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SOME BEST POSSIBLE PROPHET INEQUALITIES FOR CONVEX FUNCTIONS OF SUMS OF INDEPENDENT VARIATES AND UNORDERED MARTINGALE DIFFERENCE SEQUENCES

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Let $\Phi(\cdot)$ be a nondecreasing convex function on $[0,\infty)$. We show that for any integer $n \geq 1$ and real a,

$$E\Phi((M_n - a)^+) \leq 2E\Phi((S_n - a)^+) - \Phi(0)$$

and

$$E(M_n \vee \mathrm{med}\, S_n) \leq E|S_n - \mathrm{med}\, S_n|.$$

where $X_1, X_2, \ldots$ are any independent mean zero random variables with partial sums $S_0 = 0$, $S_k = X_1 + \cdots + X_k$ and partial sum maxima $M_n = \max_{0 \leq k \leq n} S_k$. There are various instances in which these inequalities are best possible for fixed n and/or as $n \to \infty$. These inequalities remain valid if $\{X_k\}$ is a martingale difference sequence such that $E(X_k \mid \{X_i: i \neq k\}) = 0$ a.s. for each $k \geq 1$. Modified versions of these inequalities hold if the variates have arbitrary means but are independent.

1. Introduction. Let $S_1, S_2, \ldots$ be a sequence of random variables. Put $S_0 = M_0 = 0$ and $M_n = \max_{0 \leq k \leq n} S_k$. We first want to describe the general notion of a prophet problem. Consider any fixed $n \geq 1$ and any nondecreasing convex function $\Phi(\cdot)$ on $[0, \infty)$. Able to foresee the future, a prophet would know the entire sequence $S_1^+, \ldots, S_n^+$ beforehand. As these variates unfolded s/he would therefore be able to select the index j for which $S_j^+ = M_n$. Thereby, the prophet would acquire a real-time reward of $E\Phi(M_n)$, on the average. By contrast, a mere mortal is limited to stopping times. Consequently, s/he can at best achieve an average reward of $\sup\{E\Phi(S_\tau^+)$: τ is a stopping time bounded by $n\}$. Whenever $S_1^+, \ldots, S_n^+$ is a submartingale (which will always be the case in the sequel), this supremum is $E\Phi(S_n^+)$. A so-called prophet problem result ideally delineates the set of possible ordered pairs $(E\Phi(M_n), E\Phi(S_n^+))$ and at least specifies some guaranteed aspect of this set, such as bounding how the ratios or differences of the components of these points can vary among all $\Phi(\cdot)$ and $(S_1, \ldots, S_n)$ in some given family. An excellent survey of various prophet inequality problems and results may be found in Hill and Kertz (1992).

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For the purposes of the present paper, let $X_j = S_j - S_{j-1}$ (for $j \geq 1$). Partly inspired by earlier work of Doob, Klass (1989) proved that for i.i.d. mean zero X_j's,

$$EM_n \leq (2 - (1/n))ES_n^+. \tag{1.1}$$

Subsequently, (1.1) was extended [Klass (1993)] to nondecreasing convex nonnegative functions $\Phi(\cdot)$ on $[0, \infty)$. It was shown that for $(X_1, \dots, X_n)$ having independent components but otherwise arbitrary mean zero marginal distributions that

$$E\Phi(M_n) \leq cE\Phi(S_n^+) \tag{1.2}$$

for $c = 5$, with $c = 3 - (1/n)$ in the i.i.d. case.

We want to improve (1.2) to $c = 2$. How might this be accomplished? Letting $\Phi(x) = (x - y)^+$, (1.2) for $c = 2$ certainly implies that for $y \geq 0$,

$$E(M_n - y)^+ \leq 2E(S_n - y)^+. \tag{1.3}$$

Conversely, all nondecreasing convex functions can be represented as $\Phi(0)$ plus an integral of $(x - y)^+$ against a positive measure. To be more explicit, letting $\Phi'(\cdot)$ denote the right-hand derivative of $\Phi(\cdot)$, we may write (for $x \geq 0$)

$$\Phi(x) = \Phi(0) + x\Phi'(0) + \int_0^\infty (x - y)^+ \, d\Phi'(y). \tag{1.4}$$

Substituting M_n for x, taking expectations and appealing to Fubini's theorem,

$$E\Phi(M_n) = \Phi(0) + EM_n\Phi'(0) + \int_0^\infty E(M_n - y)^+ \, d\Phi'(y).$$

Assuming (1.3) for $y \geq 0$,

$$\begin{aligned} E\Phi(M_n) &\leq \Phi(0) + 2ES_n^+\Phi'(0) + \int_0^\infty 2E(S_n - y)^+ \, d\Phi'(y) \\ &\qquad\qquad [\text{since } \Phi'(0) \geq 0 \text{ and } d\Phi'(y) \geq 0] \\ &= -\Phi(0) + 2E\{\Phi(0) + S_n^+\Phi'(0) + \int_0^\infty (S_n^+ - y)^+ \, d\Phi'(y)\} \\ &= -\Phi(0) + 2E\Phi(S_n^+). \end{aligned} \tag{1.5}$$

Thus (1.5), a slight improvement of (1.2) even with $c = 2$, is equivalent to the subfamily of less imposing inequalities (1.3) for $y \geq 0$.

2. Identifying the approach.

Suppose we attempt to establish (1.3) for $y \geq 0$ by a direct induction. What difficulties lurk? Let $S_{(j,k]} = \sum_{j<i\leq k} X_i$ and $M_{(j,k]} = \max_{j\leq m\leq k} S_{(j,m]}$. Note that $S_{(j,k]} = M_{(j,k]} = 0$ for $j \geq k$. Fix any $y \geq 0$. Then $E(M_n - y)^+ = E(M_{(1,n]} - (y - X_1))^+$. Since $y - X_1$ could easily be negative, we suddenly discover that our induction hypothesis need not be preserved. Clearly, a different hypothesis is required, giving due consideration to negative values. What makes this development vexing is that (1.3) for $y \geq 0$ is the *minimal* hypothesis that must be proved. Moreover, if we strengthen it to (1.3) for all y, we again lose control of our hypothesis form. To see that it

too is unstable, note that for $y < 0$,

$$E(M_n - y)^+ = E(-y \vee (M_{(1,n]} - (y - X_1))^+).$$

Thus the need to extend the validity of inequality (1.3) from $y \geq 0$ to all y evidently necessitates the concomitant introduction of a *less stringent* version. What might it be? Note that when $y < 0$, (1.3) can be expressed as

$$-y + EM_n \leq 2E(S_n - y)^+. \tag{2.1}$$

Though (2.1) is formally weaker than (1.3) when $y \geq 0$, a straightforward stopping time argument produces (1.3) from it. Hence, the crux of our paper depends on proving (2.1) for all y.

3. Results.

For easy reference we record the following well-known fact, which will be used repeatedly: for every finite mean random variable Y,

$$2EY^+ = E|Y| + EY. \tag{3.1}$$

THEOREM 3.1. *Let $n \geq 1$. Let $X_1, \ldots, X_n$ be independent random variables with zero means. Then, for every $y \in \mathbb{R}$, we have*

$$-y + EM_n \leq 2E(S_n - y)^+. \tag{3.2}$$

PROOF. We shall prove (3.2) by induction. Take $n = 1$ and suppose $y \leq 0$. Then $EX_1^+ \leq E(X_1 - y)^+$ and $-y = (E(X_1 - y))^+ \leq E(X_1 - y)^+$ by Jensen's inequality. Summing these two inequalities, it follows that $-y + EM_1 \leq 2E(X_1 - y)^+$. For $y > 0$ it is clear that $-y + EM_1 \leq E(M_1 - y)^+ = E(X_1 - y)^+ \leq 2E(X_1 - y)^+$.

We assume that (3.2) holds for $1 \leq k \leq n$ and all y and all sequences of independent mean zero random variables. Define $S_{(k,j]}$ and $M_{(k,j]}$ as in Section 2. Define $\tau = \inf\{k \geq 1: S_k > y\}$. Then, for $y \geq 0$,

$$\begin{aligned}
-y + EM_{n+1} &\leq E(M_{n+1} - y)^+ \\
&= E(M_{n+1} - y)I(\tau \leq n+1) \\
&= E((S_\tau - y) + M_{(\tau, n+1]})I(\tau \leq n+1) \\
&= E(E[(S_\tau - y) + M_{(\tau, n+1]} \mid \tau, S_\tau]I(\tau \leq n+1)) \\
&\leq 2E((S_\tau - y) + S_{(\tau, n+1]})^+ I(\tau \leq n+1) \\
&= 2E(S_{n+1} - y)^+,
\end{aligned}$$

where we make use of (3.2) for $1 \leq k \leq n$ in the second-to-last inequality. Using (3.1), this implies

$$EM_{n+1} \leq 2E(S_{n+1} - y)^+ + y = E|S_{n+1} - y|, \qquad y \geq 0. \tag{3.3}$$

Hence (3.2) holds for $n + 1$ if $y \geq 0$. For any random variable Z, let $\operatorname{med} Z$ denote the midpoint of the medians of Z, so that $\operatorname{med}(-Z) = -\operatorname{med} Z$. To extend (3.3) and (3.2) to all y, note that it suffices to establish (3.3) for $y =$

$\operatorname{med} S_{n+1}$, since $\operatorname{med} S_{n+1}$ minimizes $E|S_{n+1} - y|$. We need to consider two cases.

Case (i): $\operatorname{med} S_{n+1} \geq 0$. Putting $y = \operatorname{med} S_{n+1}$, (3.3) gives

$$EM_{n+1} \leq E|S_{n+1} - \operatorname{med} S_{n+1}|,$$

which proves (3.2) for $n+1$ in Case (i).

Case (ii): $\operatorname{med} S_{n+1} \leq 0$. For $1 \leq k \leq n+1$, let $\tilde{X}_k = -X_{n+2-k}$, $\tilde{S}_0 = 0$ and for $1 \leq k \leq n+1$, $\tilde{S}_k = \sum_{j=1}^{k} \tilde{X}_j$. Note that $\operatorname{med} \tilde{S}_{n+1} = -\operatorname{med} S_{n+1} \geq 0$. Applying Case (i) to the $\tilde{X}_k$'s, we see that

$$\begin{aligned} EM_{n+1} = E(M_{n+1} - S_{n+1}) &= E \max_{0 \leq k \leq n+1} \tilde{S}_k \\ &\leq E|\tilde{S}_{n+1} - \operatorname{med} \tilde{S}_{n+1}| \\ &= E|S_{n+1} - \operatorname{med} S_{n+1}|, \end{aligned}$$

which implies (3.2) as in Case (i). This completes the proof of Theorem 3.1. □

As a byproduct of the above proof, we have the following somewhat stronger result. How it may be used to prove Theorem 3.4 below was already described in Section 2.

COROLLARY 3.2. *Let $X_1, \ldots, X_n$ be independent random variables with zero means. Then*

$$E(M_n - y)^+ \leq 2E(S_n - y)^+, \qquad y \in \mathbb{R} \tag{3.4}$$

and so

$$E(M_n \vee \operatorname{med} S_n) \leq E|S_n - \operatorname{med} S_n|. \tag{3.5}$$

PROOF. We see that when $y \leq 0$, (3.2) and (3.4) are the same. That (3.4) holds for $y > 0$ is contained in the proof of Theorem 3.1.

Secondly, combining (3.4) and (3.1), for every $y \in \mathbb{R}$,

$$E(M_n \vee y) = E(M_n - y)^+ + y \leq 2E(S_n - y)^+ + y = E|S_n - y|, \tag{3.6}$$

which implies (3.5). □

REMARK. We could have proved (3.3) for $y \leq 0$ without using medians by noting that

$$\begin{aligned} E(M_{n+1} - y)^+ &= -y + EM_{n+1} \\ &= -y + E(M_{n+1} - S_{n+1}) \\ &= -y + E \max_{0 \leq k \leq n+1} (S_k - S_{n+1}) \end{aligned}$$

$$
\begin{aligned}
&= -y + E \max_{0 \le k \le n+1} \tilde{S}_k \\
&\le -y + E|\tilde{S}_{n+1} + y| \quad \text{[by (3.3)]} \\
&= -y + E|S_{n+1} - y| \\
&= 2E(S_{n+1} - y)^+.
\end{aligned}
$$

Using linearity of the expectations, we can upper bound the expected length of the convex hull of $\{S_k\colon 0 \le k \le n\}$ as a subset of $\mathbb{R}$.

COROLLARY 3.3. *Let $X_1, \dots, X_n$ be independent random variables with zero means. Then*

$$
E\Big(\max_{0 \le k \le n} S_k - \min_{0 \le k \le n} S_k\Big) \le 2E|S_n - \operatorname{med} S_n|. \tag{3.7}
$$

Observe that for any real a and any $y \ge 0$,

$$
\begin{aligned}
E((M_n - a)^+ - y)^+ &= E(M_n - (a + y))^+ \\
&\le 2E(S_n - (a + y))^+ \quad \text{[by (3.4)]} \\
&= 2E((S_n - a)^+ - y)^+.
\end{aligned} \tag{3.8}
$$

Using the integral representation (1.4) and proceeding much as in (1.5), we obtain the following refinement of Theorem 3.1 and of Corollary 3.3.

THEOREM 3.4. *Let $\Phi(\cdot)$ be a nondecreasing convex function defined on $[0, \infty)$. Let $X_1, X_2, \dots$ be independent random variables with zero means. Then, for any integer $n \ge 1$ and real a,*

$$
E\Phi((M_n - a)^+) \le 2E\Phi((S_n - a)^+) - \Phi(0). \tag{3.9}
$$

COROLLARY 3.5. *If $\Phi(\cdot)$ is any nondecreasing convex function on $[0, \infty)$ with $\Phi(0) = 0$ and if $M_{n,-} = \max_{0 \le k \le n}(-S_k)$, then*

$$
E(\Phi(M_n) + \Phi(M_{n,-})) \le 2E\Phi(|S_n|). \tag{3.10}
$$

REMARKS. (a) Notice that for Brownian motion $\{B_t\colon t \ge 0\}$ and its maximal process $M_t = \sup_{0 \le s \le t} B(s)$, we have $P(M(t) \ge y) = 2P(B(t) \ge y)$, for $y \ge 0$. Hence for any nondecreasing function Φ on $[0, \infty)$, with $\Phi(0) = 0$,

$$
E\Phi(M_t) = 2E\Phi(B_t^+).
$$

By an obvious weak convergence argument, the inequalities in (3.4) to (3.10) are best possible as $n \to \infty$.

(b) There is also a family of highly asymmetric distributions for which (3.5) and (3.7) are asymptotically best possible, having some extensions to (3.8) and (3.9). These distributions barely have first moments and their partial sums converge to $+\infty$ or $-\infty$ in probability. To be specific, let X be a nonconstant mean zero random variable such that (i) $E(X^2 \wedge y^2)/[yL(y)] \to 0$

where $L(y) = E|X|I(|X| \geq y)$, and (ii) $EX^+I(X \geq y)/E|X|I(|X| \geq y) \to \lambda$ where $\lambda = 0$ or 1. Then (i) implies that $L(y)$ and $E|S_n|/n$ are slowly varying in y and in n, respectively. Combining (ii) and (i), we have

$$\frac{S_n}{ES_n^+} \to_P 1 - 2\lambda,$$

and the fact that whenever a_n's satisfy $nL(a_n)/a_n \to 1$, then $a_n/E|S_n| \to 1$. It follows that

$$EM_n = \sum_{k=1}^{n} \frac{ES_k^+}{k} \sim \sum_{k=1}^{n} \frac{L(a_k)}{2} \sim \frac{nL(a_n)}{2} \sim ES_n^+.$$

To approximate the right-hand side of (3.5) it suffices to consider the case $\lambda = 0$. It follows that $\operatorname{med} S_n/ES_n^+ \to 1$ and there exist $\varepsilon_n \searrow 0$ such that $P(S_n \leq (1-\varepsilon_n) \operatorname{med} S_n) \to 0$. Moreover, it can be shown that $E(S_n - \operatorname{med} S_n)^+ = o(ES_n^+)$. Hence $EM_n \sim ES_n^+ \sim E|S_n - \operatorname{med} S_n|$, showing that (3.5) is best possible as $n \to \infty$. For more detailed calculations, see Theorem 5 and Corollary 4 of Klass and Teicher (1977).

(c) These results have certain optimality properties *even* for fixed n, as we now illustrate for Theorem 3.4. For each integer $k \geq 1$, there exist i.i.d. mean zero two-point random variables $X_{k1}, X_{k2}, \ldots$ with $E(X_{k,j})^+ \equiv 1$ such that for any $n \geq 1$ and any convex nondecreasing function $\Phi(\cdot)$ on $[0, \infty)$,

$$\lim_{k\to\infty} 2E\Phi((S_{k,n} - \operatorname{med} S_{k,n})^+) - \Phi(0) - E\Phi((M_{k,n} - \operatorname{med} S_{k,n})^+) = 0. \tag{3.11}$$

To see this, let $p_k = 2^{-k}$, and

$$X_{k,j} = \begin{cases} \dfrac{1}{1-p_k}, & \text{w.p. } 1 - p_k \\ -\dfrac{1}{p_k}, & \text{w.p. } p_k. \end{cases}$$

Let k_n equal first $k \geq 1$: $p_k < 1 - 2^{-1/n}$. For $k \geq k_n$,

$$\operatorname{med} S_{k,n} = \frac{n}{1-p_k} = \operatorname{ess\,sup} M_{k,n}.$$

Therefore,

$$\begin{aligned} &2\Phi((S_{k,n} - \operatorname{med} S_{k,n})^+) - \Phi(0) - \Phi((M_{k,n} - \operatorname{med} S_{k,n})^+) \\ &\quad\equiv 2\Phi(0) - \Phi(0) - \Phi(0) \\ &\quad= 0 \quad \text{for all } k \geq k_n, \end{aligned}$$

which establishes our claim in trivial fashion.

Whenever $\Phi(x) = Lx$ $(L > 0)$, the inequality in (3.5) is also optimal for each $n \geq 1$ for the related family of random variables $\hat{X}_{k,j} = -X_{k,j}$. To verify this claim, put $\hat{S}_{k,0} = \hat{M}_{k,0} = 0$, $\hat{S}_{k,j} = \sum_{i=1}^{j} \hat{X}_{k,i}$ and $\hat{M}_{k,j} = \max_{0 \le i \le j} \hat{S}_{k,i}$. Note that

$$\begin{aligned}\lim_{k\to\infty} E(\hat{M}_{k,n} - \operatorname{med} \hat{S}_{k,n})^+ &= n + \lim_{k\to\infty} \sum_{j=1}^{n} \frac{E(\hat{S}_{k,j})^+}{j} \\ &= n + \lim_{k\to\infty} \sum_{j=1}^{n} \frac{E(S_{k,j})^+}{j} \quad \text{(since } E\hat{S}_{k,j} = 0\text{)} \\ &= 2n \quad \left(\text{since } \frac{E(S_{k,j})^+}{j} \to 1 \text{ as } k \to \infty\right).\end{aligned}$$

Moreover,

$$\begin{aligned}&\lim_{k\to\infty} 2E(\hat{S}_{k,n} - \operatorname{med} \hat{S}_{k,n})^+ \\ &\quad= \lim_{k\to\infty} (E|\hat{S}_{k,n} - \operatorname{med} \hat{S}_{k,n}| + E(\hat{S}_{k,n} - \operatorname{med} \hat{S}_{k,n})) \\ &\quad= \lim_{k\to\infty} E|S_{k,n} - \operatorname{med} S_{k,n}| + n \\ &\quad= \lim_{k\to\infty} (2E(S_{k,n} - \operatorname{med} S_{k,n})^+ - E(S_{k,n} - \operatorname{med} S_{k,n}) + n) \\ &\quad= \lim_{k\to\infty} (E(M_{k,n} - \operatorname{med} S_{k,n})^+ + 2n) \quad \text{[by (3.11)]} \\ &\quad= 2n.\end{aligned}$$

Therefore

$$\lim_{k\to\infty} (2E(\hat{S}_{k,n} - \operatorname{med} \hat{S}_{k,n})^+ - E(\hat{M}_{k,n} - \operatorname{med} \hat{S}_{k,n})^+) = 0. \tag{3.12}$$

4. Extension to nonzero means and unordered martingale difference sequences. We generalize the results of Section 3 in two ways.

THEOREM 4.1. *Let $X_1, \dots X_n$ be independent random variables with finite but otherwise arbitrary means. Let $s_k = ES_k$, $0 \le k \le n$ and $s_n^* = \max_{0\le k\le n} s_k$. Then for every $y \in \mathbb{R}$, we have*

$$E(M_n - y)^+ \le 2E(S_n - s_n + s_n^* - y)^+ \tag{4.1}$$

from which it follows that

$$E(M_n \vee y) \le E|S_n - s_n + s_n^* - y| + s_n^*, \tag{4.2}$$

$$E(M_n \vee (s_n^* - s_n + \operatorname{med} S_n)) \le E|S_n - \operatorname{med} S_n| + s_n^* \tag{4.3}$$

and for any real a,

$$E\Phi((M_n - a)^+) \le 2E\Phi((S_n - s_n + s_n^* - a)^+) - \Phi(0), \tag{4.4}$$

where Φ is any nondecreasing convex function defined on $[0, \infty)$.

PROOF. Since $s_n^* - s_k \geq 0$,

$$E(M_n - y)^+ \leq E\Big[\max_{0\leq k\leq n}\{S_k + s_n^* - s_k\} - y\Big]^+$$
$$= E\Big[\max_{0\leq k\leq n}\{S_k - s_k\} - (y - s_n^*)\Big]^+$$
$$\leq 2E(S_n - s_n + s_n^* - y)^+,$$

where we apply (3.4) to $\{X_k - EX_k\colon 1 \leq k \leq n\}$ in the last inequality. This gives (4.1). □

REMARK. Observe that $E\Phi(M_n) \leq 2E\Phi(S_n^+) - \Phi(0)$ whenever $s_n = s_n^*$, which includes the case of independent random variables with nonnegative means.

Applying (4.3) to $\{X_k\colon 1 \leq k \leq n\}$ and $\{-X_k\colon 1 \leq k \leq n\}$, we deduce the following.

COROLLARY 4.2. *Let $X_1, \dots, X_n$ be independent random variables with finite but otherwise arbitrary means. Let $s_k = ES_k$, $0 \leq k \leq n$. Then*

$$E\Big(\max_{0\leq k\leq n} S_k - \min_{0\leq k\leq n} S_k\Big) \leq 2E|S_n - \operatorname{med} S_n| + \max_{0\leq k\leq n} s_k - \min_{0\leq k\leq n} s_k. \tag{4.5}$$

A reexamination of (3.2) and (3.4) reveals that their proofs do not depend on the full strength of our independence assumptions. What is needed, rather, is that the increments following any stopping time be a martingale in the forward direction and the reverse direction. To produce sequences with this property we make the following definition.

DEFINITION. Let $d = (d_1, d_2, \dots)$ be a sequence of integrable random variables. It is said to be an *unordered martingale difference sequence* if for each $k \geq 1$ the following condition is satisfied: $E(d_i \mid \mathscr{F}_i^0) = 0$ almost surely, where $\mathscr{F}_k^0$ is the σ-field generated by all the d_i for $i \neq k$.

The proofs of Theorems 3.1 and 3.4 carry over to show the following theorem.

THEOREM 4.3. *Let $d = (d_1, d_2, \dots)$ be an unordered martingale difference sequence.*

Let $f_0 = 0$, $f_n = \sum_{k=1}^n d_k$ for $n \geq 1$. Then

$$E\Big(\max_{0\leq k\leq n} f_k - y\Big)^+ \leq 2E(f_n - y)^+, \qquad y \in \mathbb{R}. \tag{4.6}$$

Consequently

$$E\Big(\max_{0\leq k\leq n} f_k \vee y\Big) \leq E|f_n - y|, \tag{4.7}$$

$$E\Big(\max_{0\leq k\leq n} f_k \vee \operatorname{med} f_n\Big) \leq E|f_n - \operatorname{med} f_n| \tag{4.8}$$

and for any real a,

$$E\Phi\Big(\max_{0\le k\le n}(f_k-a)^+\Big)\le 2E\Phi((f_n-a)^+)-\Phi(0), \tag{4.9}$$

where Φ is any nondecreasing convex function defined on $[0,\infty)$.

COROLLARY 4.4. *Let $f=(f_1,f_2,\dots)$ be an L^1-bounded martingale. Suppose that the martingale difference sequence of f is unordered. Then f is uniformly integrable.*

PROOF. Write $f_n^*=\sup_{1\le k\le n}|f_k|$ and put $\Phi(x)=x$ in (4.9). Then

$$Ef_n^*\le E\Big(\max_{0\le k\le n}f_k+\max_{0\le k\le n}(-f_k)\Big)\le 2E|f_n|\le 2\|f\|_1.$$

Hence, by monotone convergence, $E\sup_{k\ge 1}|f_k|<\infty$ and so f is uniformly integrable. □

REMARK. As an example of such a sequence, let $d_k=X_kY_k$, where the X_k's are arbitrary random variables having finite means and the Y_k's are mutually independent mean zero variates, independent of $\{X_j\}$.

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REFERENCES

HILL, T. P. and KERTZ, R. P. (1992). A survey of prophet inequalities in optimal stopping theory. *Contemp. Math.* 125 191–207.

KLASS, M. J. (1989). Maximizing $E\max_{1\le k\le n}S_k^+/ES_n^+$: a prophet inequality for sums of i.i.d. mean zero variates. *Ann. Probab.* 17 1243–1247.

KLASS, M. J. (1993). Ratio prophet inequalities for convex functions of partial sums. *Statist. Probab. Lett.* 17 205–209.

KLASS, M. J. and TEICHER, H. (1977). Iterated logarithm laws for asymmetric random variables barely with or without finite mean. *Ann. Probab.* 5 861–874.

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