The Annals of Probability
1996, Vol. 24, No. 2, 788–824

MINIMA OF *H*-VALUED GAUSSIAN PROCESSES[1]

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We study low local extremes of Gaussian random fields with values in a separable Hilbert space and constant variance. Our results are sharp for certain stationary processes on the line and for these processes we also prove global limits.

1. Introduction. Let T be a separable topological space and let $(H, \langle\cdot|\cdot\rangle, \|\cdot\|)$ be a real separable Hilbert space with unit ball H_1 and bounded linear operators $\mathscr{L} = \mathscr{L}(H)$. Further let $\{X(t)\}_{t\in T}$ be a centered separable $\mathbf{P}$-continuous H-valued Gaussian random field, with respect to a complete probability $\mathbf{P}$, such that $X(t)$ has variance $R \in \mathscr{L}$ independent of t. Also let R have spectrum (eigenvalues) $\{\lambda_k\}_{k=1}^{\infty} \subseteq (0,\infty)$ and choose a complete orthonormal system $\{e_k\}_{k=1}^{\infty}$ satisfying $Re_k = \lambda_k e_k$.

In Section 3 we study the tail $\mathbf{P}\{\|X(s_0)\|^2 < \varepsilon\}$ as $\varepsilon \downarrow 0$ for a fixed $s_0 \in T$.

In Sections 4 and 5 we derive upper and lower bounds for

$$\mathbf{P}\Big\{\inf_{t\in S} \|X(t)\|^2 < \varepsilon\Big\} \quad \text{as } \varepsilon \downarrow 0, \text{ for } S \subseteq T,$$

and in Section 6 we give several applications of these results.

Let $\{\xi_k(t)\}_{t\in\mathbb{R}}$, $k = 1, 2, \ldots$, be independent $\mathbb{R}$-valued stationary standardized Gaussian processes and let $Y(t)$ be a separable version of $\sum_{k=1}^{\infty}\sqrt{\lambda_k}\,\xi_k(t)e_k$. In Section 7 we sharpen the results of Sections 4 and 5 and find the exact asymptotic behavior of $\mathbf{P}\{\inf_{t\in[0,1]}\|Y(t)\|^2 < \varepsilon\}$ under conditions on $r_k(t) \equiv \mathbf{E}\{\xi_k(0)\xi_k(t)\}$. In Section 8 we prove global limits for Y when $r_k(t) \to 0$ not too slowly as $t \to \infty$.

Low extremes for the stationary finite case (when the sequence $\{\lambda_k\}$ terminates) have been studied for $T = \mathbb{R}$ by, for example, Aronowich and Adler (1986) and Albin (1990, 1992a). The finite case behaves radically differently from than the infinite case.

Large values of $\|X\|$ were investigated for X an Ornstein–Uhlenbeck process on the line by, for example, Iscoe and McDonald (1989), Iscoe, Marcus, McDonald, Talagrand and Zinn (1990), Albin (1992b) and Csáki and Csörgő (1992). Albin (1992a) studied stationary $\mathbb{L}^2$-differentiable X. The (not so differently behaved) finite cases were first studied by Sharpe (1978).

2. Regular variation. For easy reference we now state some facts from the theory of regular variation which will be needed in the sequel. These facts

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Received November 1994; revised June 1995.

[1]Research supported by NFR Contracts F-FU 9207-304 and F-AA/MA 9207-307.

AMS 1991 *subject classifications*. Primary 60F10, 60G10, 60G15, 60G60, 60G70.

Key words and phrases. Extrema, Gaussian process, Gaussian random field, entropy, Esscher transform, stationary process, Ornstein–Uhlenbeck process, probability on Hilbert space.

can all be found (with proofs and notes on priority) in Bingham, Goldie and Teugels (1987). [Other excellent general references for this area are Geluk and de Haan (1987) and Resnick (1987).]

We study regular variation of a strictly positive function $f(x)$ both as $x \downarrow 0$ and as $x \to \infty$. Therefore, we use the symbol $x \to U$, where either $U = 0^+$ or $U = \infty$. We assume that f is defined on $(0, \infty)$ if $U = 0^+$ and on $(\hat{x}, \infty)$ for some $\hat{x} \geq 0$ if $U = \infty$. Further we write $I_U \equiv (0, 1]$ if $U = 0^+$, whereas $I_U \equiv [1, \infty)$ if $U = \infty$.

Let $I_U(\lambda) \equiv [\min\{1, \lambda\}, \max\{1, \lambda\}]$ for $\lambda \in I_U$ and define

$$\begin{aligned} f^\star(U;\lambda) &\equiv \limsup_{x\to U} f(\lambda x)/f(x),\\ f_\star(U;\lambda) &\equiv \liminf_{x\to U} f(\lambda x)/f(x),\\ \Psi^\star(U,f;\lambda) &\equiv \limsup_{x\to U} \sup_{\mu\in I_U(\lambda)} f(\mu x) \,/\, f(x),\\ \Psi_\star(U,f;\lambda) &\equiv \liminf_{x\to U} \inf_{\mu\in I_U(\lambda)} f(\mu x)/f(x) \end{aligned}$$

for $\lambda \in I_U$. We say that $f(x)$ is O-regularly varying as $x \to U$ if

$$0 < f_\star(U;\lambda) \leq f^\star(U;\lambda) < \infty \quad \text{for each } \lambda \in I_U.$$

The upper and lower Matuszewska indices $\alpha(f)$ and $\beta(f)$ are given by

$$\begin{aligned} &\alpha(U;f) \equiv \infty, && \text{if } \Psi^\star(U,f;\lambda) = \infty, \text{ for some } \lambda \in I_U,\\ &\alpha(U;f) \equiv \lim_{x\to U} \ln(f^\star(U;x))/|\ln(x)|, && \text{if } \Psi^\star(U,f;\lambda) < \infty \text{ for each } \lambda \in I_U,\\ &\beta(U;f) \equiv -\infty, && \text{if } \Psi_\star(U,f;\lambda) = 0 \text{ for some } \lambda \in I_U,\\ &\beta(U;f) \equiv \lim_{x\to U} \ln(f_\star(U;x))/|\ln(x)|, && \text{if } \Psi_\star(U,f;\lambda) > 0 \text{ for each } \lambda \in I_U. \end{aligned}$$

Writing $\hat{f}(x) \equiv f(1/x)$, it is then easily seen that

$$\alpha(U;\hat{f}\,) = \alpha(1/U;f), \qquad \beta(U;\hat{f}\,) = \beta(1/U;f), \qquad \beta(U;1/f) = -\alpha(U;f). \tag{2.1}$$

Further we have, by the Matuszewska indices theorem,

$$\begin{aligned} \alpha(U;f) < \infty \quad &\Leftrightarrow \quad \Psi^\star(U,f;\lambda) < \infty \quad \text{for each } \lambda \in I_U,\\ \beta(U;f) > -\infty \quad &\Leftrightarrow \quad \Psi_\star(U,f;\lambda) > 0 \quad \text{for each } \lambda \in I_U,\\ f \text{ is } O\text{-regularly varying as } x \to U&\\ &\Leftrightarrow \quad -\infty < \beta(U;f) \leq \alpha(U;f) < \infty. \end{aligned} \tag{2.2}$$

Let f be O-regularly varying and take $\nu \in (1, \infty)$ if $U = \infty$ and $\nu \in (-\infty, -1)$ if $U = 0^+$. Then there are constants $C_1 = C_1(\nu) > 1$ and $\tilde{x} = \tilde{x}(\nu) \in I_U$ such that

$$C_1(\nu)^{-1}\lambda^{\beta(f)/\nu} \le f(\lambda x)/f(x) \le C_1(\nu)\lambda^{\alpha(f)\nu} \quad \text{for } \lambda \in I_U \text{ and } x \in I_U - I_U(\tilde{x}). \tag{2.3}$$

We say that f has bounded increase if $\alpha(f) < \infty$, bounded decrease if $\beta(f) > -\infty$, positive increase if $\beta(f) > 0$ and positive decrease if $\alpha(f) < 0$.

Let f be nondecreasing and unbounded above and write $f^{\leftarrow}(x) \equiv \inf\{y > 0: f(y) > x\}$. Then we have [Bingham, Goldie and Teugels (1987), Exercise 2.12.8]

$$\begin{gathered} \alpha(\infty; f) = 1/\beta(\infty; f^{\leftarrow}) \quad \text{so that} \\ \alpha(\infty; f) < \infty \quad \Leftrightarrow \quad \beta(\infty; f^{\leftarrow}) > 0. \end{gathered} \tag{2.4}$$

The upper order $\rho(f)$ and lower order $\mu(f)$ of $f(x)$ as $x \to U$ are given by

$$\rho(U; f) \equiv \limsup_{x\to U} \ln(f(x))/|\ln(x)| \quad \text{and} \quad \mu(U; f) \equiv \liminf_{x\to U} \ln(f(x))/|\ln(x)|.$$

The relations with the Matuszewska indices are

$$\beta(U; f) \le \mu(U; f) \le \rho(U; f) \le \alpha(U; f). \tag{2.5}$$

We say that f belongs to the class Γ as $x \to U$ with auxiliary function $w: (0, \infty) \to (0, \infty)$ if f is nondecreasing and

$$\lim_{x\to U} f(x + yw(x))/f(x) = e^y \quad \text{for each } y \in \mathbb{R}.$$

In that case, for each $y \in \mathbb{R}$,

$$w \text{ is self-neglecting, that is,} \quad \lim_{x\to U} w(x + yw(x))/w(x) = 1, \tag{2.6}$$

$$g(x) \sim x + yw(x) \quad \Rightarrow \quad \lim_{x\to U} f(g(x))/f(x) = e^y. \tag{2.7}$$

For f in Γ we further have

$$\lim_{x\to U} f(\lambda x)/f(x) = 0 \quad \text{for } \lambda \in (0, 1), \tag{2.8}$$

$$\lim_{x\to U} x^\gamma f(x) = U \quad \text{for } \gamma \in \mathbb{R}. \tag{2.9}$$

3. The tail of $\|X(s_0)\|^2$. Fix an $s_0 \in T$ and write $W \equiv X(s_0)$. In Propositions 1 and 2 we adapt an argument of Davis and Resnick (1991) (D&R) to study aspects of the asymptotic behavior of $\mathbf{P}\{\|W\|^2 < \varepsilon\}$ as $\varepsilon \downarrow 0$. Since the emphasis of our work is on fields and processes rather than single random variables, we will only, as briefly as possible, state and prove results needed in later sections.

The idea of D&R is to note that a central limit argument should give the tail of $\|W\|^2$ since $\|W\|^2$ is small only when the independent components $\{\langle e_k|W\rangle\}_{k=1}^{\infty}$ all are small simultaneously. An adequate central limit theorem is then obtained using (Laplace transform related) Esscher transform techniques.

The Esscher transform of $\|W\|^2$ at $s \geq 0$ is a random variable $\|W\|_s^2$ with distribution $dF_{\|W\|_s^2}(x) = e^{-sx}\, dF_{\|W\|^2}(x)/\phi(s)$, where $\phi(s) \equiv \mathbf{E}\{\exp[-s\|W\|^2]\} = \prod_{l=1}^{\infty}(1+2\lambda_l s)^{-1/2}$. It follows that

$$m(s) \equiv \mathbf{E}\{\|W\|_s^2\} = \sum_{k=1}^{\infty} \frac{\lambda_k}{1+2\lambda_k s},$$

$$V(s) \equiv \mathrm{Var}\{\|W\|_s^2\} = \sum_{k=1}^{\infty} \frac{2\,\lambda_k^2}{(1+2\lambda_k s)^2}.$$

PROPOSITION 1. *Writing $q(\varepsilon) \equiv \inf\{s > 0 : m(s) < \varepsilon\}$, we have*

$$\begin{aligned}\mathbf{P}\left\{\|W\|^2 < \varepsilon + \frac{x}{q(\varepsilon)}\right\} &\sim (\exp x)\mathbf{P}\{\|W\|^2 < \varepsilon\} \\ &\sim \frac{(\exp x)\exp\{q(\varepsilon)\varepsilon\}\phi(q(\varepsilon))}{q(\varepsilon)\sqrt{2\pi\, V(q(\varepsilon))}} \quad \textit{as } \varepsilon \downarrow 0\end{aligned} \tag{3.1}$$

for each $x \in \mathbb{R}$. Writing $f_{\|W\|^2}$ for the density function of $\|W\|^2$ we further have

$$f_{\|W\|^2}(\varepsilon + x/q(\varepsilon)) \sim e^x q(\varepsilon)\mathbf{P}\{\|W\|^2 < \varepsilon\} \quad \textit{locally uniformly for } x \in \mathbb{R}. \tag{3.2}$$

PROOF. The only part of the argument for (3.1) and (3.2) in (D&R, Section 3) which does not carry over to our setting is the verification that

$$\lim_{s\to\infty} \int_{|t|>\delta s\sqrt{V(s)}} \left| \prod_{i=1}^{\infty} g_{\lambda_i s}\left(\frac{t}{s\sqrt{V(s)}}\right)\right| dt = 0 \quad \text{for } \delta > 0, \tag{3.3}$$

where $g_s(t) \equiv \mathbf{E}\{\exp[its([N(0,1)^2]_s - \mathbf{E}\{[N(0,1)^2]_s\})]\}$ and $N(0,1)$ denotes a standardized Gaussian random variable. The reason that D&R, proof of (3.3), does not carry over is that their conditions 3.7 and 3.8 do not hold.

Let $n(s) \equiv \{i\colon\ \lambda_i s \geq 1\}$ and choose $c > 0$ such that $\ln[1 + \frac{4}{9}\delta^2 x] \geq cx$ for $x \in [0,1]$. Since $\int_\delta^\infty (1+t^2)^{-\nu}\, dt \leq \frac{1}{2}\pi(1+\delta^2)^{1-\nu}$ for $\nu > 1$ and since $\lim_{s\to\infty} n(s) = \infty$, the fact that (3.3) holds then follows from noting that

$$\begin{aligned}&\int_{|t|>\delta s\sqrt{V(s)}} \prod_{i=1}^{\infty}\left| g_{\lambda_i s}\left(\frac{t}{s\sqrt{V(s)}}\right)\right| dt \\ &\quad = 2\int_{\delta s\sqrt{V(s)}}^{\infty} \exp\left\{-\frac{1}{4}\sum_{i=1}^{\infty}\ln\left[1+\frac{4\,\lambda_i^2 t^2}{V(s)(1+2\lambda_i s)^2}\right]\right\} dt \\ &\quad \leq 2\int_{\delta s\sqrt{V(s)}}^{\infty} \exp\left\{-\frac{n(s)}{4}\ln\left[1+\frac{4\,t^2}{9\,s^2 V(s)}\right]\right. \\ &\qquad\qquad \left. -\frac{1}{4}\sum_{\{i\colon\ \lambda_i s<1\}} \ln\left[1+\frac{4\,\lambda_i^2 t^2}{9\,V(s)}\right]\right\} dt\end{aligned}$$

$$\leq 3\,s\sqrt{V(s)}\left[\int_{2\delta/3}^{\infty}(1+t^2)^{-n(s)/4}\,dt\right]\exp\left\{-\frac{c}{4}\sum_{\{i:\ \lambda_i s<1\}}\lambda_i^2 s^2\right\}$$

$$\leq \frac{3\pi}{2}\left(\sum_{i=1}^{\infty}\frac{2\,\lambda_i^2 s^2}{(1+2\lambda_i s)^2}\right)^{1/2}\left[1+\frac{4}{9}\delta^2\right]^{1-n(s)/4}\exp\left\{-\frac{c}{4}\sum_{\{i:\ \lambda_i s<1\}}\lambda_i^2 s^2\right\}$$

$$\leq \frac{3\pi}{\sqrt{2}}\left[\frac{\sqrt{n(s)}}{2}+\sqrt{\sum_{\{i:\ \lambda_i s<1\}}\lambda_i^2 s^2}\right]\left[1+\frac{4}{9}\delta^2\right]^{1-n(s)/4}$$

$$\times\exp\left\{-\frac{c}{4}\sum_{\{i:\ \lambda_i s<1\}}\lambda_i^2 s^2\right\}. \qquad \square$$

REMARK 1. Ibragimov (1982) proved (3.1) and (3.2) for $x = 0$, but gave partial priority to G. N. Sytaya [Ibragimov (1982), page 2165]. Since we need (3.1) and (3.2) for $x \neq 0$, as well as some facts [e.g., (3.4) and (3.5)] concerning the quality of convergence provided by the method of D&R, the results of Ibragimov and Sytaya are not sufficient for us. (Note that Ibragimov's result is Gaussian, whereas that of D&R is general.)

PROPOSITION 2. *If m has positive decrease as $s \to \infty$, then $q(\varepsilon)$ is O-regularly varying as $\varepsilon \downarrow 0$. Further $W_k \equiv W - \sum_{l=1}^{k}\langle W|e_l\rangle e_l$ satisfies*

$$\mathbf{P}\{\|W_k\|^2 < \varepsilon\} \sim \left(\prod_{l=1}^{k}\sqrt{2\lambda_l}\right) q(\varepsilon)^{k/2}\mathbf{P}\{\|W\|^2 < \varepsilon\} \quad \text{as } \varepsilon \downarrow 0. \tag{3.4}$$

Moreover there exist constants $C_2(k)$, $\varepsilon_1(k) > 0$ such that

$$f_{\|W_k\|^2}(\varepsilon - x/q(\varepsilon)) \leq C_2\,q(\varepsilon)^{k/2+1}e^{-x/2}\mathbf{P}\{\|W\|^2 < \varepsilon\} \tag{3.5}$$

for $0 \leq x < q(\varepsilon)\varepsilon$ and $\varepsilon \in (0, \varepsilon_1]$.

PROOF. Applying (2.4) to $f \equiv (1/m)^{\leftarrow}$: $(1/m(0), \infty) \to (0, \infty)$ we get, by (2.1),

$$\alpha(0^+;\hat{f}) = \alpha(\infty; f) = \beta(\infty; f^{\leftarrow})^{-1} = \beta(\infty; 1/m)^{-1} = -\alpha(\infty; m)^{-1} < \infty.$$

Hence $q = \hat{f}$ has bounded increase. Since q (being nonincreasing) also has bounded decrease, (2.2) shows that $q(\varepsilon)$ is O-regularly varying.

Putting $m_k(s) \equiv \sum_{l=k+1}^{\infty}\lambda_l/[1+2\lambda_l s]$ and $\hat{q}_x(\varepsilon) \equiv q(\varepsilon + \frac{1}{2}x/q(\varepsilon))$, we have

$$m_k(\hat{q}_x(\varepsilon))\begin{cases}\leq \varepsilon + \frac{1}{2}(x-k)/q(\varepsilon) + \sum_{l=1}^{k}(2\hat{q}_x(\varepsilon)[1+2\lambda_l\hat{q}_x(\varepsilon)])^{-1} \\ \geq \varepsilon + \frac{1}{2}x/q(\varepsilon) - \frac{1}{2}k/\hat{q}_x(\varepsilon)\end{cases} \quad \text{for } x \geq 0.$$

Since, by (3.1) and (2.6), q is self-neglecting, it follows that, given a $\delta \in (0, 1)$,

$$\hat{q}_{(1+\delta)k}(\varepsilon) \leq q_k(\varepsilon) \equiv \inf\{s > 0 : m_k(s) < \varepsilon\} \leq \hat{q}_{(1-\delta)k}(\varepsilon) \quad \text{for } \varepsilon \text{ small.}$$

Hence we have $q_k(\varepsilon) \sim q(\varepsilon)$, so that $m(q_k(\varepsilon)) \sim \varepsilon + \frac{1}{2}k/q(\varepsilon) + o(1/q(\varepsilon))$. Consequently, (2.7) and (3.1) yield that

$$\begin{aligned} &V(q_k(\varepsilon))^{-1/2} \exp\{q_k(\varepsilon)\varepsilon\}\phi(q_k(\varepsilon)) \\ &\quad\sim V(q_k(\varepsilon))^{-1/2} \exp\{q_k(\varepsilon)m(q_k(\varepsilon)) - \tfrac{1}{2}k\}\phi(q_k(\varepsilon)) \\ &\quad\sim \sqrt{2\pi}\, e^{-k/2} q_k(\varepsilon)\, \mathbf{P}\{\|W\|^2 < m(q_k(\varepsilon))\} \\ &\quad\sim \sqrt{2\pi}\, q(\varepsilon)\, \mathbf{P}\{\|W\|^2 < \varepsilon\} \\ &\quad\sim V(q(\varepsilon))^{-1/2} \exp\{q(\varepsilon)\varepsilon\}\phi(q(\varepsilon)). \end{aligned}$$

Observing that $V_k(s) \equiv \sum_{l=k+1}^{\infty} 2\lambda_l^2/[1+2\lambda_l s]^2 \sim V(s)$ as $s \to \infty$, the fact that (3.4) holds now readily follows from (3.1) and from noting that

$$\begin{aligned} \mathbf{E}\{\exp[-s\|W_k\|^2]\} &= \left(\prod_{l=1}^{k} \sqrt{1+2\lambda_l s}\right)\phi(s) \\ &\sim s^{k/2}\left(\prod_{l=1}^{k} \sqrt{2\lambda_l}\right)\phi(s) \quad \text{as } s \to \infty. \end{aligned}$$

Since $q_k(\varepsilon - x/q(\varepsilon)) \le 2\Psi^\star(0^+, q; \frac{1}{2})q(\varepsilon)$ for $0 \le x \le \frac{1}{2}q(\varepsilon)\varepsilon$, for ε small, (3.2) and (3.4) readily yield that there are $C_3, \varepsilon_2 > 0$ such that

$$f_{\|W_k\|^2}(\varepsilon - x/q(\varepsilon)) \le C_3 q(\varepsilon)^{k/2} f_{\|W\|^2}(\varepsilon - x/q(\varepsilon)) \tag{3.6}$$

for $0 \le x \le \frac{1}{2}q(\varepsilon)\varepsilon$ and $\varepsilon \in (0, \varepsilon_2]$. Moreover, by D&R (equation 3.20) there exist $s_1, C_4 > 0$ such that

$$\begin{aligned} &f_{\|W\|^2}(V(s)^{1/2}y + m(s)) \\ &\quad\le C_4 \exp\{-\tfrac{1}{2}y^2\} V(s)^{-1/2}\phi(s)\exp\{s(V(s)^{1/2}y + m(s))\} \end{aligned}$$

for $s \ge s_1$ and $y \in \mathbb{R}$. Taking $y = -x(sV(s)^{1/2})^{-1}$ and $s = q(\varepsilon)$ we thus obtain

$$f_{\|W\|^2}(\varepsilon - x/q(\varepsilon)) \le C_4 V(q(\varepsilon))^{-1/2}\phi(q(\varepsilon))\exp\{q(\varepsilon)\varepsilon - x\} \quad \text{for } x \in \mathbb{R} \text{ and } \varepsilon \text{ small.}$$

Combining this with (3.1) and (3.6) we deduce that (3.5) holds for $0 \le x \le \frac{1}{2}q(\varepsilon)\varepsilon$ and with the factor $e^{-x/2}$ replaced by e^{-x}. Using (3.2), (3.4) and the proven part of (3.5), it follows that there are $C_5 > 1$ and $\varepsilon_3 > 0$ such that

$$\begin{aligned} f_{\|W_k\|^2}(\varepsilon - x/q(\varepsilon)) &\le C_5\, q_k(\varepsilon - x/q(\varepsilon))\, \mathbf{P}\{\|W_k\|^2 < \varepsilon - x/q(\varepsilon)\} \\ &\le C_5^2\, \mathbf{P}\{\|W_{k+2}\|^2 < \varepsilon - x/q(\varepsilon)\}\big/\sqrt{4\lambda_{k+1}\lambda_{k+2}} \\ &\le C_5^2\, \mathbf{P}\{\|W_{k+2}\|^2 < \tfrac{1}{2}\varepsilon\}\big/\sqrt{4\lambda_{k+1}\lambda_{k+2}} \\ &\le C_5^3\, f_{\|W_k\|^2}(\tfrac{1}{2}\varepsilon) \\ &\le C_5^4\, q(\varepsilon)^{k/2+1}\, e^{-x/2}\, \mathbf{P}\{\|W\|^2 < \varepsilon\} \end{aligned}$$

for $\frac{1}{2}q(\varepsilon)\varepsilon \le x < q(\varepsilon)\varepsilon$ and $\varepsilon \in (0, \varepsilon_3]$. $\square$

4. An upper bound for local extremes of fields. For an $\mathbb{R}$-valued Gaussian process $\{\zeta(t)\}_{t\in T}$ the entropy $\mathscr{N}_\zeta(S;\varepsilon)$ is the minimum number of closed balls of radius ε in the pseudometric $d_\zeta(s,t) \equiv \sqrt{\mathbf{E}\{[\zeta(t)-\zeta(s)]^2\}}$ needed to cover $S \subseteq T$. Then $\int_0^1 \sqrt{\ln \mathscr{N}_\zeta(S;\varepsilon)}\, d\varepsilon < \infty$ is sufficient [Dudley (1967)] and, assuming stationarity, also necessary [Fernique (1975)] for a.s. continuity of $\{\zeta(t)\}_{t\in S}$. The influence of $\mathscr{N}_\zeta$ on the tail $\mathbf{P}\{\sup_{t\in[0,1]} \zeta(t) > x\}$ as $x \to \infty$ is also significant, for example, Weber (1989), Adler (1990), Samorodnitsky (1991) and Albin (1994).

A stationary H-valued Gaussian process $Z(t)$ is a.s. continuous when $\sup_{y\in H_1} \int_0^1 \sqrt{\ln \mathscr{N}_{\langle y|Z\rangle}(S;\varepsilon)}\, d\varepsilon < \infty$ [Fernique (1989), Théorème 3.3].

In Theorem 1 we give an upper estimate of the tail $\mathbf{P}\{\inf_{t\in S} \|X(t)\|^2 < \varepsilon\}$ as $\varepsilon \downarrow 0$ expressed in terms of $\mathscr{N}_{\langle y|X\rangle}(S;\cdot)$, $\{\lambda_k\}_{k=1}^\infty$ [through $q(\varepsilon) \equiv \inf\{s > 0\colon m(s) < \varepsilon\}$] and $\mathbf{P}\{\|X(s_0)\|^2 < \varepsilon\}$. [The behavior of $\mathbf{P}\{\|X(s_0)\|^2 < \varepsilon\}$ in turn was expressed in terms of $\{\lambda_k\}_{k=1}^\infty$ in (3.1).] It is required that $\mathscr{N}_{\langle y|X\rangle}(S;\varepsilon)$ is O-regularly varying as $\varepsilon \downarrow 0$. [By the proof of Theorem 1, open $d_{\langle y|X\rangle}$ balls are T-open, so $\mathscr{N}_{\langle y|X\rangle}(S;\varepsilon)$ is finite if, e.g., S is T-compact.] Our estimate is essentially sharp in the stationary case (and other not too unbalanced cases) since, under mild additional conditions, a lower bound of the same order will be derived in Section 5.

The proof of Theorem 1 relies on two transparent ideas. The first idea has its origins in the treatments of continuity by Dudley (1967) and Fernique (1975, 1989) and concerns how to use the concept of entropy to sample the process $X(t)$ sparsely enough not to obtain redundant information, but yet often enough not to overestimate $\inf_{t\in S} \|X(t)\|^2$. We find it surprising that the influence of $\mathscr{N}_{\langle y|X\rangle}(S;\cdot)$ on the left tail of $\inf_{t\in S} \|X(t)\|^2$ is equally direct as that on the right tail of $\sup_{t\in S} \|X(t)\|^2$.

Define the covariance $R_{s,t} \in \mathscr{L}$ by $\langle R_{s,t}x|y\rangle = \mathbf{E}\{\langle x|X(s)\rangle\langle y|X(t)\rangle\}$ so that $R = R_{t,t}$. The second idea concerns estimation of $\sup_{t\in N(s_0)} \|X(t) - W\|^2$ conditional on $\|W\|^2 (= \|X(s_0)\|^2) < \varepsilon$ for a neighborhood $N(s_0)$ of $s_0 \in S$. If $Q \equiv R_{s_0,t}R^{-1}$ exists in a suitable sense [cf. (4.1)], one has

$$\|X(t) - W\|^2 = \|X(t) - QW\|^2 + 2\sum_{k=1}^{\infty} \langle X(t) - QW|\hat{e}_k\rangle\langle \hat{e}_k|[Q-1]W\rangle + \|[Q-1]W\|^2$$

(where $\{\hat{e}_k\}$ is a suitable basis in H). Since $X(t) - QW$ and W are independent, only $[Q-1]W$ is affected by the conditioning. Thus the size of $X(t) - QW$ and $\langle X(t) - QW|\hat{e}_k\rangle$ can be controlled via calculus of covariance operators. The major part of the proof consists of an analysis of the size of the components $\{\langle \hat{e}_k|[Q-1]W\rangle\}$ conditional on $\|W\|^2 < \varepsilon$. The central limit theorem of D&R discussed in the beginning of Section 3 indicates that the (weak) limit behavior of $(\langle \hat{e}_k|[Q-1]W\rangle \mid \|W\|^2 < \varepsilon)$ (suitably normalized) as $\varepsilon \downarrow 0$ is Gaussian. [In a special case this statement is made precise in Lemma 4 of Section 7 and used in the proof of Theorem 4.] Here we shall estimate the "simultaneous" sizes of $\{\langle \hat{e}_k|[Q-1]W\rangle\}_{k=1}^\infty$ "before" (but close to) the limit.

Via input from Samorodnitsky (1991) we then find a suitable separant $\hat{S}$ for X along which to compute $\sup_{t\in\hat{S}\cap N(s_0)} \|X(t) - W\|^2$.

Although the classic concept of entropy is present in our result, our arguments are mainly nonclassic: Dudley's and Fernique's direct connection between entropy and continuity/boundedness/suprema is unique for the study of "high levels/large values" of Gaussian and some closely related processes. Low levels are something entirely different, and although the last part of our proof [the part after (4.12)] is more or less classic, the major part that precedes it is new.

Let $d(s,t) \equiv \sup_{z\in H_1} d_{\langle z|X\rangle}(s,t)$ and choose an $S \subseteq T$. Also write $S_\varepsilon \equiv \{t \in T: d(t,S) \le \varepsilon\}$ and $\mathbb{S}_\varepsilon \equiv \{(s,t) \in S\times S: 0 < d(s,t) \le \varepsilon\}$, and assume that

$$\text{there exists (a correlation) } r_{s,t} \in \mathscr{L} \text{ such that} \quad R_{s,t} = r_{s,t}R \quad \text{for } (s,t)\in \mathbb{S}_{\varepsilon_0} \tag{4.1}$$

for some $\varepsilon_0 > 0$. Further suppose that

$$\text{there is a } y\in H_1 \text{ such that} \quad M_1(y) \equiv \sup\left\{\frac{\|1-r_{s,t}\|}{d_{\langle y|X\rangle}(s,t)^2}\colon\ (s,t)\in\mathbb{S}_{\varepsilon_0}\right\} < \infty. \tag{4.2}$$

The requirement (4.1) holds when $R_{s,t}$ does not have too many too wild off-diagonal elements. In particular, if the component processes $\{\langle X(t)|e_k\rangle\}_{k=1}^\infty$ are independent, then (4.1) holds with $r_{s,t}e_k = \rho_{s,t}^{(k)}e_k$, where

$$\rho_{s,t}^{(k)} = \mathrm{Corr}\{\langle e_k|X(s)\rangle\langle e_k|X(t)\rangle\}.$$

Moreover, for independent components, $d_{\langle e_l|X\rangle}(s,t)^2 = 2\lambda_l(1-\rho_{s,t}^{(l)})$, so that (4.2) holds when there is an $l\in\mathbb{N}$ for which $\rho_{s,t}^{(l)}$ is minimal, that is, when there is an $l\in\mathbb{N}$ such that

$$\sup\big\{(1-\rho_{s,t}^{(k)})/(1-\rho_{s,t}^{(l)})\colon\ k\in\mathbb{N},\ (s,t)\in\mathbb{S}_{\varepsilon_0}\big\} < \infty.$$

REMARK 2. Recall that a variance $C \in \mathscr{L}$ is positive; that is, C is self-adjoint and $\langle Cz|z\rangle \ge 0$ for $z\in H$. Thus the eigenvalues $\{c_k\}_{k=1}^\infty$ of C are nonnegative. Further C is trace class, so that $\mathrm{tr}(C) = \sum_{k=1}^\infty c_k < \infty$ and $\sum_{k=1}^\infty \langle Cf_k|f_k\rangle = \mathrm{tr}(C)$ for any complete orthonormal system $\{f_k\}_{k=1}^\infty$. Moreover, there is a complete orthonormal system $\{g_k\}_{k=1}^\infty$ of eigenvectors to C (satisfying $Cg_k = c_kg_k$).

THEOREM 1. *Assume that $m(s) = \sum_{k=1}^\infty \lambda_k/(1+2\lambda_k s)$ has positive decrease as $s\to\infty$ and that* (4.1) *holds. Further assume that* (4.2) *holds for a $y\in H_1$ such that $\mathscr{M}_X(S;\varepsilon) \equiv \mathscr{N}_{\langle y|X\rangle}(S;\varepsilon)$ is O-regularly varying as $\varepsilon\downarrow 0$. Then we have*

$$\limsup_{\varepsilon\downarrow 0} \frac{\mathbf{P}\{\inf_{t\in S}\|X(t)\|^2 < \varepsilon\}}{\mathscr{M}_X(S;q(\varepsilon)^{-1/2})\,\mathbf{P}\{\|X(s_0)\|^2<\varepsilon\}} < \infty.$$

As described in Albin [1992(a), Section 5], the following lemma is an easy consequence of a result due to Fernique (1971).

LEMMA 1. *For every H-valued centered Gaussian random variable N we have*

$$\mathbf{P}\{\|N\| > u\} \le \exp\left\{\frac{1}{24} - \frac{u^2}{96\,\mathbf{E}\{\|N\|^2\}}\right\} \quad \textit{for } u \ge 0.$$

PROOF OF THEOREM 1. Write $L^\star$ for the adjoint of $L \in \mathscr{L}$, put $\varepsilon_4 \equiv \varepsilon_0 \wedge M_1(y)^{-1/2}$ and take $(s,t) \in \mathbb{S}_{\varepsilon_4}$. Note that, by (4.1), $Y_{s,t} \equiv \frac{1}{2}[1 + r_{s,t}]X(s)$ has variance $\hat{R} \equiv \frac{1}{4}[1 + r_{s,t}]R[1 + r^\star_{s,t}]$. Further write $\{\hat{\lambda}_k\}$ for the spectrum of $\hat{R}$ and let $\{\hat{e}_k\}$ be a corresponding complete orthonormal system of eigenvectors. Since by (4.2), $\|1 - r_{s,t}\| \le M_1 d_{\langle y|X\rangle}(s,t)^2 \le 1$ for $(s,t) \in \mathbb{S}_{\varepsilon_4}$, we have

$$\operatorname{tr}(\hat{R}) = \mathbf{E}\{\|Y_{s,t}\|^2\} \le \tfrac{1}{4}(2 + \|r_{s,t} - 1\|)^2 \mathbf{E}\{\|X(s)\|^2\} \le \tfrac{9}{4}\operatorname{tr}(R). \tag{4.3}$$

Moreover an easy computation shows that

$$1 - r_{s,t} = 2(\tfrac{1}{2}[1 - r_{s,t}])^{N+1} + \sum_{l=1}^{N}(\tfrac{1}{2}[1 - r_{s,t}])^l[1 + r_{s,t}] \to \sum_{l=1}^{\infty}(\tfrac{1}{2}[1 - r_{s,t}])^l[1 + r_{s,t}]$$

as $N \to \infty$, with convergence in operator norm. In view of (4.1), and again using that $\|1 - r_{s,t}\| \le M_1 d_{\langle y|X\rangle}(s,t)^2 \le 1$, it follows that

$$\begin{aligned}\mathbf{E}\{\langle X(t) - r_{s,t}X(s)|\hat{e}_k\rangle^2\} &= \big\langle [R - 2R_{s,t}r^\star_{s,t} + r_{s,t}Rr^\star_{s,t}]\hat{e}_k \bigm| \hat{e}_k\big\rangle \\ &= \big\langle [1 - r_{s,t}]R[1 + r^\star_{s,t}]\hat{e}_k \bigm| \hat{e}_k\big\rangle \\ &= 4\hat{\lambda}_k\left\langle \sum_{l=1}^{\infty}(\tfrac{1}{2}[1 - r_{s,t}])^l\hat{e}_k \biggm| \hat{e}_k\right\rangle \\ &\le 4\hat{\lambda}_k\tfrac{1}{2}\|1 - r_{s,t}\|/[1 - \tfrac{1}{2}\|1 - r_{s,t}\|] \\ &\le 4\hat{\lambda}_k M_1 d_{\langle y|X\rangle}(s,t)^2.\end{aligned} \tag{4.4}$$

Consequently [and by (4.3)], $X_{s,t} \equiv X(t) - r_{s,t}X(s)$ satisfies

$$\begin{aligned}\operatorname{tr}(\operatorname{Var}\{X_{s,t}\}) &= \sum_{k=1}^{\infty}\big\langle \operatorname{Var}\{X_{s,t}\}\hat{e}_k \bigm| \hat{e}_k\big\rangle \le 4M_1 d_{\langle y|X\rangle}(s,t)^2 \operatorname{tr}(\hat{R}) \\ &\le 9M_1 d_{\langle y|X\rangle}(s,t)^2 \operatorname{tr}(R).\end{aligned} \tag{4.5}$$

Further we have

$$\begin{aligned}\|X(s) - Y_{s,t}\| &= \tfrac{1}{2}\|[1 - r_{s,t}]X(s)\| \\ &\le \tfrac{1}{2}M_1\|X(s)\|/q \quad \text{for } d_{\langle y|X\rangle}(s,t)^2 q \le 1,\end{aligned} \tag{4.6}$$

but here $\|X(s)\| \le \|X(s) - Y_{s,t}\| + \|Y_{s,t}\|$, so that, by (4.6),

$$\|X(s) - Y_{s,t}\| \le (\tfrac{1}{2}M_1\|Y_{s,t}\|/q)\big/(1 - \tfrac{1}{2}M_1/q) \le M_1\|Y_{s,t}\|/q \tag{4.7}$$

for $d_{\langle y|X\rangle}(s,t)^2 q \le 1$ and ε small. Combining (4.6) and (4.7) we conclude that

$$\begin{aligned} \|X(s)\|^2 < \varepsilon \quad &\Rightarrow \quad \|Y_{s,t}\|^2 \le \varepsilon + M_1\varepsilon/q + \tfrac{1}{4}M_1^2\varepsilon/q^2 < \varepsilon + M_1/q \\ &\Rightarrow \quad \|X(s)\|^2 \le (1 + M_1/q)(\varepsilon + M_1/q) < \varepsilon + 2M_1/q \end{aligned} \tag{4.8}$$

for $d_{\langle y|X\rangle}(s,t)^2 q \le 1$ and ε small.

Clearly we have

$$\begin{aligned} \|X(t)\|^2 &= \|X_{s,t} - \tfrac{1}{2}[1 - r_{s,t}]X(s)\|^2 + \|r_{s,t}X(s)\|^2 \\ &\qquad - \tfrac{1}{4}\|[1 - r_{s,t}]X(s)\|^2 + 2\langle X_{s,t}|Y_{s,t}\rangle \\ &\le (\|X_{s,t}\| + \tfrac{1}{2}\|[1 - r_{s,t}]X(s)\|)^2 + \|r_{s,t}\|^2\|X(s)\|^2 + 2\langle X_{s,t}|Y_{s,t}\rangle. \end{aligned}$$

When there is a $C_6 > 0$ [not depending on (s,t)] such that $d_{\langle y|X\rangle}(s,t)^2 q \le C_6\eta$, this readily yields that

$$\begin{aligned} &\|X(t)\|^2 \ge \varepsilon - \frac{\nu}{q}, \qquad \|X(s)\|^2 < \varepsilon - \frac{\nu + \eta}{q} \quad \text{and} \quad 2\langle X_{s,t}|Y_{s,t}\rangle \le \frac{1}{2}\frac{\eta}{q} \\ &\qquad \Rightarrow \quad \|X_{s,t}\| \ge \left(\frac{1}{2}\frac{\eta}{q} - [\|r_{s,t}\|^2 - 1]\,\|X(s)\|\right)^{1/2} \\ &\qquad\qquad - \frac{1}{2}\|1 - r_{s,t}\|\,\|X(s)\| > \left(\frac{1}{4}\frac{\eta}{q}\right)^{1/2} \end{aligned} \tag{4.9}$$

for $\eta > 0$, $\nu + \eta \ge 0$, $d_{\langle y|X\rangle}(s,t)^2 q \le 1 \wedge C_6\eta$ and ε small.

Now observe that $X_{s,t}$ is independent of $X(s)$ and $Y_{s,t}$. Furthermore, the variables $\{\mathrm{sign}(\langle Y_{s,t}|\hat{e}_k\rangle)\}_{k=1}^{\infty}$ are independent, identically distributed Rademacher variables and independent of the sequence $\{|\langle Y_{s,t}|\hat{e}_k\rangle|\}_{k=1}^{\infty}$. It follows that the variables $\{\langle X_{s,t}|\hat{e}_k\rangle \mathrm{sign}(\langle Y_{s,t}|\hat{e}_k\rangle)\}_{k=1}^{\infty}$ are uncorrelated and independent of the sequence $\{|\langle Y_{s,t}|\hat{e}_k\rangle|\}_{k=1}^{\infty}$. Writing $\tilde{\varepsilon} \equiv \varepsilon + M_1/q$ and $N(0,1)$ for a standardized Gaussian random variable which is independent of $Y_{s,t}$, we thus have

$$\begin{aligned} &\mathbf{P}\left\{q\sum_{k=1}^{\infty}\langle X_{s,t}|\hat{e}_k\rangle\langle Y_{s,t}|\hat{e}_k\rangle > \tfrac{1}{4}\eta,\, \|Y_{s,t}\|^2 < \tilde{\varepsilon}\right\} \\ &\qquad \le \mathbf{P}\left\{N(0,1)\left(q^2\sum_{k=1}^{\infty}\mathbf{E}\{\langle X_{s,t}|\hat{e}_k\rangle^2\}\,\langle Y_{s,t}|\hat{e}_k\rangle^2\right)^{1/2} > \tfrac{1}{4}\eta,\ \|Y_{s,t}\|^2 < \tilde{\varepsilon}\right\}. \end{aligned}$$

Using Lemma 1, (4.4), (4.5), (4.8) and (4.9), we therefore obtain

$$
\begin{aligned}
&\mathbf{P}\left\{\|X(t)\|^2 \geq \varepsilon - \frac{\nu}{q},\, \|X(s)\|^2 < \varepsilon - \frac{\nu+\eta}{q}\right\}\\
&\quad \leq \mathbf{P}\left\{q\|X_{s,t}\|^2 > \frac{1}{4}\eta\right\}\mathbf{P}\{\|X(s_0)\|^2 < \varepsilon\}\\
&\qquad + \mathbf{P}\left\{q\sum_{k=1}^{\infty}\langle X_{s,t}|\hat{e}_k\rangle\langle Y_{s,t}|\hat{e}_k\rangle > \frac{1}{4}\eta,\, \|X(s)\|^2 < \varepsilon\right\}\\
&\quad \leq \exp\left\{\frac{1}{24} - \frac{\eta}{3456 M_1 \operatorname{tr}(R) d_{\langle y|X\rangle}(s,t)^2 q}\right\}\mathbf{P}\{\|X(s_0)\|^2 < \varepsilon\}\\
&\qquad + \mathbf{P}\left\{N(0,1)\left(q\sum_{k=1}^{\infty}\hat{\lambda}_k\langle Y_{s,t}|\hat{e}_k\rangle^2\right)^{1/2} > \frac{\eta}{8(M_1 q)^{1/2} d_{\langle y|X\rangle}(s,t)},\right.\\
&\qquad\qquad \left.\|Y_{s,t}\|^2 < \tilde{\varepsilon}\right\}.
\end{aligned} \tag{4.10}
$$

Let $P_J \in \mathscr{L}$ be the projection on $E_J \equiv \operatorname{span}\{e_i\colon i \in J\}$ for $J \subseteq \mathbb{N}$, $\hat{P}_J$ the projection on $\operatorname{span}\{\hat{e}_i\colon i \in J\}$, $P_J^{\perp} \equiv 1 - P_J$ and $\hat{P}_J^{\perp} \equiv 1 - \hat{P}_J$. Since the density function of a $\chi^2(1)$-distributed random variable is decreasing, (3.5) and (4.8) then combine with elementary computations to show that

$$
\begin{aligned}
&\mathbf{P}\{q\langle Y_{s,t}|\hat{e}_k\rangle^2 > x,\, \|Y_{s,t}\|^2 < \tilde{\varepsilon}\}\\
&\quad = \int_{z=x}^{z=\tilde{\varepsilon}q}\int_{y=0}^{y=\tilde{\varepsilon}q-z} f_{\|\hat{P}_{\{k\}}^{\perp}Y_{s,t}\|^2}\left(\tilde{\varepsilon} - \frac{y+z}{q}\right) f_{\|\hat{P}_{\{k\}}Y_{s,t}\|^2}\left(\frac{y+x}{q}\right)\frac{dy\,dz}{q^2}\\
&\quad \leq \int_{z=x}^{z=\tilde{\varepsilon}q}\int_{y=0}^{y=\tilde{\varepsilon}q-z} f_{\|\hat{P}_{\{k\}}^{\perp}Y_{s,t}\|^2}\left(\tilde{\varepsilon} - \frac{y+z}{q}\right) f_{\|\hat{P}_{\{k\}}Y_{s,t}\|^2}\left(\frac{y}{q}\right)\frac{dy\,dz}{q^2}\\
&\quad = \mathbf{P}\left\{\|Y_{s,t}\|^2 < \tilde{\varepsilon} - \frac{x}{q}\right\}\\
&\quad \leq \mathbf{P}\left\{\|X(s)\|^2 < \varepsilon - \frac{x-2M_1}{q}\right\}\\
&\quad = \int_{z=x-2M_1}^{z=\varepsilon q}\int_{y=0}^{y=\varepsilon q-z} f_{\|P_{\{1\}}^{\perp}X(s_0)\|^2}\left(\varepsilon - \frac{y+z}{q}\right) f_{\|P_{\{1\}}X(s_0)\|^2}\left(\frac{y}{q}\right)\frac{dy\,dz}{q^2}\\
&\quad \leq \frac{C_2\mathbf{P}\{\|X(s_0)\|^2 < \varepsilon\}}{\sqrt{2\pi\lambda_1}}\int_{z=x-2M_1}^{z=\infty}\int_{y=0}^{y=\infty} y^{-1/2}\exp\left(-\frac{y+z}{2}\right)dy\,dz\\
&\quad \leq C_7\exp\left(-\frac{x}{2}\right)\mathbf{P}\{\|X(s_0)\|^2 < \varepsilon\} \quad \text{for } x \geq 2M_1 \text{ and } \varepsilon \text{ small}
\end{aligned}
$$

and for some constant $C_7 > 0$. In view of (4.8) [and (3.1)] it follows that

$$\begin{aligned}\frac{\mathbf{E}\{q^p\langle Y_{s,t}|\hat{e}_k\rangle^{2p} I_{\{\|Y_{s,t}\|^2<\tilde{\varepsilon}\}}\}}{\mathbf{P}\{\|X(s_0)\|^2<\varepsilon\}} &\le \frac{2M_1\mathbf{P}\{\|Y_{s,t}\|^2<\tilde{\varepsilon}\}}{\mathbf{P}\{\|X(s_0)\|^2<\varepsilon\}}\\ &\quad+\int_{2M_1}^{\infty} C_7\exp\left(-\frac{x^{1/p}}{2}\right)dx \le C_8\end{aligned} \tag{4.11}$$

for $p > 1$ and ε small, for some $C_8 = C_8(p) > 0$. Hence Hölder's inequality gives

$$\begin{aligned}&\mathbf{P}\left\{N(0,1)\left(q\sum_{k=1}^{\infty}\hat{\lambda}_k\langle Y_{s,t}|\hat{e}_k\rangle^2\right)^{1/2} > x,\ \|Y_{s,t}\|^2<\tilde{\varepsilon}\right\}\\ &\quad\le x^{-2p}\,\mathbf{E}\{[N(0,1)]^{2p}\}\,\mathbf{E}\left\{\left(\sum_{k=1}^{\infty}\hat{\lambda}_k q\langle Y_{s,t}|\hat{e}_k\rangle^2\right)^{p} I_{\{\|Y_{s,t}\|^2<\tilde{\varepsilon}\}}\right\}\\ &\quad\le x^{-2p}\,\mathbf{E}\{[N(0,1)]^{2p}\}\ \mathrm{tr}(\hat{R})^{p-1}\\ &\qquad\times\mathbf{E}\left\{\sum_{k=1}^{\infty}\hat{\lambda}_k q^p\langle Y_{s,t}|\hat{e}_k\rangle^{2p} I_{\{\|Y_{s,t}\|^2<\tilde{\varepsilon}\}}\right\}\\ &\quad\le C_8\,x^{-2p}\,\mathbf{E}\{[N(0,1)]^{2p}\}\ \mathrm{tr}(\hat{R})^p\,\mathbf{P}\{\|X(s_0)\|^2<\varepsilon\}.\end{aligned}$$

Recalling (4.3) and (4.10) we conclude that there is a $C_9 = C_9(p) > 0$ such that

$$\begin{aligned}&\mathbf{P}\left\{\|X(t)\|^2\ge\varepsilon-\frac{\nu}{q},\ \|X(s)\|^2<\varepsilon-\frac{\nu+\eta}{q}\right\}\\ &\quad\le\exp\left\{\frac{1}{24}-\frac{\eta}{3456M_1\,\mathrm{tr}(R)d_{\langle y|X\rangle}(s,t)^2q}\right\}\mathbf{P}\{\|X(s_0)\|^2<\varepsilon\}\\ &\qquad+(12)^{2p}M_1^pC_8\,\mathrm{tr}(R)^p\mathbf{E}\{[N(0,1)]^{2p}\}\\ &\qquad\quad\times d_{\langle y|X\rangle}(s,t)^{2p}q^p\eta^{-2p}\mathbf{P}\{\|X(s_0)\|^2<\varepsilon\}\\ &\quad\le C_9d_{\langle y|X\rangle}(s,t)^{2p}q^p\eta^{-2p}\mathbf{P}\{\|X(s_0)\|^2<\varepsilon\}\end{aligned} \tag{4.12}$$

for $d_{\langle y|X\rangle}(s,t)^2q \le 1\wedge C_6\eta$ and ε small.

Put $\mathscr{C}_0 \equiv \{s_0\}$ and write m_n for the maximal cardinality of a set $\mathscr{C}_n \equiv \{s_1,\dots,s_{m_n}\}$ which is contained in S and satisfies $d_{\langle y|X\rangle}(s_i,s_j) > q^{-1/2}2^{-n}$ for $i \ne j$. Then we must have $m_n \le \mathscr{M}_X(S;q^{-1/2}2^{-(n+1)})$. Further it is clear that

$$d_{\langle y|X\rangle}(s,t_n(s)) \le q^{-1/2}2^{-n} \quad \text{for some } t_n(s)\in\mathscr{C}_n \text{ for each } s\in S. \tag{4.13}$$

By $\mathbb{L}^2$-continuity of X, given $s \in T$ and $\varepsilon > 0$, there is a T-neighborhood U of s such that $d_{\langle y|X\rangle}(s,t)^2 = \mathbf{E}\{\langle y|X(t)-X(s)\rangle^2\} < \varepsilon^2$ for $t \in U$. Thus

$d_{\langle y|X\rangle}$ balls are T-open and X is $d_{\langle y|X\rangle}$-separable. Further we have, by (4.1) and (4.2),

$$\begin{aligned}
\mathbf{E}\{\|X(t)-X(s)\|^2\} &= \mathrm{tr}(\mathrm{Var}\{X(t)-X(s)\}) \\
&= \mathrm{tr}(2[1-r_{s,t}]R) \\
&= \sum_{k=1}^{\infty} 2\lambda_k\langle[1-r_{s,t}]e_k \mid e_k\rangle \\
&\le 2\,\mathrm{tr}(R)M_1 d_{\langle y|X\rangle}(s,t)^2 \\
&\to 0 \quad \text{as } d_{\langle y|X\rangle}(s,t)\to 0.
\end{aligned}$$

Hence (4.3), $\{X(t)\}_{t\in S}$ is $\mathbb{L}^2$- and $\mathbf{P}$-continuous in the $d_{\langle y|X\rangle}$ topology. It follows (from a well-known argument) that each $d_{\langle y|X\rangle}$-dense subset of S separates X.

Now let $\nu_n \equiv (\sqrt{2}-1)\sum_{k=1}^{n} 2^{-k/2}/q$ and take $p \equiv 3\alpha(0^+;\mathscr{M}_X(S;\cdot))$ [$<\infty$ by (2.4)] in (4.12). Since $\bigcup_{n=1}^{\infty}\mathscr{C}_n$ separates X, (4.12) and (4.13) then yield that

$$\begin{aligned}
&\mathbf{P}\Big\{\inf_{s\in S}\|X(s)\|^2 < \varepsilon - 1/q\Big\} \\
&\quad\le \mathbf{P}\{\|X(s_0)\|^2 < \varepsilon\} \\
&\qquad + \mathbf{P}\bigg\{\bigcup_{n=1}^{\infty}\Big\{\inf_{s\in\mathscr{C}_n}\|X(s)\|^2 < \varepsilon-\nu_n\Big\}\cap\Big\{\inf_{t\in\mathscr{C}_{n-1}}\|X(t)\|^2 \ge \varepsilon-\nu_{n-1}\Big\}\bigg\} \\
&\quad\le \mathbf{P}\{\|X(s_0)\|^2 < \varepsilon\} \\
&\qquad + \sum_{n=1}^{\infty}\mathbf{P}\bigg\{\bigcup_{s\in\mathscr{C}_n}\{\|X(s)\|^2 < \varepsilon-\nu_n,\, \|X(t_{n-1}(s))\|^2 \ge \varepsilon-\nu_{n-1}\}\bigg\} \\
&\quad\le \bigg[1+\sum_{n=1}^{\infty} C_9(\sqrt{2}-1)^{-6\alpha(\mathscr{M}_X)}2^{3(2-n)\alpha(\mathscr{M}_X)}\mathscr{M}_X(S;q^{-1/2}2^{-(n+1)})\bigg] \\
&\qquad \times \mathbf{P}\{\|X(s_0)\|^2 < \varepsilon\}.
\end{aligned}$$

[Here we used an idea of Samorodnitsky (1991), Section 3.] Further, for ε small, by (2.6), $\hat{\varepsilon} = \varepsilon - 1/q$ satisfies $\varepsilon \le \hat{\varepsilon} + 2/q(\hat{\varepsilon})$, whereas, by (3.1), $\mathbf{P}\{\|X(s_0)\|^2 < \hat{\varepsilon}+2/q(\hat{\varepsilon})\} \le 2e^2\mathbf{P}\{\|X(s_0)\|^2 < \hat{\varepsilon}\}$. Applying (2.3) to $\mathscr{M}_X(q^{-1/2}2^{-(n+1)})/\mathscr{M}_X(q^{-1/2})$ [and noting that $\mathscr{M}_X(q(\varepsilon)^{-1/2}) \le \mathscr{M}_X(q(\hat{\varepsilon})^{-1/2})$] it thus follows that (for ε small)

$$\begin{aligned}
&\mathbf{P}\Big\{\inf_{s\in S}\|X(s)\|^2 < \hat{\varepsilon}\Big\} \\
&\quad\le \bigg[1+\sum_{n=1}^{\infty} C_9(\sqrt{2}-1)^{-6\alpha}C_1(-2)2^{(8-n)\alpha}\mathscr{M}_X(S;q(\hat{\varepsilon})^{-1/2})\bigg] \\
&\qquad \times 2e^2\mathbf{P}\{\|X(s_0)\|^2 < \hat{\varepsilon}\}. \qquad \square
\end{aligned}$$

5. Local lower bounds for fields. To derive lower bounds we assume that there is a subspace $E \subseteq H$ of a certain minimal dimension such that $d_{\langle x|X\rangle}(s,t)/d_{\langle z|X\rangle}(s,t)$ is bounded for $x, z \in H_1 \cap E$. Writing $\mathscr{P}$ for the projection on E and $\mathscr{P}^{\perp} \equiv 1 - \mathscr{P}$ one then observes that

$$\begin{aligned}&\mathbf{P}\{\|X(t)\|^2 < \varepsilon,\ \|X(s)\|^2 < \varepsilon\}\\&\qquad \le 2\mathbf{P}\{\|\mathscr{P}X(t)\|^2 + \|\mathscr{P}^{\perp}X(s)\|^2 < \varepsilon,\ \|\mathscr{P}X(s)\|^2 + \|\mathscr{P}^{\perp}X(s)\|^2 < \varepsilon\}.\end{aligned}$$

Here $(\mathscr{P}X(t), \mathscr{P}X(s))$ yields to finite Gaussian methods after a decoupling of the dependence between $\mathscr{P}X(t)$ and $\mathscr{P}^{\perp}X(s)$, whereas Section 3 takes care of $\mathscr{P}^{\perp}X(s)$.

As in Section 4, although our result involves the classic concept of entropy, it is only a minor part of the proof [essentially (5.9) and (5.10)] that is classic.

Now let $E_J \equiv \operatorname{span}\{e_i : i \in J\}$ for $J \subseteq \mathbb{N}$ and assume that

there is an $I \subseteq \mathbb{N}$ such that

$$M_2(I) \equiv \sup\left\{\frac{d_{\langle x|X\rangle}(s,t)}{d_{\langle z|X\rangle}(s,t)} : x, z \in H_1 \cap E_I, (s,t) \in \mathbb{S}_{\varepsilon_0}\right\} < \infty. \tag{5.1}$$

If the components $\{\langle X(t)|e_k\rangle\}_{k=1}^{\infty}$ are independent, then (5.1) holds when

$$\sup\bigl\{(1-\rho_{s,t}^{(i)})/(1-\rho_{s,t}^{(j)}) : i, j \in I,\ (s,t) \in \mathbb{S}_{\varepsilon_0}\bigr\} < \infty.$$

THEOREM 2. *Assume that $m(s)$ has positive decrease as $s \to \infty$ and that* (4.1) *holds. Further assume that* (4.2) *and* (5.1) *hold with $y \in H_1 \cap E_I$. If, in addition, $\mathscr{M}_X(S;\varepsilon) \equiv \mathscr{N}_{\langle y|X\rangle}(S;\varepsilon)$ is O-regularly varying as $\varepsilon \downarrow 0$,*

$$n \equiv \#I > \rho(0^+; q)\,\rho(0^+; \mathscr{M}_X(S;\cdot)) \tag{5.2}$$

and

$$\begin{aligned}\kappa > {}&\tfrac{1}{2}\rho(0^+;q)\rho(0^+;\mathscr{M}_X(S;\cdot))\alpha(0^+;\mathscr{M}_X(S;\cdot))\\&\times[n + \rho(0^+;\mathscr{M}_X(S;\cdot))]^{-1},\end{aligned} \tag{5.3}$$

then we have

$$\liminf_{\varepsilon\downarrow 0} \frac{\mathbf{P}\{\inf_{t\in S}\|X(t)\|^2 < \varepsilon\}}{\varepsilon^{\kappa}\mathscr{M}_X(S; q(\varepsilon)^{-1/2})\mathbf{P}\{\|X(s_0)\|^2 < \varepsilon\}} = \infty.$$

REMARK 3. Since $\mathscr{M}_X$ and, by Proposition 2, q are O-regularly varying, (2.2) and (2.5) imply that $\rho(q)$, $\rho(\mathscr{M}_X)$ and $\alpha(\mathscr{M}_X)$ all are finite. Furthermore, (5.2) and (5.3) hold for any $\kappa > 0$ if I can be choosen with $\#I$ arbitrarily large in (5.1).

PROOF OF THEOREM 2. Choose $\varepsilon_5 \in (0, \varepsilon_0 \wedge \varepsilon_1]$ such that $M_1(y)d(s,t)^2 \leq 1$ and $2M_1(y)^2 M_2(I)^2 \|R\| d(s,t)^2 \leq 1$ for $(s,t) \in \mathbb{S}_{\varepsilon_5}$, where $\varepsilon_1 = \varepsilon_1(n)$ is given in Proposition 2. Since $d_{\langle z|X\rangle}(s,t)^2 = 2\langle [1 - r_{s,t}]Rz | z\rangle$, (4.1), (4.2) and (5.1) yield that

$$\begin{aligned}\langle [R - r_{s,t} R r^{\star}_{s,t}]z \mid z\rangle &= d_{\langle z|X\rangle}(s,t)^2 - \langle [1 - r^{\star}_{s,t}]z \mid R[1 - r^{\star}_{s,t}]z\rangle \\ &\geq d_{\langle z|X\rangle}(s,t)^2 - M_1^2 \|R\| d_{\langle y|X\rangle}(s,t)^4 \\ &\geq \tfrac{1}{2} M_2^{-2} d_{\langle y|X\rangle}(s,t)^2 \quad \text{for } z \in H_1 \cap E_I \text{ and } (s,t) \in \mathbb{S}_{\varepsilon_5}.\end{aligned}$$

Using elementary rules for the computation of determinants we thus deduce that

$$\begin{aligned}&\det\left(\mathbf{Var}\begin{pmatrix} P_I[X(t) + (1 - r_{s,t})X(s)] \\ P_I X(s)\end{pmatrix}\right) \\ &\quad = \det\begin{pmatrix} P_I[2R - r_{s,t} R r^{\star}_{s,t}]\, P_I & P_I R P_I \\ P_I R P_I & P_I R P_I\end{pmatrix} \\ &\quad = \det(P_I R P_I)\det\big(P_I[R - r_{s,t} R r^{\star}_{s,t}]P_I\big) \\ &\quad \geq \left(\prod_{i \in I} \lambda_i\right)\Big(\inf_{z \in H_1 \cap E_I} \langle [R - r_{s,t} R r^{\star}_{s,t}]z | z\rangle\Big)^n \\ &\quad \geq \left(\prod_{i \in I} \lambda_i\right)(2M_2^2)^{-n} d_{\langle y|X\rangle}(s,t)^{2n}\end{aligned}$$

for $(s,t) \in \mathbb{S}_{\varepsilon_5}$. An elementary Gaussian argument therefore shows that

$$\begin{aligned}&\mathbf{P}\{\|P_I[X(t) + (1 - r_{s,t})X(s)]\|^2 < x,\ \|P_I X(s)\|^2 < y\} \\ &\qquad \leq \frac{(2M_2^2)^{n/2}(4\sqrt{xy})^n}{(2\pi)^n (\prod_{i\in I} \lambda_i^{1/2}) d_{\langle y|X\rangle}(s,t)^n}\end{aligned} \tag{5.4}$$

[for $(s,t) \in \mathbb{S}_{\varepsilon_5}$]. Further we observe that, by (4.2),

$$\begin{aligned}&\|X(t)\|^2 < \varepsilon, \qquad \|X(s)\|^2 < \varepsilon \quad \text{and} \quad \|P_I^{\perp} X(s)\|^2 \leq \|P_I^{\perp} X(t)\|^2 \\ &\quad \Rightarrow \quad \big\|P_I[X(t) + (1 - r_{s,t})X(s)]\big\|^2 \\ &\qquad\qquad \leq \|P_I X(t)\|^2 + 2\,\|X(t)\|\,\|1 - r_{s,t}\|\,\|X(s)\| \\ &\qquad\qquad\quad + \|1 - r_{s,t}\|^2 \|X(s)\|^2 \\ &\qquad\qquad \leq \varepsilon - \|P_I^{\perp} X(s)\|^2 + 3M_1 d_{\langle y|X\rangle}(s,t)^2 \varepsilon.\end{aligned} \tag{5.5}$$

Since $X(t)-r_{s,t}X(s)$ is independent of $X(s)$ and $P_I X(s)$ of $P_I^{\perp}X(s)$, (5.4) and (5.5) now combine with (3.5) to show that, for $(s,t)\in \mathbb{S}_{\varepsilon_5}$ and $\varepsilon\in(0,\varepsilon_5]$,

$$\begin{aligned}
&\mathbf{P}\{\|X(t)\|^2<\varepsilon,\ \|X(s)\|^2<\varepsilon\}\\
&\quad=2\mathbf{P}\big\{\|X(t)\|^2<\varepsilon,\ \|P_I X(s)\|^2+\|P_I^{\perp}X(s)\|^2<\varepsilon,\\
&\qquad\qquad \|P_I^{\perp}X(s)\|^2\le\|P_I^{\perp}X(t)\|^2\big\}\\
&\quad\le 2\int_0^{\varepsilon q}\mathbf{P}\big\{\|P_I[X(t)+(1-r_{s,t})X(s)]\big\|^2<3M_1 d_{\langle y|X\rangle}(s,t)^2\varepsilon+x/q,\\
&\qquad\qquad \|P_I X(s)\|^2<x/q\big\}\\
&\qquad\times f_{\|P_I^{\perp}X(s)\|^2}(\varepsilon-x/q)\,dx/q\\
&\quad\le\frac{2^{1+3n/2}M_2^n C_2\mathbf{P}\{\|X(s_0)\|^2<\varepsilon\}}{\pi^n(\prod_{i\in I}\lambda_i^{1/2})d_{\langle y|X\rangle}(s,t)^n}\\
&\qquad\times\int_0^{\infty}\big(3M_1 d_{\langle y|X\rangle}(s,t)^2\varepsilon+x/q\big)^{n/2}x^{n/2}e^{-x/2}\,dx\\
&\quad\le\frac{2^{3n+2}M_2^n C_2\mathbf{P}\{\|X(s_0)\|^2<\varepsilon\}}{\pi^n(\prod_{i\in I}\lambda_i^{1/2})}\\
&\qquad\times\left(\Gamma\left(\frac{n+2}{2}\right)\left(\frac{3}{2}M_1\varepsilon\right)^{n/2}+\frac{\Gamma(n+1)}{q^{n/2}d_{\langle y|X\rangle}(s,t)^n}\right).
\end{aligned}\tag{5.6}$$

Clearly, by (5.2) and (5.3), there exist $\varrho\in(0,\frac{1}{2})$ and $\nu\in(1,\infty)$ such that

$$n>\nu^2\rho(q)\rho(\mathscr{M}_X),\qquad n\varrho>\nu(\tfrac{1}{2}-\varrho)\rho(\mathscr{M}_X),\qquad \kappa>\nu^2\varrho\rho(q)\alpha(\mathscr{M}_X).\tag{5.7}$$

Further observe that, writing $B_y(s,\varepsilon)\equiv\{t\in T\colon d_{\langle y|X\rangle}(s,t)\le\varepsilon\}$ and given a $\delta\in(0,\varepsilon_5]$, there exists $s(\delta)\in S_{\varepsilon_5}$ such that

$$\mathscr{M}_X\Big(S\cap B_y(s(\delta),\varepsilon_5);\delta\Big)\ge\mathscr{M}_X(S;\delta)\big/\mathscr{M}_X(S;\varepsilon_5).\tag{5.8}$$

It is now an easy exercise in covering numbers/entropy to conclude that there is a set $N_\delta\subseteq S\cap B_y(s(\delta),\varepsilon_5)$ satisfying

$$\mathscr{M}_X\Big(S\cap B_y(s(\delta),\varepsilon_5);\delta\Big)\le\#N_\delta\le\mathscr{M}_X\Big(S\cap B_y(s(\delta),\varepsilon_5);\tfrac{1}{2}\delta\Big)\tag{5.9}$$

such that $d_{\langle y|X\rangle}(s,t)\ge\delta$ for $N_\delta\ni s\ne t\in N_\delta$. Taking $\delta=q(\varepsilon)^{\varrho-1/2}$, (5.6) thus gives

$$\begin{aligned}
&\mathbf{P}\Big\{\inf_{t\in N_\delta}\|X(t)\|^2<\varepsilon\Big\}\\
&\quad\ge\#N_\delta\,\mathbf{P}\{\|X(s_0)\|^2<\varepsilon\}-\sum_{N_\delta\ni s\ne t\in N_\delta}\mathbf{P}\{\|X(s)\|^2<\varepsilon,\ \|X(t)\|^2<\varepsilon\}\\
&\quad\ge\#N_\delta\,\mathbf{P}\{\|X(s_0)\|^2<\varepsilon\}\big[1-C_{10}\mathscr{M}_X(S;\tfrac{1}{2}q^{\varrho-1/2})(\varepsilon^{n/2}+q^{-n\varrho})\big]
\end{aligned}\tag{5.10}$$

for ε small and for some $C_{10} > 0$. Here we have, by (5.7), choosing a $C_{11} > 0$ such that $q(\varepsilon) \le C_{11}\varepsilon^{-\nu\rho(q)}$ and $\mathscr{M}_X(S;\varepsilon) \le C_{11}\varepsilon^{-\nu\rho(\mathscr{M}_X)}$,

$$\begin{aligned}&\mathscr{M}_X(S;\tfrac{1}{2}q^{\varrho-1/2})[\varepsilon^{n/2}+q^{-n\varrho}]\\ &\quad\le C_{11}2^{\nu\rho(\mathscr{M}_X)(1/2-\varrho)}\Big(C_{11}\varepsilon^{-\nu^2\rho(q)\rho(\mathscr{M}_X)(1/2-\varrho)}\varepsilon^{n/2}\\ &\qquad\qquad+q^{\nu\rho(\mathscr{M}_X)(1/2-\varrho)}q^{-n\varrho}\Big)\to 0\end{aligned}\tag{5.11}$$

as $\varepsilon \downarrow 0$. For ε small (5.8)–(5.10) therefore show that

$$\mathbf{P}\Big\{\inf_{t\in S}\|X(t)\|^2<\varepsilon\Big\}\ge\tfrac{1}{2}\mathscr{M}_X(S;\varepsilon_5)^{-1}\mathscr{M}_X(S;q^{\varrho-1/2})\mathbf{P}\{\|X(s_0)\|^2<\varepsilon\}.$$

The theorem thus follows from (5.7) and the fact that, by applying (2.3) to $\mathscr{M}_X$,

$$\mathscr{M}_X(S;q^{-1/2})\le C_1(-\nu)C_{11}^{\nu\varrho\alpha(\mathscr{M}_X)}\varepsilon^{-\nu^2\varrho\rho(q)\alpha(\mathscr{M}_X)}\mathscr{M}_X(S;q^{\varrho-1/2}).\qquad\Box$$

REMARK 4. By (4.1) and (4.2) we have $d_{\langle z|X\rangle}(s,t)^2 \le 2M_1\|R\|d_{\langle y|X\rangle}(s,t)^2$ for $z\in H_1$, so that $B_y(s,\ \varepsilon/\sqrt{2M_1\|R\|}\,)\subseteq B_z(s,\varepsilon)$. Consequently,

$$\mathscr{N}_{\langle y|X\rangle}(S;\varepsilon)\le\sup_{z\in H_1}\mathscr{N}_{\langle z|X\rangle}(S;\varepsilon)\le\mathscr{N}_{\langle y|X\rangle}\big(S;\varepsilon/\sqrt{2M_1\|R\|}\,\big).$$

Since $\mathscr{N}_{\langle y|X\rangle}(S;\cdot)$ is O-regularly varying, it follows that one can replace $\mathscr{M}_X = \mathscr{N}_{\langle y|X\rangle}$ with $\sup_{z\in H_1}\mathscr{N}_{\langle z|X\rangle}$ in Theorems 1–3. $\Box$

To get a sharp(er) lower bound we require that the $y\in H_1$ given in (4.2) satisfies

$$\begin{aligned}M_3(y)\equiv\sup\Bigg\{&\frac{\mathscr{N}_{\langle y|X\rangle}(S\cap B_y(s,\hat\varepsilon);\varepsilon)\mathscr{N}_{\langle y|X\rangle}(S;\hat\varepsilon)}{\mathscr{N}_{\langle y|X\rangle}(S;\varepsilon)}:\\ &\qquad 0<\varepsilon\le\hat\varepsilon\le\varepsilon_0,\ s\in S_{\varepsilon_0}\Bigg\}<\infty.\end{aligned}\tag{5.12}$$

The requirement (5.12) essentially means that the ratio between the maximum entropy and the average entropy for a $d_{\langle y|X\rangle}$ ball is bounded.

We say that $\{X(t)\}_{t\in T}$ is stationary if $T=(T,+)$ is an Abelian group and if $R_{s,\,s+t}=R_{s+\tau,\,s+t+\tau}$ for $s,t,\tau\in T$. Then $\mathscr{N}_{\langle y|X\rangle}(B_y(s,\hat\varepsilon);\varepsilon)$ does not depend on s.

PROPOSITION 3. *Assume that* $\{X(t)\}_{t\in T}$ *is stationary and that* $\mathscr{N}_{\langle y|X\rangle}(S,\varepsilon)$ *is O-regularly vaying as* $\varepsilon\downarrow 0$. *If, in addition, S has nonempty* $d_{\langle y|X\rangle}$*-interior, then* (5.12) *holds.*

REMARK 5. If for example, T is metric and S is T-compact with $\mathbf{E}\{\langle y|X(t)-X(s)\rangle^2\}>0$ for $S\ni s\ne t\in S$, then $d_{\langle y|X\rangle}(s,t_i)\to 0\Rightarrow t_i\to_T s$ for $\{(s,t_i)\}_{i=1}^{\infty}\subseteq S\times S$. If S also has nonempty T-interior, it follows that S has nonempty $d_{\langle y|X\rangle}$-interior.

PROOF OF PROPOSITION 3. Take $\hat{s} \in S$ and $\varepsilon_6 > 0$ such that $B_y(\hat{s}, 2\varepsilon_6) \subseteq S$. Then it is an easy exercise in covering numbers/entropy to show that

$$\mathscr{N}_{\langle y|X\rangle}(S, \varepsilon) \geq \mathscr{N}_{\langle y|X\rangle}(B_y(\hat{s}, \varepsilon_6); 4\hat{\varepsilon})\mathscr{N}_{\langle y|X\rangle}(B_y(\cdot, \hat{\varepsilon}); \varepsilon) \quad \text{for } \varepsilon \leq \hat{\varepsilon} \leq \varepsilon_6.$$

Here we have, by (5.8) (and Section 2), for $\hat{\varepsilon} \leq \frac{1}{4}\varepsilon_6$ sufficiently small,

$$\begin{aligned}\mathscr{N}_{\langle y|X\rangle}(B_y(\hat{s}, \varepsilon_6); 4\hat{\varepsilon}) &\geq \mathscr{N}_{\langle y|X\rangle}(S; 4\hat{\varepsilon})/\mathscr{N}_{\langle y|X\rangle}(S; \varepsilon_6)\\ &\geq [2\,\Psi(0^+, \mathscr{N}_{\langle y|X\rangle}(S, \cdot); \tfrac{1}{4})]^{-1}\\ &\qquad \times \mathscr{N}_{\langle y|X\rangle}(S; \hat{\varepsilon})\big/\mathscr{N}_{\langle y|X\rangle}(S; \varepsilon_6).\end{aligned}$$

Hence (5.12) holds with $M_3(y) = 2\,\Psi(0^+, \mathscr{N}_{\langle y|X\rangle}; \frac{1}{4})\mathscr{N}_{\langle y|X\rangle}(S; \varepsilon_6)$. $\square$

THEOREM 3. *Assume that $m(s)$ has positive decrease as $s \to \infty$ and that* (4.1) *holds. Further asssume that* (4.2), (5.1) *and* (5.12) *hold with $y \in H_1 \cap E_I$. If, in addition, $\mathscr{M}_X(S; \varepsilon) \equiv \mathscr{N}_{\langle y|X\rangle}(S; \varepsilon)$ is O-regularly varying as $\varepsilon \downarrow 0$ and*

$$n \equiv \#I > \max\{\rho(0^+; q)\,\rho(0^+; \mathscr{M}_X(S; \cdot)),\ \alpha(0^+; \mathscr{M}_X(S; \cdot))\}, \tag{5.13}$$

then we have

$$\liminf_{\varepsilon\downarrow 0} \frac{\mathbf{P}\{\inf_{t\in S} \|X(t)\|^2 < \varepsilon\}}{\mathscr{M}_X(S; q(\varepsilon)^{-1/2})\,\mathbf{P}\{\|X(s_0)\|^2 < \varepsilon\}} > 0.$$

PROOF. Take $\eta, \nu > 1$ such that $\eta[n - \nu\alpha(\mathscr{M}_X)] > 1$ and $n > \nu^2\rho(q)\rho(\mathscr{M}_X)$ and let $K > 0$. Also choose N_δ as in the proof of Theorem 2, where now $\delta = Kq(\varepsilon)^{-1/2}$. Then it is easily seen that $\varepsilon^{n/2}\#N_\delta \to 0$ [consider (5.11) with $\varrho = 0$ but with the factor $q^{-n\varrho}$ omitted]. Further we have, by (5.12) and (2.3) (for ε small),

$$\begin{aligned}&\sup_s \#\{t \in N_\delta\colon\ d_{\langle y|X\rangle}(s, t) < q^{-1/2}(k+1)^\eta K\}\\ &\quad\leq \sup_s \mathscr{M}_X\big(S \cap B_y(s, q^{-1/2}(k+1)^\eta K); \tfrac{1}{2}q^{-1/2}K\big)\\ &\quad\leq M_3(y)\mathscr{M}_X(S; \tfrac{1}{2}q^{-1/2}K)/\mathscr{M}_X(S; q^{-1/2}(k+1)^\eta K)\\ &\quad\leq M_3(y)\,C_1(-\nu)\,2^{\nu\alpha(\mathscr{M}_X)}\,(k+1)^{\eta\nu\alpha(\mathscr{M}_X)}.\end{aligned}$$

Hence (5.6), (5.8) and (5.9) show that there is a $C_{12} > 0$ such that [cf. (5.10)]

$$\begin{aligned}&\mathbf{P}\Big\{\inf_{t\in N_\delta} \|X(t)\|^2 < \varepsilon\Big\}\\ &\quad\geq \#N_\delta\,\mathbf{P}\{\|X(s_0)\|^2 < \varepsilon\}\\ &\qquad - \sum_{s\in N_\delta}\sum_{k\in\mathbb{N}}\ \sum_{\{t\in N_\delta\colon\ k^\eta K \leq q^{1/2}d_{\langle y|X\rangle}(s,t) < (k+1)^\eta K\}} \mathbf{P}\{\|X(t)\|^2 < \varepsilon, \|X(s)\|^2 < \varepsilon\}\end{aligned}$$

$$\geq \#N_\delta \mathbf{P}\{\|X(s_0)\|^2 < \varepsilon\}$$
$$\times \left[1 - C_{12}\varepsilon^{n/2}\#N_\delta - C_{12}\sum_{k=1}^{\infty}(k+1)^{\eta\nu\alpha(\mathscr{M}_X)}(k^\eta K)^{-n}\right]$$
$$\geq \tfrac{1}{2}\mathscr{M}_X(S;\varepsilon_5)^{-1}\mathscr{M}_X(S;Kq^{-1/2})\mathbf{P}\{\|X(s_0)\|^2 < \varepsilon\}$$

for ε and K large. Thus the theorem follows from applying (2.3) to $\mathscr{M}_X$. $\square$

6. Examples. In Examples 1–6 that follow it is assumed that $\{\lambda_k\}_{k=1}^{\infty} \subseteq (0,\infty)$ have been choosen so that $m(s) = \sum_{k=1}^{\infty}\lambda_k/(1+2\lambda_k s)$ has positive decrease as $s\to\infty$. Further, in Examples 1, 4 and 5, S, $\hat{S} \subseteq \mathbb{R}^n$ are compact with nonempty interior. There we also write $\|t\| \equiv \sup_{1\leq i\leq n}|t_i|$, $D_S \equiv \sup_{s,\,t\in S}\|t-s\|$ and, choosing an n-dimensional hypercube $K\subseteq S$, $d_S \equiv \sup_{s,\,t\in K}\|t-s\|$.

EXAMPLE 1 (An Ornstein–Uhlenbeck process). Let

$$X(t) \equiv \sum_{k=1}^{\infty}\sqrt{\lambda_k}\,\xi_k(t)e_k \quad \text{for } t\in\mathbb{R}^n,$$

where $\{\xi_k\}_{k=1}^{\infty}$ are independent standardized (Gaussian) such that

$$\mathbf{E}\{\xi_k(s)\xi_k(t+s)\} = \exp\left\{-a_k\sum_{i=1}^{n}|t_i|\right\}$$
$$\text{where } 0 < a_k \leq \sup_{l\in\mathbb{N}} a_l \equiv \bar{a} < \infty.$$

Then independence of the components yields (4.1). Furthermore, we have

$$\exp(-a_k n D_S)a_k\|t-s\| \leq 1-\rho_{s,\,t}^{(k)} = 1-\exp\left\{-a_k\sum_{i=1}^{n}|t_i-s_i|\right\}$$
$$\leq n\bar{a}\,\|t-s\|.$$

Hence (4.2) holds for $y = e_k$ for any $k\in\mathbb{N}$, whereas (5.1) holds for any finite $I\subseteq\mathbb{N}$. Recalling that $d_{\langle e_k|X\rangle}(s,t)^2 = 2\lambda_k(1-\rho_{s,\,t}^{(k)})$, we also conclude that

$$\{s\in\mathbb{R}^n\colon \|s\| \leq \varepsilon^2/(2\lambda_k\bar{a}\,n)\} \subseteq B_{e_k}(0;\varepsilon)$$
$$\subseteq \{s\in\mathbb{R}^n\colon\ \|s\| \leq \exp(a_k n D_S)\varepsilon^2/(2\lambda_k a_k)\}.$$

It follows that $\mathscr{N}_{\langle e_k|X\rangle}(S;\varepsilon)$ is (O-) regularly varying with

$$\left(\exp(-a_k n D_S)\lambda_k a_k d_S\varepsilon^{-2}-1\right)^n \leq \mathscr{N}_{\langle e_k|X\rangle}(S;\varepsilon) \leq \left(\lambda_k\bar{a}\,nD_S\varepsilon^{-2}+1\right)^n.$$

Since by Proposition 3, (5.12) also holds, Theorems 1 and 3 combine to show that

$$0 < \liminf_{\varepsilon\downarrow 0}\frac{\mathbf{P}\{\inf_{t\in S}\|X(t)\|^2 < \varepsilon\}}{q(\varepsilon)^n\mathbf{P}\{\|X(s_0)\|^2 < \varepsilon\}} \leq \limsup_{\varepsilon\downarrow 0}\frac{\mathbf{P}\{\inf_{t\in S}\|X(t)\|^2 < \varepsilon\}}{q(\varepsilon)^n\mathbf{P}\{\|X(s_0)\|^2 < \varepsilon\}} < \infty.$$

EXAMPLE 2 (Independent identically distributed components). Let

$$X(t) \equiv \sum_{k=1}^{\infty} \sqrt{\lambda_k}\,\xi_k(t) e_k \quad \text{for } t \in T,$$

where the processes $\{\xi_1(t)\}_{t\in T}$, $\{\xi_2(t)\}_{t\in T}, \ldots$ are independent identically distributed and standardized. Then (4.1) holds, (4.2) holds for $y = e_k$ for any $k \in \mathbb{N}$ and (5.1) holds for any finite $I \subseteq \mathbb{N}$. Provided that, given an $S \subseteq T$, $\mathscr{N}_{\langle e_k|X\rangle}(S;\varepsilon)$ is O-regularly varying, Theorems 1 and 2 now show that

$$\begin{aligned}
&\limsup_{\varepsilon\downarrow 0} \frac{\mathbf{P}\{\inf_{t\in S}\|X(t)\|^2 < \varepsilon\}}{\mathscr{N}_{\langle e_k|X\rangle}(S; q(\varepsilon)^{-1/2})\mathbf{P}\{\|X(s_0)\|^2 < \varepsilon\}} < \infty,\\
&\liminf_{\varepsilon\downarrow 0} \frac{\mathbf{P}\{\inf_{t\in S}\|X(t)\|^2 < \varepsilon\}}{\varepsilon^{\kappa}\mathscr{N}_{\langle e_k|X\rangle}(S; q(\varepsilon)^{-1/2})\mathbf{P}\{\|X(s_0)\|^2 < \varepsilon\}} = \infty \quad \text{for each } \kappa > 0.
\end{aligned} \tag{6.1}$$

If the processes $\{\xi_k\}_{k=1}^{\infty}$ are stationary and S has nonempty $d_{\langle e_k|X\rangle}$-interior, then (5.12) holds by Proposition 3, and Theorem 3 gives a sharp lower bound.

EXAMPLE 3. Let $X(t) \equiv \sum_{k=1}^{\infty} \sqrt{\lambda_k}\,\xi_k(t) e_k$, where $\{\xi_k\}_{k=1}^{\infty}$ are independent and standardized. Assume that there is an $J \subseteq \mathbb{N}$ such that $\{\xi_j\}_{j\in J}$ are identically distributed and satisfy $\rho_{s,t}^{(j)} \le \rho_{s,t}^{(k)}$ for each choice of $(j,k) \in J \times (\mathbb{N} - J)$. Then (4.1) holds, (4.2) holds for $y = e_j$ for any $j \in J$ and (5.1) holds for finite $I \subseteq J$. If $\mathscr{N}_{\langle e_j|X\rangle}(S;\varepsilon)$ is O-regularly varying for $j \in J$, then Theorem 1 gives an upper bound for $\mathbf{P}\{\inf_{t\in S}\|X(t)\|^2 < \varepsilon\}$. If, in addition, $\#J$ is sufficiently large [to satisfy (5.2)], then Theorem 2 gives a lower bound.

EXAMPLE 4 (A nonstationary Ornstein–Uhlenbeck process generated from fractional Brownian motion; space dependence in T). Choose an $\alpha \in (0,1)$ and let

$$\begin{aligned}
\varrho_{s,t} \equiv{}& \frac{1}{2}\biggl(\sum_{i=1}^{n}\exp(s_i)\biggr)^{-\alpha/2}\biggl(\sum_{i=1}^{n}\exp(t_i)\biggr)^{-\alpha/2}\\
&\times\Biggl[\biggl(\sum_{i=1}^{n}\exp(s_i)\biggr)^{\alpha} + \biggl(\sum_{i=1}^{n}\exp(t_i)\biggr)^{\alpha} - \biggl(\sum_{i=1}^{n}\biggl(\exp\Bigl(\frac{t_i}{2}\Bigr) - \exp\Bigl(\frac{s_i}{2}\Bigr)\biggr)^2\biggr)^{\alpha}\Biggr]
\end{aligned}$$

for $s,t \in \mathbb{R}^n$. Define X as in Example 2, where $\langle e_k|X(t)\rangle = \sqrt{\lambda_k}\,\xi_k(t)$ has correlation $\rho_{s,t}^{(k)} = \varrho_{s,t}$. [Equivalently one may define $\xi_k(t) \equiv (\sum_i \exp t_i)^{-\alpha/2} \cdot W_k(\exp(t_1/2),\ldots,\exp(t_n/2))$, where $\{W_k\}_{k=1}^{\infty}$ are independent fractional Brownian motions satisfying $\mathbf{E}\{W_k(s)W_k(t)\} = |s|^{2\alpha} + |t|^{2\alpha} - |t-s|^{2\alpha}$.] By Example 2, (4.1) holds, (4.2) holds for $y = e_1$ and (5.1) holds for finite $I \subseteq \mathbb{N}$. Furthermore, a Taylor expansion reveals that

$$1 - \varrho_{s,t} = 2^{-(1+2\alpha)}\biggl(\sum_i (t_i - s_i)^2 \exp(s_i)\biggr)^{\alpha} \bigg/ \biggl(\sum_i \exp(s_i)\biggr)^{\alpha} + \Delta_{s,t}\,|t-s|^{\min\{2,1+2\alpha\}},$$

where $\sup\{|\Delta_{s,t}|: |t-s| \le \sigma_1, |s| \le M_4\} < \infty$ for $\sigma_1, M_4 > 0$. It follows that

$$C_{13}^{-1}\|t-s\|^{1/\alpha} \le d_{\langle e_1|X\rangle}(s,t) \le C_{13}\|t-s\|^{1/\alpha} \quad \text{for } \|t-s\| \le \sigma_2$$

for some constants C_{13}, $\sigma_2 > 0$. Now take an $\hat{s}$ in the interior of S and write $\hat{S} = S(\hat{s}) \equiv \{t \in \mathbb{R}^n\colon \|\hat{s}-t\| \le \frac{1}{2}\sigma_2\}$. Then we have

$$\hat{S} \cap B_{e_1}(s;\hat{\varepsilon}) \subseteq \{t \in \mathbb{R}^n\colon \|s-t\| \le (C_{13}\hat{\varepsilon})^\alpha\},$$
$$\{t \in \mathbb{R}^n\colon \|s-t\| \le (C_{13}^{-1}\varepsilon)^\alpha\} \subseteq B_{e_1}(s;\varepsilon) \quad \text{for } \varepsilon \le \varepsilon_7 \equiv C_{13}\sigma_2^{1/\alpha}.$$

Consequently, $\mathscr{N}_{\langle e_1|X\rangle}(\hat{S};\varepsilon)$ is (O-) regularly varying with

$$\left(\tfrac{1}{2}C_{13}^{-\alpha}d_{\hat{S}}\varepsilon^{-\alpha}-1\right)^n \le \mathscr{N}_{\langle e_1|X\rangle}(\hat{S};\varepsilon) \le \left(\tfrac{1}{2}C_{13}^{\alpha}D_{\hat{S}}\varepsilon^{-\alpha}+1\right)^n \quad \text{for } \varepsilon \le \varepsilon_7.$$

The fact that (5.12) also holds (for $\{X(t)\}_{t\in\hat{S}}$) thus follows from observing that

$$\begin{aligned}\mathscr{N}_{\langle e_1|X\rangle}(\hat{S} \cap B_{e_1}(s;\hat{\varepsilon});\varepsilon) &\le \mathscr{N}_{\langle e_1|X\rangle}\big(\{t \in \mathbb{R}^n\colon \|s-t\| \le (C_{13}\hat{\varepsilon})^\alpha\};\varepsilon\big)\\ &\le (C_{13}^{2\alpha}(\hat{\varepsilon}/\varepsilon)^\alpha+1)^n\end{aligned}$$

for $\varepsilon \le \varepsilon_7$. Applying Theorems 1 and 3 we conclude that

$$\begin{aligned}0 &< \liminf_{\varepsilon\downarrow 0}\frac{\mathbf{P}\{\inf_{t\in\hat{S}}\|X(t)\|^2 < \varepsilon\}}{q(\varepsilon)^{n\alpha/2}\mathbf{P}\{\|X(s_0)\|^2 < \varepsilon\}}\\ &\le \limsup_{\varepsilon\downarrow 0}\frac{\mathbf{P}\{\inf_{t\in\hat{S}}\|X(t)\|^2 < \varepsilon\}}{q(\varepsilon)^{n\alpha/2}\mathbf{P}\{\|X(s_0)\|^2 < \varepsilon\}} < \infty.\end{aligned} \tag{6.2}$$

Since $\hat{S} = S(\hat{s}) \subseteq S$ for σ_2 small, while $S \subseteq \bigcup_{i=1}^N S(\hat{s}_i)$ for some choice of $\{\hat{s}_i\}_{i=1}^N \subseteq S$, Boole's inequality shows that (6.2) also holds when $\hat{S}$ is replaced by S.

EXAMPLE 5 ($\mathbb{L}^2$-differentiability; space dependence in H). Let $X(t) \equiv \sum_{i=1}^n X_i(t_i)$ for $t \in \mathbb{R}^n$, where $\{X_1(\tau)\}_{\tau\in\mathbb{R}}, \dots, \{X_n(\tau)\}_{\tau\in\mathbb{R}}$ are independent stationary H-valued variables with common variance $\hat{R}$. Assume that each $X_i(\tau)$ satisfies (4.1) with

$$R_{\tau_1,\tau_2}^{(i)} = r_{\tau_1,\tau_2}^{(i)}\hat{R} \quad \text{where } r_{0,\tau}^{(i)} = 1 - \tfrac{1}{2}r_i''\tau^2 + \Delta_\tau^{(i)}\tau^2 \text{ and } \lim_{\tau\to 0}\|\Delta_\tau^{(i)}\| = 0,$$

for some r_i'', $\Delta_\tau^{(i)} \in \mathscr{L}$. Then $X(t)$ possesses a derivative $X'(t)$ such that $\lim_{h\to 0}\mathbf{E}\{\|h^{-1}[X(t+h)-X(t)]-X'(t)\|^2\} = 0$ and $X(0)$ and $X'(0)$ are independent. Furthermore, $R = n\hat{R}$ and $R_{s,t} = ((1/n)\sum_i r_{s_i,t_i}^{(i)})R$, so that (4.1) holds. It follows that

$$\begin{aligned}d_{\langle z|X\rangle}(0,t)^2 &= \Big\langle \sum_i [r_i'' - 2\Delta_{t_i}^{(i)}]\hat{R}t_i^2 z \;\Big|\; z\Big\rangle\\ &\ge \Big[\inf_i \langle r_i''\hat{R}z|z\rangle - 2\sup_i \|\Delta_{t_i}^{(i)}\hat{R}\|\Big]|t|^2,\\ d_{\langle z|X\rangle}(0,t)^2 &\le \|\hat{R}\|\sup_i[\|r_i''\| + 2\|\Delta_{t_i}^{(i)}\|]\,|t|^2\end{aligned} \tag{6.3}$$

for $z \in H_1$ and, in particular, $\langle r_i'' \hat{R} z | z\rangle \geq 0$. [In fact $r_i'' \hat{R} = \mathrm{Var}\{X_i'(\tau)\}$, so that $r_i'' \hat{R}$ is positive.] We now make the additional assumption that

$$\text{there is a finite } I \subseteq \mathbb{N} \text{ such that } \#I > n\rho(0^+; q) \text{ and}$$
$$\inf_i \inf_{z \in H_1 \cap E_I} \langle r_i'' \hat{R} z | z\rangle > 0.$$

Then (6.3) implies (5.1). Furthermore, we have

$$\|1 - r_{0,t}\| = (2n)^{-1} \left\| \sum_i [r_i'' - 2\Delta_{t_i}^{(i)}] t_i^2 \right\| \leq (2n)^{-1} \sup_i [\|r_i''\| + 2\|\Delta_{t_i}^{(i)}\|] |t|^2.$$

Hence, by (6.3), (4.2) holds for $y \in E_I$. For $D_{\hat{S}}$ sufficiently small we also obtain

$$C_{14}^{-1} \varepsilon^{-n} \leq \mathscr{N}_{\langle y|X\rangle}(\hat{S}; \varepsilon) \leq 1 + C_{14} \varepsilon^{-n}$$

for some $C_{14} > 0$. Thus $\mathscr{N}_{\langle y|X\rangle}(\hat{S}; \cdot)$ is (O-) regularly varying with $\rho(0^+; \mathscr{N}_{\langle y|X\rangle}) = \alpha(0^+; \mathscr{N}_{\langle y|X\rangle}) = n$. [As in Example 4 we can first study a sufficiently small (in terms of $D_{\hat{S}}$) set $\hat{S} \subseteq \mathbb{R}^n$ and then easily extend results to a larger S afterwards.] It is also clear that $\hat{S}$ has nonempty $d_{\langle y|X\rangle}$-interior. Hence, by Proposition 3, (5.12) holds. Since $\rho(0^+; q) \geq 1$ it also follows that (5.13) holds when $\#I > n\rho(0^+; q)$. [If $\rho(0^+; q) < 1$, then there is a $\delta < 1$ such that $\varepsilon^\delta q(\varepsilon)$ is bounded for ε small. However, then $m(s)^\delta s$ is bounded for s large, which contradicts the fact that $m(s)s \to \infty$.] In conclusion, by Theorems 1 and 3, X satisfies (6.2) with $\alpha = 1$.

EXAMPLE 6 (A nonstationary Ornstein–Uhlenbeck process generated from set-indexed Brownian motion). Take X as in Example 2 with $T = \{A \subseteq [0,1]^n\colon A \text{ Borel set}\}$ and $\rho_{A,B} = (|A||B|)^{-1/2} \times |A \cap B|$ for $A, B \in T$, where $|\cdot|$ is the Lebesgue measure. Take a $\delta \in (0, \frac{1}{2})$ and let $S = S(\delta) \equiv \{A \in T\colon |A| \geq \delta^2\}$. Then we have $\langle e_k | X(A)\rangle = \sqrt{\lambda_k/|A|}\, W(A)$, where W is Brownian motion on T with $\mathbf{E}\{W(A)W(B)\} = |A \cap B|$ and $d_W(A,B)^2 = |A| + |B| - 2|A \cap B|$. It follows that

$$\begin{aligned} d_{\langle e_k|X\rangle}(A,B)^2 &= \lambda_k (|A||B|)^{-1/2} \big[2\sqrt{|A||B|} - 2|A \cap B|\big] \\ &\leq \lambda_k \delta^{-2} d_W(A,B)^2 \end{aligned}$$

for $A, B \in S$. On the other hand, we have

$$\begin{aligned} \big(\sqrt{|A|} - \sqrt{|B|}\big)^2 &= \frac{[(|A| \vee |B|) - (|A| \wedge |B|)]^2}{(\sqrt{|A| \vee |B|} + \sqrt{|A| \wedge |B|})^2} \\ &\leq \frac{(|A| \vee |B|)[(|A| \vee |B|) - (|A| \wedge |B|)]}{(|A| \vee |B|) + \delta^2 + 2\delta\sqrt{|A| \vee |B|}} \\ &\leq (1 + 2\delta)^{-1} [(|A| \vee |B|) - (|A| \wedge |B|)] \\ &\leq (1 + 2\delta)^{-1} d_W(A,B)^2 \end{aligned}$$

for $A, B \in S$, which implies that

$$\begin{aligned} d_{\langle e_k|X\rangle}(A,B)^2 &= \frac{\lambda_k[d_W(A,B)^2 - (\sqrt{|A|}-\sqrt{|B|})^2]}{\sqrt{|A||B|}} \\ &\geq \frac{2\delta\lambda_k d_W(A,B)^2}{\delta^2(1+2\delta)} \\ &\geq \delta^{-1}\lambda_k d_W(A,B)^2 \end{aligned}$$

for A, $B \in S$. Given an $\mathbb{R}$-valued Gaussian process $\{\zeta(t)\}_{t\in T}$, if we write $\mathscr{N}_\zeta^{\,0}(S;\varepsilon)$ for the minimum number of closed d_ζ balls of radius ε centered at S needed to cover $S \subseteq T$, then it is an easy exercise in covering numbers to see that $\mathscr{N}_\zeta(S;\varepsilon) \leq \mathscr{N}_\zeta^{\,0}(S;\varepsilon) \leq \mathscr{N}_\zeta(S;\frac{1}{2}\varepsilon)$. Combining this with the facts above, we conclude that

$$\begin{aligned} \mathscr{N}_W\big(S;2\sqrt{\delta/\lambda_k}\,\varepsilon\big) &\leq \mathscr{N}_W^{0}\big(S;2\sqrt{\delta/\lambda_k}\varepsilon\big) \leq \mathscr{N}^{0}_{\langle e_k|X\rangle}(S;2\varepsilon) \\ &\leq \mathscr{N}_{\langle e_k|X\rangle}(S;\varepsilon), \\ \mathscr{N}_{\langle e_k|X\rangle}(S;\varepsilon) &\leq \mathscr{N}^{0}_{\langle e_k|X\rangle}(S;\varepsilon) \leq \mathscr{N}_W^{0}(S;\delta\lambda_k^{-1/2}\varepsilon) \\ &\leq \mathscr{N}_W\big(S;\tfrac{1}{2}\delta\lambda_k^{-1/2}\varepsilon\big). \end{aligned} \tag{6.4}$$

Now consider the pinned Brownian motion $\overline{W}(A) \equiv W(A) - |A|W([0,1]^n)$. Since $(|A|-|B|)^2 \leq (1-\delta^2)d_W(A,B)^2$ for $A, B \in S$, we have

$$\delta^2 d_W(A,B)^2 \leq d_W(A,B)^2 - (|A|-|B|)^2 = d_{\overline{W}}(A,B)^2 \leq d_W(A,B)^2$$

for A, $B \in S$. It follows that

$$\begin{aligned} \mathscr{N}_W(S;\varepsilon) &\geq \mathscr{N}_W^{0}(S;2\varepsilon) \geq \mathscr{N}_{\overline{W}}^{0}(S;2\varepsilon) \geq \mathscr{N}_{\overline{W}}(S;2\varepsilon), \\ \mathscr{N}_W(S;\varepsilon) &\leq \mathscr{N}_W^{0}(S;\varepsilon) \leq \mathscr{N}_{\overline{W}}^{0}(S;\delta\varepsilon) \leq \mathscr{N}_{\overline{W}}\big(S;\tfrac{1}{2}\delta\varepsilon\big). \end{aligned}$$

In view of (6.4) we thus obtain

$$\mathscr{N}_{\overline{W}}\big(S;4\sqrt{\delta/\lambda_k}\varepsilon\big) \leq \mathscr{N}_{\langle e_k|X\rangle}(S;\varepsilon) \leq \mathscr{N}_{\overline{W}}\big(S;\tfrac{1}{4}\delta^2\lambda_k^{-1/2}\varepsilon\big). \tag{6.5}$$

The next lemma is proved in Samorodnitsky [(1991), Remark 6.2 and Example 6.2].

LEMMA 2. *Let* $\overline{S}(\varrho) \equiv \{A \in T\colon \mathbf{E}\{\overline{W}(A)^2\} > \varrho^2\}$ *for* $\varrho \in [0^-, \frac{1}{2}]$. *Then there is a constant* $C_{15} > 0$ *such that*

$$\mathscr{N}_{\overline{W}}\big(\overline{S}(\tfrac{1}{2}-\delta_0);\varepsilon\big) \geq C_{15}^{-1}\varepsilon^{-4n}\sqrt{\delta_0} \quad \textit{for } \delta_0 \in \big[0,\tfrac{1}{2}\big], \tag{6.6}$$

$$\mathscr{N}_{\overline{W}}\big(\overline{S}(\delta_1)-\overline{S}(\delta_2);\varepsilon\big) \leq C_{15}\varepsilon^{-4n}\Big[\sqrt{\delta_2-\delta_1}+\varepsilon^2\Big] \quad \textit{for } 0^- \leq \delta_1 \leq \delta_2 \leq \tfrac{1}{2}. \tag{6.7}$$

In view of the easily established facts that $\overline{S}(\delta\sqrt{1-\delta^2}) = S(\delta^+) - S(\sqrt{1-\delta^2})$ and $T = \overline{S}(0^-) - \overline{S}(\frac{1}{2})$, (6.5)–(6.7) readily combine to show that

$$\begin{aligned} C_{15}^{-1}&\big[\tfrac{1}{2} - \delta\sqrt{1-\delta^2}\,\big]^{1/2}(16\delta/\lambda_k)^{-2n}\varepsilon^{-4n} \\ &\le \mathscr{N}_{\overline{W}}\big(\overline{S}(\delta\sqrt{1-\delta^2});4\sqrt{\delta/\lambda_k}\,\varepsilon\big) \\ &\le \mathscr{N}_{\langle e_k|X\rangle}(S;\varepsilon) \\ &\le \mathscr{N}_{\overline{W}}\big(\overline{S}(0^-)-\overline{S}(\tfrac{1}{2});\tfrac{1}{4}\delta^2\lambda_k^{-1/2}\varepsilon\big) \\ &\le C_{15}(16\lambda_k/\delta^4)^{2n}\big[\sqrt{1/2}+\tfrac{1}{16}\delta^4\varepsilon^2/\lambda_k\big]\varepsilon^{-4n}. \end{aligned}$$

Hence X satisfies (6.1) with $S = S(\delta)$ and $C_{16}^{-1}\varepsilon^{-4n} \le \mathscr{N}_{\langle e_k|X\rangle}(S;\varepsilon) \le C_{16}\varepsilon^{-4n}$.

7. Local extremes for stationary processes. The treatment is specialized to the process $Y(t) = \sum_{k=1}^{\infty}\sqrt{\lambda_k}\,\xi_k(t)e_k$, $t \in T = \mathbb{R}$, where $\{\xi_k\}$ are independent ($\mathbb{R}$-valued) stationary and standardized with covariance functions $\{r_k\}$. Using Albin (1990, 1992b) the upper and lower estimates of Sections 4 and 5 combine with a weak convergence argument linked to the linear structure of $\mathbb{R}$ to give the exact asymptotic behavior of $\mathbf{P}\{\inf_{t\in[0,h]}\|Y(t)\|^2 < \varepsilon\}$ under conditions on $\{r_k\}$.

For easy reference we now state the needed results from Albin (1990, 1992b): Let $\{\kappa(t)\}_{t\ge 0}$ be an $(0,\infty)$-valued separable stationary stochastic process such that $0 < \mathbf{P}\{\kappa(0) < \varepsilon\} \to 0$ as $\varepsilon \downarrow 0$. Assume that there are constants $-\infty \le x_0 < 0 < x_1 \le \infty$, a function $w\colon(0,\infty) \to (0,\infty)$ and a strictly decreasing continuous function $F\colon(x_0,x_1) \to (0,\infty)$ such that

$$\limsup_{\varepsilon\downarrow 0}\mathbf{P}\{\kappa(0) < \varepsilon - xw(\varepsilon)\}\big/\mathbf{P}\{\kappa(0) < \varepsilon\} = F(x) \quad \text{for } x \in (x_0,x_1). \tag{7.1}$$

Further assume that there is a nondecreasing function $p\colon(0,\infty) \to (0,\infty)$ and a stochastic process $\{\chi(t)\}_{t>0}$ such that

$$\begin{aligned} &\text{the finite-dimensional distributions of } \left\{\left(\frac{\kappa(p(\varepsilon)t)-\varepsilon}{w(\varepsilon)}\,\Big|\,\kappa(0) < \varepsilon\right)\right\}_{t>0} \\ &\qquad \to_{\mathscr{D}} \text{ those of } \{\chi(t)\}_{t>0} \text{ as } \varepsilon \downarrow 0. \end{aligned} \tag{7.2}$$

We shall make the additional requirements that, given an $h > 0$,

$$\lim_{N\to\infty}\limsup_{\varepsilon\downarrow 0}\sum_{k=N}^{[h/ap(\varepsilon)]}\mathbf{P}\{\kappa(ap(\varepsilon)k) < \varepsilon \mid \kappa(0) < \varepsilon\} = 0 \quad \text{for each } a > 0, \tag{7.3}$$

and that there are constants A, b, c, d, ε_8, $\eta_1 > 0$ such that

$$\mathbf{P}\{\kappa(p(\varepsilon)t) < \varepsilon - \eta w(\varepsilon),\ \kappa(0) \ge \varepsilon\} \le A\,t^{b+1}\eta^{-c}\,\mathbf{P}\{\kappa(0) < \varepsilon\} \tag{7.4}$$

for $0 < t^d < \eta < \eta_1$ and $\varepsilon \in (0,\varepsilon_8]$. [Here (7.1) implies that $w(\varepsilon) = o(\varepsilon)$.]

LEMMA 3. *Assume that* (7.1)–(7.4) *hold. Then the limit*

$$\lim_{a\downarrow 0} a^{-1}\mathbf{P}\Big\{\inf_{k\geq 1}\chi(ak)\geq 0\Big\}\equiv L$$

exists and $0<L<\infty$. *Furthermore, we have*

$$\mathbf{P}\Big\{\inf_{t\in[0,\,h]}\kappa(t)<\varepsilon\Big\}\sim h\,L\,p(\varepsilon)^{-1}\mathbf{P}\{\kappa(0)<\varepsilon\}\quad \textit{as } \varepsilon\downarrow 0.$$

PROOF. This follows from combining Albin [(1990), Theorem 1] and Albin [(1992b), Proposition 2(i) and the remark following that result]. Of course, although results in Albin (1990, 1992b) are stated for suprema, they are trivially adapted to deal with the infimum of a process $\kappa(t)$ by considering the supremum of $-\kappa(t)$. □

Suppose that there are constants $\alpha\in(0,2]$ and $c_1,c_2,\ldots\in[0,\infty)$ such that

$$r_k(t)\sim 1-c_k|t|^{\alpha}\quad \text{as } t\to 0 \text{ for } k\in\mathbb{N}. \tag{7.5}$$

For the convergence in (7.5) we need a property reminiscent of uniformity for $k\in\mathbb{N}$:

$$M_5\equiv\sup_{k\in\mathbb{N}}\sup_{t\in\mathbb{R}}|t|^{-\alpha}[1-r_k(t)]<\infty. \tag{7.6}$$

Of course, (7.5) implies that there exists an $h>0$ such that

$$\sup_{k\in K} r_k(t)<1\quad \text{for } t\in(0,h]. \tag{7.7}$$

THEOREM 4. *Let* $\{\zeta_k\}_{k=1}^{\infty}$ *be independent zero-mean* $\mathbb{R}$*-valued Gaussian processes with covariances* $\mathbf{E}\{\zeta_k(s)\zeta_k(t)\}=c_k[|s|^{\alpha}+|t|^{\alpha}-|t-s|^{\alpha}]$ *and let* $\mathscr{E}$ *be a unit-mean exponential random variable such that* Y, $\{\zeta_k\}_{k=1}^{\infty}$ *and* $\mathscr{E}$ *are independent. Further put* $Z(t)\equiv\sum_{k=1}^{\infty}\sqrt{\lambda_k}\zeta_k(t)e_k$ *and assume that* $m(s)=\sum_{k=1}^{\infty}\lambda_k/(1+2\lambda_k s)$ *has positive decrease as* $s\to\infty$ *and that* (7.5)–(7.7) *hold. If, in addition,*

$$K\equiv\{k\in\mathbb{N}\colon c_k>0\}\quad \textit{satisfies } \#K>2\rho(0^+;q)/\alpha \tag{7.8}$$

(*with* q *defined as before*), *then the limit*

$$\begin{aligned}&L(\{\lambda_k\},\alpha,\{c_k\})\\&\qquad\equiv\lim_{a\downarrow 0}\frac{1}{a}\mathbf{P}\left\{\inf_{k\geq 1}\left[\|Z(ak)\|^2+\sum_{l=1}^{\infty}\sqrt{\frac{2}{\lambda_l}}\,\langle e_l|Z(ak)\rangle\langle e_l|Y(0)\rangle\right]\geq\mathscr{E}\right\}\end{aligned}$$

exists with $L\in(0,\infty)$. *Moreover, we have*

$$\mathbf{P}\Big\{\inf_{t\in[0,h]}\|Y(t)\|^2<\varepsilon\Big\}\sim h\,L\,q(\varepsilon)^{1/\alpha}\mathbf{P}\{\|Y(0)\|^2<\varepsilon\}\quad \textit{as } \varepsilon\downarrow 0.$$

REMARK 6. Here $Z(t)$ is a well-defined Gaussian process since, by (7.6),

$$\sum_{l=1}^{\infty}\mathbf{Var}\{\sqrt{\lambda_l}\,\zeta_l(t)\} = \sum_{l=1}^{\infty} 2\lambda_l c_l |t|^{\alpha} \le 2\ \mathrm{tr}(R) M_5 |t|^{\alpha} < \infty.$$

Furthermore, $\sum_{l=1}^{\infty}\sqrt{2/\lambda_l}\langle e_l|Z(t)\rangle\langle e_l|Y(0)\rangle$ is well defined since

$$\sum_{l=1}^{\infty}\mathbf{Var}\{\sqrt{2/\lambda_l}\,\langle e_l|Z(t)\rangle\langle e_l|Y(0)\rangle\}$$
$$= \sum_{l=1}^{\infty} 4\lambda_l c_l |t|^{\alpha} \le 4\ \mathrm{tr}(R) M_5 |t|^{\alpha}.$$

LEMMA 4. *Let* $p \equiv q^{-1/\alpha}$, $w \equiv q^{-1}$ *and* $\kappa(t) \equiv \|Y(t)\|^2$. *Then* (7.2) *holds with*

$$\chi(t) \equiv \|Z(t)\|^2 + \sum_{l=1}^{\infty}\sqrt{2/\lambda_l}\langle e_l|Z(t)\rangle\langle e_l|Y(0)\rangle - \mathscr{E}.$$

PROOF. Let $\{\tilde{\xi}_k\}_{k=1}^{\infty}$ be an independent copy of $\{\xi_k\}_{k=1}^{\infty}$ and write $\tilde{\zeta}_k(t) \equiv \tilde{\xi}_k(t) - r_k(t)\tilde{\xi}_k(0)$. Since $\xi_k(t) - r_k(t)\xi_k(0)$ and $\xi_k(0)$ are independent,

$$\begin{aligned} &\text{the finite-dimensional distributions of}\quad q(\|Y(pt)\|^2 - \varepsilon)\\ &=_{\mathscr{D}}\ \text{those of}\ \sum_{k=1}^{\infty} q\lambda_k\big(\tilde{\zeta}_k(pt)^2 + 2\tilde{\zeta}_k(pt) r_k(pt)\xi_k(0)\\ &\qquad - [1 - r_k(pt)^2]\xi_k(0)^2\big) + q(\|Y(0)\|^2 - \varepsilon). \end{aligned} \tag{7.9}$$

Here we have, by (7.5), $q[1 - r_k(pt)^2] \to 2c_k|t|^{\alpha}$, and since

$$\mathrm{Cov}\{q^{1/2}\tilde{\zeta}_k(ps),\ q^{1/2}\tilde{\zeta}_k(pt)\} = q[r_k(p(t-s)) - r_k(ps)r_k(pt)], \tag{7.10}$$

it follows that

$$\text{the finite-dimensional distributions of } q^{1/2}\tilde{\zeta}_k(pt) \to_{\mathscr{D}} \text{ those of } \zeta_k(t). \tag{7.11}$$

Writing $N(0,1)$ for a standardized Gaussian random variable independent of $Y(0)$ [and recalling that $\mathbf{E}\{[N(0,1)]^2\} = \sqrt{2/\pi}$] we further have, by (7.10) and Jensen's inequality and by an application of (4.11) (with $s = t$),

$$\mathbf{E}\left\{\left|\sum_{k=l+1}^{\infty} q\lambda_k\tilde{\zeta}_k(pt) r_k(pt)\xi_k(0)\right| \,\middle|\, \|Y(0)\|^2 < \varepsilon\right\}$$
$$= \mathbf{E}\left\{\left(\sum_{k=l+1}^{\infty} q^2\lambda_k^2[1 - r_k(pt)^2] r_k(pt)^2\xi_k(0)^2\right)^{1/2} |N(0,1)| \,\middle|\, \|Y(0)\|^2 < \varepsilon\right\}$$

$$\leq \sqrt{\frac{2}{\pi}}\biggl(\sum_{k=l+1}^{\infty} q\lambda_k[1-r_k(pt)^2]r_k(pt)^2\mathbf{E}\{q\lambda_k\xi_k(0)^2 \mid \|Y(0)\|^2<\varepsilon\}\biggr)^{1/2}$$

$$\leq \biggl(\frac{2}{\pi}\sum_{k=l+1}^{\infty} q\lambda_k[1-r_k(pt)^2]C_8(1)\biggr)^{1/2}.$$

Since by (7.6), $\sup_{k\in\mathbb{N}} q[1-r_k(pt)^2] \leq 2M_5|t|^\alpha$, it follows that

$$\begin{aligned}
&\limsup_{\varepsilon\downarrow 0}\mathbf{E}\biggl\{\biggl|\sum_{k=l+1}^{\infty} q\lambda_k\bigl(\tilde{\zeta}_k(pt)^2+2\tilde{\zeta}_k(pt)r_k(pt)\xi_k(0)\bigr)\\
&\qquad\qquad -\sum_{k=1}^{\infty} q\lambda_k[1-r_k(pt)^2]\xi_k(0)^2\biggr| \biggm| \|Y(0)\|^2<\varepsilon\biggr\}\\
&\quad\leq \limsup_{\varepsilon\downarrow 0}\sum_{k=l+1}^{\infty} q\lambda_k[1-r_k(pt)^2]\\
&\qquad + \limsup_{\varepsilon\downarrow 0}\biggl((8/\pi)\sum_{k=l+1}^{\infty} q\lambda_k[1-r_k(pt)^2]C_8\biggr)^{1/2}\\
&\qquad + \limsup_{\varepsilon\downarrow 0} q\sup_{k\in\mathbb{N}}[1-r_k(pt)^2]\mathbf{E}\{\|Y(0)\|^2 \mid \|Y(0)\|^2<\varepsilon\}\\
&\quad\leq 2M_5|t|^\alpha\sum_{k=l+1}^{\infty}\lambda_k+\sqrt{16C_8M_5|t|^\alpha/\pi}\biggl(\sum_{k=l+1}^{\infty}\lambda_k\biggr)^{1/2}\\
&\qquad + 2M_5|t|^\alpha\limsup_{\varepsilon\downarrow 0}\varepsilon,
\end{aligned}$$

where the right-hand side tends to zero as $l\to\infty$. In view of (7.9) and (7.11), (7.2) will therefore follow if we can prove that $W\equiv Y(0)$ satisfies

$$\begin{aligned}
&\bigl(q^{1/2}\langle W|e_1\rangle,\ldots,\ q^{1/2}\langle W|e_l\rangle,\ q(\|W\|^2-\varepsilon)\bigm| \|W\|^2<\varepsilon\bigr)\\
&\qquad \to_{\mathscr{D}} \bigl((2\lambda_1)^{-1/2}\langle W|e_1\rangle,\ldots,(2\lambda_l)^{-1/2}\langle W|e_l\rangle,\ -\mathscr{E}\bigr)
\end{aligned}\tag{7.12}$$

for each $l\in\mathbb{N}$. However, by (3.2) and (3.4), we have

$$\begin{aligned}
&f_{q^{1/2}\langle W|e_1\rangle,\ldots,\,q^{1/2}\langle W|e_l\rangle\,|\,q(\|W\|^2-\varepsilon)}(y_1,\ldots,y_l \mid x)\\
&\quad=\frac{f_{\|W_l\|^2}(\varepsilon+(x-\sum_{k=1}^l y_k^2)/q)\prod_{k=1}^l f_{\sqrt{\lambda_k}\,N(0,1)}(y_k/q^{1/2})}{q^{l/2}f_{\|W\|^2}(\varepsilon+x/q)}\\
&\quad\to\prod_{k=1}^{l} f_{\sqrt{1/2}\,N(0,1)}(y_k)\quad\text{for}\quad (y_1,\ldots,y_l,x)\in\mathbb{R}^{l+1}.
\end{aligned}$$

Since, by (3.2), $f_{q(\|W\|^2-\varepsilon)\,|\,\|W\|^2<\varepsilon}(x) \to e^x$ for $x<0$, it follows that

$$\begin{aligned} &f_{q^{1/2}\langle W|e_1\rangle,\dots,q^{1/2}\langle W|e_l\rangle,\,q(\|W\|^2-\varepsilon)\,|\,\|W\|^2<\varepsilon}(y_1,\dots,y_l,x)\\ &\quad= f_{q^{1/2}\langle W|e_1\rangle,\dots,q^{1/2}\langle W|e_l\rangle\,|\,q(\|W\|^2-\varepsilon)}(y_1,\dots,y_l\,|\,x)\, f_{q(\|W\|^2-\varepsilon)\,|\,\|W\|^2<\varepsilon}(x)\\ &\quad\to \prod_{k=1}^{l} f_{\sqrt{1/2}\,N(0,1)}(y_k)\, e^x\\ &\quad= f_{(2\lambda_1)^{-1/2}\langle W|e_1\rangle,\dots,(2\lambda_l)^{-1/2}\langle W|e_l\rangle,\,-\mathscr{E}}(y_1,\dots,y_l,x) \end{aligned}$$

for $(y_1,\dots,y_l,x)\in\mathbb{R}^l\times\mathbb{R}^+$. Hence (7.12) holds. $\square$

LEMMA 5. *The condition* (7.3) *holds for* $\kappa(t)=\|Y(t)\|^2$.

PROOF. First we observe that, by (7.6),

$$\begin{aligned} d_{\langle y|Y\rangle}(s,t)^2 &= 2\sum_{k=1}^{\infty}\lambda_k\langle y|e_k\rangle^2[1-r_k(t-s)]\\ &\le 2M_5|t-s|^{\alpha}\Big(\sup_{k\in\mathbb{N}}\lambda_k\Big) \end{aligned}\tag{7.13}$$

for $y\in H_1$. On the other hand, by (7.5), there is an $h_1>0$ such that

$$d_{\langle y|Y\rangle}(s,t)^2 \ge 2\inf_{k\in K}\lambda_k[1-r_k(t-s)] \ge |t-s|^{\alpha}\Big(\inf_{k\in K}\lambda_k c_k\Big)\tag{7.14}$$

for $y\in H_1\cap E_K$ and $|t-s|\le h_1$. Thus (5.1) holds for $I=K$. Moreover, (4.1) holds with $r_{s,t}e_k=r_k(t-s)e_k$. In view of (7.5) and (7.14) it follows that

$$\frac{\|1-r_{s,t}\|}{d_{\langle y|Y\rangle}(s,t)^2} \le \frac{M_5}{(\inf_{k\in K}\lambda_k c_k)} \quad \text{for } y\in H_1\cap E_K \text{ and } |t-s|\le h_1.$$

Hence (4.2) also holds. Writing $n\equiv \#K$, an application of (5.6) [which requires (4.1), (4.2) and (5.1), but not (5.2), (5.3) or O-regular variation] now shows that

$$\begin{aligned} &\mathbf{P}\{\|Y(t)\|^2<\varepsilon\,|\,\|Y(s)\|^2<\varepsilon\}\\ &\quad\le C_{17}[\varepsilon^{n/2}+p^{n\alpha/2}|t-s|^{-n\alpha/2}] \quad \text{for } |t-s|\le h_2 \end{aligned}$$

for some constants $h_2, C_{17}>0$. Furthermore, by (7.7), there is a $C_{18}\ge C_{17}$ such that

$$\begin{aligned} &\mathbf{P}\{\|Y(t)\|^2<\varepsilon\;\big|\;\|Y(0)\|^2<\varepsilon\}\\ &\quad= \mathbf{P}\bigg\{\sum_{k=1}^{\infty}\lambda_k[\tilde{\zeta}_k(t)+r_k(0)\xi_k(0)]^2<\varepsilon\;\Big|\;\|Y(0)\|^2<\varepsilon\bigg\} \end{aligned}$$

$$\leq \mathbf{P}\bigg\{ \sum_{k\in K} \lambda_k \tilde{\zeta}_k(t)^2 < (\sqrt{\varepsilon} + \|Y(0)\|)^2 \,\Big|\, \|Y(0)\|^2 < \varepsilon \bigg\}$$

$$\leq \prod_{k\in K} \mathbf{P}\{\lambda_k [1 - r_k(0)^2][N(0,1)]^2 < 4\varepsilon\}$$

$$\leq C_{18}\, \varepsilon^{n/2} \quad \text{for } \min\{h, h_2\} \leq |t-s| \leq h$$

[where $\tilde{\zeta}_k(t)$ is defined as in the proof of Lemma 4]. It follows readily that

$$\sum_{k=N}^{[h/(ap)]} \mathbf{P}\{\|Y(apk)\|^2 < \varepsilon \,|\, \|Y(0)\|^2 < \varepsilon\}$$

$$\leq C_{18}\bigg[(h/a)\varepsilon^{n/2} q^{1/\alpha} + \sum_{k=N}^{\infty} (ak)^{-n\alpha/2}\bigg].$$

Here, since by (7.8), $n\alpha/2 > \rho(q) \geq 1$ (cf. Example 5), the right-hand side tends to $\sum_{k=N}^{\infty}(ak)^{-n\alpha/2}$ as $\varepsilon \downarrow 0$, which in turn tends to 0 as, $N \to \infty$. □

PROOF OF THEOREM 4. Clearly, by (3.1), (7.1) holds with $F(x) \equiv e^{-x}$ (and $w = q^{-1}$). To verify (7.4) we take a $j \in \mathbb{N}$ such that $j\alpha > 1$. Using (7.13), an application of (4.12) with $\nu = 0$ then shows that there is a $\eta_2 > 0$ such that

$$\mathbf{P}\{\|Y(pt)\|^2 < \varepsilon - \eta/q,\ \|Y(0)\|^2 \geq \varepsilon\}$$

$$\leq C_9(j) d_{\langle y|Y\rangle}(0, p(\varepsilon)t)^{2j} q^j \eta^{-2j} \mathbf{P}\{\|Y(0)\|^2 < \varepsilon\}$$

$$\leq 2^j C_9 M_5^j t^{j\alpha} \eta^{-2j} \Big(\sup_{k\in K} \lambda_k\Big)^j \mathbf{P}\{\|Y(0)\|^2 < \varepsilon\}$$

for $0 < t^\alpha < \eta < \eta_2$ and ε small. [Note that, by (7.13), the condition $d_{\langle y|X\rangle}(0, pt)^2 q \leq C_6 \eta$ required for the validity of (4.12) is satisfied when $t^\alpha < \eta$.] In view of Lemmas 4 and 5, Theorem 4 now follows from Lemma 3. □

8. Global limits for stationary processes. By well-established principles in extremal theory, control of local extremes combines with a suitable global mixing property to imply one of three possible global limit results. However, it is often difficult to prove mixing since that involves manipulation of and verification of sharp quantitative results for finite-dimensional distributions of arbitrary order. In Lemma 11 we will show that a careful adaption of finite Gaussian ideas of Berman (1964) and Sharpe [(1978), pages 384–387] [in the sequence of estimates (8.5)] combine with estimates needed to allow passage to infinite dimensions to verify the required mixing property. Then we use Albin (1990) to prove global limits.

For easy reference we now state the needed results from Albin (1990). Assume that (7.1)–(7.4) hold and let $T(\varepsilon) \sim p(\varepsilon)/(L\,\mathbf{P}\{\kappa(0) < \varepsilon\})$ as $\varepsilon \downarrow 0$ (where L is given in Lemma 3). Further assume that

(8.1) $$p(\varepsilon + xw(\varepsilon)) \sim p(\varepsilon) \quad \text{as } \varepsilon \downarrow 0 \text{ for } x \in (x_0, x_1).$$

CONDITION D. This condition holds if, for any choice of $a > 0$ and $\tau \in (0, 1)$ and for any choice of points $s_1 < \cdots < s_I < t_1 < \cdots < t_J$ belonging to $\{akp(\varepsilon)\colon k \in \mathbb{Z}, 0 \le akp(\varepsilon) \le T(\varepsilon)\}$ and satisfying $t_1 - s_I \ge \tau T(\varepsilon)$, we have, as $\varepsilon \downarrow 0$,

$$\begin{aligned}\Bigg|\mathbf{P}\Bigg\{\bigcap_{i=1}^{I}\{\kappa(s_i) \ge \varepsilon\} \cap \bigcap_{j=1}^{J}\{\kappa(t_j) \ge \varepsilon\}\Bigg\}\\ - \mathbf{P}\Bigg\{\bigcap_{i=1}^{I}\{\kappa(s_i) \ge \varepsilon\}\Bigg\}\mathbf{P}\Bigg\{\bigcap_{j=1}^{J}\{\kappa(t_j) \ge \varepsilon\}\Bigg\}\Bigg| \to 0.\end{aligned}$$

CONDITION D$'$. This condition holds if, for any choice of $a > 0$, we have

$$\lim_{\tau\downarrow 0}\limsup_{\varepsilon\downarrow 0}\sum_{k=1+[h/ap(\varepsilon)]}^{[\tau T(\varepsilon)/ap(\varepsilon)]}\mathbf{P}\{\kappa(akp(\varepsilon)) < \varepsilon \mid \kappa(0) < \varepsilon\} = 0.$$

DEFINITION 1. We say that κ has a δ-downcrossing of the level ε at t if $\kappa(t) = \varepsilon$ and $\kappa(s) > \varepsilon$ for $s \in (t - \delta, t)$.

LEMMA 6. *If* (7.1)–(7.4), (8.1) *and Conditions* D *and* D$'$ *hold, then we have*

$$\begin{aligned}\lim_{\varepsilon\downarrow 0}\mathbf{P}\Big\{w(\varepsilon)^{-1}\Big[\inf_{t\in[0,\,T(\varepsilon)]}\kappa(t) - \varepsilon\Big] \le x\Big\}\\ = 1 - \exp\{-F(-x)\} \quad \textit{for } x \in (x_0, x_1).\end{aligned}$$

If, in addition, $\kappa(t)$ is a.s. continuous, then

$$N_\varepsilon(\kappa; A) \equiv \#\{t \in T(\varepsilon)A\colon \kappa \textit{ has a } \delta\textit{-downcrossing of } \varepsilon \textit{ at } t\} \quad \textit{for } A \subseteq \mathbb{R}^+$$

converges weakly as a random measure to a Poisson process with intensity 1.

PROOF. Since, by (8.1), Albin [(1990), equation 2.15] holds with $c = 0$, the lemma follows from Albin [(1990), Theorems 2(c) and 10]. $\square$

THEOREM 5. *Let $m(s)$ have positive decrease as $s \to \infty$. Further let $T(\varepsilon) \sim [Lq(\varepsilon)^{1/\alpha}\mathbf{P}\{\|Y(0)\|^2 < \varepsilon\}]^{-1}$ (where L is given in Theorem* 4) *and assume that*

$$r(t) \equiv \sup_{s\ge t,\, k\in\mathbb{N}}|r_k(s)| \quad \textit{satisfies} \quad \lim_{\varepsilon\downarrow 0}q(\varepsilon)^2 r(T(\varepsilon)) = 0. \tag{8.2}$$

If, in addition, (7.5), (7.6) *and* (7.8) *hold, then we have*

$$\lim_{\varepsilon\downarrow 0}\mathbf{P}\Big\{q(\varepsilon)\Big[\inf_{t\in[0,\,T(\varepsilon)]}\|Y(t)\|^2 - \varepsilon\Big] \le x\Big\} = 1 - \exp\{-e^x\} \quad \textit{for } x \in \mathbb{R}.$$

Moreover, $N_\varepsilon(\|Y\|^2; \cdot)$ converges weakly to a Poisson process with intensity 1.

LEMMA 7. (i) $\lim_{\varepsilon\downarrow 0} q(\varepsilon)^2 r(\tau_1 T(\tau_2\varepsilon)) = 0$ *for* $\tau_1, \tau_2 > 0$.
(ii) *The right inverse* $r^{\to}(t) \equiv \sup\{s > 0: r(s) > t\}$ *satisfies*

$$\lim_{\varepsilon\downarrow 0} q(\varepsilon)^{1/\alpha}\,\mathbf{P}\{\|Y(0)\| < \tau\varepsilon\}\, r^{\to}([\varepsilon q(\varepsilon)+1]^{-1/2}) = 0 \quad \textit{for each } \tau > 0.$$

PROOF. (i) By (2.6) and (3.1) we have $q(\varepsilon - \ln(\frac{1}{2}\tau_1)/q(\varepsilon)) \sim q(\varepsilon)$ and $T(\varepsilon - \ln(\frac{1}{2}\tau_1)/q(\varepsilon)) \sim \frac{1}{2}\tau_1 T(\varepsilon)$, so that $T(\varepsilon - \ln(\frac{1}{2}\tau_1)/q(\varepsilon)) \le \tau_1 T(\varepsilon)$ for ε small. Since q is O-regularly varying we get, by [the change of variable $\varepsilon \to \varepsilon/\tau_2$ and] (8.2),

$$\begin{aligned}
&\limsup_{\varepsilon\downarrow 0} q(\varepsilon)^2 r(\tau_1 T(\tau_2\varepsilon))\\
&\qquad \le \Psi^\star(0^+, q; 1\wedge \tau_2^{-1})^2 \limsup_{\varepsilon\downarrow 0} q(\varepsilon)^2 r(\tau_1 T(\varepsilon))\\
&\qquad \le \Psi^\star(0^+, q; 1\wedge \tau_2^{-1})^2 \limsup_{\varepsilon\downarrow 0} q(\varepsilon - \ln(\tfrac{1}{2}\tau_1)/q)^2 r(T[\varepsilon - \ln(\tfrac{1}{2}\tau_1)/q])\\
&\qquad = 0.
\end{aligned}$$

(ii) Since by (i), $r(T(\frac{1}{2}\tau\varepsilon)) < q(\varepsilon)^{-2} \le [\varepsilon q(\varepsilon)+1]^{-1/2}$ for ε small, we have $r^{\to}([\varepsilon q(\varepsilon)+1]^{-1/2}) \le T(\frac{1}{2}\tau\varepsilon)$. Consequently,

$$\begin{aligned}
&q\left(\frac{1}{2}\tau\varepsilon\right)^{1/\alpha}\mathbf{P}\left\{\|Y(0)\| < \frac{1}{2}\tau\varepsilon\right\} r^{\to}([\varepsilon q(\varepsilon)+1]^{-1/2})\\
&\qquad \sim \frac{r^{\to}([\varepsilon q(\varepsilon)+1]^{-1/2})}{L\,T(\frac{1}{2}\tau\varepsilon)} \le \frac{2}{L}
\end{aligned}$$

for ε small. Since by (2.8) and (3.1), $\mathbf{P}\{\|W\| < \tau\varepsilon\}/\mathbf{P}\{\|W\| < \frac{1}{2}\tau\varepsilon\} \to 0$, (ii) now follows using that $q(\frac{1}{2}\tau\varepsilon) \ge \frac{1}{2}\Psi^\star(0^+, q; 1\wedge(2/\tau))^{-1} q(\varepsilon)$ for ε small. □

The next lemma is contained in Hoffmann-Jørgensen, Shepp and Dudley [(1979), Theorem 2.1].

LEMMA 8. *For every H-valued centered Gaussian random variable N we have*

$$\mathbf{P}\{\|N\| < \varepsilon\} \ge \mathbf{P}\{\|N+z\| < \varepsilon\} \quad \textit{for each } \varepsilon > 0 \textit{ and } z \in H.$$

LEMMA 9. *If* $\lim_{t\to\infty} r(t) = 0$, *then* $r(t) < 1$ *for* $t > 0$ *and* (7.7) *holds.*

PROOF. Assume that $r(t) = 1$ for some $t > 0$. Then there exists $\{t_k\}_{k=1}^{\infty} \subseteq [t, \infty)$ satisfying $r_k(t_k) > 1 - k^{-1}$. Now choose $\hat{t} \in (0, \infty)$ and $\varepsilon > 0$. Further, take $k, n \in \mathbb{N}$ such that $nt \geq \hat{t}$ and $\sqrt{2/k} < \varepsilon/n$. Then we have

$$\begin{aligned} r_k(nt_k) &= r_k((n-1)t_k) + \mathbf{E}\{[\xi_k(nt_k) - \xi_k((n-1)t_k)]\xi_k(0)\} \\ &\geq r_k((n-1)t_k) - \sqrt{\mathbf{E}\{[\xi_k(nt_k) - \xi_k((n-1)t_k)]^2\}} \\ &\geq r_k((n-1)t_k) - \varepsilon/n \\ &\geq \cdots \geq 1 - \varepsilon, \end{aligned}$$

so that $r(\hat{t}) \geq 1 - \varepsilon$. Hence $r(\hat{t}) = 1$, which contradicts the fact that $r(t) \to 0$. □

LEMMA 10. *Condition* D′ *holds*.

PROOF. Since $\hat{Y}(t) \equiv \sum_{k=1}^{\infty} \langle e_k | Y(t) - r_k(t) Y(0) \rangle e_k$ is independent of $Y(0)$ with $\operatorname{Var}\{\langle e_k | Y(t) - r_k(t) Y(0) \rangle\} \geq [1 - r(t)^2] \operatorname{Var}\{\langle e_k | Y(0) \rangle\}$, we have, by Lemma 8,

$$\begin{aligned} &\mathbf{P}\{\|Y(t)\|^2 < \varepsilon, \|Y(0)\|^2 < \varepsilon\} \\ &\quad = \mathbf{E}\left\{\mathbf{P}\left\{\left\|\hat{Y}(t) + \sum_{k=1}^{\infty} r_k(t) \langle e_k | Y(0) \rangle e_k\right\|^2 < \varepsilon \,\middle|\, Y(0)\right\} I_{\{\|Y(0)\|^2 < \varepsilon\}}\right\} \\ &\quad \leq \mathbf{E}\big\{\mathbf{P}\{\|\hat{Y}(t)\|^2 < \varepsilon\} I_{\{\|Y(0)\|^2 < \varepsilon\}}\big\} \\ &\quad \leq \mathbf{P}\big\{\|Y(0)\|^2 < \varepsilon\big/[1 - r(t)^2]\big\} \mathbf{P}\{\|Y(0)\|^2 < \varepsilon\} \quad \text{for } t > 0. \end{aligned} \tag{8.3}$$

[Here $r(t) < 1$ by (8.2) and Lemma 9.] Since $\varepsilon/[1 - r(t)^2] \geq \varepsilon + 1/q(\varepsilon) \Rightarrow t \leq r^{\rightarrow}([\varepsilon q(\varepsilon) + 1]^{-1/2})$, Lemma 7(ii) and (3.1) yield

$$\begin{aligned} &\limsup_{\varepsilon \downarrow 0} \sum_{k=1+[h/(ap)]}^{[\tau T/(ap)]} \mathbf{P}\{\|Y(akp)\|^2 < \varepsilon \mid \|Y(0)\|^2 < \varepsilon\} \\ &\quad \leq \limsup_{\varepsilon \downarrow 0} (ap)^{-1} r^{\rightarrow}([\varepsilon q(\varepsilon) + 1]^{-1/2}) \mathbf{P}\{\|Y(0)\|^2 < \varepsilon/[1 - r(h)^2]\} \\ &\qquad + \limsup_{\varepsilon \downarrow 0} (\tau T/(ap)) \mathbf{P}\{\|Y(0)\|^2 < \varepsilon + 1/q(\varepsilon)\} \\ &\quad = \tau e/(aL) \to 0 \quad \text{as } \tau \downarrow 0. \end{aligned}$$

□

LEMMA 11. *Condition* D *holds*.

PROOF. Choose $\rho = (\rho_1, \ldots, \rho_n) \in [-1, 1]^n$ and let $\{\zeta_k\}_{k=1}^n$ be independent standard Gaussian processes with parameter space $\{1, 2\}$ and such that $\mathbf{E}\{\zeta_k(1)\zeta_k(2)\} = \rho_k$. Further write $\varrho \equiv \sup_{1 \leq k \leq n} \rho_k$, let k_0 satisfy $\lambda_{k_0} = \sup_{k \geq 2} \lambda_k$ and define

$$Z_k^{\rho, n}(t) \equiv \sum_{\substack{2 \leq l \leq n \\ l \neq k}} \lambda_l \zeta_l(t)^2 \quad \text{and} \quad X_k^{\rho, n}(t) \equiv Z_k^{\rho, n}(t) + \lambda_1 \zeta_1(t)^2$$

for $n \geq k \geq 2$. Since $e^{-x/2} \leq (ex)^{-1/2}$ and

$$f_{\lambda_1\zeta_1(1)^2,\,\lambda_1\zeta_1(2)^2}(x_1,\,x_2) \leq \Big[2\pi\lambda_1\sqrt{(1-\rho_1{}^2)x_1x_2}\,\Big]^{-1},$$

we obtain, writing $B(\cdot,\cdot)$ for the beta function and using (8.3),

$$\begin{aligned}
&\mathbf{E}\bigg\{\prod_{l=1}^{2}\exp\bigg[-\frac{\varepsilon - X_k^{\rho,n}(l)}{2\lambda_k(1+|\varrho|)}\bigg]I_{\{X_k^{\rho,n}(l)<\varepsilon\}}\bigg\}\\
&\quad\leq \frac{\lambda_k(1+|\varrho|)}{e}\\
&\qquad\times \mathbf{E}\big\{(\varepsilon - X_k^{\rho,n}(1))^{-1/2}(\varepsilon - X_k^{\rho,n}(2))^{-1/2}I_{\{X_k^{\rho,n}(1)<\varepsilon,\,X_k^{\rho,n}(2)<\varepsilon\}}\big\}\\
&\quad= \frac{\lambda_k(1+|\varrho|)}{e}\\
&\qquad\times \int_{\substack{0<x_1<\varepsilon-z_1\\ 0<x_2<\varepsilon-z_2\\ 0<z_1,\,z_2<\varepsilon}} \frac{f_{\lambda_1\zeta_1(1)^2,\,\lambda_1\zeta_1(2)^2}(x_1,\,x_2)\,f_{Z_k^{\rho,n}(1),\,Z_k^{\rho,n}(2)}(z_1,\,z_2)}{(\varepsilon - x_1 - z_1)^{1/2}(\varepsilon - x_2 - z_2)^{1/2}}\,dx\,dz\\
&\quad= \frac{\lambda_k(1+|\varrho|)}{e}\int_{0<x_1,\,x_2<1,\,0<z_1,\,z_2<\varepsilon}\frac{z_1^{1/2}z_2^{1/2}f_{\lambda_1\zeta_1(1)^2,\,\lambda_1\zeta_1(2)^2}(x_1z_1,\,x_2z_2)}{(1-x_1)^{1/2}(1-x_2)^{1/2}} \qquad (8.4)\\
&\qquad\qquad\times f_{Z_k^{\rho,n}(1),\,Z_k^{\rho,n}(2)}(\varepsilon - z_1,\varepsilon - z_2)\,dx\,dz\\
&\quad\leq \frac{B(\frac{1}{2},\frac{1}{2})^2\lambda_k\sqrt{1+|\varrho|}}{2\pi e\lambda_1\sqrt{1-|\varrho|}}\int_{0<z_1,\,z_2<\varepsilon} f_{Z_k^{\rho,n}(1),\,Z_k^{\rho,n}(2)}(\varepsilon - z_1,\varepsilon - z_2)\,dz\\
&\quad\leq \frac{B(\frac{1}{2},\frac{1}{2})^2\lambda_k}{\pi e\lambda_1\sqrt{2(1-|\varrho|)}}\mathbf{P}\{Z_k^{\rho,n}(1)<\varepsilon,\ Z_k^{\rho,n}(2)<\varepsilon\}\\
&\quad\leq \frac{B(\frac{1}{2},\frac{1}{2})^2\lambda_k}{\pi e\lambda_1\sqrt{2(1-|\varrho|)}}\mathbf{P}\bigg\{Z_{k_0}^{\hat\rho,n}(1)<\frac{\varepsilon}{1-\varrho^2}\bigg\}\mathbf{P}\{Z_{k_0}^{\hat\rho,n}(1)<\varepsilon\}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{for any } \hat\rho\in[-1,1]^n.
\end{aligned}$$

Let $\eta_1(t), \nu_1(t), \ldots, \eta_n(t), \nu_n(t)$ be independent centered stationary Gaussian processes and choose $u_1, \ldots, u_N \in \mathbb{R}$. Define the $N\,|\,N$ matrices $\underline{\Gamma}_k, \underline{\Sigma}_k \in \mathbb{R}_{N|N}$ by

$$(\underline{\Gamma}_k)_{ij} \equiv \mathrm{Cov}\{\eta_k(u_i), \eta_k(u_j)\} \quad\text{and}\quad (\underline{\Sigma}_k)_{ij} \equiv \mathrm{Cov}\{\nu_k(u_i), \nu_k(u_j)\}$$

and assume that $(\underline{\Gamma}_k)_{ii} = (\underline{\Sigma}_k)_{ii} = \lambda_k$. Also define $\underline{C}_k^h \equiv h\underline{\Gamma}_k + (1-h)\underline{\Sigma}_k \equiv \lambda_k\underline{c}_k^h$. Further write $f(\underline{\Lambda};\cdot)$ for a centered Gaussian density based on the covariance matrix $\underline{\Lambda} \in \mathbb{R}_{k|k}$ and define ${}^{ij}\underline{\Lambda} \in \mathbb{R}_{2|2}$ by $({}^{ij}\underline{\Lambda})_{11} \equiv \underline{\Lambda}_{ii}$, $({}^{ij}\underline{\Lambda})_{12} \equiv \underline{\Lambda}_{ij}$, $({}^{ij}\underline{\Lambda})_{21} \equiv \underline{\Lambda}_{ji}$ and $({}^{ij}\underline{\Lambda})_{22} \equiv \underline{\Lambda}_{jj}$. For the χ^2 processes $Y^{(n)}(t) \equiv \sum_{k=1}^n \eta_k(t)^2$ and $Z^{(n)}(t) \equiv$

$\sum_{k=1}^n \nu_k(t)^2$ we then obtain, using the elementary facts that $(\partial/\partial\underline{\Lambda}_{ij})f(\underline{\Lambda};x) = (\partial/\partial x_i)(\partial/\partial x_j)f(\underline{\Lambda};x)$ and that $(y^2+z^2-2\hat{\varrho}yz)/(1-\hat{\varrho}^2) \geq (y^2+z^2)/(1+|\hat{\varrho}|)$,

$$\begin{aligned}
&\mathbf{P}\Big\{\bigcap_{i=1}^N \{Y^{(n)}(u_i)\geq\varepsilon\}\Big\} - \mathbf{P}\Big\{\bigcap_{i=1}^N \{Z^{(n)}(u_i)\geq\varepsilon\}\Big\}\\
&\quad= \int_{\substack{h\in[0,1]\\ \bigcap_{m=1}^N\{\sum_{l=1}^n (x_m^l)^2\geq\varepsilon\}}} \frac{\partial}{\partial h}\prod_{l=1}^n f(\underline{C}_l^h;x^l)\,dx\,dh\\
&\quad= \int_{\substack{h\in[0,1]\\ \bigcap_{m=1}^N\{\sum_{l=1}^n (x_m^l)^2\geq\varepsilon\}}} \sum_{\substack{1\leq i<j\leq N\\ 1\leq k\leq n}} ((\underline{\Gamma}_k)_{ij}-(\underline{\Sigma}_k)_{ij})\partial_{ij} f(\underline{C}_k^h;x^k)\\
&\qquad\qquad\times\prod_{l\neq k} f(\underline{C}_l^h;x^l)\,dx\,dh\\
&\quad= \sum_{\substack{1\leq i<j\leq N\\ 1\leq k\leq n}} \int_{\substack{h\in[0,1]\\ \bigcap_{m\neq i,j}\{\sum_{l=1}^n (x_m^l)^2\geq\varepsilon\}\\ \sum_{l=1}^n (x_i^l)^2<\varepsilon,\\ \sum_{l=1}^n (x_j^l)^2<\varepsilon}} ((\underline{\Gamma}_k)_{ij}-(\underline{\Sigma}_k)_{ij})\partial_{ij} f(\underline{C}_k^h;x^k)\\
&\qquad\qquad\times\prod_{l\neq k} f(\underline{C}_l^h;x^l)\,dx\,dh
\end{aligned}$$

$$\begin{aligned}
(8.5)\quad&= \sum_{\substack{1\leq i<j\leq N\\ 1\leq k\leq n\\ \sigma_1,\sigma_2\in\{-1,1\}}} \int_{\substack{h\in[0,1]\\ \bigcap_{m\neq i,j}\{\sum_{l=1}^n (x_m^l)^2\geq\varepsilon\}\\ \sum_{l\neq k}(x_i^l)^2<\varepsilon\\ \sum_{l\neq k}(x_j^l)^2<\varepsilon}} \sigma_1\sigma_2((\underline{\Gamma}_k)_{ij}-(\underline{\Sigma}_k)_{ij})\Big(\prod_{l\neq k} f(\underline{C}_l^h;x^l)\Big)\\
&\qquad\times f\Big(\underline{C}_k^h;x^k,\ x_i^k=\sigma_1\sqrt{\varepsilon-\sum_{l\neq k}(x_i^l)^2},\ x_j^k=\sigma_2\sqrt{\varepsilon-\sum_{l\neq k}(x_j^l)^2}\Big)\,dx\,dh\\
&\leq \sum_{\substack{1\leq i<j\leq N\\ 1\leq k\leq n\\ \sigma_1,\sigma_2\in\{-1,1\}}} \int_{\substack{h\in[0,1]\\ \sum_{l\neq k}(x_i^l)^2<\varepsilon,\\ \sum_{l\neq k}(x_j^l)^2<\varepsilon}} |(\underline{\Gamma}_k)_{ij}-(\underline{\Sigma}_k)_{ij}|\Big(\prod_{l\neq k} f({}^{ij}\underline{C}_l^h;x^l)\Big)\\
&\qquad\times f\Big({}^{ij}\underline{C}_k^h;\sigma_1\sqrt{\varepsilon-\sum_{l\neq k}(x_i^l)^2},\ \sigma_2\sqrt{\varepsilon-\sum_{l\neq k}(x_j^l)^2}\Big)\,dx\,dh\\
&\leq 4\sum_{\substack{1\leq i<j\leq N\\ 1\leq k\leq n}} \sup_{h\in[0,1]} \int_{\substack{\sum_{l\neq k}(x_i^l)^2<\varepsilon,\\ \sum_{l\neq k}(x_j^l)^2<\varepsilon}} |(\underline{\Gamma}_k)_{ij}-(\underline{\Sigma}_k)_{ij}|\Big(\prod_{l\neq k} f({}^{ij}\underline{C}_l^h;x^l)\Big)\\
&\qquad\times\frac{1}{2\pi\lambda_k\sqrt{1-|(\underline{c}_k^h)_{ij}|}}\exp\Big\{-\frac{2\varepsilon-\sum_{l\neq k}(x_i^l)^2-\sum_{l\neq k}(x_j^l)^2}{2\lambda_k(1+|(\underline{c}_k^h)_{ij}|)}\Big\}\,dx.
\end{aligned}$$

(When one or more of the $\underline{C}_k^h$'s are singular, a continuity argument shows that this inequality still holds.) Using (8.4) it follows that (in the notation of

Condition D)

$$\begin{aligned}
&\left|\mathbf{P}\left\{\bigcap_{i=1}^{I}\left\{\sum_{k=1}^{n}\lambda_k\xi_k(s_i)^2\ge\varepsilon\right\}\cap\bigcap_{j=1}^{J}\left\{\sum_{k=1}^{n}\lambda_k\xi_k(t_j)^2\ge\varepsilon\right\}\right\}\right.\\
&\qquad\left.-\mathbf{P}\left\{\bigcap_{i=1}^{I}\left\{\sum_{k=1}^{n}\lambda_k\xi_k(s_i)^2\ge\varepsilon\right\}\right\}\mathbf{P}\left\{\bigcap_{j=1}^{J}\left\{\sum_{k=1}^{n}\lambda_k\xi_k(t_j)^2\ge\varepsilon\right\}\right\}\right|\\
&\qquad\le\sum_{\substack{1\le i\le I\\1\le j\le J\\1\le k\le n}}\sup_{h\in[0,1]}\left[\frac{2|r(t_j-s_i)|}{\pi\sqrt{1-h|r(t_j-s_i)|}}\right.\\
&\qquad\qquad\times\mathbf{E}\left\{\prod_{l=1}^{2}\exp\left[-\frac{\varepsilon-X_k^{(hr(t_j-s_i),\dots,hr(t_j-s_i)),n}(l)}{2\lambda_k(1+|hr(t_j-s_i)|)}\right]\right.\\
&\qquad\qquad\qquad\left.\left.\times I_{\{X_k^{(hr(t_j-s_i),\dots,hr(t_j-s_i)),n}(l)<\varepsilon\}}\right\}\right]
\end{aligned}$$

(8.6)
$$\begin{aligned}
&\le\sum_{k=1}^{n}\frac{\sqrt{2}B(\frac12,\frac12)^2[T/(ap)]^2r(\tau T)\lambda_k}{e\pi^2\lambda_1(1-r(\tau T))}\\
&\qquad\times\mathbf{P}\left\{Z_{k_0}^{\cdot,n}(1)<\frac{\varepsilon}{1-r(\tau T)^2}\right\}\mathbf{P}\{Z_{k_0}^{\cdot,n}(1)<\varepsilon\}\\
&\to\frac{\sqrt{2}B(\frac12,\frac12)^2T^2r(\tau T)\ \mathrm{tr}(R)}{e\pi^2\lambda_1(1-r(\tau T))(ap)^2}\\
&\qquad\times\mathbf{P}\left\{\left\|\sum_{\substack{k\ne1\\k\ne k_0}}\lambda_k\xi_k(0)^2\right\|^2<\frac{\varepsilon}{1-r(\tau T)^2}\right\}\\
&\qquad\times\mathbf{P}\left\{\left\|\sum_{\substack{k\ne1\\k\ne k_0}}\lambda_k\xi_k(0)^2\right\|^2<\varepsilon\right\}\quad\text{as } n\to\infty.
\end{aligned}$$

Here $\varepsilon/[1-r(\tau T)^2]\le\varepsilon+1/q(\varepsilon)$ for ε small since, by Lemma 7(i), $\sqrt{\varepsilon q(\varepsilon)+1}\,r(\tau T)\to 0$. In view of Lemma 7(i), the fact that the Condition D holds thus follows from observing that, by (3.1), (3.4) and (8.6),

$$\begin{aligned}
\limsup_{\varepsilon\downarrow0}&\left|\mathbf{P}\left\{\bigcap_{i=1}^{I}\{\|Y(s_i)\|^2\ge\varepsilon\}\cap\bigcap_{j=1}^{J}\{\|Y(t_j)\|^2\ge\varepsilon\}\right\}\right.\\
&\qquad\left.-\mathbf{P}\left\{\bigcap_{i=1}^{I}\{\|Y(s_i)\|^2\ge\varepsilon\}\right\}\mathbf{P}\left\{\bigcap_{j=1}^{J}\{\|Y(t_j)\|^2\ge\varepsilon\}\right\}\right|\\
&\le\frac{4\sqrt{2}\lambda_{k_0}B(\frac12,\frac12)^2\ \mathrm{tr}(R)}{(\pi aL)^2}\limsup_{\varepsilon\downarrow0}q(\varepsilon)^2r(\tau T).\qquad\square
\end{aligned}$$

PROOF OF THEOREM 5. By (7.13) we have

$$\Big\{s \in \mathbb{R}: |s| \leq \Big(2M_5\Big[\sup_{k\in\mathbb{N}} \lambda_k\Big]\Big)^{-1/\alpha} \varepsilon^{2/\alpha}\Big\} \subseteq B_y(0;\varepsilon) \quad \text{for } y \in H_1,$$

so that $\mathscr{N}_{\langle y|Y\rangle}([0,1];\varepsilon) \leq 1 + \frac{1}{2}(2M_5[\sup_{k\in\mathbb{N}} \lambda_k])^{1/\alpha}\varepsilon^{-2/\alpha}$. Consequently, $\sup_{y\in H_1} \int_0^1 \sqrt{\ln \mathscr{N}_{\langle y|Y\rangle}([0,1];\varepsilon)}\, d\varepsilon < \infty$ so that Y is a.s. continuous (see the beginning of Section 4). In view of Lemma 6 [and the fact that, by Section 7, (7.1)–(7.4) and (8.1) hold] the theorem thus follows from Lemmas 10 and 11. □

PROPOSITION 4. *Equation* (8.2) *holds when* $t^\sigma \sup_{k\in\mathbb{N}} |r_k(t)| \to 0$ *for some* $\sigma > 0$.

PROOF. Choose $C_{19} > 0$ such that $\sup_{k\in\mathbb{N}} |r_k(t)| \leq C_{19}t^{-\sigma}$ for $t > 0$. Then we also have $r(t) \leq C_{19}t^{-\sigma}$. Consequently,

$$q(\varepsilon)^2 r(T(\varepsilon)) \leq C_{19}L^\sigma q(\varepsilon)^{2+\sigma/\alpha}[\mathbf{P}\{\|W\|^2 < \varepsilon\}]^\sigma,$$

where, by (2.9), the right-hand side tends to zero [since $\varepsilon^{2\rho(q)}q(\varepsilon) \to 0$]. □

In the finite case with precisely n nonzero λ_k's and $n > 2/\alpha$, sufficient mixing requires that $t^\sigma r(t) \to 0$ for some $\sigma > (n/2 - 1/\alpha)^{-1}$; compare Albin [(1992b), Section 3].

REFERENCES

ADLER, R. J. (1990). *An Introduction to Continuity, Extrema, and Related Topics for General Gaussian Processes*. IMS, Hayward, CA.

ALBIN, J. M. P. (1990). On extremal theory for stationary processes. *Ann. Probab.* **18** 92–128.

ALBIN, J. M. P. (1992a). Extremes and crossings for differentiable stationary processes with application to Gaussian processes in $\Re^m$ and Hilbert space. *Stochastic Process. Appl.* **42** 119–147.

ALBIN, J. M. P. (1992b). On the general law of iterated logarithm with application to self-similar processes and to Gaussian processes in $\Re^n$ and Hilbert space. *Stochastic Process. Appl.* **41** 1–31.

ALBIN, J. M. P. (1994). On the upper and lower classes for a stationary Gaussian stochastic process. *Ann. Probab.* **22** 77–93.

ARONOWICH, M. and ADLER, R. J. (1986). Extrema and level crossings of χ^2 processes. *Adv. in Appl. Probab.* **18** 901–920.

BERMAN, S. M. (1964). Limit theorems for the maximum term in stationary sequences. *Ann. Math. Statist.* **35** 502–516.

BINGHAM, N. H., GOLDIE, C. M. and TEUGELS, J. L. (1987). *Regular Variation.* Cambridge Univ. Press.

CSÁKI, E. and CSÖRGŐ, M. (1992). Inequalities for increments of stochastic processes and moduli of continuity. *Ann. Probab.* **20** 1031–1052.

DAVIS, R. A. and RESNICK, S. I. (1991). Extremes of moving averages of random variables with finite endpoint. *Ann. Probab.* **19** 312–328.

DUDLEY, R. M. (1967). The size of compact subsets of Hilbert space and continuity of Gaussian processes. *J. Funct. Anal.* **1** 290–330.

FERNIQUE, X. (1971). Régularité de processus gaussiens. *Invent. Math.* **12** 304–320.

FERNIQUE, X. (1975). *Régularité des Trajectoires des Fonctions Aléatoires Gaussiennes. Lecture Notes in Math.* **480** 1–96. Springer, Berlin.

FERNIQUE, X. (1989). Régularité de fonctions aléatoires gaussiennes stationnaires à valeurs vectorielles. *Probability Theory on Vector Spaces IV. Lecture Notes in Math.* **1391** 66–73. Springer, Berlin.

GELUK, J. L. and DE HAAN, L. (1987). *Regular variation, extensions and Tauberian theorems*. *CWI Tract.* **40**. Centrum voor Wiskunde en Informatica, Amsterdam.

HOFFMANN-JØRGENSEN, J., SHEPP, L. A. and DUDLEY, R. M. (1979). On the lower tail of Gaussian seminorms. *Ann. Probab.* **7** 319–342.

IBRAGIMOV, A. (1982). Hitting probability of a Gaussian vector with values in a Hilbert space in a sphere of small radius. *J. Sov. Math.* **20** 2164–2175.

ISCOE, I. and MCDONALD, D. (1989). Large deviations for l^2-valued Ornstein–Uhlenbeck processes. *Ann. Probab.* **17** 58–73.

ISCOE, I., MARCUS, M. B., MCDONALD, D., TALAGRAND, M. and ZINN, J. (1990). Continuity of l^2-valued Ornstein–Uhlenbeck processes. *Ann. Probab.* **18** 68–84.

RESNICK, S. I. (1987). *Extreme Values, Regular Variation, and Point Processes.* Springer, New York.

SAMORODNITSKY, G. (1991). Probability tails of Gaussian extrema. *Stochastic Process. Appl.* **38** 55–84.

SHARPE, K. (1978). Some properties of the crossings process generated by a stationary χ^2 process. *Adv. in Appl. Probab.* **10** 373–391.

WEBER, M. (1989). The supremum of Gaussian processes with a constant variance. *Probab. Theory Related Fields* **81** 585–591.

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