\eta)$ be a local function such that $\|f\|_2 < \infty$. Let L be the generator for the symmetric finite range simple exclusion or those symmetric nearest neighbor zero-range processes with rates satisfying* (LG) *and* (M). *Then $f(\eta)$ is admissible for L acting on $L^2(\Sigma, P_\theta)$ if and only if*

$$E^\theta\left[f(\eta)\left\{\sum_{i\in\Lambda}(\eta_i - \theta)\right\}^n\right] = 0$$

for

$$n = \begin{cases} 0, 1, 2, & \text{in } d = 1, \\ 0, 1, & \text{in } d = 2, \\ 0, & \text{in } d \geq 3. \end{cases}$$

Furthermore, we have the invariance principle with respect to P_θ,

$$\lambda^{-1/2} S(\lambda t) \to B(\delta^2(f)t)$$

in all dimensions for simple exclusion and in $d = 1, 2$ for attractive zero-range.

REMARK 1.7. We note that in dimensions $d \geq 3$, simply a mean-zero condition on $f(\eta)$ is sufficient for admissibility. This mimics the well known result for finite state irreducible Markov chains (see [2]). Similar invariance principles also hold for a wider class of zero-range systems (see Remark 1.1), and the generalized exclusion process.

Let us consider a few examples for the nearest-neighbor simple exclusion process in dimension $d = 1$:

EXAMPLE 1.1. Let $f(\eta) = \eta_1 - \eta_2$. We will give a direct proof that $f(\eta)$ is admissible. Clearly, $E_\theta[f] = 0$ for any θ. Furthermore,

$$\begin{aligned} E_\theta[f\phi] &= E_\theta[(\eta_1 - \eta_2)\phi(\eta)] \\ &= E_\theta[\eta_1\phi(\eta)] - E_\theta[\eta_2\phi(\eta)] \\ &= E_\theta[\eta_1\phi(\eta)] - E_\theta[\eta_1\phi(\eta^{12})] \\ &= E_\theta[\eta_1(\phi(\eta) - \phi(\eta^{12}))] \\ &\leq \left(E_\theta[\eta_1^2]\right)^{1/2}\left(E_\theta\left[(\phi(\eta^{12}) - \phi(\eta))^2\right]\right)^{1/2} \\ &\leq c(f) D_\theta(\phi)^{1/2}, \end{aligned}$$

where we have used the Schwarz inequality in the penultimate line and have replaced the variation on the bond $1 \leftrightarrow 2$ by the full Dirichlet form in the last line. This shows that f is admissible.

EXAMPLE 1.2. Let $f(\eta) = \eta_1 - 1/2$. Here $E_{1/2}[f] = 0$. However, the second condition is not satisfied. We may construct a sequence of local functions $\{\phi_n\}$ so that $E_{1/2}[f\phi_n] = 1$, but $D_{1/2}(\phi_n) \to 0$ as $n \to \infty$. This is a consequence of the recurrence of random walk in dimension 1. In fact, take $\phi_n = 4\sum_{i=-n}^{n}(\eta_i - 1/2)$. We will see that these functions play an important role in the "necessary" part of the proof of Theorems 1.1–1.3 given in the next section. We note, however, that $f(\eta)$ is admissible in dimensions $d \geq 3$. This is due to the transience of random walk in three or more dimensions, which prevents the construction of a sequence $\{\phi_n\}$.

EXAMPLE 1.3. The previous discussion enables us to calculate the distribution of the occupation time at the origin, say, for dimensions $d \geq 3$:

$$\frac{1}{\sqrt{t}}\int_0^t (\eta_0(s) - \theta)\, ds \to N(0, \sigma^2(\theta)),$$

where initial configurations are distributed according to P_θ. The correct scalings in dimensions $d = 1, 2$ for the occupation time integral to converge to a nondegenerate Gaussian are, respectively, $t^{-3/4}$ and $\sqrt{t \log(t)}$ (see [5]).

We may regard the object $E_y[f]$, for fixed f, as a function of y. In fact, defne $A_f(y) = E_y[f]$. When f is a polynomial, $A_f(y)$ is a polynomial in y. Hence, as indicated by the differing examples above, $A_f(y)$ should be "flat" up to second degree near some point θ in dimension $d = 1$ in order for f to be admissible with respect to $L^4(P_\theta)$.

These admissibility conditions, on the other hand, are also measures of the asymptotic independence of the state variables. Consider (3): if the correlations $E_\theta[f(\eta(s))f(\eta(0))]$ decrease fast enough to be integrable, then f, by definition, is admissible. If we understand simple exclusion to be not so different from random walk, then we might argue for a general mean-zero $f(\eta)$ that its correlations, taking into account the linear, quadratic and higher order expansions, behave like $a_1 s^{-1/2} + a_2 s^{-1} + O(s^{-3/2})$, clearly not integrable if $a_1, a_2 \neq 0$. Consequently, in $d = 1$, these conditions are extra criteria on f so that the slower modes a_1 and a_2 vanish. This is made more precise in Section 3.

In Section 2, we outline an "integrations by parts" method, applicable to all three processes considered, which has been used before in different forms in a few settings [16, 3]. This approach relies on two estimates. First, we make use of the estimate on the spectral gap for the process defined on a finite range of coordinates afforded by condition (A1). Second, we utilize nice regularity properties of the invariant measures of the process.

In Section 3, we will give special arguments for Theorem 1.1 utilizing the condition (3) in the case of nearest-neighbor interactions. The method of proof here applies only to the simple exclusion process and relies heavily on the nearest-neighbor condition. The fact that the state space $\Sigma = \{0, 1\}^\Lambda$ for finite subsets $\Lambda \subset Z^d$ is of finite cardinality is also useful.

In Section 4, we will demonstrate the robustness of the method given in Section 2 by modifying the arguments given there to prove Theorem 1.3 for a one-dimensional generalized exclusion process. The key idea is that the Markovian invariant measures behave like product measures with exponential error.

2. The main argument. In this section, we prove Theorems 1.1 and 1.2 for the simple exclusion and zero-range processes. The proof we will outline applies in a variety of situations in which the spectral gap estimate (A1) holds and when the invariant measures possess certain regularity. For the simple exclusion and zero-range processes, the invariant measures are product measures with well behaved marginals. Accordingly, several estimates required for the characteristic functions of these marginals are immediate. As discussed earlier, the spectral gap estimate (A1) is known for exclusion processes. The condition is also true for a wide class of zero-range models under some conditions; however, as remarked, we expect (A1) to be satisfied by all constructable zero-range processess.

In order to fix ideas, we work in dimension $d = 1$; the higher dimensional cases are analogous.

PROOF OF THEOREMS 1.1 AND 1.2. *Sufficiency.* Let $f(\eta)$ be given dependent only on a finite number of coordinates indexed in $\Lambda \in Z$. We assume f satisfies the criteria (9) with respect to P_θ. To prove the admissibility of f, we will establish (2).

Let $g_n(y) = E_\theta[f(\eta) \mid (2n+1)^{-1}\sum_{-n}^{n}(\eta_i - \theta) = y]$ for $n > |\Lambda|$. Then we can solve $u_{n,K}$ in any hyperplane $\{(2n+3)^{-1}\sum_{i=-n-1}^{n+1} = K\}$, where

$$(-L_n)u_{n,K} = g_n - g_{n+1}.$$

Here $(-L_n)$ is the generator on bonds $\{-n-1, -n\}, \dots, \{n, n+1\}$. The solution $u_{n,K}$ exists because $(-L_n)$, on the hyperplane $\{\sum_{-n-1}^{n+1} \eta_i = (2n+3)K\}$, is a finite-dimensional operator and $g_n - g_{n+1}$ is orthogonal to all invariant measures of this truncated finite-dimensional process, that is, $E_\theta[g_n - g_{n+1} \mid \sum_{-n-1}^{n+1}\eta_i = z] = 0$ for all z. This is best seen by expressing, for $n \geq |\Lambda|$,

$$g_n(y) = E_\theta\left[f(\eta) \mid \sum_{-n}^{n} \eta_i = y, \eta_{-n-1}, \eta_{n+1}, \cdots\right]$$

as a martingale. Therefore, we may write, for a test function ϕ,

$$\begin{aligned} E_\theta[f\phi] &= E_\theta\big[(f - g_{|\Lambda|+1})\phi\big] \\ &\quad + E_\theta\left[\sum_{n=|\Lambda|+2}^{\infty} (g_n - g_{n+1})\phi\right] + \lim_{n\to\infty} E_\theta[g_n\phi]. \end{aligned} \tag{10}$$

By the martingale property, $\|g_n\|_{L^2(P_\theta)} \to E_\theta[f] = 0$ as $n \to \infty$. Therefore, the last term in (10) vanishes. Lumping the first term into the second by abuse of notation, we have

$$
\begin{aligned}
E_\theta[f\phi] &= E_\theta\Big[\sum (g_n - g_{n+1})\phi\Big] \\
&= \sum E_\theta\left[E_\theta\left[(g_n - g_{n+1}\phi)\,\middle|\,\frac{1}{2n+3}\sum_{i=-n-1}^{n+1} \eta_i = K\right]\right] \\
&= \sum E_\theta\big[E_\theta[(-L_n)u_{n,K}\phi|K]\big] \\
&= \sum E_\theta\Big[E_\theta\big[(-L_n)^{1/2}u_{n,K}\cdot(-L_n)^{1/2}\phi\big|K\big]\Big] \\
&\le \sum E_\theta\left[E_\theta\Big[\big((-L_n)^{1/2}u_{n,K}\big)^2\Big|K\Big]^{1/2}\cdot E_\theta\Big[\big((-L)_n^{1/2}\phi\big)^2\Big|K\Big]^{1/2}\right] \\
&= \sum E_\theta\left[E_\theta\Big[\big((-L_n)^{-1/2}(g_n - g_{n+1})\big)^2\Big|K\Big]^{1/2}\right. \\
&\qquad\qquad \left.\cdot E_\theta\Big[\big((-L)_n^{1/2}\phi\big)^2\Big|K\Big]^{1/2}\right] \qquad (11)\\
&\le \sum E_\theta\left[(W(n,K))^{1/2}E_\theta\Big[(g_n - g_{n+1})^2\Big|K\Big]^{1/2}\right. \\
&\qquad\qquad \left.\cdot E_\theta\Big[\big((-L)_n^{1/2}\phi\big)^2\Big|K\Big]^{1/2}\right] \\
&\le \sum E_\theta\Big[W(n,K)E_\theta\big[(g_n - g_{n+1})^2\big|K\big]\Big]^{1/2}\cdot D_\theta(\phi)^{1/2} \\
&\le D_\theta(\phi)^{1/2}\sum E_\theta\Big[W(n,K)^2\Big]^{1/4}\cdot E_\theta\left[E_\theta\Big[(g_n - g_{n+1})^2\Big|K\Big]^2\right]^{1/4} \\
&\le D_\theta(\phi)^{1/2}\sum E_\theta\Big[W(n,K)^2\Big]^{1/4}\cdot E_\theta\Big[(g_n - g_{n+1})^4\Big]^{1/4} \\
&\le 8D_\theta(\phi)^{1/2}\sum E_\theta\Big[W(n,K)^2\Big]^{1/4}\cdot E_\theta\Big[(g_n)^4\Big]^{1/4} \\
&\le 8D_\theta(\phi)^{1/2}\sum C\cdot n\cdot E_\theta\Big[(g_n)^4\Big]^{1/4}.
\end{aligned}
$$

Here in the third step, we use the equality $g_n - g_{n+1} = (-L_n)u_{n,K}$; in the fourth step, we use symmetry; in the fifth, Schwarz inequality; in the sixth, the above equality again; in the seventh, we use that the operator norm of $(-L_n)^{-1/2}$ on the space of mean-zero functions on the hyperplane $H_{n,K}$ is bounded by $\sqrt{W(n,K)}$; in the next step, we overestimate by the Schwarz inequality and the full Dirichlet form $D_\theta(\phi)$; to obtain the last line, we recall our condition (A1).

Observe now that if we can show that $\|g_n\|_{L^4(P_\theta)}$ decays fast enough, then the proof is finished. In particular, a decay of $\|g_n\|_4 \sim n^{1+\varepsilon}$ for $\varepsilon > 0$ [we will abbreviate the $L^p(P_\theta)$ norm by $\|\cdot\|_p$ as θ is fixed] gives that the above sum

(11) diverges as $O(n^{1-\varepsilon})$. However, this is sufficient because instead of the index sequence $\{n\}$, we may substitute the sequence $\{2^n\}$. In this spacing, the same decay $\|g_{2^n}\|_4 \sim 2^{-(1+\varepsilon)n}$ is enough for the last sum in (11), $\Sigma 2^n 2^{-(1+\varepsilon)n}$, to converge.

We will use the full force of the assumptions (9) to prove the following lemma, which establishes this decay with $\varepsilon = 1/2$ and, therefore, the sufficiency part of Theorems 1.1 and 1.2. To simplify the exposition, we prove the lemma for the zero-range processes.

LEMMA 2.1. *Let $f(\eta) = f(\eta_1, \eta_2, \dots, \eta_{|\Lambda|})$ be a local function. Under the conditions* (1.9), *we have that*

$$\left\| g_n\left(\frac{1}{2n+1} \sum_{-n}^{n} (\eta_i - \theta) \right) \right\|_4 \le K \|f\|_4 n^{-3/2},$$

where $K = K(|\Lambda|, \theta)$ is a finite constant.

PROOF. Let $y = (2n+1)^{-1} \Sigma_{-n}^{n} (\eta_i - \theta)$. In what follows, we apply the convention that C and K represent finite constants which may vary on application; typically these constants depend on $|\Lambda|$ and θ.

Step 1. To prove the lemma, we employ an Edgeworth expansion or Cramér's trick to evaluate $g_n(y)$. The idea is to "modify" P_θ so that the variables $\eta_i - \theta$ have mean y:

$$\begin{aligned} &E_\theta\left[f(\eta_1, \dots, \eta_{|\Lambda|}) \,\middle|\, \frac{1}{2n+1} \sum_{-n}^{n} (\eta_i - \theta) = y \right] \\ &\quad = \frac{E_\theta[fI(1/(2n+1)\Sigma_{-n}^{n}(\eta_i - \theta) = y)]}{E_\theta[I(1/(2n+1)\Sigma_{-n}^{n}(\eta_i - \theta) = y)]} \qquad (12) \\ &\quad = E_{m(\lambda)}\left[f \frac{\sqrt{2n+1}\,\Theta_{2n+1-|\Lambda|}\big(-\Sigma_1^{|\Lambda|}(\eta_i - \theta - y)\big)}{\sqrt{2n+1-|\Lambda|}\,\Theta_{2n+1}(0)} \right], \end{aligned}$$

where $E_{m(\lambda)}$ refers to expectation with respect to the modified product measure $P_{m(\lambda)}$ with marginal $\mu^\lambda = (\exp(\lambda(\eta_i - \theta))/M(\lambda))\mu_{\alpha(\theta)}$, whose mean is the density $m(\lambda)$. Also we define

$$\Theta_l(x) = \sqrt{l}\, P_{m(\lambda)}\left\{ \sum_{1}^{l} (\eta_i - \theta - y) = x \right\}.$$

Step 2. We now choose the parameter λ so that $m(\lambda) = y + \theta$. With this choice, $\Theta_l(x)$ obeys a classical local central limit theorem (see [12]). More precisely, when $\theta > 0$,

$$\begin{aligned} \lim_{l\to\infty} \Theta_l(x) &= \lim_{l\to\infty} \sqrt{l}\, P_{m(\lambda)}\left\{ \sum_{1}^{l} (\eta_i - \theta - y) = x \right\} \qquad (13) \\ &= \frac{1}{\sqrt{2\pi}\,\sigma(y)} \exp\left\{ \frac{-\bar{x}^2}{2\sigma(y)^2} \right\}, \end{aligned}$$

where $\sigma^2(y) = E_{m(\lambda)}[\eta_i - \theta - y]^2$ is the variance and $\bar{x} = \lim_{m\to\infty} x/\sqrt{m}$. This limit allows us to conclude, for y small [when $\sigma^2(y) > 0$], that $\Theta_l(0) > C(\theta) > 0$.

Also $\Theta_l(x)$ may be developed in powers of $x/\sqrt{l}$ by the expansion in terms of the characteristic function: $\nu_y(t) = E_{y+\theta}[\exp\{it(\eta - \theta - y)\}]$. First, we write the equality

$$\Theta_l(x) = \sqrt{l}\int_{-\pi}^{\pi} e^{itx}\nu_y(t)^l\, dt.$$

By expanding $\exp\{itx\}$ in the usual way, we have that $\Theta_l(x)$ equals

$$\begin{aligned}(14)\qquad &\int_{-\pi\sqrt{l}}^{\pi\sqrt{l}} \nu_y(t)^l\, dt + \frac{ix}{\sqrt{l}}\int_{-\pi\sqrt{l}}^{\pi\sqrt{l}} t\nu_y(t)^l\, dt \\ &\quad + \frac{-x^2}{l}\int_{-\pi\sqrt{l}}^{\pi\sqrt{l}} t^2\nu_y(t)^l\, dt + r(x)l^{-3/2},\end{aligned}$$

where $r(x)$ represents the error term.

Now, for $\theta > 0$, the zero-range marginals are quite regular and we may compute, for t small, that

$$\nu_y(t) = 1 - t^2(\sigma^2(y)/2) + O(t^3),$$

where, for y small, the variance $\sigma^2(y) > 0$. For $|t| > \varepsilon > 0$, we then have that

$$|\nu_y(t)| < K(\varepsilon) < 1.$$

Also it is not hard to determine that for y and $|t| < \varepsilon$ small, we have that

$$|\nu_y(t)|^n < \exp\{-Ct^2\}$$

for all $n \geq 1$ and some $C(\theta) > 0$.

These observations allow us to bound the error: $|r(x)| < C(\theta)|x|^3$.

Step 3. Let $J = [-\delta, \delta]$ for small $\delta > 0$. We will bound $\|g_n(y)\|_4$ as follows. First write

$$\begin{aligned}\|g_n(y)\|_4 &\leq \Big\{E_\theta\Big[(g_n(y))^4 I(y \in J)\Big]\Big\}^{1/4} + \Big\{E_\theta\Big[(g_n(y))^4 I(y \notin J)\Big]\Big\}^{1/4} \\ &= F_1 + F_2.\end{aligned}$$

Now, by large deviation estimates, or direct computation, we have that

$$F_2 < C\|f\|_4 n^{-3/2}.$$

Step 4. Therefore, what remains is to bound F_1. Our immediate plan is to obtain a better estimate of $g_n(y)$ by substituting the Taylor expansion and lower bound estimate for $\Theta_l(\cdot)$ of Step 2 into (12). We have that

$$\begin{aligned}
&E_{\theta+y}\left[f \frac{\sqrt{2n+1}\,\Theta_{2n+1-|\Lambda|}\big(\sum_1^{|\Lambda|}(\eta_i-\theta-y)\big)}{\sqrt{2n+1-|\Lambda|}\,\Theta_{2n+1}(0)} \right] \\
&\quad = K_0 E_{\theta+y}[f(\eta)] + \frac{K_1}{\sqrt{n}} E_{\theta+y}\left[f(\eta) \sum_1^{|\Lambda|}(\eta_i-\theta) \right] \\
&\qquad + \frac{K_2}{n} E_{\theta+y}\left[f(\eta) \left(\sum_1^{|\Lambda|}(\eta_i-\theta) \right)^2 \right] + n^{-3/2} E_{\theta+y}[f(\eta) r(\eta)] \\
&\quad = V_0 + V_1 + V_2 + V_3,
\end{aligned}$$

where uniformly, for $y \in J$, $|K_i| = |K_i(n, y)| < K(|\Lambda|, \delta)$ and $\|r\|_p < K(|\Lambda|, \theta, p)$ for any $p > 0$. The strategy now is to investigate the $\|V_i I(y \in J)\|_4$ separately.

It is not difficult to bound the last term:

$$\|V_3 I(y \in J)\|_4 < K \|f\|_4 n^{-3/2}.$$

The estimation of $\|V_0\|_4$ requires a closer analysis. As $y \in J = [-\delta, \delta]$ with δ small, we may expand V_0 in powers of y. We write

$$\begin{aligned}
V_0 &= C_0 E_\theta[f(\eta)] + C_1 E_\theta\left[f(\eta) \cdot \sum_1^{|\Lambda|}(\eta_i-\theta) \right] y \\
&\quad + C_2 E_\theta\left[f(\eta) \cdot \left(\sum_1^{|\Lambda|}(\eta_i-\theta) \right)^2 \right] y^2 + r(\eta) y^3,
\end{aligned} \tag{15}$$

where again, for $y \in J$, after considering the derivatives of $\lambda(y)$ at 0, some regrouping and bounding the constant K_0, we have $C_i < C$ and $\|r\|_4 < C(|\Lambda|, \theta)\|f\|_4$. Each y factor, by the central limit theorem, represents a decay of $n^{-1/2}$. Therefore, to estimate $\|V_0\|_4$ we need only investigate the L^4 norms of the zeroth, first and second order terms in (15). These terms, however, vanish from our assumptions (9). This completes the estimation of $\|V_0\|_4$.

In the same manner, we may expand the terms $\|V_1\|_4$ and $\|V_2\|_4$ in powers of y. Here, however, we need only investigate up to the second and third term, respectively, because V_1 and V_2 possess decay factors of $n^{-1/2}$ and n^{-1} already. The lower order terms now vanish from assumptions (9).

This completes the proof of Lemma 2.1 and, consequently, the "sufficiency" part of Theorems 1.1 and 1.2. □

In fact, much the same proof of the lemma yields the stronger result in dimension d which we state:

LEMMA 2.2. *Let $f(\eta) = f(\eta_1, \dots, \eta_{|\Lambda|})$ be a local function. Assume $\|f\|_p < \infty$. Then, given that*

$$E_\theta\left[f\left(\sum_{i\in\Lambda}(\eta_i - \theta)\right)^j\right] = 0 \quad \text{for } j = 0, 1, \dots, k,$$

we have

$$\left\|E_\theta\left[f\,\middle|\,\frac{1}{|B_n^d|}\sum_{i\in B_n^d}(\eta_i - \theta)\right]\right\|_p \le C\|f\|_p n^{(k+1)d/2},$$

where $C = C(|\Lambda|, \theta)$ is a finite constant.

Necessity. We show by contradiction the necessity of the admissibility conditions (9). Our proof will be for the zero-range process, but a straightforward modification (see Section 4 for the arguments given for the generalized exclusion process) can be made applicable for the simple exclusion model.

Step 1. Suppose f is admissible but that $E_\theta[f] \ne 0$. Then we may choose $\phi(\eta) = 1$. With this choice, $E_\theta[f\phi] \ne 0$, but $D_\theta(\phi) = 0$, contradicting the admissibility of f. This establishes the necessity of the first condition in (9).

Step 2. Let f be admissible again. By Step 1 we know then that $E_\theta[f] = 0$. However, suppose that $E_\theta[f\sum_1^{|\Lambda|}(\eta_i - \theta)] \ne 0$. Then we may choose

$$\phi_n(\eta) = \sum_{-n}^{n} J\left(\frac{i}{2n+1}\right)(\eta_i - \theta),$$

where J is a test function supported on $[-1, 1]$, vanishing at the boundary, and $J(x) \equiv 1$ for $x \in [-\varepsilon, \varepsilon]$. For large n, we then have $E_\theta[f\phi_n] = E_\theta[f\sum_1^{|\Lambda|}(\eta_i - \theta)] \ne 0$. However,

$$\begin{aligned}
2D_\theta(\phi_n) &= E_\theta\left[c(\eta_i)\sum\left(\phi_n(\eta^{ii+1}) - \phi_n(\eta)\right)^2\right]\\
&= E_\theta\left[\sum c(\eta_i)\left(J\left(\frac{i+1}{2n+1}\right) - J\left(\frac{i}{2n+1}\right)\right)^2\right]\\
&\cong C\sum_{-n}^{n}\frac{1}{(2n+1)^2}J'\left(\frac{i}{2n+1}\right)\\
&\to 0 \quad \text{as } n \to \infty,
\end{aligned}$$

which would contradict admissibility. Therefore, the second condition in (9) is also necessary.

Step 3. Again let f be admissible. We have then, from Steps 1 and 2, that $E_\theta[f] = 0$ and $E_\theta[f\Sigma_1^{|\Lambda|}(\eta_i - \theta)] = 0$. However, suppose that $E_\theta[f\Sigma_1^{|\Lambda|}(\eta_i - \theta)^2] \neq 0$. In this case, we may choose

$$\phi_n = \sum_{|i|, |l| \le n} J\left(\frac{i}{2n+1}, \frac{l}{2n+1}\right)(\eta_i - \theta)(\eta_l - \theta)$$

with the condition that J is a test function, $J \equiv 1$ in a small ball around the origin.

Then by independence of coordinates, $E_\theta[f\phi_n] = E_\theta[f(\Sigma^{|\Lambda|_1}(\eta_i - \theta))^2] \neq 0$ for large n. However, after some calculation,

$$\begin{aligned}
2D_\theta(\phi_n) &= E_\theta\Big[\sum c(\eta_i)\big(\phi_n(\eta^{ii+1}) - \phi_n(\eta)\big)^2\Big] \\
&\le CE_\theta\left[\sum_i c(\eta_i)\left\{\sum_l \left(J\left(\frac{i+1}{2n+1}, \frac{l}{2n+1}\right)\right.\right.\right. \\
&\qquad\qquad \left.-J\left(\frac{i}{2n+1}, \frac{l}{2n+1}\right)\right)^2 (\eta_l - \theta)^2 \\
&\qquad \left.\left. + \sum_j \left(J\left(\frac{j}{2n+1}, \frac{i+1}{2n+1}\right) - J\left(\frac{j}{2n+1}, \frac{i}{2n+1}\right)\right)^2 (\eta_j - \theta)^2\right\}\right] \\
&\cong C\int |\nabla J|^2\, dx.
\end{aligned}$$

By varying J, $D_\theta(\phi_n)$ may be made as small as desired (because two-dimensional Brownian motion is transient). This violates our admissibility assumption on f. Consequently, we conclude that the third condition in (9) is also necessary.

This concludes the proof of "necessity" and therefore the proof of Theorems 1.1 and 1.2 for the exclusion and zero-range models. □

3. Case of nearest-neighbor simple exclusion process. In Example 1.2, we showed that $f(\eta) = \eta_1 - 1/2$ is not admissible for the exclusion process in dimension $d = 1$ by constructing a sequence of functions $\{\phi_n(\eta)\}$ for which (2) cannot be satisfied for any constant $c(f)$. We can demonstrate the inadmissibility of f in a different way by contradicting condition (3) instead. This is accomplished by computing explicitly the correlations $E_{1/2}[f(\eta(s))f(\eta(0))]$ to determine that they are not integrable. A more expansive version of this method of computation will enable us to prove Theorem 1.1 for the nearest-neighbor symmetric simple exclusion process. Our proof here makes strong use of the nearest-neighbor condition and the form of the state space $\Sigma = \{0, 1\}^{Z^d}$.

Let us now introduce the dual process for the nearest-neighbor exclusion process. Due to the structure of Σ, the following set is an orthogonal basis for $L^2(\Sigma, P_{1/2})$:

$$L^2(\Sigma, P_{1/2}) = \operatorname{span}\Big\{1, \big\{(\eta_i - \tfrac{1}{2}) \colon i \in Z^d\big\}, \big\{(\eta_i - \tfrac{1}{2})(\eta_j - \tfrac{1}{2}) \colon i \neq j\big\}, \dots\Big\}.$$

The generator L restricted to the span of $\{(\eta_{i_1} - \frac{1}{2}) \cdots (\eta_{i_k} - \frac{1}{2}) \colon i_1 \neq i_2 \neq \cdots \neq i_k\}$ behaves as excluded random walk with k particles on the indices. This dual action, applied to the case $k = 1$, where L behaves as the generator for simple random walk, allows us to write

$$\begin{aligned} E_{1/2}[f(\eta(s))f(\eta(0))] &= E_{1/2}[f(\eta(0))T_s(\eta(0))] \\ &= E_{1/2}\big[(\eta_1(0) - \tfrac{1}{2})T_s(\eta_1(0) - \tfrac{1}{2})\big] \\ &= E_{1/2}\Big[(\eta_1(0) - \tfrac{1}{2}) \sum_{j \in Z} P_{1j}(s)(\eta_j(0) - \tfrac{1}{2})\Big], \end{aligned}$$

where T_s is the exclusion semigroup and $P_{xy}(s)$ are random walk probabilities. By independence of coordinates, it is easy to calculate the right-hand side as $(1/4)P_{11}(s)$.

Then the integral in (3) may be expressed as

$$\begin{aligned} \int_0^\infty E_{1/2}[f(\eta(0))f(\eta(s))]\,ds &= \tfrac{1}{4}\int_0^\infty P_{11}(s)\,ds \\ &= \infty \quad (\text{for } d = 1, 2) \\ &< \infty \quad (\text{for } d \geq 3), \end{aligned}$$

recalling that d-dimensional random walk is recurrent for $d = 1, 2$ and transient for $d \geq 3$. This argument will be extended to prove Theorem 1.1.

We first state the crucial lemma.

LEMMA 3.1. *Symmetric nearest neighbor simple exclusion on Z^d with k particles is recurrent in $d = 1$ for $k = 1, 2$ and in $d = 2$ for $k = 1$; otherwise the process is transient.*

PROOF. We shall prove the result for $d = 1$; the arguments for $d \geq 2$ are analogous. For one particle, $k = 1$, the claim is clear. We consider now the case $k = 2$.

Two particle simple exclusion on Z may be represented as a jump process on $\{(i, j) \colon i > j\}$, where the point (i, j) corresponds to the positions of the particles on the integer line. Two particle exclusion in this characterization is a nearest-neighbor process whose jump rates all equal $1/2$ irrespective of the starting point. In order to demonstrate recurrence, we will show that this process is a random function of recurrent two-dimensional random walk.

Let us call the points $\{(m + 1, m) \colon m \in Z\}$ as boundary points and the others as interior points. The distinction between these types is that boundary points have two neighbors and interior points have four. Define the mapping Ψ: $Z^2 \to \{(i,j)\colon i > j\}$, where

$$\Psi(i,j) = \begin{cases} (i,j), & \text{if } i > j, \\ (j+1, i-1), & \text{otherwise.} \end{cases}$$

This map is reflection across the line $\{(m+1, m)\colon m \in Z\}$. Let $x(t)$ be two-dimensional random walk with generator L^{RW}. Then $\Psi(x_t)$ is a Markov process on $\{(i,j)\colon i > j\}$. Let us compute the generator $\tilde{L}$ of the process. Write for a test function h,

$$\begin{aligned}\tilde{L}h &= L^{\mathrm{RW}} h(\Psi(i,j)) \\ &= \tfrac{1}{4}\big[h(\Psi(i+1,j)) + h(\Psi(i-1,j)) + h(\Psi(i,j+1)) \\ &\qquad\qquad + h(\Psi(i,j-1))\big] - h(\Psi(i,j)).\end{aligned}$$

When (i,j) is an interior point, the right-hand side is evaluated as

$$\begin{aligned}&\tfrac{1}{4}\big[h(\Psi(i,j) + (1,0)) + h(\Psi(i,j) + (-1,0)) \\ &\quad + h(\Psi(i,j) + (0,1)) + h(\Psi(i,j) + (0,-1))\big] - h(\Psi(i,j)).\end{aligned}$$

If (i,j) is a boundary point, $(i,j) = (m+1, m)$, then we calculate the right-hand side as

$$\begin{aligned}&\tfrac{1}{2}\big[h(\Psi(m+1,m) + (0,1)) + h(\Psi(m+1,m) + (1,0))\big] \\ &\quad - h(\Psi(m+1,m)).\end{aligned}$$

Hence, for interior points, $\tilde{L}$ gives rates $1/4$ to its four neighbors; for boundary points, $\tilde{L}$ gives rates $1/2$ to each of two neighbors. As remarked earlier, however, two particle exclusion gives rates $1/2$ to all neighbors irrespective of interior or boundary point distinctions, but we notice both $\Psi(x(t))$ and simple exclusion share the same embedded jump probabilities. This suggests introducing the following random time change. Define v: $Z^2 \to R^1$ by

$$v(i,j) = \begin{cases} 1, & \text{if } (i,j) \text{ is a boundary point,} \\ 1/2, & \text{otherwise.} \end{cases} \tag{16}$$

For a fixed trajectory $x(\cdot)$ let

$$A(t) = \int_0^t v(\Psi(x(s)))\, ds.$$

This function increases; hence $A^{-1}(t)$ exists. We claim now that $y(t) = \Psi(x(A^{-1}(t)))$ is two particle exclusion. This claim is established by verifying that $y(t)$ and the exclusion process share the same generator. We will make use of the fact that $\phi(z(t)) - \int_0^t L\phi(z(s))\, ds$ is a martingale for test functions ϕ if and only if $z(t)$ is a Markov process with generator L. Because $\Psi(x(\tau))$ is a Markov process with generator $\tilde{L}$, we can write $\phi(\Psi(x(\tau))) -$

$\int_0^\tau \tilde{L}\phi(\Psi(x(s)))\,ds$ as a martingale. Let $\tau = A^{-1}(t)$. Then by changing variables,

$$\phi(\Psi(x(\tau))) - \int_0^\tau L^{\mathrm{RW}}\phi(\Psi(x(s)))\,ds$$
$$= \phi\big(\Psi\big(x\big(A^{-1}(t)\big)\big)\big) - \int_0^{A^{-1}(t)} L^{\mathrm{RW}}\phi\big(\Psi\big(x\big(A^{-1}A(s)\big)\big)\big)\,ds$$
$$= \phi(y(t)) - \int_0^t \tilde{L}\phi(y(u))\frac{1}{v(y(u))}\,du \quad \text{if } u = A(s).$$

This implies that $y(t)$ is a Markov process with generator

$$L\phi(y) = \frac{\tilde{L}\phi(y)}{v(y)}.$$

The time change factor $v(\cdot)$ adjusts the rates so that L is the simple exclusion generator. This establishes the recurrence of two particle exclusion.

To show the transience of three or more particle simple exclusion on one dimension, we use the following potential theoretic technique. Let L be the generator of a Markov process $x(t)$. If there exists a function f defined on the state space, such that

(17) $\quad (Lf)(x) \le 0 \quad \text{for } |x| \text{ large}, \qquad f(x) > 0 \quad \text{and} \quad f(x) \to 0 \text{ for } |x| \to \infty,$

then $x(t)$ is transient.

This is seen as follows. Let $B_r = \{|x| < r\}$ and $A_{r,R} = \{|x| > r \text{ and } |x| < R\}$. Let τ_R be the exit time from the annulus $A_{r,R}$. Then by Doob's theorem and Fatou's lemma,

$$E_y f(x(\tau_R)) - E_y\int_0^{\tau_R} Lf(x(s))\,ds \le f(y).$$

Now (17) allows us to write, for a given point $y \in A_{r,R}$, that

$$f(y) \ge E_y f(x(\tau_R))$$
$$\ge \inf_{x\in B_r} f(x) P_y\{\tau_R < \infty, x(\tau_R) \in B_r\}.$$

Allowing $R \to \infty$, by Fatou's lemma again, we then conclude, if $x(t)$ is recurrent, that there exists an $\varepsilon > 0$ so that $f(y) > \varepsilon$. However, this prevents f from vanishing at ∞. Hence the chain must be transient.

Therefore, to prove transience of three or more particle simple exclusion, we need only exhibit an f satisfying (17). Let

$$f(m) = \big(1 + |m|^2\big)^{(2-k)/2},$$

where $m = (m_1, \dots, m_k)$ is the ordered position of k particles. Clearly f is positive and vanishes at ∞. To finish, we will demonstrate that $(Lf)(m) \le 0$ for $|m|$ large.

For interior points m, calculate

$$(Lf)(m) = \sum_{i=1}^{k} \left[\tfrac{1}{2} f(m_1, \dots, m_i + 1, \dots, m_k) - f(m) + \tfrac{1}{2} f(m_1, \dots, m_i - 1, \dots, m_k)\right].$$

It is not difficult to show this quantity is negative for $|m|$ large and $k \geq 3$.

We may represent boundary points m as $m = (s_1, \dots, s_\nu)$, where the s_i are strings of consecutive integers, representing ν clumps of adjacent particles. Let l_i be the length of the ith string; we note that l_i may be 1. Let us denote $s_i + 1$ to be the new string obtained from s_i by shifting the rightmost particle by 1 to the right. The string $s_i - 1$ is analogously defined. Then

$$(Lf)(m) = \sum_{i=1}^{v} \left[\tfrac{1}{2} f(s_1, \dots, s_i + 1, \dots, s_\nu) - f(m) + \tfrac{1}{2} f(s_1, \dots, s_i - 1, \dots, s_\nu)\right]$$

also is negative for large $|m|$ and $k \geq 3$. We omit the details. This concludes the proof of the lemma. □

REMARK 3.1. If we denote P^{RW} as the transition probability for random walk, it is clear that one particle simple exclusion is a random walk and that its transition probabilities behave accordingly: In dimension $d = 1$, standard local central limit theorems give

$$|\sqrt{2\pi}\, \sigma_p P_{ij}^{\mathrm{RW}}(s) - s^{-1/2}| < C s^{-3/2},$$

where σ_p^2 is the variance of the jump probability p and $C = C(|j - i|)$ is a finite constant depending on the moments of p.

Two particle simple exclusion transition probabilities behave somewhat differently. Formally we know $P_{lm}^{\mathrm{SE}}(s) = \sum_{\Psi(m^*)=m} P_{lm^*}^{\mathrm{RW}}(A^{-1}(s))$. Also, we note that $t \leq A(t) \leq 2t$, so, to first order, ignoring the usual normalization $2\pi\sigma_p$, we have

$$P_{lm}^{\mathrm{SE}}(s) \sim \begin{cases} 2\left(A^{-1}(s)\right)^{-1} + O(s^{-3}), & \text{if } m \text{ is an interior point,} \\ \left(A^{-1}(s)\right)^{-1} + O(s^{-3}), & \text{if } m \text{ is a boundary point.} \end{cases}$$

This remark will be used in what follows. We now prove Theorem 1.1 in the case of nearest-neighbor symmetric interactions.

PROOF OF THEOREM 1.1. We prove the result for $d = 1$; arguments for higher dimensions are analogous. Let $f(\eta)$ be supported only on a finite number of coordinates, $\Lambda \subset Z$. We will show that conditions (9) are equivalent to the condition (3) to complete the proof. Fix the invariant measure as the Bernoulli product measure P_θ. Because each coordinate is either occupied or not, that is, takes values either 1 or 0, we may represent $f(\eta)$ as a polynomial in the dual basis:

$$f(\eta) = c + \sum_{i\in\Lambda} c_i(\eta_i - \theta) + \sum_{(i,j)\in\Lambda^2} c_{ij}(\eta_i - \theta)(\eta_j - \theta)$$
$$+ \text{finite sum of higher order terms.}$$

A simple computation, noting the independence of coordinates η_i, reduces (9) to the following conditions on the coefficients:

$$(18)\quad E_\theta\left[f(\eta)\left(\sum_1^{|\Lambda|}(\eta_i - \theta)\right)^n\right] = 0 \quad \text{for } n = 0,1,2 \Leftrightarrow c, \sum c_i, \sum c_{ij} = 0.$$

We now compute the correlations. Due to the orthogonality of the dual basis and the duality relation, we have

$$\begin{aligned}
(19)\quad & E_\theta[f(\eta(0))f(\eta(s))] \\
&= c^2 + E_\theta\left\{\left[\sum c_i(\eta_i(0) - \theta)\right]\left[\sum c_j(\eta_j(s) - \theta)\right]\right\} \\
&\quad + E_\theta\left\{\left[\sum c_{ij}(\eta_i(0) - \theta)(\eta_j(0) - \theta)\right]\right. \\
&\qquad\qquad \left.\times\left[\sum c_{lm}(\eta_l(s) - \theta)(\eta_m(s) - \theta)\right]\right\} \\
&\quad + E_\theta[H(0)H(s)] \\
&= c^2 + \sum c_i c_j E_\theta\left[(\eta_i(0) - \theta)(\eta_j(s) - \theta)\right] \\
&\quad + \sum c_{ij}c_{lm}E_\theta\left[(\eta_i(0) - \theta)(\eta_j(0) - \theta)(\eta_l(s) - \theta)(\eta_m(s) - \theta)\right] \\
&\quad + E_\theta[H(0)H(s)] \\
(20)\quad &= c^2 + \tfrac{1}{2}\left(\theta^2 + (1-\theta)^2\right)\sum c_i c_j P_{ij}(s) \\
&\quad + \tfrac{1}{4}\left(\theta^2 + (1-\theta)^2\right)^2 \sum c_{ij}c_{lm}P_{(i,j)(l,m)}(s) + E_\theta[H(0)H(s)],
\end{aligned}$$

where H denotes the "finite sum of higher order terms." We observe that $E_\theta[H(0)H(s)]$ is a finite linear combination of transition probabilities of three or more particle simple exclusion.

To finish the proof, we verify that (20) is integrable if and only if the conditions (18) hold. Clearly we must have $c = 0$. In addition, Lemma 3.1 implies that $E_\theta[H(0)H(s)]$, composed of a finite linear combination of transition probabilities from transient chains, is integrable. The remaining second and third terms, however, correspond to recurrent chains and bear further scrutiny.

Ignoring the constant factor, the second term is given by

$$\sum c_i c_j P_{ij}^{\mathrm{RW}}(s) = \sum c_i c_j\left(P_{ij}^{\mathrm{RW}}(s) - \frac{1}{\sqrt{2\pi}\,\sigma_p\sqrt{s}}\right) + |\Lambda|\left(\frac{1}{\sqrt{2\pi}\,\sigma_p\sqrt{s}}\right)\left(\sum c_i\right)^2.$$

Remark 3.1 allows us to conclude, therefore, because the $P_{ij}^{\mathrm{RW}}(s)$ are transition probabilities for one-dimensional random walk, that the right-hand side is integrable if and only if $\sum c_i = 0$.

The third term is reconfigured after the calculation

$$\int_0^t P_{(i,j)(l,m)}^{\mathrm{SE}}(s)\,ds = E_{(i,j)}^{\mathrm{RW}} \int_0^t I\big(\Psi\big(x\big(A^{-1}(s)\big)\big) = (l,m)\big)\,ds$$
$$= E_{(i,j)}^{\mathrm{RW}} \int_0^{A^{-1}(t)} I(\Psi(x(s)) = (l,m))\,v(l,m)\,ds,$$

where E^{RW} is expectation with respect to P^{RW}. This allows us to compute

$$\lim_{t\to\infty} \int_0^t \sum c_{ij} c_{lm} P_{(i,j)(l,m)}(s)\,ds$$
$$= \lim_{t\to\infty} \sum c_{ij} c_{lm} E_{(i,j)}^{\mathrm{RW}} \int_0^{A^{-1}(t)} I(\Psi(x(s)) = (l,m))\,v(l,m)\,ds.$$

The observation that $t \le A^{-1}(t) \le 2t$, Remark 3.1 and the definition of v enable us to conclude that the left-hand side is finite if and only if $\sum c_{ij} = 0$.

This concludes the proof of Theorem 1.1 for the nearest-neighbor symmetric simple exclusion process. □

4. Extension to a generalized exclusion process. The important features of the zero-range and simple exclusion dynamics used in the proof of Theorems 1.1 and 1.2 given in Section 2 are certain regularity properties for the product invariant measures and the finite volume spectral gap estimate (A1). Such regularity is, however, available for more general measures, for example, Markovian measures. In addition, the required spectral gap bound is available for processes whose invariant measures are Gibbs states (see Remark 1.1). These observations allow us to prove Theorem 1.3 for the one-dimensional nearest-neighbor exclusion process with Markovian invariant measures by following the arguments given in Section 2.

PROOF OF THEOREM 1.3.

Sufficiency. As in Section 2, define the conditional expectation $g_{n,a,b}$ by

$$g_{n,a,b}(y) = E_0\left[f \,\middle|\, \frac{1}{2n+1}\sum_{-n}^{n} \eta_i = y,\, \eta_{-n-1} = a,\, \eta_{n+1} = b\right],$$

where $a, b \in \{-1, 1\}$. Our definition here differs slightly from the product invariant measure case in that we condition additionally with respect to the nearest-neighbor boundary coordinates η_{-n-1}, η_{n+1} in order to ensure ergodicity of the finite coordinate process.

Because $E_0[g_{n,a,b} - g_{n+1,a,b} \mid \sum_{i=-n-1}^{n+1} \eta_i, \eta_{-n-2}, \eta_{n+2}] = 0$, we may apply the same initial arguments for sufficiency given in Section 2 to the generalized exclusion process. Recall that we need only show that $\|g_n\|_2$ is $O(n^{1+\varepsilon})$ for some $\varepsilon > 0$ [where $\|\cdot\|_2$ is the $L^2(P_0)$-norm] to finish the proof of sufficiency. A modification of the proof of Lemma 2.1 is now used to prove the following lemma.

LEMMA 4.1. *Let $f(\eta)$ be a local function supported in $\Lambda = \{1, 2, \dots, |\Lambda|\} \subset Z^d$. Under the assumptions on f in Theorem* 1.3 *and any $\varepsilon > 0$, we have that*

$$\|g_n(y, a, b)\|_2 \le K\|f\|_2 n^{-(3/2-\varepsilon)}$$

for $K = K(|\Lambda|)$, a finite constant.

PROOF.

Step 1. Let $y = (2n+1)^{-1}\sum_{-n}^{n} \eta_i$ and $\eta_{I(n,\gamma)} = (\eta_{-n^\gamma}, \dots, \eta_{n^\gamma})$, where $0 < \gamma < 1/2$ will be chosen later.

Recall the notation from Section 1. Through Cramér's trick or the Edgeworth expansion, as in Section 2, we may express $g_n(y, a, b)$ as

$$\begin{aligned} &E_0\Bigg\{ f \exp\Bigg(\lambda \sum_{-n^\gamma}^{n^\gamma} \eta_i\Bigg) E_0\Bigg[\exp\Bigg|\Bigg(\lambda \sum_{i \notin I(n,\gamma)} \eta_i\Bigg)\Bigg| \eta_{I(n,\gamma)}, a, b\Bigg] \\ &\qquad \times \Theta^{I(n,\gamma)}_{n,a,b,\lambda}\Bigg(-\sum_{-n^\gamma}^{n^\gamma} (\eta_i - y)\Bigg) \mid a, b\Bigg\} \\ &\quad \times \Bigg\{\sqrt{1 - (2n^\gamma + 1)/(2n+1)}\, E_0\Bigg[\exp\Bigg(\lambda \sum_{-n}^{n} \eta_i\Bigg)\Bigg| a, b\Bigg] \Theta_{n,a,b,\lambda}(0)\Bigg\}^{-1} \end{aligned} \tag{21}$$

where we define

$$\Theta^{I(n,\gamma)}_{n,a,b,\lambda}(x) = \sqrt{2(n - n^\gamma)}\, P_0^\lambda\Bigg\{\sum_{i \notin I(n,\gamma)} (\eta_i - y) = x \Bigg| \eta_{I(n,\gamma)}, a, b\Bigg\}$$

and

$$\Theta_{n,a,b,\lambda}(x) = \sqrt{2n+1}\, P_0^\lambda\Bigg\{\sum_{-n}^{n} (\eta_i - y) = x \Bigg| a, b\Bigg\}.$$

Step 2. Call $n' = 2n - 2n^\gamma$. By choosing $\lambda = \lambda(y)$ so that $m(\lambda(y)) = y$, the quantity $\Theta_{n,a,b,\lambda}$ satisfies a local central limit theorem (see [7]) for Markovian variables:

$$\lim_{n \to \infty} \Theta_{n,a,b,\lambda}(x) = \frac{1}{\sqrt{2\pi}\,\sigma(y)} \exp\Bigg\{\frac{-\bar{x}^2}{2\sigma^2(y)}\Bigg\},$$

where $\sigma^2(y)$ is the limiting variance with respect to the measure P_0^λ. Hence, for $y \in (-1, 1)$, we have the lower bound

$$\Theta_{n,a,b,\lambda}(0) > C > 0.$$

As in Section 2, we may develop $\Theta^{I(n,\gamma)}_{n,a,b,\lambda}(x)$ in powers of $x/\sqrt{n'}$ by expanding the characteristic function

$$\Psi_n(t) = E_0\Bigg[\exp\Bigg\{it \sum_{i \notin I(n,\gamma)} (\eta_i - y)\Bigg\}\Bigg| \eta_{I(n,\gamma)}, a, b\Bigg].$$

We may write

$$\Theta_{n,a,b,\lambda}^{I(n,\gamma)}(x) = \sqrt{n'}\int_{-\pi}^{\pi} e^{itx}\Psi_n(t)\,dt.$$

Expanding $\exp\{itx\}$, we have, as before,

$$\begin{aligned}\Theta_{n,a,b,\lambda}^{I(n,\gamma)}(x) &= \int_{-\pi\sqrt{n'}}^{\pi\sqrt{n'}} \Psi_n(t)\,dt + \frac{ix}{\sqrt{n'}}\int_{-\pi\sqrt{n'}}^{\pi\sqrt{n'}} t\Psi_n(t)\,dt \\ &\quad + \frac{-x^2}{n'}\int_{-\pi\sqrt{n'}}^{\pi\sqrt{n'}} t^2\Psi_n(t)\,dt + r(x)n^{-3/2},\end{aligned} \tag{22}$$

where $r(\cdot)$ expresses the error.

It is now a messy computation to work out (by diagonalizing 2×2 transfer matrices, for example) that, for y small and t small, $|t| < \varepsilon$, we have

$$|\Psi_n(t)| < \exp\{-Ct^2\}$$

for some $C > 0$. At the same time, for y small, for $|t| > \varepsilon$ we have the bound

$$|\Psi_n(t)| < (K(\varepsilon))^n < 1$$

for $K(\varepsilon) < 1$. These estimates allow us to control the error $|r(x)| < C|x|^3$.

Step 3. Let $J = [-\delta, \delta]$ for δ small. As in Section 2, we estimate $\|g_{n,a,b}\|_2$ as

$$\begin{aligned}\|g_{n,a,b}(y)\|_2 &\le \{E_0[g_{n,a,b}(y)I(y \in J)]\}^{1/2} + \{E_0[g_{n,a,b}(y)I(y \notin J)]\}^{1/2} \\ &= F_1 + F_2.\end{aligned}$$

The second term F_2, as before for the zero-range process, is bounded through large deviation estimates or direct computation:

$$\|g_{n,a,b}(y)I(y \notin J)\|_2 < \|f\|_2 n^{-3/2}.$$

Step 4. Our strategy to estimate the first term F_1, similar to that employed in Section 2, is to substitute the expansion and lower bound determined for $\Theta_{n,a,b,\lambda}(\cdot)$ in Step 2 into (21), obtaining a more refined expression for $g_{n,a,b}$ to be treated through the conditions (8). We calculate that $g_{n,a,b}$ equals

$$\begin{aligned}&E_0^\lambda\left[f(\eta)\left\{c_0 + \frac{c_1}{\sqrt{n}}\sum_{i\in I(n,\gamma)}\eta_i + \frac{c_2}{n}\left(\sum_{i\in I(n,\gamma)}\eta_i\right)^2 + r(\eta)n^{-3/2}\right\}\middle| a, b\right] \\ &\quad = V_0 + V_1 + V_2 + V_3,\end{aligned}$$

where the constants $c_i = c_i(n, y, \eta_{-n^\gamma}, \eta_{n^\gamma}, a, b)$ are uniformly bounded for $y \in J$.

Step 5. We now bound the norms $\|V_i I(y \in J)\|_2$ to estimate F_1. Recalling the error bound for $r(\eta)$ and the size of $I(n,\gamma)$, the last term is bounded as

$$\{E_0[V_3^2 I(y \in J)]\}^{1/2} \le C\|f\|_2 n^{-3/2+3\gamma}.$$

The norms $\|V_i I(y \in J)\|_2$ for $i = 0, 1, 2$, as before in Section 2, will be analyzed through Taylor expansion in y. Each y factor in the expansion, by the central limit theorem for finite-state Markov chains, represents a decay of $n^{-1/2}$. We will show that $\|V_i I(y \in J)\|_2$ decays as fast as $n^{-3/2+3\gamma}$.

After regrouping covariance terms resulting in the expansion (which vanish for product invariant measures), we have

$$\begin{aligned} V_0 = {} & C_0 E_0[K_0 f(\eta) \mid a, b] + C_1 E_0\left[K_1 f(\eta) \sum_{-n}^{n} \eta_i \middle| a, b\right] y \\ & + C_2 E_0\left[K_2 f(\eta)\left(\sum_{-n}^{n} \eta_i\right)^2\right] y^2 + r(n, z, a, b) y^3, \end{aligned} \tag{23}$$

where r is given in terms of $f^1 = K_3 f - E_0^{\lambda(z)}[K_3 f \mid a, b]$ and $\kappa(\eta) = \sum_{-n}^{n} \eta_i$ by

$$E_0^{\lambda(z)}\Big[\Big\{\big(f^1\kappa - E_0^{\lambda(z)}[f^1\kappa \mid a, b]\big)\kappa - E_0^{\lambda(z)}\big[\big(f^1\kappa - E_0^{\lambda(z)}[f^1\kappa \mid a, b]\big)\kappa\big]\Big\}\kappa \mid a, b\Big]$$

and $|z| \le |y|$. The "constants" K_i are uniformly bounded and depend on the boundary of $I(n, \gamma)$ as well as a and b; the "constants" $C_i = C_i(n, y)$ involve quantities such as $yE_0[\sum_{-n}^{n}\eta_i \mid a, b] \le n^\gamma y$ (from splitting the sum into two parts: one away from the boundary and one near) and are uniformly bounded in L^2.

We now apply our assumptions (8), standard Markov chain estimates and conditioning on the boundary of $I(n, \gamma)$ (to pull out the constants K_i) to show the rapid decay of the first three terms in (23).

The difference between $E_0[f \mid \eta_{I(n,\gamma)^c}, a, b]$ and $E_0[f]$ clearly decays exponentially in $n^\gamma - |\Lambda|$.

To estimate $E_0[f \cdot \sum_{-n}^{n}\eta_i \mid \eta_{I(n,\gamma)^c}, a, b]$, split the sum in the expectation into sums over $I(n, \gamma, \varepsilon) = \{-n^\gamma\varepsilon, \dots, n^\gamma\varepsilon\}$ for $0 < \varepsilon < 1$ and its complement. The resulting first term is different from its respective unconditioned expectation exponentially by an amount small in n^γ. The unconditioned expectation differs from $E[f \cdot \sum_{-\infty}^{\infty} \eta_i]$ by $\|f\|_\infty$ times an exponentially decaying factor in n^γ. The second resulting term is bounded by $CnE_0[f \mid \eta_{-n^\gamma\varepsilon}, \eta_{n^\gamma\varepsilon}]$, which also decays exponentially by the previous discussion for the first term in (23).

By similar arguments, by splitting the region of summation several times, the third term in (23) is shown to decay exponentially.

To bound the remainder term, recall that $\|f^1\|_\infty < C$ and that the difference between $P_0^{\lambda(z)}\{\eta_{I(n,\gamma)} \mid \eta_{I(n,\gamma,\alpha)^c}, a, b\}$ and $P_0^{\lambda(z)}\{\eta_{I(n,\gamma)} \mid a, b\}$, for $\alpha > 1$, decays exponentially. Now by splitting the sums in the remainder into sums over $I(n, \gamma, \alpha)$ and its complement for some fixed $\alpha > 1$, then by carefully conditioning on $I(n, \gamma, \alpha)$ and using the bounds above on f^1 and the probabilities, we may show that $r(n, z, a, b)$ is $O(n^{3\gamma})$. Hence, the L^2-norm of the remainder is $O(n^{-3/2+3\gamma})$. We omit the details.

Because the terms $\|V_1\|_2$ and $\|V_2\|_2$ possess factors $n^{-1/2}$ and n^{-1}, respectively, we need only take the Taylor expansion in y up to second and first order. Under the assumptions (8), the terms in these expansions decay, by similar arguments given for the terms in (23), $O(n^{-3/2+3\gamma})$.

We now have that

$$\|g_{n,a,b} I(y \in J)\|_2 \le C n^{-3/2+3\gamma}.$$

Choose γ and α appropriately to complete the proof of Lemma 4.1 and, consequently, the "sufficiency" part of Theorem 1.3 for the generalized exclusion process. □

As in Section 2, a similar proof of the lemma gives the stronger result in dimension $d = 1$:

LEMMA 4.2. *Let $f(\eta)$ be a function depending on a finite set of coordinates η_Λ. Then, given that*

$$E_0\left[f\left(\sum_{i=-\infty}^{\infty} \eta_i\right)^j\right] = 0 \quad \text{for } j = 0, 1, \dots, k,$$

we have, for any $\varepsilon > 0$, that

$$\left\| E_0\left[f \,\middle|\, \frac{1}{2n+1} \sum_{i=-n}^{n} \eta_i, \eta_{-n-1}, \eta_{n+1}\right]\right\|_2 \le C n^{-(k+1)/2+\varepsilon},$$

where $C = C(|\Lambda|)$ is a finite constant.

We now prove the necessity of the conditions (8).

Necessity. We show that the admissibility conditions (8) for the generalized exclusion process are required. As in Section 2, this is accomplished by contradiction.

Step 1. Let f be admissible and suppose that $E_0[f] \neq 0$. Then with the choice of $\phi(\eta) = 1$, we note that $E_0[f\phi] \neq 0$, but that $D_0(\phi) = 0$. This contradicts the admissibility of f; hence the first condition of (8) is necessary.

Step 2. Let f be again admissible. We know, therefore, that $E_0[f] = 0$. However, suppose that $E_0[f \sum_{i=-\infty}^{\infty} \eta_i] \neq 0$. In this case we may specify the sequence

$$\phi_n(\eta) = \sum_{i=-n}^{n} J\left(\frac{i}{2n+1}\right)\eta_i,$$

where J is as before in Section 2. Clearly, $E_0[f\phi_n] \to E_0[f \sum_{-\infty}^{\infty} \eta_i] \neq 0$. However,

$$\begin{aligned} D_0(\phi_n) &= E_0\left[\sum_{-n}^{n} \left(J\left(\frac{i+1}{2n+1}\right) - J\left(\frac{i}{2n+1}\right)\right)^2 (\eta_{i+1} - \eta_i)^2\right] \\ &\to 0 \quad \text{as } n \to \infty \text{ for any } J. \end{aligned}$$

We conclude the necessity of the second condition in (8).

Step 3. Let f be admissible once more. Steps 1 and 2 yield then that $E_0[f] = 0$ and that $E_0[f\sum_{i=-\infty}^{\infty}\eta_i] = 0$. Now suppose that $E_0[f(\sum_{-\infty}^{\infty}\eta_i)^2] \neq 0$. We may then consider the sequence

$$\phi_n(\eta) = \sum_{|i|,\,|l|\le n} J\left(\frac{i}{2n+1}, \frac{l}{2n+1}\right)\eta_i\eta_l$$

for a J as in Step 3 in the proof of necessity in Section 2. Immediately we have that $E_0[f\phi_n] \to E_0[f(\sum_{-\infty}^{\infty}\eta_i)^2] \neq 0$. However, by the Markov property and standard inequalities we can bound the Dirichlet form $D_0(\phi_n)$ by

$$\frac{C}{(2n+1)^2}\sum_i E_0\left[\left(\sum_l J_1\left(\frac{i}{2n+1}, \frac{l}{2n+1}\right)\eta_l\right)^2 \right.$$
$$\left. + \left(\sum_j J_2\left(\frac{j}{2n+1}, \frac{i}{2n+1}\right)\eta_j\right)^2 (\eta_i - \eta_{i+1})^2\right],$$

where the symbols J_1 and J_2 refer to partial derivation.

To evaluate this quantity, rewrite the first square as a double sum over indices, say l and m. The correlations $E_0[\eta_m\eta_l]$ vanish exponentially in the difference $|l-m|$. Now split the double sum over regions $|l-m| \le n^{1/2}$ and its complement. The second part now decreases faster than any polynomial. This procedure is repeated for the second square. Now, it is a small calculation to conclude that the Dirichlet form may be bounded by $C\int|\nabla J|^2\,dx$. By adjusting J, this can be made as small as possible, demonstrating the necessity of the third condition in (8).

This finishes the proof of "necessity" and consequently the proof of Theorem 1.3 for the generalized exclusion process. □

Acknowledgment. We would like to thank Professor S. R. S. Varadhan for suggesting the problem and for many helpful conversations, as well as his encouragement. Some of this work appears in the first author's Ph.D dissertation under Prof. Varadhan.

REFERENCES

[1] ANDJEL, E. D. (1982). Invariant measures for the zero range process. *Ann. Probab.* **10** 525–547.

[2] BILLINGSLEY, P. (1971). *Weak Convergence of Probability Measures: Applications in Probability*. SIAM, Philadelphia.

[3] ESPOSITO, R., MARRA, R. and YAU, H. T. (1994). Diffusive limit of asymmetric simple exclusion. *Reviews in Mathematical Physics* **6** 1233–1267.

[4] KIPNIS, C. (1986). Central limit theorems for infinite series of queues and applications to simple exclusion. *Ann. Probab.* **14** 397–408.

[5] KIPNIS, C. (1987). Fluctuations des temps d'occupation d'un site dans l'exclusion simple symmetrique. *Ann. Inst. H. Poincaré Probab. Statist.* **23** 21–35.

[6] KIPNIS, C. and VARADHAN, S. R. S. (1986). Central limit theorem for additive functionals of reversible Markov processes. *Comm. Math. Phys.* **104** 1–19.

[7] KOLMOROGOV, A. N. (1962). A local limit theorem for Markov chains. *Selected Translations in Mathematical Statistics and Probability* **2** 109–129.
[8] LANDIM, C., SETHURAMAN, S. and VARADHAN, S. (1996). Spectral gap for zero-range dynamics. *Ann. Probab.* **24** 1651–1682.
[9] LEBOWITZ, J. L. and SPOHN, H. (1982). A microscopical basis for Fick's law. *J. Statist. Phys.* **28** 539–556.
[10] LIGGETT, T. M. (1985). *Interacting Particle Systems.* Springer, New York.
[11] LU, S. L. and YAU, H. T. (1993). Spectral gap and logarithmic Sobolev inequality for Kawasaki and Glauber dynamics. *Comm. Math. Phys.* **156** 399–433.
[12] PETROV, V. V. (1975). *Sums of Independent Random Variables.* Springer, New York.
[13] QUASTEL, J. (1992). Diffusion of color in the simple exclusion process. *Comm. Pure Appl. Math.* **45** 623–679.
[14] SPITZER, F. (1970). Interaction of Markov processes. *Adv. Math.* **5** 142–154.
[15] SPOHN, H. (1991). *Large Scale Dynamics of Interacting Particles.* Springer, New York.
[16] VARADHAN, S. R. S. (1994). Nonlinear diffusion limit for a system with nearest neighbor interactions II. In *Asymptotic Problems in Probability Theory: Stochastic Models and Diffusion on Fractals* (K. Elworthy and N. Ikeda, eds.). Wiley, New York.

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