The Annals of Probability
1997, Vol. 25, No. 4, 1872–1894

A MARTINGALE APPROACH TO HOMOGENIZATION OF UNBOUNDED RANDOM FLOWS

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We study the asymptotic behavior of Brownian motion in steady, unbounded incompressible random flows. We prove an invariance principle for almost all realizations of random flows. The key compactness result is obtained by Moser's iterative scheme in PDE theory.

1. Introduction. Let $\{\mathbf{x}(t)\}_{t\geq 0}$ be a solution of the following Itô stochastic differential equation

$$d\mathbf{x}(t) = \mathbf{b}(\mathbf{x}(t))\,dt + \sqrt{2}\,d\mathbf{w}(t), \qquad \mathbf{x}(0) = \mathbf{0}, \tag{1}$$

where $\{\mathbf{w}(t)\}_{t\geq 0}$ is the standard d-dimensional Brownian motion and $\mathbf{b}(\mathbf{x})$, $\mathbf{x} \in R^d$ is an random d-dimensional drift independent of $\mathbf{w}(t)$. Furthermore, we assume that the random field $\mathbf{b}(\mathbf{x})$, $\mathbf{x} \in R^d$ is zero mean, homogeneous and divergence-free:

$$\nabla \cdot \mathbf{b}(\mathbf{x}) = 0. \tag{2}$$

Due to (2), there exists a *skew symmetric stream* matrix $\mathbf{H}(\mathbf{x})$, $H_{i,j}(\mathbf{x}) = -H_{j,i}(\mathbf{x})$, $\forall i, j$, such that

$$\nabla \cdot \mathbf{H}(\mathbf{x}) = \mathbf{b}(\mathbf{x}). \tag{3}$$

The stream matrix is not determined uniquely by (2) but up to a gauge. The stream matrix $\mathbf{H}(\mathbf{x})$ is not homogeneous in general due to the randomness of the velocity field. However, if the dimension is bigger than two and the velocity correlation decays sufficiently fast, then there exists a square integrable, homogeneous stream matrix such that (2) holds [cf. Fannjiang and Papanicolaou (1996)]. In two dimensions the stream matrix in general is not homogeneous, regardless of decay in velocity correlation, and has logarithmically divergent variances [cf. Fannjiang (1997)]. In this paper, we assume that $\mathbf{H}(\mathbf{x})$ is a homogeneous square integrable process.

To study the long-time, large-scale asymptotics of the solutions of (1), we take the diffusive scaling and consider the family of processes

$$\mathbf{x}_\varepsilon(t) = \varepsilon\,\mathbf{x}(t/\varepsilon^2), \qquad \varepsilon > 0. \tag{4}$$

Received April 1996; revised October 1996.
[1]Research supported in part by NSF Grant DMS-96-00119.
AMS 1991 *subject classifications*. Primary 60F17, 35B27; secondary 60G44.
Key words and phrases. Convection-diffusion, invariance principle, homogenization, martingale.

The probability distributions of these processes are identical with those of the solutions of the following Itô stochastic differential equations:

$$d\mathbf{x}_{\varepsilon}(t) = \frac{1}{\varepsilon}\mathbf{b}\left(\frac{\mathbf{x}_{\varepsilon}(t)}{\varepsilon}\right) dt + \sqrt{2}\, d\mathbf{w}(t), \tag{5}$$

$$\mathbf{x}_{\varepsilon}(0) = \mathbf{0}. \tag{6}$$

When both the drift $\mathbf{b}(\mathbf{x})$ and $\mathbf{H}(\mathbf{x})$ are *bounded*, it has been shown [see Papanicolaou and Varadhan (1982) and Osada (1982)] that for almost all realizations of the drift the distributions of the processes $\{\mathbf{x}_{\varepsilon}(t)\}_{t \geq 0}$, $\varepsilon > 0$ converge weakly in the space of probability measures on the path space $C([0, +\infty);\ R^{d})$ to the law of a Brownian motion with generally enhanced variances called the *effective diffusivity*. This limiting process eliminates the inhomogeneity of $\{\mathbf{x}_{\varepsilon}(t)\}_{t \geq 0}$, $\varepsilon > 0$ and is thus called *homogenization*.

If one is only concerned with the boundary value problems associated with the generator L^{ε} of (5):

$$L^{\varepsilon} u_{\varepsilon} = \frac{1}{\varepsilon}\mathbf{b}\left(\frac{\mathbf{x}}{\varepsilon}\right) \cdot \nabla u_{\varepsilon}(t, \mathbf{x}) + \Delta u_{\varepsilon}(t, \mathbf{x}) = \nabla \cdot \left(I + \mathbf{H}\left(\frac{\mathbf{x}}{\varepsilon}\right)\right)\nabla u_{\varepsilon}, \tag{7}$$

then it has long been shown [Papanicolaou and Varadhan (1982)] that the weak convergence to the corresponding boundary value problems associated with the Laplacian with the effective diffusivity as coefficients holds for bounded stream matrices.

However, many important examples of random drifts, such as Gaussian fields, do not fall into the category of bounded coefficients. For generalization to unbounded coefficients, Oelschläger (1988) proves that probabilistic homogenization holds for coefficients with finite second moments in $\mathbf{b}(\mathbf{x})$ and $\mathbf{H}(\mathbf{x})$ in addition to the usual linear growth condition on the first derivatives of the drift. However, the convergence is defined in the sense of a Vasserstein metric on the space of d-dimensional random proceses with continuous trajectories, which is weaker than the metric associated with the weak convergence of probability measures [see Oelschläger (1988), page 1090]. Avellaneda and Majda (1991) generalize the PDE aspect of homogenization to unbounded coefficients with finite pth moment, $p > d$, in $\mathbf{H}(\mathbf{x})$ and finite qth moment, $q > d/2$, in $\mathbf{b}(\mathbf{x})$. The PDE aspect of homogenization is roughly equivalent to the convergence of finite dimensional distributions of the processes and should require less regularity on $\mathbf{b}(\mathbf{x})$ or $\mathbf{H}(\mathbf{x})$ than would the almost sure convergence in law studied in the present paper. Indeed, it turns out that only the square integrability for the stream matrix is necessary for the convergence of the semigroups [Fannjiang and Papanicolaou (1996)]. Fannjiang and Papanicolaou (1996) also prove the probabilistic convergence in measure but not almost surely. To this end they obtain the crucial resolvent estimates by the variational principles.

The obstacle to proving usual almost sure (with respect to the ensemble of random drifts) convergence is law has been, as observed by Oelschläger

(1988) on page 1085, the lack of Nash–Aronson estimates in the case of unbounded coefficients. In this paper, we take up the martingale approach initiated by Osada (1982) which consists essentially of the decomposition of the trajectory $\mathbf{x}_\varepsilon(t)$ into a martingale part $\mathbf{y}_\varepsilon(t)$ and a "corrector" part $\mathbf{z}_\varepsilon(t)$. We show, by Moser's iterative scheme, the vanishing of the corrector $\mathbf{z}_\varepsilon(t)$ in the limit $\varepsilon \to 0$ and obtain the weak convergence of distribution of the processes $\{\mathbf{x}_\varepsilon(t)\}_{t\ge 0}$, $\varepsilon > 0$ to the limiting Wiener measure. The strengthening of the sense of convergence is due to the corrector estimates stated in Lemma 3. The almost sure convergence then follows from a standard martingale invariance principle [see Brown (1971) and Helland (1982)].

Our key assumption, in addition to the standard ones, is that the stream matrix is homogeneous and has finite pth moment $\mathbf{E}|\mathbf{H}|^p < \infty$ where $p > d$. In addition to stronger convergence results, our method is also considerably simpler than that of either Oelschläger (1988) or Fannjiang and Papanicolaou (1996).

2. Notation and formulation of results. Let us start with some customary notation from linear algebra and vector calculus. For a pair of vectors $\mathbf{a}, \mathbf{b} \in R^d$, we shall denote by $\mathbf{a} \otimes \mathbf{b}$ the tensor given by the matrix $[a_i b_j]_{i,j=1,\dots,d}$, where $\mathbf{a} = (a_1, \dots, a_d)$, $\mathbf{b} = (b_1, \dots, b_d)$. Let $\mathbf{A} = [a_{ij}]$, $\mathbf{B} = [b_{ij}]$ be two $d \times d$ matrices. We shall make use of the notation $(\mathbf{a}, \mathbf{b})$ and $(\mathbf{A}, \mathbf{B})$ for the standard scalar product of vectors and matrices, respectively; that is, $(\mathbf{a}, \mathbf{b}) = \sum_{i=1}^d a_i b_i$, and $(\mathbf{A}, \mathbf{B}) = \operatorname{tr} \mathbf{A}\mathbf{B}^T$. For a vector field $\mathbf{f} = (f_1, \dots, f_d)$, $\nabla \cdot \mathbf{f} = \sum_{k=1}^d \partial_k f_k$; for a function f, $\nabla f = (\partial_1 f, \dots, \partial_d f)$.

Let us denote by (Ω, V, P) a probability space. The expectation computed with respect to probability measure P will be denoted by $\mathbf{E}$. Let $\mathbf{b}(\mathbf{x}; \omega)$ be a d-dimensional random vector field defined on R^d. We shall assume that it satisfies the following conditions.

(B1) For any $\omega \in \Omega$, $\mathbf{b}(\mathbf{x}; \omega)$ has C^1-smooth path realizations and it grows linearly. That is, there is $C(\omega)$ so that $|\mathbf{b}(\mathbf{x}; \omega)| \le C(\omega)(1 + |x|)$.
(B2) The field $\mathbf{b}$ is strictly homogeneous, zero mean and square integrable. That is all its finite-dimensional distributions do not depend on translations in R^d, $\mathbf{E}\mathbf{b}(\mathbf{0}) = \mathbf{0}$ and $\mathbf{E}|\mathbf{b}(\mathbf{0})|^2 < +\infty$.
(B3) The field $\mathbf{b}$ satisfies $\nabla \cdot \mathbf{b}(\mathbf{x}) = \sum_{k=1}^d \partial_{x_k} b_k(\mathbf{x}) = 0$.

REMARK. Condition (B1) suffices to claim the global existence of solutions of relevant stochastic differential equations.

As is well known [see, e.g., Rozanov (1969)], under these conditions there exists a family of transformations defined on the probability space $\tau_{\mathbf{x}}: \Omega \to \Omega$, $\mathbf{x} \in R^d$, such that we have the following.

(T1) $\tau_0 = Id_\Omega$ and $\tau_{\mathbf{x}+\mathbf{y}} = \tau_{\mathbf{x}}\tau_{\mathbf{y}}$ for all $\mathbf{x}, \mathbf{y} \in R^d$.
(T2) $P[\tau_{\mathbf{x}}(A)] = P[A]$, for $\mathbf{x} \in R^d$, $A \in \tilde{B}$, where $\tilde{B}$ is the σ-algebra generated by $\mathbf{b}(\mathbf{x}; \omega)$, $\mathbf{x} \in R^d$.

(T3) The mapping $(\omega, \mathbf{x}) \mapsto \tau_{\mathbf{x}}(\omega)$ is $\tilde{B} \otimes B_{R^d}$ to $\tilde{B}$ measurable. Here B_{R^d} denotes the σ-algebra of Borel subsets of R^d.

(T4) $\mathbf{b}(\mathbf{x}; \omega) = \mathbf{b}(\mathbf{0}; \tau_{\mathbf{x}}(\omega))$.

In addition to these properties we will assume that the field $\mathbf{b}$ is ergodic, which is reflected in the following property of the flow of transformations.

(T5) The group $\{\tau_{\mathbf{x}}\}_{\mathbf{x} \in R^d}$ is ergodic. That is, if for a certain $A \in \tilde{B}$, $P[\tau_{\mathbf{x}}(A) \triangle A] = 0$ ($\triangle$ here means the set operation of symmetric difference) for all $\mathbf{x} \in R^d$ then $P[A] = 0$ or 1.

Given $1 \le p < \infty$, let us denote by $L^p(\Omega)$ the space of all $\tilde{B}$-measurable random variables $X: \Omega \to R$ such that $\int_\Omega |X|^p \, dP < +\infty$ with norm defined as $|||X|||_p = (\int_\Omega |X|^p \, dP)^{1/p}$. In cases when $p = +\infty$, the space consists of random variables with finite essential supremum with the norm

$$|||X|||_\infty = \operatorname{ess\,sup}|X|.$$

Consider the group of unitary linear operators defined on the space $L^2(\Omega)$ by the formula $U^x f(\omega) = f(\tau_{\mathbf{x}}(\omega))$. The group has d anti-self-adjoint generators D_k: $D_k \to L^2(\Omega)$, $k = 1, \dots, d$, corresponding to the subgroups $U^{t\mathbf{e}_k}$, $t \in R$, $k = 1, \dots, d$ where $\mathbf{e}_1 = (1, 0, \dots, 0), \dots, \mathbf{e}_d = (0, \dots, 0, 1)$. There exists a spectral measure U defined on σ-algebra B_{R^d} with values in the space of orthogonal projections on $L^2(\Omega)$ such that

$$D_k = \int i\lambda_k U(d\boldsymbol{\lambda}), \qquad k = 1, \dots, d,$$

$$U^x = \int e^{i(\boldsymbol{\lambda}, \mathbf{x})} U(d\boldsymbol{\lambda}), \qquad \mathbf{x} \in R^d,$$

where the integrals are understood as spectral integrals (see Dunford and Schwartz (1988)]. According to Rozanov (1969) we can write the field $\mathbf{b}$ in the form of a stochastic integral

$$\mathbf{b}(\mathbf{x}) = U^{\mathbf{x}} \tilde{\mathbf{b}} = \int e^{i(\boldsymbol{\lambda}, \mathbf{x})} \hat{\mathbf{b}}(d\boldsymbol{\lambda}), \tag{8}$$

where $\tilde{\mathbf{b}} = (b_1(0), \dots, b_d(0))$ and $\hat{\mathbf{b}}$ is a d-dimensional random vector-valued measure such that for any set $A \in B_{R^d}$, $\hat{\mathbf{b}}(A) = (U(A)\tilde{b}_1, \dots, U(A)\tilde{b}_d)$. The structural measure S of $\hat{\mathbf{b}}(A)$ defined as $S(A) = \mathbf{E}[\hat{\mathbf{b}}(A) \otimes \hat{\mathbf{b}}(A)]$ is a nonnegative symmetric matrix-valued measure which is the spectral measure of the field $\mathbf{b}$ [see Adler (1981)]. We can write then also

$$\mathbf{R}(\mathbf{x}) = \int e^{i(\boldsymbol{\lambda}, \mathbf{x})} S(d\boldsymbol{\lambda}),$$

where $\mathbf{R}(\mathbf{x}) = \mathbf{E}\{\mathbf{b}(\mathbf{x}) \otimes \mathbf{b}(\mathbf{0})\}$ is the correlation matrix of the field $\mathbf{b}$. The measure E given on B_R by

$$E(A) = \int_A d\lambda \int_{|\boldsymbol{\lambda}| = \lambda} \operatorname{tr} S(d\boldsymbol{\lambda})$$

is called [see Chorin (1994)] the power-energy spectrum of the field $\mathbf{b}$. We define also a stream matrix $\mathbf{H}(\mathbf{x};\omega) = [H_{ij}(\mathbf{x};\omega)]_{i,j=1,\dots,d}$ as a random antisymmetric matrix-valued field defined on R^d such that $\mathbf{b} = \nabla \cdot \mathbf{H}$. More precisely,

$$\text{(H1)} \qquad H_{ij}(\mathbf{x};\omega) = -H_{ji}(\mathbf{x};\omega) \quad \text{for } i, j = 1, \dots, d,$$

$$\text{(H2)} \qquad \sum_{j=1}^{d} \partial_i H_{ij}(\mathbf{x};\omega) = b_i(\mathbf{x};\omega), \qquad i = 1, \dots, d.$$

We shall also assume that $\mathbf{H}$ is homogeneous, zero mean and has pth absolute moment for some $p > d$, that is,

$$\text{(H3)(a)} \qquad \mathbf{H}(\mathbf{x};\omega) = \tilde{\mathbf{H}}(\tau_{\mathbf{x}}(\omega)),$$

where $\tilde{\mathbf{H}}(\omega) = \mathbf{H}(\mathbf{0};\omega)$,

$$\mathbf{E}\tilde{\mathbf{H}} = \mathbf{0}$$

and

$$\text{(H3)(b)} \qquad \mathbf{E}|\tilde{\mathbf{H}}|^p < +\infty \quad \text{for some } p > d.$$

Finally we suppose that

$$\text{(H4)} \qquad \mathbf{H}(\mathbf{x};\omega) \text{ is } C^2\text{-smooth } P \text{ a.s.}$$

Consider now $\{\mathbf{x}(t)\}_{t \ge 0}$, the solution of the following Itô stochastic differential equation:

$$d\mathbf{x}(t) = \mathbf{b}(\mathbf{x}(t))\,dt + \sqrt{2}\,d\mathbf{w}(t),$$
$$\mathbf{x}(0) = \mathbf{0}.$$

Here $\mathbf{w}(t)$ is the standard Brownian motion defined on another probability space (Σ, M, Q) and $\mathbf{v} \in R^d$ is a constant vector. Let us denote by $\mathbf{M}$ the expectation calculated with respect to the probability measure Q. The expression $\{\mathbf{x}(t)\}_{t \ge 0}$ is considered a stochastic process on the product probability space $(\Omega \times \Sigma, V \otimes M, P \otimes Q)$. The scaled trajectories are defined by $\mathbf{x}_\varepsilon(t) = \varepsilon\mathbf{x}(t/\varepsilon^2)$ for $\varepsilon > 0$, the scaling parameter. We shall consider the following two families of measures on the Polish space $X = C([0, +\infty); R^d)$. First, Q_ε^ω is the family of probability distributions of the trajectories $\mathbf{x}_\varepsilon^\omega(t;\sigma) = \mathbf{x}_\varepsilon(t;\omega,\sigma)$, where $\sigma \in \Sigma$ and with $\omega \in \Omega$ fixed. The second family, which we denote by Q_ε, is the family of probability laws of the processes $\{\mathbf{x}_\varepsilon(t)\}_{t \ge 0}$ considered over the space $\Sigma \otimes \Omega$. We will suppress the subscript for the processes $\{\mathbf{x}_1^\omega(t)\}_{t \ge 0}$ and $\{\mathbf{x}_1(t)\}_{t \ge 0}$.

DEFINITION 1. We say that a family of continuous trajectory stochastic processes satisfies the invariance principle if their probability laws converge weakly over the space X to a Wiener measure.

The main results of our paper are stated in the following theorem.

THEOREM 1. *Suppose that a vector field* $\mathbf{b}$ *satisfies assumptions* (B1)–(B3) *and its stream matrix* $\mathbf{H}$ *meets assumptions* (H1)–(H4). *Then the following hold*:

(i) *For* P *almost every* $\omega \in \Omega$, *the limits*

$$\lim_{t \uparrow +\infty} \mathbf{M} \frac{x_i^{\omega}(t)\, x_j^{\omega}(t)}{t} = d_{ij}, \qquad i, j = 1, \dots, d$$

exist and are deterministic constants.

(ii) *For* P *almost every* $\omega \in \Omega$, *the family of processes* $\{\mathbf{x}_{\varepsilon}^{\omega}(t)\}_{t \ge 0}$, $\varepsilon > 0$ *satisfies the invariance principle with the limiting Wiener measure having the covariance matrix equal to* $\mathbf{D} = [d_{ij}]$.

(iii) *The family of processes* $\{\mathbf{x}_{\varepsilon}(t)\}_{t \ge 0}$, $\varepsilon > 0$ *considered over the product probability space* $(\Omega \times \Sigma,\ V \otimes M,\ P \otimes Q)$ *satisfies the invariance principle with the limiting Wiener measure as in* (ii).

REMARK. Next we explain how to construct $\mathbf{H}$ that are suitable for Theorem 1 without using the second (or higher) moment of velocity.

First, we can point to homogeneous Gaussian stream matrices which have finite moments of all orders. Condition (b1) also follows from some elementary properties of Gaussian random fields [see, e.g., Adler (1981)].

To give non-Gaussian examples, we begin with any random stationary, zero mean, skew-symmetric matrix $\mathbf{H}$ with finite pth moments satisfying the condition

$$\begin{aligned} &\mathbf{E}|\mathbf{H}_{ij}(\mathbf{x} + \mathbf{h}) - \mathbf{H}_{ij}(\mathbf{x})|^{\alpha} \\ &\qquad \le \frac{c|\mathbf{h}|^{2d}}{|\log|\mathbf{h}||^{1+\eta}} \quad \text{for sufficiency small } \mathbf{h}, \forall\ i, j \end{aligned} \tag{12}$$

for $\eta > \alpha > 0$. This is the case, for example, when the covariances of all entries are C^{2d+1}-smooth and have zero derivatives up to the $2d$th order. Then the random field $\mathbf{H}$ is *uniformly continuous sample-wise* [see Theorem 3.2.5 and Corollary in Adler (1981)]. Without loss of generality, we may assume that $\mathbf{H}_{ij}(0) = 0$. Using the sample-wise uniform continuity of $\mathbf{H}$, by covering the line between the origin and $\boldsymbol{x}$ with sufficiently small balls, it is easy to show that $\mathbf{H}$ satisfies the linear bound

$$|\mathbf{H}_{ij}(\mathbf{x})| \le c(1 + |\mathbf{x}|) \quad \text{for } |\mathbf{x}| \gg 1, \forall\ i, j. \tag{13}$$

Our goal is to modify the sample stream matrix so that it satisfies (H4) and its derivatives satisfy condition (B1). To improve the regularity of the stream matrix, we apply twice the Steklov averaging procedure consisting of integrating the sample matrix at any point $\mathbf{x}$ over a unit ball centered at $\mathbf{x}$. The sample matrix gains one differentiability with one local averaging procedure. Hence the twice locally averaged stream matrix is twice differentiable sample-wise. To check (B1), simple calculus shows that the modulus of the derivatives, that is, the velocity field, at point $\mathbf{x}$ is bounded by a constant, independent of the sample and the location, times the maximum of the moduli of the stream matrix $\mathbf{H}$ in the disk centered at $\mathbf{x}$ with radius two. Thus (B1) follows from the linear bound (13) on $\mathbf{H}$.

An alternative to the local averaging for ensuring that the velocity field has property (B1) is via the covariances of the stream matrix. Starting with a C^2-smooth stream matrix $\mathbf{H}$ with finite pth moments, suppose that the covariances of the entries of $\mathbf{H}$ are C^{2d+3}-smooth. Then the covariances of the velocity field, which are the second derivatives of those of the stream matrix, are C^{2d+1}-smooth. In view of the remark made after (12), we know that the velocity is C^1 and uniformly continuous sample-wise and consequently has a linear bound like (13).

3. Proof of main results. We start with the following lemma stating the existence of a random change of variables such that the motion in the new set of coordinates is a martingale.

LEMMA 1. *There exists a random change of variables* $\mathbf{y}\colon \Omega \times R^d \to R^d$, $\mathbf{y} = (y_1, \dots, y_d)$, *such that we have the following*:

(Y1)
$$\Delta_{\mathbf{x}} \mathbf{y} + \big(\mathbf{b}(\mathbf{x}), \nabla_{\mathbf{x}}\big)\mathbf{y} = 0,$$

(Y2)
$$\mathbf{y}(0) = 0,$$
$$\nabla_{\mathbf{x}} \mathbf{y}(\mathbf{x}; \omega) = \nabla_{\mathbf{x}} \mathbf{y}\big(\mathbf{0}; \tau_{\mathbf{x}}(\omega)\big) \quad \textit{for } P \textit{ a.s. } \omega$$

and

$$\mathbf{E}\nabla_{\mathbf{x}} \mathbf{y}(\mathbf{0}) = \mathbf{I}.$$

(Y3) *For any* $R > 0$,

$$\lim_{\varepsilon \downarrow 0} \sup_{|\mathbf{x}| \le R} |\mathbf{y}_\varepsilon(\mathbf{x}) - \mathbf{x}| = 0, \quad P \textit{ a.s.},$$

where $\mathbf{y}_\varepsilon(\mathbf{x}) = \varepsilon \mathbf{y}(\mathbf{x}/\varepsilon)$.

PROOF. Let $R > 0$ be fixed. The existence of such coordinates $\mathbf{y}(\mathbf{x})$ satisfying (Y1), (Y2) and (Y3′):

$$\lim_{\varepsilon \downarrow 0} \| \mathbf{y}_\varepsilon(\mathbf{x}) - \mathbf{x} \|_{L^2(B_{4R})} = 0$$

and

$$\limsup_{\varepsilon \downarrow 0} \|\nabla_{\mathbf{x}} \mathbf{y}_\varepsilon\|_{L^2(B_{4R})} < +\infty, \quad P \text{ a.s.}$$

is standard and its proof is postponed until Section 5.

We will show now the transition from (Y3′) to (Y3), which is the major obstacle to be overcome. By the classical Sobolev embedding theorem [see Gilbarg and Trudinger (1983)], (Y3′) implies that

$$\limsup_{\varepsilon \downarrow 0} \| \mathbf{y}_\varepsilon \|_{L^{2q}(B_{4R})} < +\infty, \tag{14}$$

where $q = p/(p-2)$. Denote $\mathbf{z}(\mathbf{x}) = \mathbf{y}(\mathbf{x}) - \mathbf{x}$ and $\mathbf{z}_\varepsilon(\mathbf{x}) = \varepsilon \mathbf{z}(\mathbf{x}/\varepsilon)$. It is easy to see that the kth component of $\mathbf{z}_\varepsilon$, $z_\varepsilon^{(k)}$, satisfies the equation

$$\nabla_{\mathbf{x}} \cdot \left[\left(\mathbf{I} + \mathbf{H}(\mathbf{x}/\varepsilon), \nabla_{\mathbf{x}} z_\varepsilon^{(k)}(\mathbf{x}) \right) \right] = -\nabla_{\mathbf{x}} \cdot \mathbf{H}_k(\mathbf{x}/\varepsilon),$$

where $\mathbf{H}_k = (H_{k1}, \dots, H_{kd})$ and $k = 1, \dots, d$. Note that

$$\lim_{\varepsilon \downarrow 0} \int_{B_{4R}} |\mathbf{H}(\mathbf{x}/\varepsilon)|^p \, d\mathbf{x} = |B_{4R}| \, |||\tilde{\mathbf{H}}|||_p^p \tag{15}$$

by the ergodicity of $\mathbf{H}(\mathbf{x}; \omega)$. Now the gap between (Y3′) and (Y3) can be bridged by the following lemma.

LEMMA 2. *For arbitrary $r > 0$ there exist constants $C > 0$ depending only on d and r and $\gamma > 0$, $1 \geq \mu > 0$, depending only on d such that for all $\varepsilon > 0$,*

$$\|\mathbf{z}_\varepsilon\|_{L^\infty(B_r)} \leq C \left[1 + \|\mathbf{H}_\varepsilon\|_{L^p(B_{2r})} \right]^\gamma \|\mathbf{z}_\varepsilon\|_{L^{2q}(B_{2r})}^\mu. \tag{16}$$

Here $\mathbf{H}_\varepsilon(\mathbf{x}) = \mathbf{H}(\mathbf{x}/\varepsilon)$ and $q = p/(p-2)$.

From the second statement of (Y3′) and the classical compact embedding theorem [see Gilbarg and Trudinger (1983)] from $W^{2,1}(B_{4R})$ to $L^r(B_{4R})$, $r < 2d/(d-2)$, it follows that the family $\{\mathbf{z}^\varepsilon\}$ is compact in $L^r(B_{4R})$. Its limiting points in $L^r(B_{4R})$ coincide with those in $L^2(B_{4R})$, namely, $\mathbf{0}$ [the first statement in (Y3′)]. Note that

$$\frac{2p}{p-2} < \frac{2d}{d-2} \tag{17}$$

since $p > d$. Thus we have that the right-hand side of (16) tends to zero in the limit $\varepsilon \downarrow 0$ and from this (Y3) follows. □

PROOF OF THEOREM 1. *Proof of part* 1. Since the field $\mathbf{b}(\mathbf{x})$ is divergence-free the process $\{\nabla_{\mathbf{x}} \mathbf{y}(\mathbf{x}(t))\}_{t \geq 0}$ is strictly stationary and ergodic if considered on the product probability space $(\Omega \times \Sigma, \tilde{V} \otimes M, P \otimes Q)$ [see Papanicolaou and Varadhan (1982), Oelschläger (1988), Osada (1982) and Port and Stone (1976)]. This implies that

$$\frac{1}{t} \int_0^t \left(\nabla_{\mathbf{x}} y_i(\mathbf{x}(s)), \nabla_{\mathbf{x}} y_j(\mathbf{x}(s)) \right) ds$$

tends to $\mathbf{e}(\nabla_{\mathbf{x}} \mathbf{y}_i(\mathbf{0}), \nabla_{\mathbf{x}} \mathbf{y}_j(\mathbf{0}))$ both in L^1 and in the almost sure sense with respect to the measure $P \otimes Q$, when $t \uparrow +\infty$. An application of Itô's formula then leads to

$$\mathbf{M} \frac{y_i(\mathbf{x}^\omega(t)) \, y_j(\mathbf{x}^\omega(t))}{t} = \frac{2}{t} \int_0^t \mathbf{M}\left(\nabla_{\mathbf{x}} y_i(\mathbf{x}^\omega(s)) \nabla_{\mathbf{x}} y_j(\mathbf{x}^\omega(s)) \right) ds. \tag{18}$$

According to Papanicolaou and Varadhan (1982), $\eta^\omega(t) = \tau_{\mathbf{x}^\omega(t;\, \omega,\, \xi)}(\omega)$ is a Markov process on an abstract state space $(\Omega, \tilde{\nu})$. Osada (1982) guarantees that $\mathbf{P}^t \chi_A = \chi_A$, for all $t \geq 0$ iff $P(A) = 0$ or 1, where $\mathbf{P}^t$, $t \geq 0$, is the

$L^2(\Omega, \tilde{\nu}, P)$ semigroup of Markov operators associated with the process; that is, the measure P is ergodic. Thus, by the ergodic theorem of Birkhoff and Khinchine,

$$\frac{1}{t}\int_0^t \mathbf{P}^s f(\omega)\,ds \to \mathbf{E} f$$

P a.s. and in consequence the right-hand side of (18) converges to $2\mathbf{E}(\nabla_{\mathbf{x}} y_i(0)\nabla_{\mathbf{x}} y_j(0))$ P almost surely, when $t\uparrow +\infty$. Note that for almost all $\omega\in\Omega$,

$$\limsup_{t\uparrow+\infty} \mathbf{M}\frac{|\mathbf{x}^{\omega}(t)|^2}{t} < +\infty. \tag{19}$$

Indeed, we can write that

$$\begin{aligned} \mathbf{M}\frac{|\mathbf{x}^{\omega}(t)|^2}{t} &= \mathbf{M}\left\{\frac{|\mathbf{x}^{\omega}(t)|^2}{t}\chi_{[|\mathbf{x}^{\omega}(t)|\ge\sqrt[3]{t}]}\right\} + \mathbf{M}\left\{\frac{|\mathbf{x}^{\omega}(t)|^2}{t}\chi_{[|\mathbf{x}^{\omega}(t)|<\sqrt[3]{t}]}\right\}\\ &\le \mathbf{M}\left\{\frac{|\mathbf{x}^{\omega}(t)|^2}{t}\chi_{[|\mathbf{x}^{\omega}(t)|\ge\sqrt[3]{t}]}\right\} + 1/\sqrt[3]{t}. \end{aligned}\tag{20}$$

To estimate the first term on the right-hand side of the inequality in (20) let us observe that

$$\begin{aligned} \frac{|\mathbf{x}^{\omega}(t)|^2}{t}\chi_{[|\mathbf{x}^{\omega}(t)|\ge\sqrt[3]{t}]} = {}& \frac{|\mathbf{x}^{\omega}(t)-\mathbf{y}(\mathbf{x}^{\omega}(t))|^2}{t}\chi_{[|\mathbf{x}^{\omega}(t)|\ge\sqrt[3]{t}]}\\ &+ \frac{2\big(\mathbf{y}(\mathbf{x}^{\omega}(t)),\mathbf{x}^{\omega}(t)-\mathbf{y}(\mathbf{x}^{\omega}(t))\big)}{t}\chi_{[|\mathbf{x}^{\omega}(t)|\ge\sqrt[3]{t}]}\\ &+ \frac{|\mathbf{y}(\mathbf{x}^{\omega}(t))|^2}{t}\chi_{[|\mathbf{x}^{\omega}(t)|\ge\sqrt[3]{t}]}. \end{aligned}\tag{21}$$

However,

$$\begin{aligned} &\frac{|\mathbf{x}^{\omega}(t)-\mathbf{y}(\mathbf{x}^{\omega}(t))|^2}{t}\chi_{[|\mathbf{x}^{\omega}(t)|\ge\sqrt[3]{t}]}\\ &\quad\le \sup_{0<\varepsilon<1/\sqrt[3]{t}}\ \sup_{|\mathbf{x}|\le 1}|\mathbf{z}_{\varepsilon}(\mathbf{x})|^2\frac{|\mathbf{x}^{\omega}(t)|^2}{t}\chi_{[|\mathbf{x}^{\omega}(t)|\ge\sqrt[3]{t}]}. \end{aligned}\tag{22}$$

Using (21) and (22) we can write that

$$\begin{aligned} \frac{|\mathbf{x}^{\omega}(t)|^2}{t}\chi_{[|\mathbf{x}^{\omega}(t)|\ge\sqrt[3]{t}]} \le {}& \sup_{0<\varepsilon<1/\sqrt[3]{t}}\ \sup_{|\mathbf{x}|\le 1}|\mathbf{z}_{\varepsilon}(\mathbf{x})|^2\frac{|\mathbf{x}^{\omega}(t)|^2}{t}\chi_{[|\mathbf{x}^{\omega}(t)|\ge\sqrt[3]{t}]}\\ &+ \frac{2\big|\big(\mathbf{y}(\mathbf{x}^{\omega}(t)),\mathbf{x}^{\omega}(t)-\mathbf{y}(\mathbf{x}^{\omega}(t))\big)\big|}{t}\chi_{[|\mathbf{x}^{\omega}(t)|\ge\sqrt[3]{t}]}\\ &+ \frac{|\mathbf{y}(\mathbf{x}^{\omega}(t))|^2}{t}\chi_{[|\mathbf{x}^{\omega}(t)|\ge\sqrt[3]{t}]}. \end{aligned}\tag{23}$$

Moving over the first term on the right-hand side of (23) to the left-hand side, we obtain

$$\frac{|\mathbf{x}^{\omega}(t)|^2}{t}\left[1-\sup_{0<\varepsilon<1/\sqrt[3]{t}}\sup_{|\mathbf{x}|\le 1}|\mathbf{z}_{\varepsilon}(\mathbf{x})|^2\right]^{+}\chi_{[|\mathbf{x}^{\omega}(t)|\ge\sqrt[3]{t}]}$$

$$\le\frac{2|(\mathbf{y}(\mathbf{x}^{\omega}(t)),\mathbf{x}^{\omega}(t)-\mathbf{y}(\mathbf{x}^{\omega}(t)))|}{t}\chi_{[|\mathbf{x}^{\omega}(t)|\ge\sqrt[3]{t}]} \tag{24}$$

$$+\frac{|\mathbf{y}(\mathbf{x}^{\omega}(t))|^2}{t}\chi_{[|\mathbf{x}^{\omega}(t)|\ge\sqrt[3]{t}]}.$$

The superscript $+$ denotes as usual the positive part of an expression.

According to part (Y3) of Lemma 1 for P a.s. ω, we can find $t_0(\omega)$ such that for all $t\ge t_0(\omega)$ we have

$$\sup_{0<\varepsilon<1/\sqrt[3]{t}}\sup_{|\mathbf{x}|\le 1}|\mathbf{z}_{\varepsilon}(\mathbf{x})|^2\le\tfrac{1}{2}.$$

After performing simple algebraic manipulations in (24) and then applying $\mathbf{M}$ and averaging we obtain the following estimate, valid for all $t\ge t_0(\omega)$:

$$\mathbf{M}\left\{\frac{|\mathbf{x}^{\omega}(t)|^2}{t}\chi_{[|\mathbf{x}^{\omega}(t)|\ge\sqrt[3]{t}]}\right\}$$

$$\le 4\mathbf{M}\left\{\frac{|(\mathbf{y}(\mathbf{x}^{\omega}(t)),x^{\omega}(t)-\mathbf{y}(\mathbf{x}^{\omega}(t)))|}{t}\chi_{[|\mathbf{x}^{\omega}(t)|\ge\sqrt[3]{t}]}\right\}$$

$$+2\mathbf{M}\left\{\frac{|\mathbf{y}(\mathbf{x}^{\omega}(t))|^2}{t}\chi_{[|\mathbf{x}^{\omega}(t)|\ge\sqrt[3]{t}]}\right\}.$$

Applying Schwarz's inequality to estimate the term involving the scalar product, we can write that

$$M\left\{\frac{|(\mathbf{y}(\mathbf{x}^{\omega}(t)),\mathbf{x}^{\omega}(t)+\mathbf{y}(\mathbf{x}^{\omega}(t)))|}{t}\chi_{[|\mathbf{x}^{\omega}(t)|\ge\sqrt[3]{t}]}\right\}$$

$$\le\left[\mathbf{M}\frac{|\mathbf{y}(\mathbf{x}^{\omega}(t))|^2}{t}\right]^{1/2}\left\{\mathbf{M}\left\{\frac{|\mathbf{x}^{\omega}(t)-\mathbf{y}(\mathbf{x}^{\omega}(t))|^2}{t}\chi_{[|\mathbf{x}^{\omega}(t)|\ge\sqrt[3]{t}]}\right\}\right\}^{1/2}. \tag{25}$$

Using (22) again, we can write that the left-hand side of (25) is less than or equal to

$$\mathbf{M}\left\{\frac{|(\mathbf{y}(\mathbf{x}^{\omega}(t)),\mathbf{x}^{\omega}(t)-\mathbf{y}(\mathbf{x}^{\omega}(t)))|}{t}\chi_{[|\mathbf{x}^{\omega}(t)|\ge\sqrt[3]{t}]}\right\}$$

$$\le\sup_{0<\varepsilon<1/\sqrt[3]{t}}\sup_{|\mathbf{x}|\le 1}|\mathbf{z}_{\varepsilon}(\mathbf{x})|\left[\mathbf{M}\frac{|\mathbf{y}(\mathbf{x}^{\omega}(t))|^2}{t}\right]^{1/2}\left\{\mathbf{M}\frac{|\mathbf{x}^{\omega}(t)|^2}{t}\chi_{[|\mathbf{x}^{\omega}(t)|\ge\sqrt[3]{t}]}\right\}^{1/2}.$$

The above estimate allows us to rewrite (20) in the following form:

$$\mathbf{M}\frac{|\mathbf{x}^{\omega}(t)|^2}{t} \le 1/\sqrt[3]{t} + 4 \sup_{0<\varepsilon<1/\sqrt[3]{t}} \sup_{|\mathbf{x}|\le 1} |\mathbf{z}^{\varepsilon}(\mathbf{x})| \left[\mathbf{M}\frac{|\mathbf{y}(\mathbf{x}^{\omega}(t))|^2}{t}\right]^{1/2}$$
$$\times \left\{\mathbf{M}\frac{|\mathbf{x}^{\omega}(t)|^2}{t}\right\}^{1/2} + 2\mathbf{M}\left[\frac{|\mathbf{y}(\mathbf{x}^{\omega}(t))|^2}{t}\right].$$

Allowing t to be large, we can make an estimate of the second term on the right-hand side of the above inequality, using again part (Y3) of Lemma 1, by

$$4\varrho\left[\mathbf{M}\frac{|\mathbf{y}(\mathbf{x}^{\omega}(t))|^2}{t}\right]^{1/2}\left\{\mathbf{M}\left[\frac{|\mathbf{x}^{\omega}(t)|^2}{t}\right]\right\}^{1/2},$$

where $\varrho > 0$ can be chosen arbitrarily small. This clearly proves the statement made in (19).

Note also that

$$\mathbf{M}\left\{\frac{x_i^{\omega}(t)\,x_j^{\omega}(t)}{t}\right\} = \mathbf{M}\left\{\frac{[x_i^{\omega}(t) - y_i(\mathbf{x}^{\omega}(t))][x_j^{\omega}(t) - y_j(\mathbf{x}^{\omega}(t))}{t}\right\}$$
$$+ \mathbf{M}\left\{\frac{[y_i(\mathbf{x}^{\omega}(t))][x_j^{\omega}(t) - y_j(\mathbf{x}^{\omega}(t))]}{t}\right\}$$
$$+ \mathbf{M}\left\{\frac{[y_j(\mathbf{x}^{\omega}(t))][x_i^{\omega}(t) - y_i(\mathbf{x}^{\omega}(t))]}{t}\right\}$$
$$+ \mathbf{M}\left\{\frac{y_i(\mathbf{x}^{\omega}(t))\,y_j(\mathbf{x}^{\omega}(t))}{t}\right\}.$$

Similarly to the preceding analysis, the first term can be estimated by

$$\max\left[1/\sqrt[3]{t}, \sup_{0<\varepsilon<1/\sqrt[3]{t}} \sup_{|\mathbf{x}|\le 1} |\mathbf{z}_{\varepsilon}(\mathbf{x})|^2\right]\mathbf{M}\left\{\frac{x_i^{\omega}(t)\,x_j^{\omega}(t)}{t}\right\}.$$

The second and the third terms can be estimated correspondingly. Thus the sum of the first three terms tends to 0 when t tends to infinity. The first part of the theorem follows from the remark we made at the beginning of the proof.

Proof of parts 2 *and* 3. Consider the following two families of stochastic processes:

$$\mathbf{y}_{\varepsilon}^{\omega}(t) = \varepsilon\,\mathbf{y}\left(\mathbf{x}^{\omega}\left(\frac{t}{\varepsilon^2}\right)\right), \qquad \varepsilon > 0$$

and

$$\mathbf{z}_{\varepsilon}^{\omega}(t) = \varepsilon\,\mathbf{z}\left(\mathbf{x}^{\omega}\left(\frac{t}{\varepsilon^2}\right)\right), \qquad \varepsilon > 0.$$

Obviously $\mathbf{x}_\varepsilon^\omega(t) = \mathbf{y}_\varepsilon^\omega(t) - \mathbf{z}_\varepsilon^\omega(t)$. By Itô's formula [see, e.g., Karatzas and Shreve (1991)]

$$\mathbf{y}_\varepsilon^\omega(t) = \varepsilon \int_0^{t/\varepsilon^2} \left(\nabla_{\mathbf{x}} \mathbf{y}(\mathbf{x}^\omega(s)), \sqrt{2}\, d\mathbf{w}(s)\right).$$

Therefore $\{\mathbf{y}_\varepsilon^\omega(t)\}_{t \geq 0}$ are continuous local martingales, for fixed ω and all $\varepsilon > 0$. In fact, we can easily show that they are continuous martingales for P a.s. ω. To see that, it suffices to show that there exists a set N such that $P(N) = 0$ and for $\omega \notin N$,

$$\mathbf{M} \int_0^T |\nabla_{\mathbf{x}} \mathbf{y}(\mathbf{x}^\omega(t))|^2 \, dt < +\infty \quad \text{for arbitrary } T \geq 0. \tag{26}$$

The martingale property of the relevant stochastic integrals follows then from the definition of the stochastic integral for the class of square integrable, nonanticipative processes [see Karatzas and Shreve (1991), page 139, Definition 2.9)].

To prove (26) let us observe that according to the statement made in the proof of part 1, the process $\{\nabla_{\mathbf{x}} \mathbf{y}(\mathbf{x}(t))\}_{t \geq 0}$ is strictly stationary over the product probability space $(\Omega \times \Sigma, V \otimes M, P \otimes Q)$ and it has a second absolute moment. That is,

$$\mathbf{EM}|\nabla_{\mathbf{x}} \mathbf{y}(\mathbf{x}(t))|^2 = \mathbf{E}|\nabla_{\mathbf{x}} \mathbf{y}(\mathbf{0})|^2 < +\infty \quad \text{for all } t.$$

Hence

$$\mathbf{EM} \int_0^T |\nabla_{\mathbf{x}} \mathbf{y}(\mathbf{x}(t))|^2 \, dt < +\infty,$$

for arbitrary $T > 0$ and (26) follows from the Fubini theorem combined with some elementary measure theoretic considerations.

Notice that the quadratic variations of the martingales $\{\mathbf{y}_\varepsilon^\omega(t)\}_{t \geq 0}$, $\varepsilon > 0$ are equal to

$$\langle \mathbf{y}_\varepsilon^\omega \rangle_t = \varepsilon^2 \int_0^{t/\varepsilon^2} \left(\nabla_{\mathbf{x}} \mathbf{y}(\mathbf{x}^\omega(s)), \nabla_{\mathbf{x}} \mathbf{y}(\mathbf{x}^\omega(s))\right) ds \quad \text{for } \omega \notin N.$$

Consider now, for a fixed ω, a family of processes

$$\tilde{\mathbf{y}}_\varepsilon^\omega(t) = \mathbf{D}^{-1/2} \mathbf{y}_\varepsilon^\omega(t),$$

where

$$\mathbf{D} = \mathbf{E}\left[\nabla_{\mathbf{x}} \mathbf{y}(\mathbf{0}) (\nabla_{\mathbf{x}} \mathbf{y}(\mathbf{0}))^T\right] \geq \mathbf{I}. \tag{27}$$

Let $\mathbf{v}_1, \mathbf{v}_2$ be arbitrary fixed vectors in R^d. Then for $\omega \notin N$ and arbitrary $\varepsilon > 0$, $(\mathbf{y}_\varepsilon^\omega(t), \mathbf{v}_1)$ and $(\mathbf{y}_\varepsilon^\omega(t), \mathbf{v}_2)$ are zero mean, square integrable martingales whose joint quadratic variation equals

$$\begin{aligned} &\langle (\tilde{\mathbf{y}}_\varepsilon^\omega(\cdot), \mathbf{v}_1)(\tilde{\mathbf{y}}_\varepsilon^\omega(\cdot), \mathbf{v}_2) \rangle_t \\ &\quad = \varepsilon^2 \int_0^{t/\varepsilon^2} \left(\mathbf{D}^{-1} \nabla_{\mathbf{x}} \mathbf{y}(\mathbf{x}^\omega(s)) (\nabla_{\mathbf{x}} \mathbf{y})^T (\mathbf{x}^\omega(s)), \mathbf{v}_1 \otimes \mathbf{v}_2\right) ds \end{aligned} \tag{28}$$

Since the process $\{\nabla_{\mathbf{x}}\mathbf{y}(\mathbf{s}(t))\}_{t\ge 0}$ is strictly stationary and ergodic if considered on the product probability space $(\Omega\times\Sigma, \tilde{V}\otimes M, P\otimes Q)$, we obtain therefore, by the ergodic theorem of Birkhoff and Khinchine, that the limit of the expression on the right-hand side of (28) as $\varepsilon\downarrow 0$ is equal to $t(\mathbf{v}_1,\mathbf{v}_2)$, both $P\otimes Q$ a.s. and in the L^1 sense. We have immediately from this that

$$\lim_{\varepsilon\downarrow 0}\langle(\tilde{\mathbf{y}}_\varepsilon^\omega(\cdot),\mathbf{v}_1)(\tilde{\mathbf{y}}_\varepsilon^\omega(\cdot),\mathbf{v}_2)\rangle_t = t(\mathbf{v},\mathbf{v}_2)$$

both a.s. and in the L^1 sense over the probability space $(\Omega,\tilde{V},P)$. By Theorem 5.4. of Helland [(1982), pages 92 and 93], we obtain that the family $\{\tilde{\mathbf{y}}_\varepsilon^\omega(t)\}_{t\ge 0}$ satisfies the invariance principle P a.s. with the standard Wiener measure as its limit when $\varepsilon\downarrow 0$, which in turn implies that $\{\mathbf{y}_\varepsilon^\omega(t)\}_{t\ge 0}$ satisfies the invariance principle with the limiting Wiener measure having correlation matrix $\mathbf{D}$ for P almost sure ω.

Helland's theorem asserts the convergence of measures over the Skorokhod space $D([0,+\infty); R^d)$ equipped with Stone's topology [see Port and Stone (1976)]. However, a simple application of Lemma 5 given in the Appendix allows us to extend this result to the case of weak convergence of measures over the space $C([0,+\infty); R^d)$ as claimed in the assertion of Theorem 1.

Denote by $\tau_{N,\omega}^\varepsilon$ the exit times of $\mathbf{x}_\varepsilon^\omega(t)$ from the ball B_N. Using condition (Y3), we can easily establish that

$$\lim_{\varepsilon\downarrow 0}\sup_{0\le t\le\tau_N^\varepsilon}|\mathbf{z}_\varepsilon^\omega(t)| = 0. \tag{29}$$

Let $\varepsilon_0(\omega)$ be such that for $\varepsilon<\varepsilon_0(\omega)$ we have $\sup_{0\le t\le\tau_{N,\omega}^\varepsilon}|\mathbf{z}_\varepsilon^\omega(t)|<1$. We see that for those ε,

$$\begin{aligned}
Q\Big[\sup_{0\le t\le T}|\mathbf{x}_\varepsilon^\omega(t)|\ge N\Big] &= Q\big[\tau_{N,\omega}^\varepsilon\le T\big]\\
&= Q\Big[\tau_{N,\omega}^\varepsilon\le T,\ \sup_{0\le t\le T}|\mathbf{y}_\varepsilon^\omega(t)|\ge N-1\Big]\\
&\le Q\Big[\sup_{0\le t\le T}|\mathbf{y}_\varepsilon^\omega(t)|\ge N-1\Big].
\end{aligned}$$

Since for P almost every $\omega\{\mathbf{y}_\varepsilon^\omega(t)\}_{t\ge 0}$ converges weakly to a nondegenerate Brownian motion, we have that there exist constants $\gamma, C>0$ independent of ε,ω such that

$$\limsup_{\varepsilon\downarrow 0} Q\Big[\sup_{0\le t\le T}|\mathbf{x}_\varepsilon^\omega(t)|\ge N\Big]\le Ce^{-\gamma N} \tag{30}$$

[see, e.g., Chung and Zhao (1995), Proposition 1.16, page 20]. We claim that for almost every ω,

$$\lim_{\varepsilon\downarrow 0} Q\Big[\sup_{0\le t\le T}|\mathbf{z}_\varepsilon^\omega(t)|\ge\varrho\Big] = 0. \tag{31}$$

Indeed, suppose that $N > 0$ is arbitrary. Then by (29) for almost every ω,

$$\begin{aligned}
&\limsup_{\varepsilon\downarrow 0} Q\Big[\sup_{0\le t\le T} |\mathbf{z}_\varepsilon^\omega(t)| \ge \varrho\Big] \\
&\qquad \le \limsup_{\varepsilon\downarrow 0} Q\Big[\sup_{0\le t\le \tau^\varepsilon_{N,\omega}} |\mathbf{z}_\varepsilon^\omega(t)| \ge \varrho\Big] + \limsup_{\varepsilon\downarrow 0} Q\Big[\sup_{0\le t\le T} |\mathbf{x}_\varepsilon^\omega(t)| \ge N\Big] \\
&\qquad \le Ce^{-\gamma N}.
\end{aligned}$$

The last inequality in the expression above is a consequence of (30). Since N can be chosen arbitrarily, this concludes the proof of our claim. The claim combined with the fact that the family $\{\mathbf{y}_\varepsilon^\omega(t)\}_{t\ge 0}$, $\varepsilon > 0$ satisfies the invariance principle P a.s. implies that also the family of processes $\{\mathbf{x}_\varepsilon^\omega(t)\}_{t\ge 0}$, $\varepsilon > 0$ satisfies the invariance principle P a.s. In particular, the weak compactness of the probability distributions $\{\mathbf{x}_\varepsilon^\omega(t)\}_{t\ge 0}$, $\varepsilon > 0$ over the space $C([0, +\infty);\ R^d)$ implies that for arbitrary $\varrho, T > 0$,

$$\lim_{\delta\downarrow 0} Q\Big[\sup_{0\le s\le t\le T,\ t-s\le\delta,\ \varepsilon>0} |\mathbf{x}_\varepsilon^\omega(t) - \mathbf{x}_\varepsilon^\omega(s)| \ge \varrho\Big] = 0, \quad P \text{ a.s.}$$

[see Billingsley (1968), page 55, Theorem 8.2]. Averaging over all $\omega \in \Omega$ with respect to the measure P, we have that also

$$\lim_{\delta\downarrow 0} P \otimes Q\Big[\sup_{0\le s\le t\le T,\ t-s\le\delta,\ \varepsilon>0} |\mathbf{x}_\varepsilon(t) - \mathbf{x}^\varepsilon(s)| \ge \varrho\Big] = 0,$$

which implies that $\{\mathbf{x}_\varepsilon(t)\}_{t\ge 0}$, $\varepsilon > 0$ is weakly compact. Since the limit identification follows easily from the fact that the family $\{\mathbf{x}_\varepsilon^\omega(t)\}_{t\ge 0}$, $\varepsilon > 0$ satisfies the invariance principle P a.s. with the same limiting measure independent of ω, we can conclude the weak convergence of $\{\mathbf{x}_\varepsilon(t)\}_{t\ge 0}$, $\varepsilon > 0$ to a Wiener measure with the covariance matrix $\mathbf{D}$ as in (27). □

4. Proof of Lemma 2: Uniform bound on the correctors. In this section we prove Lemma 2 in a slightly more general setting. We shall adopt Moser's iterative scheme [cf. Moser (1961)].

First we introduce some additional notation. Let B_R and B_R^o denote a closed ball in R^d with center at $\mathbf{0}$ and radius R and the interior of the ball respectively. When $R = 1$ we omit the subscript and write B for B_1.

Suppose that a_{ij}: $R^d \to R$, $i, j = 1, \dots, d$ is a family of Borel measurable functions which satisfies the following conditions.

(A1) There exists $\lambda > 0$ such that for all $\mathbf{x} \in R^d$ and $\boldsymbol{\xi} = (\xi_1, \dots, \xi_d) \in R^d$ we have

$$\sum_{i,j=1}^d a_{ij}(\mathbf{x})\xi_i\xi_j \ge \lambda|\boldsymbol{\xi}|^2.$$

Without any loss of generality we shall assume that $\lambda \le 1$.

(A2) For a certain $p > d$ and $R > 0$,

$$A_{p,R} = \left[\sum_{i,j=1}^{d} \int_{B_R} |a_{ij}|^p \, d\mathbf{x} \right]^{1/p} < +\infty,$$

We shall also consider a vector field $f = (f_1, \dots, f_d)\colon R^d \to R^d$ such that

$$f_1, \dots, f_d \in W^{1,p}(B_R^o). \tag{f}$$

As is customary, $W^{1,p}(B_R^o)$ stands for the space of those functions $f \in L^p(B_R^o)$ whose weak derivatives $\partial_1 f, \dots, \partial_d f \in L^p(B_R^o)$ with the norm

$$\|f\|_{1,p,B_R} = \left\{ \|f\|_{L^p(B_R)}^p + \sum_{k=1}^{d} \|\partial_k f\|_{L^p(B_R)}^p \right\}^{1/p}$$

LEMMA 3. *Assume that u is a classical solution of the equation*

$$-\nabla \cdot \left[\mathbf{A}(\mathbf{x}) \nabla u(\mathbf{x}) \right] = \nabla \cdot \mathbf{f}(\mathbf{x}), \tag{32}$$

with $\mathbf{A} = [a_{ij}]$ satisfying assumptions (A1) *and* (A2) *and $\mathbf{f}$ satisfying condition (f). Then there exist constants $C, \nu > 0$ and $0 < \mu \le 1$ depending only on d and R for which*

$$\|u\|_{L^\infty(B_{R/2})} \le C\left(A_{p,R} + \|\mathbf{f}\|_{L^p(B_R)} + 1 \right)^{\nu} \|u\|_{L^{2q}(B_R)}^{\mu},$$

where

$$q = \frac{p}{p-2},$$

and $\|\mathbf{f}\|_{L^p(B_R)} = \{\sum_{k=1}^{d} \|f_k\|_{L^p(B_R)}^p\}^{1/p}$.

PROOF. First we make the following reduction to positive subsolutions of equation (32). By a subsolution $v(\mathbf{x})$ of (32), we mean a function $v(\mathbf{x})$ such that

$$\int \left(\mathbf{A}(\mathbf{x}) \nabla_{\mathbf{x}} v(\mathbf{x}), \nabla_{\mathbf{x}} \phi(\mathbf{x}) \right) + \int \left(\mathbf{f}(\mathbf{x}), \nabla_{\mathbf{x}} \phi(\mathbf{x}) \right) \le 0$$

for any nonnegative smooth function ϕ with compact support.

Any solution $u(\mathbf{x})$ of (32) can be written as the difference of two positive subsolutions $v(\mathbf{x})$, $w(\mathbf{x})$:

$$u(\mathbf{x}) = v(\mathbf{x}) - w(\mathbf{x})$$

where

$$v(\mathbf{x}) = \sqrt{u^2(\mathbf{x}) + 1}$$

and

$$w(\mathbf{x}) = \sqrt{u^2(\mathbf{x}) + 1} - u(\mathbf{x}).$$

Thus, to prove Lemma 3, suffice it to prove the statement for any positive subsolution.

In the rest of the proof we assume that u is a positive subsolution of (32) and, without loss of generality, that $R = 1$.

When $d \ge 3$ we set

$$\varrho_n = \frac{1}{2} + \frac{1}{2^{n+1}},$$
$$\bar{\varrho}_n = \tfrac{1}{2}(\varrho_n + \varrho_{n+1}), \tag{33}$$
$$\alpha_n = \left[\frac{d(p-2)}{(d-2)p}\right]^n, \qquad n = 0, 1, \dots .$$

Let $\varphi \in C_0^\infty(R)$ be such that $0 \le \varphi \le 1$, $\varphi(t) \equiv 1$, for $|t| \le \frac{1}{2}$, $\varphi(t) \equiv 0$, for $|t| \ge 3/4$ and $|\varphi'| \le 2$. Define $\zeta_n(\mathbf{y}) = \varphi(2^{n+1}(|\mathbf{y}| - \frac{1}{2}))$. Observe that $\zeta_n(\mathbf{y}) \equiv 1$ in $B_{\varrho_{n+1}}$ and $\zeta_n(\mathbf{y}) \equiv 0$ outside $B_{\bar{\varrho}_n}$. Multiplying equation (32) by $|u|^{2\alpha_n - 1}\zeta_n^2$ and integrating by parts gives

$$\int_{B_{\varrho n}} \left(\mathbf{A}\nabla_{\mathbf{x}} u, \nabla_{\mathbf{x}}\left(|u|^{2\alpha_n - 1}\zeta_n^2\right)\right) d\mathbf{x} = \int_{B_{\varrho n}} \left(\mathbf{f}, \nabla_{\mathbf{x}}\left(|u|^{2\alpha_n - 1}\zeta_n^2\right)\right) d\mathbf{x}. \tag{34}$$

After a straightforward calculation, (34) yields

$$\begin{aligned}
\int_{B_{\varrho n}} \left(\mathbf{A}\nabla_{\mathbf{x}}(\zeta_n|u|^{\alpha_n}), \nabla_{\mathbf{x}}(\zeta_n|u|^{\alpha_n})\right) d\mathbf{x} &= \alpha_n \int_{B_{\varrho n}} \left(|u|^{\alpha_n - 1}\zeta_n \mathbf{f}, \nabla_{\mathbf{x}}(\zeta_n|u|^{\alpha_n})\right) dx \\
&\quad + \frac{\alpha_n}{2\alpha_n - 1}\int_{B_{\varrho n}} \left(|u|^{2\alpha_n - 1}\zeta_n \mathbf{f}, \nabla_{\mathbf{x}}\zeta_n\right) d\mathbf{x} \\
&\quad + \int_{B_{\varrho n}} \left(\mathbf{A}\nabla_{\mathbf{x}}(\zeta_n|u|^{\alpha_n}), |u|^{\alpha_n}\nabla_{\mathbf{x}}\zeta_n\right) d\mathbf{x}.
\end{aligned}$$

Using condition (A1) and the Hölder inequality, we have that

$$\begin{aligned}
&\lambda\|\nabla_{\mathbf{x}}(\zeta_n|u|^{\alpha_n})\|^2_{L^2(B_{\varrho n})} \\
&\quad \le \|\nabla_{\mathbf{x}}(\zeta_n|u|^{\alpha_n})\|_{L^2(B_{\varrho n})}\Big[A(p,1)\|\,|u|^{\alpha_n}\nabla_{\mathbf{x}}\zeta_n\|_{L^{2q}(B_{\varrho n})} + \alpha_n\|\mathbf{f}\zeta_n|u|^{\alpha_n - 1}\|_{L^2(B_{\varrho n})}\Big] \\
&\qquad + \frac{\alpha_n}{2\alpha_n - 1}\|\mathbf{f}\zeta_n\|_{L^p(B_{\varrho n})}\|\,|u|^{2\alpha_n - 1}\nabla_{\mathbf{x}}\zeta_n\|_{L^{p/(p-1)}(B_{\varrho n})} \\
&\quad \le \|\nabla_{\mathbf{x}}(\zeta_n|u|^{\alpha_n})\|_{L^2(B_{\varrho n})}\Big[A(p,1)2^{n+1}\|u\|^{\alpha_n}_{L^{2\alpha_n q}(B_{\varrho n})} \\
&\qquad\qquad + \alpha_n\|\mathbf{f}\|_{L^p(B_{\varrho n})}\|u\|^{\alpha_n - 1}_{L^{2(\alpha_n - 1)q}(B_{\varrho n})}\Big] \\
&\qquad + 2^{n+1}\frac{\alpha_n}{2\alpha_n - 1}\|\mathbf{f}\|_{L^p(B_{\varrho n})}\|u\|^{2\alpha_n - 1}_{L^{2q(\alpha_n - 1/2)(p-2)/(p-1)}(B_{\varrho n})}.
\end{aligned}$$

By an elementary calculation we have that

$$
\begin{aligned}
&\|\nabla_{\mathbf{x}}(\zeta_n|u|^{\alpha_n})\|_{L^2(B_{\varrho n})}\\
&\quad\le \frac{1}{\lambda}\Big[A(p,1)2^{n+1}\|u\|^{\alpha_n}_{L^{2\alpha_n q}(B_{\varrho n})}\\
&\qquad + \alpha_n\|\mathbf{f}\|_{L^p(B_{\varrho n})}\|u\|^{\alpha_n-1}_{L^{2(\alpha_n-1)q}(B_{\varrho n})}\Big]\\
&\qquad + \left[\frac{2^{n+1}\alpha_n}{\lambda(2\alpha_n-1)}\right]^{1/2}\|\mathbf{f}\|^{1/2}_{L^p(B_{\varrho n})}\|u\|^{\alpha_n-1/2}_{L^{2q(\alpha_n-1/2)(p-2)/(p-1)}(B_{\varrho n})}.
\end{aligned}
\tag{35}
$$

In the sequel we shall denote by C any generic positive constant depending only on d and independent of n. It follows from the Sobolev inequality that

$$\|\zeta_n|u|^{\alpha_n}\|_{L^{2d/(d-2)}(B_{\varrho n})} \le C\|\nabla_{\mathbf{x}}(\zeta_n|u|^{\alpha_n})\|_{L^2(B_{\varrho n})}, \tag{36}$$

and from the fact

$$\|u\|_{L^a(B_{\varrho n})} \le C\|u\|_{L^b(B_{\varrho n})}, \qquad 1\le a\le b, \tag{37}$$

that

$$\|u\|_{L^{2(\alpha_n-1)q}(B_{\varrho n})} \le C\|u\|_{L^{2\alpha_n q}(B_{\varrho n})}, \tag{38}$$

$$\|u\|_{L^{2(\alpha_n-1/2)q(p-2)/(p-1)}(B_{\varrho n})} \le C\|u\|_{L^{2\alpha_n q}(B_{\varrho n})}. \tag{39}$$

Thus we have from (35)–(39) and the definitions of α_n and q that

$$
\begin{aligned}
\|u\|^{\alpha_n}_{L^{2q\alpha_{n+1}}(B_{\varrho_{n+1}})} &\le C\frac{1}{\lambda}\Big[A(p,1)2^{n+1}\|u\|^{\alpha_n}_{L^{2\alpha_n q}(B_{\varrho n})} + \alpha_n\|\mathbf{f}\|_{L^p(B_{\varrho n})}\|u\|^{\alpha_n-1}_{L^{2\alpha_n q}(B_{\varrho n})}\Big]\\
&\quad + \left[\frac{2^{n+1}\alpha_n}{\lambda(2\alpha_n-1)}\right]^{1/2}\|\mathbf{f}\|^{1/2}_{L^p(B_{\varrho n})}\|u\|^{\alpha_n-1/2}_{L^{2\alpha_n q}(B_{\varrho n})}.
\end{aligned}
\tag{40}
$$

Since $2\alpha_{n+1}q = 2\alpha_n d/(d-2)$, the above inequality implies that if $\|u\|_{L^{2\alpha_n q}(B_{\varrho n})} \ge 1$, then

$$
\begin{aligned}
&\|u\|_{L^{2q\alpha_{n+1}}(B_{\varrho_{n+1}})}\\
&\quad\le \left\{\frac{C2^n\alpha_n}{\lambda}\Big[A(p,1) + \|\mathbf{f}\|_{L^p(B_{\varrho n})} + \|\mathbf{f}\|^{1/2}_{L^p(B_{\varrho n})}\Big]\right\}^{1/\alpha_n}\|u\|_{L^{2\alpha_n q}(B_{\varrho n})}.
\end{aligned}
\tag{41}
$$

If, on the other hand, $\|u\|_{L^{2\alpha_n q}(B_{\varrho n})} \le 1$, we have

$$
\begin{aligned}
&\|u\|_{L^{2\alpha_{n+1}q}(B_{\varrho n})}\\
&\quad\le C\left\{\frac{2n\alpha_n}{\lambda}\Big[A(p,1) + \|\mathbf{f}\|_{L^p(B_{\varrho n})} + \|\mathbf{f}\|^{1/2}_{L^p(B_{\varrho n})}\Big]\right\}^{1/\alpha_n}\|u\|^{\gamma_n}_{L^{2\alpha_n q}(B_{\varrho n})},
\end{aligned}
\tag{42}
$$

where $\gamma_n = 1 - 1/\alpha_n$. These two estimates together imply that

$$\|u\|_{L^{2\alpha_{n+1}q}(B)} \le \left\{C\Big[A(p,1) + \|\mathbf{f}\|_{L^p(B)} + \|\mathbf{f}\|^{1/2}_{L^p(B)} + 1\Big]\right\}^{\nu}\|u\|^{\mu}_{L^{2q}(B)} \tag{43}$$

for all $n = 0,1,\dots$. Here $0 < \mu = \prod_{k=1}^{+\infty}\tilde\gamma_k \le 1$, while $\nu = \sum_{k=1}^{+\infty} k/\alpha_k$ and $\tilde\gamma_k = 1$ if $\|u\|_{L^{2\alpha_k q}(B_{\varrho n})} \ge 1$, or $\tilde\gamma_k = \gamma_k$ if $\|u\|_{L^{2\alpha_k q}(B_{\varrho n})} \le 1$. Passing to the limit $n \to \infty$ in (43) we obtain the statement of the lemma for $d \ge 3$.

For $d = 2$, instead of (33), we set

$$\alpha_n = \left(\frac{\alpha}{2q}\right)^n, \qquad n = 0, 1, 2, 3, \dots \tag{44}$$

for any $\alpha > 2q$. The calculations up to (35) go without change. The Sobolev inequality now takes the form

$$\|\zeta_n |u|^{\alpha_n}\|_{L^\alpha(B_{\varrho n})} \le C \|\nabla_{\mathbf{x}}(\zeta_n |u|^{\alpha_n})\|_{L^2(B_{\varrho n})}, \tag{45}$$

for any $2q < \alpha < \infty$. With (45) and with $2d/(d-2)$ replaced by sufficiently large $\alpha > 2q$, we can go through the rest of the calculations and complete the proof. □

5. Proof of Lemma 1: existence of harmonic coordinates. Denote by $H^1(\Omega)$ the space of all square integrable random variables $X: \Omega \to R$ such that $X \in D_k$ for $k = 1, \dots, d$. Define the scalar product on this space by the formula

$$[X, Y]_{H^1} = \mathbf{E}XY + \sum_{k=1}^{d} \mathbf{E} D_k X D_k Y,$$

for any $X, Y \in H^1(\Omega)$. It is easy to check that H^1 equipped with this scalar product is a Hilbert space. For any $X \in H^1(\Omega)$ the standard norm is defined as $|||X|||_{H^1} = [X, X]_{H^1}^{1/2}$.

Denote by $L_d^2(\Omega)$ the space of all d-dimensional random vectors

$$\mathbf{X} = (X_1, \dots, X_d): \Omega \to R^d$$

with the norm $|||\mathbf{X}|||_2^2 = \sum_{k=1}^d |||X_k|||_2^2$. By the symbol $L_g^2(\Omega)$ we denote the subspace of gradient fields, that is, the L^2 closure of the vectors of the form $(D_1 v, \dots, D_d v)$, where $v \in H^1(\Omega)$.

Denote also by $W^{1,\infty}(\Omega)$ the subspace of L^2 consisting of those $v \in \bigcap_{k=1}^d Dk$ for which

$$|||v|||_{W^{1,\infty}} = |||v|||_\infty + \sum_{k=1}^{d} |||D_k v|||_\infty < +\infty.$$

The following lemma holds.

LEMMA 4. (i) *The subspace S consisting of random vectors of the form $(D_1 v, \dots, D_d v)$, where $v \in W^{1,\infty}(\Omega)$, is dense in $L_g^2(\Omega)$ in the L^2 norm.*

(ii) *Suppose that $\Phi = (\Phi_1, \dots, \Phi_d)$ is the random spectral measure of the vector $\mathbf{F} = (F_1, \dots, F_d) \in L_g^2(\Omega)$. That is, $\mathbf{F}(\tau_{\mathbf{x}}(\omega)) = \int e^{i(\mathbf{x}, \boldsymbol{\lambda})}\Phi(d\boldsymbol{\lambda})$. Then for any $m, n = 1, \dots, d$ we have $\lambda_m \Phi_n(d\boldsymbol{\lambda}) = \lambda_n \Phi_m(d\boldsymbol{\lambda})$. That is, for any Borel measurable and bounded function φ,*

$$\int \varphi(\boldsymbol{\lambda}) \lambda_m \Phi_n(d\boldsymbol{\lambda}) = \int \varphi(\boldsymbol{\lambda}) \lambda_n \Phi_m(d\boldsymbol{\lambda}). \tag{46}$$

PROOF. In the first part of the proof we follow closely the argument presented in Dedik and Subin (1982). From the definition of $L^2_g(\Omega)$ we can see that it suffices only to prove that $W^{1,\infty}(\Omega)$ is dense in $H^1(\Omega)$ in H^1 norm. To see that, we choose arbitrary $v \in H^1(\Omega)$. It is clear that $v_\varepsilon = \varepsilon^d \int \eta(\mathbf{x}/\varepsilon) U^{\mathbf{x}} v \, d\mathbf{x}$, where $\eta \ge 0$ and $\eta \in C_0^\infty(R^d)$, v_ε belongs to $W^{1,\infty}(\Omega)$ and approximates v, as $\varepsilon \downarrow 0$ in H^1 norm. Part (ii) follows from the fact that the vectors in S satisfy (46) and they are dense in $L^2_g(\Omega)$. $\square$

Consider now a family of bilinear forms on $H^1(\Omega)$ given by

$$B_\lambda^{(n)}(u, v) = \sum_{k,l=1}^d \mathbf{E}\Big\{\Big(\delta_{kl} D_k u + \tilde{H}_{kl}^{(n)} D_k u\Big) D_l v\Big\} + \lambda \mathbf{E}\, uv,$$

where $\lambda > 0$, n is a positive integer. Here

$$\tilde{H}_{kl}^{(n)} = \tilde{H}_{kl} \quad \text{if } |\tilde{H}_{kl}| \le n$$

and

$$\tilde{H}_{kl}^{(n)} = n \frac{\tilde{H}_{kl}}{|\tilde{H}_{kl}|} \quad \text{if } |\tilde{H}_{kl}| > n.$$

Consider also linear functionals

$$L_k(v) = \sum_{l=1}^d \mathbf{E}\Big\{\tilde{H}_{kl}^{(n)} D_l v\Big\}.$$

By the classical Lax–Milgram lemma, one can find $u_{n,\lambda}^{(k)} \in H^1(\Omega)$ such that

$$L_k(v) = B_\lambda^{(n)}\big(u_{n,\lambda}^{(k)}, v\big) \quad \text{for all } v \in H^1(\Omega). \tag{47}$$

After substituting $u_{n,\lambda}^{(k)}$ for v into (47) we have that

$$\sum_{l=1}^d |||\, D_l u_{n,\lambda}^{(k)} |||_2 \le \sum_{l,m=1}^d |||\, \tilde{H}_{kl} |||_2 .$$

Letting first λ tend to 0 and later choosing a suitable subsequence of n tending to $+\infty$, we obtain $\mathbf{F}^{(k)} = (F_1^{(k)}, \dots, F_d^{(k)}) \in L^2_g(\Omega)$ such that

$$\sum_{p,q=1}^d \mathbf{E}\Big\{\Big(\delta_{pq} F_p^{(k)} + \tilde{H}_{pq}^{(n)} F_p^{(k)}\Big) D_q v = \sum_{q=1}^d \mathbf{E}\, \tilde{H}_{kq}^{(n)} D_q v \quad \text{for all } v \in W^{1,\infty}(\Omega).$$

Following Papanicolaou and Varadhan (1982), we set

$$y_k(\mathbf{x}; \omega) = x_k - \sum_{q=1}^d \int_{R^d} i \lambda_q \frac{e^{i(\boldsymbol{\lambda}, \mathbf{x})} - 1}{|\boldsymbol{\lambda}|^2} U(d\boldsymbol{\lambda}) F_q^{(p)}.$$

Observe that the weak partials of y_k satisfy

$$\partial_{x_r} y_k(\mathbf{x}; \omega) = \delta_{r,k} + \sum_{q=1}^d \int_{R^d} \frac{e^{i(\boldsymbol{\lambda}, \mathbf{x})} \lambda_q \lambda_r}{|\boldsymbol{\lambda}|^2} U(d\boldsymbol{\lambda}) F_q^{(k)}.$$

However, from Lemma 4, we know that

$$\int_{R^d} \frac{e^{i(\boldsymbol{\lambda},\mathbf{x})}\lambda_q \lambda_r}{|\boldsymbol{\lambda}|^2} U(d\boldsymbol{\lambda}) F_q^{(k)} = \int_{R^d} \frac{e^{i(\boldsymbol{\lambda},\mathbf{x})}\lambda_q^2}{|\boldsymbol{\lambda}|^2} U(d\boldsymbol{\lambda}) F_r^{(k)}.$$

Therefore

$$\partial_{x_r} y_k(\mathbf{x};\omega) = \delta_{r,k} + F_r^{(k)}(\tau_{\mathbf{x}}(\omega)),$$

which proves assertion (Y2) of Lemma 1.

Let $\eta \in C_0^\infty(R_d)$. For any $v \in W^{1,\infty}(\Omega)$ we have

$$\begin{aligned} 0 &= \sum_{p,q=1}^{d} \int_{R^d} \eta(\mathbf{x})\, d\mathbf{x} \mathbf{E}\Big\{ D_q v \big[\delta_{pq}\partial_{x_p} y_k(\mathbf{x};\omega) + H_{pq}(\mathbf{x};\omega)\partial_{x_p} y_k(\mathbf{x};\omega)\big]\Big\} \\ &= \sum_{p,q=1}^{d} \mathbf{E}\Big\{ v\int_{R^d} \partial_{x_q}\eta(\mathbf{x})\big[\delta_{pq}\partial_{x_p} y_k(\mathbf{x};\omega) + H_{pq}(\mathbf{x};\omega)\partial_{x_p} y_k(\mathbf{x};\omega)\big]\Big\} d\mathbf{x}. \end{aligned}$$

Hence

$$\sum_{p,q=1}^{d} \int_{R^d} \partial_{x_q}\eta(\mathbf{x})\big[\delta_{pq}\partial_{x_p} y_k(\mathbf{x};\omega) + H_{pq(\mathbf{x};\omega)}\partial_{x_p} y_k(\mathbf{x};\omega)\big]\, d\mathbf{x} = 0.$$

From the classical theory of elliptic PDE's, y_k is the classical solution of

$$\sum_{p,q=1}^{d} \partial_{x_q}\big[\delta_{pq}\partial_{x_p} y_k(\mathbf{x};\omega) + H_{pq}\partial_{x_p} y_k\big] = 0, \qquad k = 1,\dots,d.$$

which ends the proof of (Y1). Using the notation from Section 3, we have, by an application of the ergodic theorem, that

$$\lim_{\varepsilon\downarrow 0} \|\nabla_{\mathbf{x}} \mathbf{y}_\varepsilon\|^2_{L^2(B_R)} = |||\mathbf{F}|||_2^2 |B_R| < +\infty, \quad P \text{ a.s.}$$

Therefore, to end the proof of assertion (Y3′) we need only to prove that for any $\eta \in C_0^\infty(B_R)$ we have

$$\lim_{\varepsilon\downarrow 0} \int_{R^d} \mathbf{y}_\varepsilon(\mathbf{x})\eta(\mathbf{x})\,d\mathbf{x} = \int_{R^d} \mathbf{x}\eta(\mathbf{x})\, d\mathbf{x}.$$

Following Kozlov (1985), we can write

$$\begin{aligned} \int_{B_R} \mathbf{y}_\varepsilon(\mathbf{x})\eta(\mathbf{x})\, d\mathbf{x} &= \sum_{p=1}^{d} \int_{B_R} d\mathbf{x} \int_0^1 \partial_{x_p}\mathbf{y}_\varepsilon(t\mathbf{x})\, x_p \eta(\mathbf{x})\, dt \\ &= \sum_{p=1}^{d} \int_0^1 dt \int_{B_R} \partial_{x_p}\mathbf{y}_\varepsilon(t\mathbf{x})\, x_p \eta(\mathbf{x})\, d\mathbf{x}. \end{aligned} \tag{48}$$

Set $\sigma > 0$ arbitrary. From the ergodic theorem we obtain that for P a.s. ω there exists $\varepsilon(\omega)$ such that

$$\sup_{\varrho \le \varepsilon(\omega)} \left| \sum_{p=1}^{d} \lim_{\varepsilon\downarrow 0} \int_{B_R} \partial_{x_p}\mathbf{y}_\varepsilon(s\mathbf{x})\, x_p \eta(\mathbf{x})\, d\mathbf{x} - \int_{B_R} \mathbf{x}\eta(\mathbf{x})\, d\mathbf{x} \right| \le \sigma. \tag{49}$$

We can rewrite the right-hand side of (48) as equal to

$$\sum_{p=1}^{d}\int_0^{\varepsilon/\varepsilon(\omega)} dt\int_{B_R}\partial_{x_p}\mathbf{y}_{\varepsilon/t}(t\mathbf{x})\,x_p\eta(\mathbf{x})\,d\mathbf{x} + \sum_{p=1}^{d}\int_{\varepsilon/\varepsilon(\omega)}^{1} dt\int_{B_R}\partial_{x_p}\mathbf{y}_{\varepsilon/t}(t\mathbf{x})\,x_p\eta(\mathbf{x})\,d\mathbf{x}. \tag{50}$$

By virtue of (49) the second integral can be written as

$$\int_{B_R} x_p\eta(\mathbf{x})\,d\mathbf{x} + r_\varepsilon,$$

where $|r_\varepsilon| < \sigma$ for $\varepsilon < \varepsilon(\omega)$.

As for the first integral notice that the integrand can be estimated by

$$\|\nabla\mathbf{y}_{\varepsilon/t}\|_{L^2(B_R)}\|x_p\eta\|_{L^2(B_R)} \le |B_R| \sup_{\varrho\le 1/\varepsilon(\omega)}\left(\frac{1}{\varrho^d}\int_{B_\varrho}\|\nabla\mathbf{y}_{\varepsilon/t}\|^2 d\mathbf{x}\right)^{1/2}\|x_p\eta\|_{L^2(B_R)}.$$

Thus, the first integral can be estimated by $\varepsilon\|B_R\|/\varepsilon(\omega)$. In conclusion, we get that

$$\limsup_{\varepsilon\downarrow 0}\left|\int_{B_R}\mathbf{y}_\varepsilon(\mathbf{x})\eta(\mathbf{x})\,d\mathbf{x} - \int_{B_R}x_p\eta(\mathbf{x})\,d\mathbf{x}\right| \le \sigma$$

for arbitrary $\sigma > 0$.

APPENDIX

A certain fact about weak convergence of measures. Let us denote by $C([0,T];\ R^d)$ and $D([0,T];\ R^d)$ the space of all continuous functions and the space of all functions having only discontinuities of the first kind correspondingly, equipped with their respective topologies. The reader is referred to Billingsley (1968) for the precise definition of these spaces. The following lemma is the main objective of this section.

LEMMA 5. *Assume that $T > 0$ is arbitrary. Suppose that $\{\mu_n\}_{n\ge 0}$ is a sequence of Borelian probability measures given on the space $D([0,T];\ R^d)$ and supported in $C([0,T];\ R^d)$. Assume also that $\{\mu_n\}_{n\ge 0}$ converges weakly over $D([0,T];\ R^d)$ space to a measure μ_* supported in $C([0,T];\ R^d)$.*

Then $\{\mu_n\}_{n\ge 0}$ converges weakly to μ_* over $C([0,T];\ R^d)$.

PROOF. The sequence $\{\mu_n\}_{n\ge 0}$ is tight in $D([0,T];\ R^d)$. Hence according to Billingsley [(1968), Theorem 15.2, page 125], for any $\varepsilon, \varrho > 0$ we can find $A, \delta > 0$ such that for all $n \ge 0$ the following holds:

(i) $$\mu_n\left[x\colon \sup_{0\le t\le T}|x(t)| \ge A\right] \le \varepsilon;$$

(ii) $$\mu_n[x\colon \omega'_x(\delta) \ge \varrho] \le \varepsilon.$$

Here

$$\omega'_x(\delta) = \inf_{t_i} \max_{0 \le i \le r} \omega_x([t_{i-1}, t_i]),$$

where the $\inf_{t_i}$ is taken over all possible partitions $0 < t_1 < , \dots , t_k < T$, $k = 1, 2, \dots$ of $[0, T]$ such that $t_i - t_{i-1} \ge \delta$ and

$$\omega_x([a, b]) = \sup_{a \le s \le t \le b} |x(t) - x(s)|,$$

for $a < b$.

According to Billingsley [(1968), (14.11), page 111], we have

$$\tfrac{1}{2}\,\omega_x(\delta) \le \omega'_x(\delta) \le \omega_x(2\delta),$$

provided that $x \in C([0, T];\ R^d)$. Here

$$\omega_x(\delta) = \sup_{0 \le s \le t \le T,\ t-s \ge \delta} |x(t) - x(s)|. \tag{49}$$

Equation (49) together with conditions (i) and (ii) imply, by virtue of Theorem 8.2, page 55 of Billingsley (1968), that the sequence of measures $\{\mu_n\}_{n \ge 0}$ is tight in $C([0, T];\ R^d)$.

Observe also that the finite-dimensional projections

$$\pi(t)\colon C([0, T];\ R^d) \to R^d, \qquad t \in [0, T],$$

given by

$$\pi(t)(x) = x(t)$$

are continuous on the support of μ_* for all $t \in [0, T]$. Therefore, the finite-dimensional distributions of the measures μ_n converge weakly, over the relevant finite-dimensional Euclidean spaces, to the corresponding finite-dimensional distributions of μ_*. This concludes the proof of the lemma. □

Acknowledgment. T.K. thanks Professor Zhengfang Zhou for several discussions on the subject of this paper.

REFERENCES

ADLER, R. J. (1981). *Geometry of Random Fields*. Wiley, New York.

AVELLANEDA, M. and MAJDA, A. J. (1991). An integral representation and bounds on the effective diffusivity in passive advection by laminar and turbulent flows. *Comm. Math. Phys.* **138** 339–391.

BILLINGSLEY, P. (1968). *Convergence of Probability Measures*. Wiley, New York.

BOUCHAUD, J.-P. and GEORGES, A. (1990). Anomalous diffusion in disordered media: statistical mechanics, models and physical applications. *Phys. Rep.* **195** 127–293.

BROWN, B. M. (1971). Martingale central limit theorems. *Ann. Math. Statist.* **42** 59–66.

CHORIN, A. (1994). *Vorticity and Turbulence*. Springer, Berlin.

CHUNG, K. L. and ZHAO, Z. (1995). *From Brownian Motion to Schrödinger's Equation*. Springer, New York.

DEDIK, P. E. and SUBIN, M. A. (1982). Random pseudodifferential operators and the stabilization of solutions of parabolic equations with random coefficients. *Math. USSR Sb.* **41** 33–52.

DUNFORD, N. and SCHWARTZ, J. T. (1988). *Linear Operators I*. Wiley, New York.

FANNJIANG, C. A. (1997). Anomalous diffusion in random flows. In *Mathematics of Multiscale Materials* (K. M. Golden, G. R. Grimmet, R. D. James, G. W. Milton and P. N. Sen, eds.). Springer. To appear.

FANNJIANG, C. A. and PAPANICOLAOU, G. C. (1996). Diffusion in turbulence. *Probab. Theory Related Fields* **105** 279–334.

GILBARG, D. and TRUDINGER, N. S. (1983). *Elliptic Partial Differential Equations of Second Order*, 2nd ed. Springer, New York.

HELLAND, I. S. (1982). Central limit theorems for martingales with discrete or continuous time. *Scand. J. Statist.* **9** 79–94.

KARATZAS, I. and SHREVE, S. (1991). *Brownian Motion and Stochastic Calculus*. Springer, New York.

KOZLOV, S. M. (1985). The method of averaging and walks in inhomogeneous environments. *Russian Math. Surveys* **40** 73–145.

MOSER, J. (1961). On Harnack's theorem for elliptic differential equations. *Comm. Pure Appl. Math.* **14** 577–591.

OELSCHLÄGER, K. (1988). Homogenization of a diffusion process in a divergence free random field. *Ann. Probab.* **16** 1084–1126.

OSADA, H. (1982). Homogenization of diffusion processes with random stationary coefficients. *Proceedings of Fourth Japan–USSR Symposium on Probability Theory. Lecture Notes in Math.* **1021** 507–517. Springer, Berlin.

PAPANICOLAOU, G. C. and VARADHAN, S. R. S. (1982). Boundary value problems with rapidly oscillating random coefficients. *Colloq. Math. Soc. János Bolyai* **27** 835–873.

PORT, S. C and STONE, C. J. (1976). Random measures and their application to motion in an imcompressible fluid. *J. Appl. Probab.* **13** 499–506.

ROZANOV, YU. A. (1969). *Stationary Random Processes*. Holden-Day, San Francisco.

STONE, C. (1963). Weak convergence of stochastic processes defined on semiinfinite time intervals. *Proc. Amer. Math. Soc.* **14** 694–696.

STROOK, D. and VARADHAN, S. R. S. (1979). *Multidimensional Diffusion Processes*. Springer, Berlin.

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