*The Annals of Probability*
1999, Vol. 27, No. 4, 2089–2108

# A BERRY–ESSEEN BOUND FOR FINITE POPULATION STUDENT'S STATISTIC[1]

BY M. BLOZNELIS

*Vilnius University and Institute of Mathematics and Informatics*

A general and precise Berry–Esseen bound is proved for the Studentized mean based on $N$ random observations drawn without replacement from a finite population. The bound yields the optimal rate $O(N^{-1/2})$ under minimal conditions. If the Erdős–Rényi condition holds this bound implies the asymptotic normality of Student's statistic and the self-normalized sum.

**1. Introduction and results.** Let $\{x\}$ denote a sequence of real numbers $x_1, \dots, x_n$ and let $X_1, \dots, X_N$, $N < n$, denote random variables with values in $\{x\}$ such that $\mathbb{X} = \{X_1, \dots, X_N\}$ represents a simple random sample of size $N$ drawn without replacement from $\{x\}$. We shall assume that $\mathbf{E}\, X_1 = 0$ and $\sigma^2 = \mathbf{E}\, X_1^2 > 0$.

Let

$$\mathbf{t} = \mathbf{t}(\mathbb{X}) = \overline{X}/\hat{\sigma}$$

denote the Student statistic, where

$$\overline{X} = N^{-1}(X_1 + \cdots + X_N) \quad \text{and} \quad \hat{\sigma}^2 = N^{-1} \sum_{i=1}^{N} (X_i - \overline{X})^2.$$

Put $\mathbf{t} = 0$ if $\hat{\sigma} = 0$. By the finite population central limit theorem (CLT) [see Erdős and Rényi (1959)] for large $N$, the distribution of $\sqrt{N}\,\mathbf{t}$ can be approximated by a normal distribution. In this paper we estimate the rate of the normal approximation. We construct a bound for

$$\delta_N = \sup_x \Big| \mathbf{P}\big\{\sqrt{N/q}\,\mathbf{t}(\mathbb{X}) < x\big\} - \Phi(x) \Big|,$$

where $\Phi(x)$ denotes the standard normal distribution function,

$$p = N/n \quad \text{and} \quad q = 1 - p.$$

THEOREM 1.1. *There exists an absolute constant $c > 0$ such that*

$$\delta_N \le \frac{c}{\sqrt{q}} \frac{\beta_3}{\sqrt{N}\,\sigma^3}, \qquad \beta_3 := \mathbf{E}\,|X_1|^3. \tag{1.1}$$

---

Received August 1997; revised May 1999.

[1]Supported in part by the Alexander von Humboldt Foundation.

*AMS* 1991 *subject classifications*. Primary 62E20; secondary 60F05.

*Key words and phrases*. Berry–Esseen bound, Student statistic, central limit theorem, finite population, self-normalized sum.

A similar Berry–Esseen bound but for the finite population sample mean was proved by Höglund (1978). The estimate of Theorem 1.1 holds for any fixed sample size $N$ and population size $n$. If $\beta_3/\sigma^3$ is bounded and $q$ is bounded away from 0 as $N \to \infty$ and $n \to \infty$, then (1.1) establishes a Berry–Esseen bound $O(N^{-1/2})$. Note that the factor $1/\sqrt{q}$ in the right-hand side of (1.1) cannot be removed or replaced by $q^\alpha$ with $\alpha > -1/2$ [cf. one-term Edgeworth expansion for $\mathbf{P}\{\sqrt{N/q}\,\mathbf{t}(\mathbb{X}) < x\}$ given in Babu and Singh (1985)].

Write $w = \sqrt{n\,p\,q}$.

THEOREM 1.2. *There exists an absolute constant* $c > 0$ *such that*

$$\delta_N \le \frac{c}{\sigma^2}\,\mathbf{E}\,X_1^2\,\mathbb{I}_{|X_1|>\sigma w} + \frac{c}{w\,\sigma^3}\mathbf{E}\,|X_1|^3\,\mathbb{I}_{|X_1|\le\sigma w}. \tag{1.2}$$

Theorems 1.1 and 1.2 can be considered as a particular extension to the case of simple random sampling of Berry–Esseen bounds for Student's statistic based on i.i.d. observations, proved recently by Bentkus and Götze (1996). Indeed, the case where $n \to \infty$ and $N$ is fixed corresponds to the i.i.d. situation and in this way we obtain Theorems 1.1 and 1.2 of Bentkus and Götze (1996) as corollaries of Theorems 1.1 and 1.2. It could be mentioned that our techniques are related to those of Bentkus and Götze (1996), Bloznelis and Götze (1997) and Höglund (1978).

Next we apply Theorem 1.2 to prove the CLT for the Studentized mean. Consider a sequence of populations $\{x\}_n = \{x_{n\,1}, \dots, x_{n\,n}\}$ such that $\sum_i x_{n\,i} = 0$, for every $n = 2, 3, \dots$. Let $\mathbb{X}_{n\,N} = \{X_{n\,1}, \dots, X_{n\,N}\}$ denote a sample of size $N = N_n$ drawn without replacement from $\{x\}_n$. Write $\sigma_n^2 = \mathbf{E}\,X_{n\,1}^2$ and assume that $\sigma_n^2 > 0$, for every $n = 2, 3, \dots$. Write $p_n = N_n/n$ and $q_n = 1 - p_n$. Erdős and Rényi (1959) proved that if

$$\forall \varepsilon > 0, \qquad \lim_{n\to\infty} \sigma_n^{-2}\mathbf{E}\,X_{n\,1}^2\mathbb{I}_{|X_{n\,1}|\ge\varepsilon\sigma_n w_n} = 0, \qquad w_n^2 = n\,p_n\,q_n, \tag{1.3}$$

then the sequence $S_n = S(\{x\}_n) = (X_{n\,1} + \dots + X_{n\,N_n})/(\sigma_n w_n)$ converges in distribution to the standard normal distribution as $n \to \infty$. Note that (1.3) implies $N_n \to \infty$ as $n \to \infty$. Hajek (1960) showed that the Erdős–Rényi condition (1.3) is also necessary for the asymptotic normality of $S_n$. One consequence of Theorem 1.2 is that this condition is sufficient also for the asymptotic normality of the Studentized mean.

COROLLARY 1.3. *Assume that* (1.3) *holds. Then* $\sqrt{N_n/q_n}\,\mathbf{t}(\mathbb{X}_{n\,N_n})$ *converges in distribution to the standard normal distribution.*

Maybe more interesting is the fact that it may happen that $\sqrt{N_n/q_n}\mathbf{t}(\mathbb{X}_{n\,N_n})$ is asymptotically standard normal when $S_n$ does not. Such a situation is exhibited in the following example.

EXAMPLE. Let $\{x\}_n$ be a sequence of populations as above. Assume that this sequence satisfies (1.3) and that $\sigma_n = 1$. Construct a new sequence of populations $\{\tilde{x}\}_{n+2}$ by putting $\{\tilde{x}\}_{n+2} = \{x\}_n \cup \{-n,\, n\}$. Choose the sequence

$N_n$ so that $N_n p_n \to 0$ and let $\tilde{\mathbb{X}}_{n\,N_n}$ denote a simple random sample of size $N_n$ drawn from the population $\{\tilde{x}\}_n$. It is easy to see that in this case (1.3) fails and $S(\{\tilde{x}\}_n)$ converges to a degenerate distribution. Furthermore, since

$$\mathbf{P}\{\{-n,\ n\} \subset \tilde{\mathbb{X}}_{n+2\,N_{n+2}}\} \le 2N_{n+2}\,p_{n+2} \to 0,$$

the limiting behavior (as $n \to \infty$) of distributions of $\mathbf{t}(\mathbb{X}_{n\,N_n})$ and $\mathbf{t}(\tilde{\mathbb{X}}_{n+2\,N_{n+2}})$ is the same, that is, both are asymptotically standard normal.

REMARK. All the results stated above remain valid if instead of the standardized Student statistic $\sqrt{N}\mathbf{t}$ one considers the self-normalized sums

$$\frac{X_1 + \cdots + X_N}{\sqrt{X_1^2 + \cdots + X_N^2}}.$$

In particular, Theorems 1.1 and 1.2 hold with $\delta_N$ replaced by $\delta'_N$, where

$$\delta'_N := \sup_x \left| \mathbf{P}\left\{ \frac{X_1 + \cdots + X_N}{\sqrt{X_1^2 + \cdots + X_N^2}} < \sqrt{q}\,x \right\} - \Phi(x) \right|.$$

In contrast to the case of independent and identically distributed observations, where the normal approximation of the Studentized mean and related statistics was studied by a number of authors [see, e.g. Chung (1946), Efron (1969), Logan, Mallows, Rice and Shepp (1973), Chibisov (1980), Helmers and van Zwet (1982), van Zwet (1984), Slavova (1985), Bhattacharya and Ghosh (1978), Hall (1988), Griffin and Mason (1991), Sharakhmetov (1995), Bentkus and Götze (1996), Bentkus, Bloznelis and Götze (1996), Gine, Götze and Mason (1997), Bentkus, Götze and van Zwet (1997), Putter and van Zwet (1998) and so on] there are only a few results concerned with the rate of the normal approximation of finite population Student's statistic. Praškova (1989) constructed a Berry–Esseen bound for the Studentized mean based on the observations drawn without replacement from a finite set of *random variables*, assuming that each of them is of zero mean. Rao and Zhao (1994) proved the Berry–Esseen bound,

$$\delta_N \le \frac{c}{\sqrt{q}} \frac{\mathbf{E}\,|X_1|^4}{\sqrt{N}\ \sigma^4},$$

which establishes the rate $O(N^{-1/2})$ but involves the fourth moment. Babu and Singh (1985) studied a higher order asymptotics of the distribution function of $\sqrt{N}\ \mathbf{t}$. Berry–Esseen bounds for some other nonlinear finite population statistics were obtained by Zhao and Chen (1990), Kokic and Weber (1990) and, as a particular case of the rate of convergence of general multivariate sampling statistics, by Bolthausen and Götze (1993).

**2. Proofs.** This section is organized as follows. In the beginning we formulate a general result; see Theorem 2.1 below. Then we give proofs of Theorems 1.1 and 1.2 and Corollary 1.3, which are simple consequences of Theorem 2.1. The proof of Theorem 2.1, is postponed to the end of the section.

Define the number $a \geq 0$ by the truncated second moment equation,

$$a^2 = \sup\{b\colon\ \mathbf{E}\, X_1^2 \mathbb{I}_{X_1^2 \leq b\, w^2} \geq b\}.$$

It is easy to check that $a \leq \sigma$ and $a$ is the largest solution of the equation

$$a^2 = \mathbf{E}\, X_1^2 \mathbb{I}_{|X_1| \leq aw}.$$

In the case where $a$ is positive we write

$$\gamma = a^{-2}\sigma^2 - 1, \quad \alpha = w^2|\mathbf{E} Y_1|, \quad \mu = w^2 \mathbf{E}|Y_1|^3, \quad Y_1 = a^{-1}w^{-1}X_1 \mathbb{I}_{|X_1| \leq aw}$$

and note that $|Y_1| \leq 1$, $\mathbf{E} Y_1^2 = w^{-2}$ and $N^{-1/2} \leq w^{-1} \leq \mu$, by Lyapunov's inequality $(\mathbf{E} Y_1^2)^3 \leq (\mathbf{E}|Y_1|^3)^2$.

THEOREM 2.1. *There exists an absolute constant $c > 0$ such that*

$$\delta_N \leq c\, w^2 \mathbf{P}\{|X_1| > a\, w\} + c(\mathscr{R} + \gamma\, \mathbb{I}_{p>q}), \qquad \mathscr{R} = \alpha + \mu, \tag{2.1}$$

*whenever $a > 0$.*

Theorem 1.1 is an immediate consequence of Theorem 1.2.

PROOF OF THEOREM 1.2. We may and shall assume without loss of generality that $\sigma = 1$. This implies $a \leq 1$.

In the case where $a^2 \geq 1/4$ we derive (1.2) from (2.1). Introduce the events $\Delta_1 = \{|X_1| > aw\}$, $\Delta_2 = \{aw < |X_1| \leq w\}$ and $\Delta_3 = \{|X_1| > w\}$. Combining the identity $\mathbb{I}_{\Delta_1} = \mathbb{I}_{\Delta_2} + \mathbb{I}_{\Delta_3}$ (here $\mathbb{I}_\Delta$ denotes the indicator function of the event $\Delta$) and Chebyshev's inequality, we get

$$\begin{aligned}
\mathbf{P}\{|X_1| > a\, w\} = \mathbf{E}\mathbb{I}_{\Delta_2} + \mathbf{E}\mathbb{I}_{\Delta_3} &\leq \frac{1}{a^3 w^3}\mathbf{E}|X_1|^3 \mathbb{I}_{\Delta_2} + \frac{1}{w^2}\mathbf{E} X_1^2 \mathbb{I}_{\Delta_3},\\
a^2\gamma = \sigma^2 - a^2 = \mathbf{E} X_1^2 \mathbb{I}_{\Delta_1} &= \mathbf{E} X_1^2 \mathbb{I}_{\Delta_2} + \mathbf{E} X_1^2 \mathbb{I}_{\Delta_3}\\
&\leq \frac{1}{aw}\mathbf{E}|X_1|^3 \mathbb{I}_{\Delta_2} + \mathbf{E} X_1^2 \mathbb{I}_{\Delta_3},\\
aw|\mathbf{E} Y_1| = |\mathbf{E} X_1 \mathbb{I}_{\Delta_1}| &\leq \mathbf{E}|X_1| \mathbb{I}_{\Delta_2} + \mathbf{E}|X_1| \mathbb{I}_{\Delta_3}\\
&\leq \frac{1}{a^2 w^2}\mathbf{E}|X_1|^3 \mathbb{I}_{\Delta_2} + \frac{1}{w}\mathbf{E} X_1^2 \mathbb{I}_{\Delta_3}.
\end{aligned}$$

In the last step we used $\mathbf{E} X_1 = 0$. Using these inequalities we obtain bounds for $\mathbf{P}\{|X_1| > aw\}$, $\alpha$, $\gamma$ and $\mu$. Substitution of these bounds in the right-hand side of (2.1) yields (1.2).

In the case where $a^2 < 1/4$ we have $\mathbf{E}\, X_1^2 \mathbb{I}_{|X_1| \leq w/2} < 1/4$ and, therefore, $\mathbf{E}\, X_1^2 \mathbb{I}_{|X_1| > w/2} \geq 3/4$. Furthermore,

$$3/4 \leq \mathbf{E}\, X_1^2 \mathbb{I}_{|X_1| > w/2} \leq 2w^{-1}\mathbf{E}\, |X_1|^3 \mathbb{I}_{w/2 < |X_1| \leq w} + \mathbf{E}\, X_1^2 \mathbb{I}_{|X_1| > w}.$$

Since $\delta_N \le 1$, we obtain

$$\delta_N \le 1 \le \tfrac{8}{3} w^{-1} \mathbf{E}\,|X_1|^3 \mathbb{I}_{w/2<|X_1|\le w} + \tfrac{4}{3} \mathbf{E}\, X_1^2 \mathbb{I}_{|X_1|>w},$$

thus completing the proof of Theorem 1.2. $\square$

PROOF OF COROLLARY 1.3. We may and shall assume without loss of generality that $\sigma_n = 1$, for $n = 2, 3, \dots$.

Introduce the events $\Delta_{n\,1} = \{|X_{n\,1}| > w_n\}$ and $\Delta_{n\,2} = \{|X_{n\,1}| \le w_n\}$. In view of Theorem 1.2 it suffices to show that for every $\varepsilon > 0$,

$$\limsup_n \big(\mathbf{E} X_{n\,1}^2 \mathbb{I}_{\Delta_{n\,1}} + w_n^{-1} \mathbf{E}|X_{n\,1}|^3 \mathbb{I}_{\Delta_{n\,2}}\big) \le \varepsilon. \tag{2.2}$$

Let us show (2.2). Given $\varepsilon > 0$, introduce the events $\Delta_{n\,3} = \{|X_{n\,1}| > \varepsilon w_n\}$ and $\Delta_{n\,4} = \{|X_{n\,1}| \le \varepsilon w_n\}$. We have

$$\mathbf{E} X_{n\,1}^2 \mathbb{I}_{\Delta_{n\,1}} + w_n^{-1} \mathbf{E}|X_{n\,1}|^3 \mathbb{I}_{\Delta_{n\,2}} \le \mathbf{E} X_{n\,1}^2 \mathbb{I}_{\Delta_{n\,3}} + \varepsilon \mathbf{E} X_{n\,1}^2 \mathbb{I}_{\Delta_{n\,4}} \le \mathbf{E} X_{n\,1}^2 \mathbb{I}_{\Delta_{n\,3}} + \varepsilon.$$

Now (2.2) follows from (1.3). $\square$

It remains to prove Theorem 2.1. We shall assume that $a > 0$ in what follows. Before the proof we introduce some notation. In what follows $c, c_1, \dots$ denote generic absolute constants. By $c(\alpha_1, \alpha_2, \dots)$ we denote constants which may depend only on the parameters $\alpha_1, \alpha_2, \dots$. We write $A \ll B$ if $A \le c\,B$. The expression $\exp\{ix\}$ is abbreviated by $\mathrm{e}\{x\}$.

For $k = 1, 2, \dots$, write $\Omega_k = \{1, \dots, k\}$. Given a sum $S = s_1 + \cdots + s_k$, denote $S^{(i)} = S - s_i$. Given $A \subset \Omega_k$, write $S_A = \sum_{j \in A} s_j$.

Let $\theta_1, \theta_2, \dots$ denote independent random variables uniformly distributed in $[0, 1]$ and independent of all other random variables considered. For a complex valued smooth function $h$ we use the Taylor expansion

$$h(x) = h(0) + h'(0)x + \cdots + h^{(n)}(0)\frac{x^n}{n!} + \mathbf{E}_{\theta_1} h^{(n+1)}(\theta_1 x)(1-\theta_1)^n \frac{x^{n+1}}{n!}.$$

Here $\mathbf{E}_{\theta_1}$ denotes the conditional expectation given all the random variables but $\theta_1$. In particular, we have the mean value formula, $h(x) - h(0) = \mathbf{E}_{\theta_1} h'(\theta_1 x)x$.

Let $g$ be a three-times differentiable real function with bounded derivatives such that

$$g(x) = x^{-1/2} \quad \text{for} \quad |x-1| \le c_1 \quad \text{and} \quad |g(x) - 1| \le c_1 \quad \text{for } x \in \mathbb{R}.$$

The (small) constant $0 < c_1 < 1$ will be specified later.

Let $\mathbb{X}^* = (X_1, \dots, X_n)$ denote a random permutation uniformly distributed over permutations of the sequence $\{x_1, \dots, x_n\}$. In particular, $X_1, \dots, X_N$ represents a simple random sample of size $N$ drawn without replacement from $\{x\}$. Let $\overline{\nu} = (\nu_1, \dots, \nu_n)$ denote a sequence of independent Bernoulli random variables independent of $\mathbb{X}^*$ and having probabilities

$$\mathbf{P}\{\nu_i = 1\} = p, \qquad \mathbf{P}\{\nu_i = 0\} = q, \qquad 1 \le i \le n.$$

Given $A = \{i_1, \dots, i_k\} \subset \Omega_n$, let $\mathbf{E}_{\{i_1,\dots,i_k\}} = \mathbf{E}_A$ (respectively, $\mathbf{E}^{(i_1,\dots,i_k)}$) denote the conditional expectation given all the random variables, but $\nu_{i_1}, \dots, \nu_{i_k}$ (respectively, $X_{i_1}, \dots, X_{i_k}$).

Write

$$Y_i = \frac{1}{aw} X_i \mathbb{I}_{|X_i| \le aw}, \qquad Z_i = Y_i^2 - \mathbf{E}Y_i^2, \qquad 1 \le i \le n,$$

$$Y = \sum_{i=1}^{N} Y_i, \qquad Z = \sum_{i=1}^{N} Z_i, \qquad Y' = \sum_{i=N+1}^{n} Y_i, \qquad Z' = \sum_{i=N+1}^{n} Z_i, \tag{2.3}$$

$$S = (Y - \mathbf{E}Y) g(1 + qZ), \qquad S' = -(Y' - \mathbf{E}Y') g(1 - qZ'),$$

and note that

$$\begin{aligned} \mathbf{E}Z_i^2 &\ll \mathbf{E}|Z_i|^{3/2} \ll \mathbf{E}|Y_i|^3 = w^{-2}\mu, \\ \mathbf{E}|Y_i - \mathbf{E}Y_i|^3 &\le 8\mathbf{E}|Y_i|^3 = 8w^{-2}\mu. \end{aligned} \tag{2.4}$$

Below we shall use the following simple inequality. Given $\{i_1, \dots, i_k\} \subset \Omega_n$ and $j \in \Omega_n \setminus \{i_1, \dots, i_k\}$ let $X_j^*$ be a measurable function of $X_j$. We have

$$\mathbf{E}^{(i_1,\dots,i_k)} |X_j^*|^\alpha \le \frac{n}{n-k} \mathbf{E}|X_j^*|^\alpha, \qquad \text{for } \alpha > 0. \tag{2.5}$$

We shall apply this inequality to random variables $Y_j$, $Z_j$, $Y_j - \mathbf{E}Y_j$, and so on.

Given a random variable $W$, write $\Delta_W = \sup_x |\mathbf{P}\{W \le x\} - \Phi(x)|$. Let $W'$ be a random variable defined on the same probability space as $W$. Then

$$\Delta_W \le \Delta_{W'} + \varepsilon \max_x |\Phi'(x)| + \mathbf{P}\{|W - W'| > \varepsilon\} \qquad \forall \varepsilon > 0, \tag{2.6}$$

$$|\Delta_W - \Delta_{W'}| \le \mathbf{P}\{|W \ne W'|\}. \tag{2.7}$$

The proof of Theorem 2.1 consists of two steps. In the first step (see Lemma 2.1) we replace $X_1, \dots, X_N$ by truncated random variables $Y_1, \dots, Y_N$ and replace the statistic $\sqrt{N/q}\mathbf{t}$ by $S$ (respectively, by $S'$) in the case where $p \le q$ (respectively, $p > q$); see (2.3). Furthermore, the Berry–Esseen smoothing lemma reduces the problem of estimation $|P\{S \le x\} - \Phi(x)|$ to that of the estimation the difference $|\mathbf{E}\exp\{itS\} - \exp\{-t^2/2\}|$. In the second step we estimate this difference by means of expansions. For $p > q$, we estimate $|P\{S' \le x\} - \Phi(x)|$ in much the same way.

LEMMA 2.1. *Assume that $a > 0$ and $N \ge 2$. Then*

$$\begin{aligned} &\delta_N \le \Delta_S \mathbb{I}_{p \le q} + \Delta_{S'} \mathbb{I}_{p > q} + c\mathscr{R}_1, \\ &\mathscr{R}_1 = w^2 \mathbf{P}\{|X_1| > a\,w\} + \alpha + \mu + \gamma\,\mathbb{I}_{p > q}. \end{aligned} \tag{2.8}$$

PROOF. We may and shall assume that $\alpha < 1$ and $\mu < 1$. Otherwise (2.8) follows from the inequality $\delta_N \le 1$.

Let us prove (2.8) in the case where $p \le q$, that is, $1/2 \le q$. Introduce the statistic $\tilde{S} = Y g(1 + qZ - qY^2/N)$ based on the sample $\mathbb{Y} = (Y_1, \dots, Y_N)$.

Since $\sqrt{N/q}\mathbf{t}(\mathbb{X}) = \sqrt{N/q}\mathbf{t}(\mathbb{Y})$ on the event $A_1 = \{\mathbb{X} = aw\mathbb{Y}\}$ and $\sqrt{N/q}\mathbf{t}(\mathbb{Y}) = \tilde{S}$ on $A_2 = \{q|Z - Y^2/N| \le c_1\}$, we have

$$\begin{aligned}\mathbf{P}\{\sqrt{N/q}\mathbf{t}(\mathbb{X}) \ne \tilde{S}\} &\le 1 - \mathbf{P}\{A_1 \cap A_2\} \\ &\le 1 - \mathbf{P}\{A_1\} + 1 - \mathbf{P}\{A_2\} \ll \mathscr{R}_1.\end{aligned} \tag{2.9}$$

Indeed, $1 - \mathbf{P}\{A_1\} \le N\,\mathbf{P}\{|X_1| > a\,w\} \le 2w^2\mathbf{P}\{|X_1| > a\,w\}$ and

$$1 - \mathbf{P}\{A_2\} \le \mathbf{P}\left\{|Z| > \frac{c_1}{2}\right\} + \mathbf{P}\left\{\frac{Y^2}{N} > \frac{c_1}{2}\right\} \le c\mathbf{E}|Z|^{3/2} + \frac{c}{N}\mathbf{E}Y^2 \ll \mu.$$

In the last step we used the inequalities

$$\mathbf{E}Y^2 \le c, \qquad \mathbf{E}|Z|^{3/2} \le c\,\mu \tag{2.10}$$

and $N^{-1/2} \le w^{-1} \le \mu$. To prove (2.10) we combine Hoeffding's (1963) Theorem 4 and the Marcinkiewicz–Zygmund inequality. It follows from (2.9) and (2.7) that

$$|\delta_N - \Delta_{\tilde{S}}| \ll \mathscr{R}_1. \tag{2.11}$$

Decompose $\tilde{S} = S + R_1 + R_2$, where $R_1 = g(1 + qZ)\mathbf{E}Y$ and $R_2 = \tilde{S} - Yg(1 + qZ)$ satisfy

$$|R_1| \le N|\mathbf{E}Y_1|(1 + c_1) \le 4\alpha \quad \text{and} \quad |R_2| \le c|Y|^3 N^{-1},$$

by the mean value theorem. Fix $\varepsilon = 5\alpha + N^{-1/2}$ and note that

$$\mathbf{P}\{|S - \tilde{S}| \ge \varepsilon\} \le \mathbf{P}\{|R_2| \ge N^{-1/2}\} \le N^{-1/2}\mathbf{E}|Y|^3 \ll N^{-1/2} \le \mu. \tag{2.12}$$

Here we used the inequality $\mathbf{E}|Y|^3 \le c$, which is proved in much the same way as (2.10). Finally, (2.6) applied to $\tilde{S}$ and $S$ in combination with (2.12) and the simple bound $\max_x |\Phi'(x)| \le c$ implies $\Delta_{\tilde{S}} \le \Delta_S + c\alpha + c\mu$. This inequality together with (2.11) yields (2.8), for $p \le q$.

Let us prove (2.8) in the case where $p > q$. We may and shall assume that $2\gamma < c_1/2$. Otherwise, (2.8) follows from the inequalities $\delta_N \le 1 \ll \gamma$.

It follows from the identities $\sum_{i=1}^n X_i = 0$ and $\sum_{i=1}^n X_i^2 = n\sigma^2$ that

$$\overline{X} = \frac{-X'}{N}, \qquad \hat{\sigma}^2 = \frac{\sigma^2}{p} - \frac{1}{N}\sum_{i=N+1}^{n} X_i^2 - \frac{(X')^2}{N^2} \quad \text{where } X' = \sum_{i=N+1}^{n} X_i.$$

Therefore, on the event $A_3 = \{(X_{N+1}, \dots, X_n) = aw(Y_{N+1}, \dots, Y_n)\}$ we have

$$\sqrt{N/q}\mathbf{t}(\mathbb{X}) = -Y'(1 - qZ' + R_3)^{-1/2} \qquad \text{where } R_3 = \gamma/p - qN^{-1}(Y')^2.$$

Furthermore, on the event $A_4 = \{q|Z' + (Y')^2/N| \le c_1/2\}$ we have $-Y'(1 - qZ' - R_3)^{-1/2} = \tilde{S}'$, where $\tilde{S}' = -Y'g(1 - qZ' + R_3)$. Hence, $\sqrt{N/q}\mathbf{t}(\mathbb{X}) = \tilde{S}'$ on the event $A_3 \cap A_4$. It is easy to show [cf. (2.9)] that $1 - \mathbf{P}\{A_3 \cap A_4\} \ll \mathscr{R}_1$. Therefore, by (2.7), $|\delta_N - \Delta_{\tilde{S}'}| \ll \mathscr{R}_1$. The remaining part of the proof is much the same as that of the case where $p \le q$. $\square$

PROOF OF THEOREM 2.1. By Lemma 2.1, it suffices to show $\Delta_S \mathbb{I}_{p\le q} \ll \mathscr{R}$ and $\Delta_{S'} \mathbb{I}_{p>q} \ll \mathscr{R}$. We give the proof of the first inequality only. The proof of the second inequality is much the same.

We shall assume that $p \le 1/2 \le q$ in what follows and show that $\Delta_S \ll \mathscr{R}$. We may and shall assume that for a small constant $c_2$,

$$\alpha < c_2, \qquad \mu < c_2. \tag{2.13}$$

Indeed, if at least one of these inequalities fails we obtain $\Delta_S \le 1 \ll \mathscr{R}$.

Denote

$$\varphi(t) = \mathbf{E}\,\mathrm{e}\{tS\}, \qquad \psi(t) = \mathbf{E}\,\mathrm{e}\{t(Y - \mathbf{E}Y)\},$$
$$\phi_r(t) = \exp\{-t^2r^2/2\}, \quad r > 0.$$

Given two complex valued functions $f$ and $h$, write

$$I_{[d;e]}(f,h) = \int_{|t|\in(d;e]} |t|^{-1}|f(t) - h(t)|dt, \qquad e > d \ge 0.$$

The Berry–Esseen smoothing inequality [see Feller (1971), page 538] yields

$$\Delta_S \ll I_{[0;H]}(\varphi,\phi_1) + H^{-1}, \qquad H = c_3 b^2 \mu_0^{-1}. \tag{2.14}$$

Here we denote

$$b^2 = w^2\mathbf{E}(Y_1 - \mathbf{E}Y_1)^2 = 1 - \alpha^2 w^{-2}, \qquad \mu_0 = w^2\mathbf{E}|Y_1 - \mathbf{E}Y_1|^3.$$

The (small) constant $c_3$ will be specified later. Since $\mu_0 \ll \mu$ and, by (2.13), $b^{-2} \le c$, we have $H^{-1} \ll \mathscr{R}$. It remains to show $I_{[0;H]}(\varphi,\phi_1) \ll \mathscr{R}$. Write

$$I_{[0;H]}(\varphi,\phi_1) \le I_{[0;H]}(\varphi,\psi) + I_{[0;H]}(\psi,\phi_b) + I_{[0;H]}(\phi_b,\phi_1).$$

Clearly, $I_{[0;H]}(\phi_b,\phi_1) \ll (1 - b^2) \ll \mathscr{R}$, by (2.13). It follows from Höglund [(1978), formula (8)] that $I_{[0;H]}(\psi,\phi_b) \ll b^{-3}\mu_0$, provided that $c_3$ is sufficiently small. By (2.13), $b^{-3}\mu_0 \ll \mu_0 \ll \mu$. Therefore, it remains to bound $I_{[0;H]}(\varphi,\psi)$. We split $I_{[0;H]}(\varphi,\psi) = I_{[0;c_4]}(\varphi,\psi) + I_{[c_4;H]}(\varphi,\psi)$ and estimate the summands separately.

Let us show

$$I_{[c_4;H]}(\varphi,\psi) \ll \mathscr{R}. \tag{2.15}$$

To this aim we represent the characteristic functions $\varphi$ and $\psi$ in Erdős–Rényi (1959) form; see (2.16) below. Write

$$T = \sum_{i=1}^n T_i, \qquad Q = \sum_{i=1}^n Q_i, \qquad S = \sum_{i=1}^n S_i,$$
$$T_i = (Y_i - \mathbf{E}Y_i)(\nu_i - p), \qquad Q_i = qZ_i(\nu_i - p), \qquad S_i = w^{-1}(\nu_i - p).$$

We have

$$\text{(2.16)}\qquad \begin{aligned} \varphi &= \lambda \int_{-\pi w}^{\pi w} \mathbf{E}\, \mathrm{e}\{tTg(1+Q)+sS\}\, ds,\\ \psi &= \lambda \int_{-\pi w}^{\pi w} \mathbf{E}\, \mathrm{e}\{tT+sS\}\, ds, \end{aligned}$$

with $\lambda^{-1} = 2\pi w \mathbf{P}\{S=0\}$. Höglund (1978) showed that $2^{-1/2}\pi \le \lambda^{-1} \le (2\pi)^{1/2}$. Given a number $L > 0$ and a complex valued bivariate function $f$, write $f \prec L$ if

$$\int_{\mathscr{Z}} |t|^{-1} |f(s,t)|\, ds\, dt \ll L \quad \text{where } \mathscr{Z} = \{(s,t)\colon c_4 \le |t| \le H,\ |s| \le \pi w\}.$$

Given two complex valued functions $f$, $h$, write $f \sim h$ if $f - h \prec \mathscr{R}$.

Introduce the integer valued function

$$\text{(2.17)}\qquad m = m(s,t) \approx 2^{-1} c_4 n u^{-1} \ln u, \qquad u = t^2 + s^2, \qquad (s,t) \in \mathscr{Z}.$$

A simple calculation shows that $10 \le m(s,t) \le n/2$, for $(s,t) \in \mathscr{Z}$, provided that $c_4$ is sufficiently large. Write $z := mpqw^{-2} = m/n \ll u^{-1} \ln u$. We shall often use the following fact. For $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \ge 0$ satisfying $\alpha_3 + \alpha_4 > \alpha_1 + \alpha_2 + 1/2$,

$$(t^2)^{\alpha_1} (s^2)^{\alpha_2} z^{\alpha_3} u^{-\alpha_4} \prec c(\alpha_1, \alpha_2, \alpha_3, \alpha_4).$$

Denote

$$A = \Omega_m, \qquad B = \Omega_n \setminus \Omega_m, \qquad g_0 = g(1+Q_B), \qquad g_1 = g'(1+Q_B).$$

Split

$$\text{(2.18)}\qquad \begin{aligned} T &= T_A + T_B, \quad Q = Q_A + Q_B,\\ T_A Q_A &= D_A + U_A, \quad T_B Q_B = D_B + U_B, \end{aligned}$$

where we denote

$$\text{(2.19)}\qquad D_G = \sum_{j \in G} T_j Q_j, \qquad U_G = \sum_{i,\, j \in G,\, i \ne j} T_i Q_j, \qquad G \subset \Omega_n.$$

Introduce the random variables

$$\begin{aligned} &v_j = v_j^* - 2^{-1} t T_j Q_j, \qquad v_j^* = t T_j g_0 + s S_j,\\ &v_j^\star = t T_j + s S_j, \qquad \tilde v_j = |t T_j| + |s S_j|, \qquad 1 \le j \le n,\\ &V = \sum_{j=1}^n v_j, \qquad V^* = \sum_{j=1}^n v_j^*, \qquad V^\star = \sum_{j=1}^n v_j^\star,\\ &H_G = |\mathbf{E}_G\ \mathrm{e}\{V_G\}|, \qquad H_G^* = |\mathbf{E}_G\ \mathrm{e}\{V_G^*\}|,\\ &H_G^\star = |\mathbf{E}_G\ \mathrm{e}\{V_G^\star\}|, \qquad G \subset \Omega_n. \end{aligned}$$

Several useful inequalities to be used below are collected in the next two lemmas.

LEMMA 2.2. *Assume that* (2.13) *holds. We have*

$$H\mu \ll 1, \qquad H^2\mathbf{E}(Y_1 - \mathbf{E}Y_1)^2 \le c_3^2, \tag{2.20}$$

$$\mathbf{E}U_A^2 \ll z^2\mu, \qquad \mathbf{E}|U_A Q_A| \ll z^{3/2}\mu, \tag{2.21}$$

$$\mathbf{E}|T_B Q_A^2|^{3/4} \ll z\mu, \qquad \mathbf{E}|T_B Q_A|^{3/2} \ll z\mu, \tag{2.22}$$

$$\mathbf{E}\Bigl|\sum_{j\in A} T_j Q_j^2\Bigr|^{3/4} \ll z\mu, \qquad \mathbf{E}\Bigl|\sum_{j\in A} T_j Q_j Q_A^{(j)}\Bigr| \ll z^{3/2}\mu^{3/2}. \tag{2.23}$$

*For any* $G \subset \Omega_n$ *and* $i_1, i_2, i_3 \in \Omega_n \setminus G$, *we have*

$$\mathbf{E}^{(i_1, i_2, i_3)}|T_G|^r \ll c, \qquad 0 < r \le 6. \tag{2.24}$$

LEMMA 2.3. *Let* $G \subset \Omega_n$ *and* $|G| \ge m/4$. *There exists a small constant* $c_* > 0$ *such that the inequality* $c_1, c_2, c_3, c_4^{-1} < c_*$ *implies*

$$\mathbf{E}^{(i,j)}H_G^2 < u^{-10}, \qquad \mathbf{E}^{(i,j)}(H_G^*)^2 < u^{-10}, \qquad \mathbf{E}^{(i,j)}(H_G^\star)^2 < u^{-10}, \tag{2.25}$$

$$\mathbf{E}^{(i,j)}H_G < u^{-5}, \qquad \mathbf{E}^{(i,j)}H_G^* < u^{-5}, \qquad \mathbf{E}^{(i,j)}H_G^\star < u^{-5}, \tag{2.26}$$

*for any* $i, j \in \Omega \setminus G$. *Furthermore,* $H_G^* \le \zeta_G^{1/2}$, $\zeta_G = \prod_{k\in G}\zeta_k$, *where* $\zeta_k$ *are given by* (3.7).

These lemmas are proved in Section 3. We shall assume that $c_1$, $c_2$, $c_3$ and $c_4^{-1}$ are choosen small enough so that (2.25) and (2.26) hold.

In view of the inequality $\lambda \le 2^{1/2}\pi^{-1}$, (2.15) follows from

$$f \sim f^* \quad \text{where } f = \mathbf{E}\,\mathrm{e}\{tTg(1+Q) + sS\}, \quad f^* = \mathbf{E}\,\mathrm{e}\{tT + sS\}. \tag{2.27}$$

Let us prove (2.27). The proof consists of the following steps:

$$f \sim f_1, \qquad f_1 = \mathbf{E}\,\mathrm{e}\{W_1\}, \qquad W_1 = V^* + tTQ_A g_1, \tag{2.28}$$

$$f_1 \sim f_2, \qquad f_2 = \mathbf{E}\,\mathrm{e}\{W_2 + tT_B Q_A g_1\}, \qquad W_2 = V_A + V_B^*, \tag{2.29}$$

$$f_2 \sim f_3, \qquad f_3 = \mathbf{E}\,\mathrm{e}\{V_A + V_B^*\}, \tag{2.30}$$

$$f_3 \sim f_4, \qquad f_4 = \mathbf{E}\,\mathrm{e}\{V^*\}, \tag{2.31}$$

$$f_4 \sim f_5, \qquad f_5 = \mathbf{E}\,\mathrm{e}\{V_A^\star + V_B^*\}, \tag{2.32}$$

$$f_5 \sim f^*. \tag{2.33}$$

PROOF OF (2.28). Expanding in powers of $Q_A$, we get $g(1+Q) = g_0 + Q_A g_1 + Q_A^2 r$, where $r$ is a bounded function of $Q_A$, $Q_B$. Substituting this expansion we obtain $tTg(1+Q) + sS = W_1 + tTQ_A^2 r$ and therefore,

$$|f - f_1| \le \mathbf{E}|\,\mathrm{e}\{tTQ_A^2 r\} - 1|. \tag{2.34}$$

By (2.18), $TQ_A^2 = R_1 + R_2 + R_3$, where $R_1 = T_B Q_A^2$, $R_2 = U_A Q_A$ and $R_3 = D_A Q_A$. Split

$$R_3 = R_{3.1} + R_{3.2}, \qquad R_{3.1} = \sum_{j\in A} T_j Q_j Q_A^{(j)}, \qquad R_{3.2} = \sum_{j\in A} T_j Q_j^2.$$

Now, applying the inequality

$$|\,\mathrm{e}\{x\}-1|\le 2|x|^{\tau},\qquad 0\le\tau\le 1,\qquad x\in\mathbb{R}, \tag{2.35}$$

several times, with $\tau=1$ and $\tau=3/4$, we get from (2.34),

$$\begin{aligned}|f-f_1| &\ll |t|(\mathbf{E}|R_2|+\mathbf{E}|R_{3.1}|)+|t|^{3/4}(\mathbf{E}|R_1|^{3/4}+\mathbf{E}|R_{3.2}|^{3/4})\\ &\ll |t|(z^{3/2}\mu+z^{3/2}\mu^{3/2})+|t|^{3/4}z\mu,\end{aligned}$$

by Lemma 2.2. We obtain $|f-f_1|\prec\mathscr{R}$, thus proving (2.28). □

PROOF OF (2.29). Write $TQ_A=T_BQ_A+D_A+U_A$ [see (2.18)] and expand $g_1=g'(1+Q_B)=-2^{-1}+Q_Br$ to get $D_Ag_1=-2^{-1}D_A+D_AQ_Br$, where $r$ is a bounded function of $Q_B$. Now we have

$$W_1=W_2+tT_BQ_Ag_1+w_1+w_2,\qquad w_1=tU_Ag_1,\qquad w_2=tD_AQ_Br.$$

First, we shall show $f_1\sim f_6$, where $f_6=\mathbf{E}\,\mathrm{e}\{W_2+tT_BQ_Ag_1+w_1\}$. By (2.35), $|f_1-f_6|\ll\mathbf{E}|w_2|$. Let us show $\mathbf{E}|w_2|\prec\mathscr{R}$. By the symmetry,

$$\mathbf{E}|w_2|\le m|t|\mathbf{E}|T_1Q_1Q_B|=m|t|\mathbf{E}|T_1Q_1|\mathbf{E}^{(1)}|Q_B|. \tag{2.36}$$

Since $\nu_j-p$, $1\le j\le n$, are independent centered random variables, we have

$$\mathbf{E}^{(1)}Q_B^2=\sum_{j\in B}\mathbf{E}^{(1)}Q_j^2=|B|pq\mathbf{E}^{(1)}Z_n^2,$$

by the symmetry. Furthermore, combining (2.5) and (2.4) we obtain $\mathbf{E}^{(1)}Q_B^2\ll\mu$ and, therefore, $\mathbf{E}^{(1)}|Q_B|\ll\mu^{1/2}$. Substituting this bound in (2.36) and estimating $\mathbf{E}|T_1Q_1|\ll pq\mathbf{E}|Y_1|^3$ we obtain $\mathbf{E}|w_2|\ll|t|z\mu^{3/2}\ll|t|^{1/2}z\mu\prec\mathscr{R}$. In the last step we used the inequality $|t|\mu\ll 1$, which holds for $|t|\le H$; see (2.20).

Let us show $f_6\sim f_2$. Expanding the exponent in powers of $iw_1$, we get

$$f_6=f_2+f_7+R,\qquad f_7=\mathbf{E}\,\mathrm{e}\{W_2+tT_BQ_Ag_1\}iw_1\quad\text{with } |R|\ll t^2\mathbf{E}U_A^2.$$

By (2.21), $|R|\ll t^2z^2\mu\prec\mathscr{R}$. Therefore, $f_1\sim f_2+f_7$. Next we show

$$f_7\sim f_8,\qquad f_8=\mathbf{E}\,\mathrm{e}\{W_2\}iw_1. \tag{2.37}$$

An application of (2.35) with $\tau=3/4$ gives

$$|f_7-f_8|\ll|t|^{7/4}\mathbf{E}|T_BQ_A|^{3/4}|U_A|\le|t|^{7/4}(\mathbf{E}|T_BQ_A|^{3/2})^{1/2}(\mathbf{E}U_A^2)^{1/2},$$

by Cauchy–Schwarz. Invoking inequalities of Lemma 2.2, we obtain $|f_7-f_8|\ll|t|^{7/4}z^{3/2}\mu\prec\mathscr{R}$ and thus (2.37) follows.

We complete the proof of (2.29) by showing $f_8\prec\mathscr{R}$. By the symmetry,

$$f_8=it(m^2-m)f_9,\qquad f_9=\mathbf{E}\,\mathrm{e}\{W_2\}T_1Q_2g_1. \tag{2.38}$$

Recall that $W_2=V_A+V_B^*$ and write

$$f_9=\mathbf{E}\,\mathrm{e}\{V_{A''}+V_B^*\}\,\mathrm{e}\{v_1+v_2\}T_1Q_2g_1,\qquad A''=A\setminus\{1,2\}.$$

Expanding

$$\begin{aligned} e\{v_1+v_2\} &= (1+v_1r_1)\,e\{v_2\} \\ &= e\{v_2\}+v_1r_1(1+v_2r_2), \qquad r_j = i\mathbf{E}_{\theta_j}\, e\{\theta_j v_j\}, \end{aligned}$$

and using the fact that the conditional expectation of $T_1$(respectively, $Q_2$) given all the random variables, but $\nu_1$(respectively, $\nu_2$) is zero, we obtain

$$f_9 = \mathbf{E}\, e\{V_{A''}+V_B^*\}Rg_1, \qquad R = T_1Q_2v_1v_2r_1r_2.$$

Since $|g_1| \le c$ we can write

$$|f_9| \ll \mathbf{E}|R|H_{A''} \ll \mathbf{E}|\tilde{R}|\mathbf{E}^{(1,2)}H_{A''}, \qquad \tilde{R} = T_1Q_2\tilde{v}_1\tilde{v}_2.$$

Combining the inequality $\mathbf{E}^{(1,2)}H_{A''} < u^{-5}$ (see Lemma 2.3) and the simple bound $\mathbf{E}|\tilde{R}| \ll p^2q^2w^{-4}u\mu$, we obtain $|f_9| \ll n^{-2}u^{-4}\mu$. Substituting this inequality in (2.38) we get $f_8 \prec \mathscr{R}$, thus completing the proof of (2.29). □

PROOF OF (2.30). Split $A = A_1 \cup A_2 \cup A_3$ so that $A_i \cap A_j = \varnothing$, for $i \ne j$, and $|A_j| \approx m/3$ and $j \in A_j$, for $j = 1, 2, 3$. Write

$$tT_BQ_Ag_1 = w_1+w_2+w_3, \qquad w_j = tT_BQ_{A_j}g_1, \;\; j = 1,2,3$$

and denote $W_3 = W_2 + w_2 + w_3$. First, we show

$$f_2 \sim f_{10}+f_{11}, \qquad f_{10} = \mathbf{E}\, e\{W_3\}, \qquad f_{11} = \mathbf{E}\, e\{W_3\}iw_1. \tag{2.39}$$

Expanding the exponent in $f_2 = \mathbf{E}\, e\{W_3+w_1\}$ in powers of $iw_1$, we obtain

$$f_2 = f_{10}+f_{11}+f_{12}, \qquad f_{12} = \mathbf{E}\, e\{W_3\}w_1^2r_1,$$

where $r_1$ is a bounded function of $w_1$.

Let us show $f_{12} \prec \mathscr{R}$. Expanding

$$e\{w_2+w_3\} = (1+w_2r_2)\,e\{w_3\} = e\{w_3\}+w_2r_2(1+w_3r_3),$$

where $r_j$ is a bounded function of $w_j$, for $j = 2, 3$, we obtain

$$\begin{aligned} f_{12} &= f_{12.1}+f_{12.2}+f_{12.3}, \qquad f_{12.1} = \mathbf{E}\, e\{W_2+w_3\}w_1^2r_1, \\ f_{12.2} &= \mathbf{E}\, e\{W_2\}w_1^2w_2r_1r_2, \qquad f_{12.3} = \mathbf{E}\, e\{W_2\}w_1^2w_2w_3r_1r_2r_3. \end{aligned}$$

We shall show that $f_{12.j} \prec \mathscr{R}$, for $j = 1, 2, 3$. Clearly,

$$|f_{12.1}| \ll \mathbf{E}H_{A_2}w_1^2, \qquad |f_{12.2}| \ll \mathbf{E}H_{A_3}w_1^2|w_2|, \qquad |f_{12.3}| \ll \mathbf{E}w_1^2|w_2w_3|.$$

Using the symmetry and the fact that conditionally, given $\mathbb{X}^*$, the random variables $Q_j$, $j \in \Omega_n$ are uncorrelated, we construct bounds for $f_{12.j}$, $j = 1, 2, 3$. We have

$$|f_{12.3}| \le t^4\mathbf{E}T_B^4Q_{A_1}^2|Q_{A_2}Q_{A_3}| = t^4|A_1|\mathbf{E}T_B^4Q_1^2|Q_{A_2}Q_{A_3}| \le t^4m^3\mathbf{E}T_B^4Q_1^2|Q_2Q_3|.$$

Combining the bound $\mathbf{E}^{(1,2,3)}T_B^4 \le c$ [see (2.24)] and the inequalities

$$\mathbf{E}Q_1^2|Q_2Q_3| \ll p^3q^3\mathbf{E}Z_1^2|Z_2Z_3| \ll p^3q^3(\mathbf{E}|Z_1|^{3/2})(\mathbf{E}|Z_2|)(\mathbf{E}|Z_3|) \ll p^3q^3w^{-6}\mu$$

[here we use (2.4) and (2.5)] we obtain $f_{12.3} \ll t^4z^3\mu \prec \mathscr{R}$. Similarly,

$$\begin{aligned} |f_{12.2}| &\ll |t|^3m^2\mathbf{E}H_{A_3}|T_B|^3Q_1^2|Q_2| \\ &\ll |t|^3m^2p^2q^2\mathbf{E}Z_1^2|Z_2|\mathbf{E}^{(1,2)}H_{A_3}|T_B|^3. \end{aligned} \tag{2.40}$$

By Hölder's inequality, (2.25) and (2.24),

$$\mathbf{E}^{(1,2)}H_{A_3}|T_B|^3 \le (\mathbf{E}^{(1,2)}H_{A_3}^2)^{1/2}(\mathbf{E}^{(1,2)}T_B^6)^{1/2} \ll u^{-5}. \tag{2.41}$$

Substituting (2.41) in (2.40) and then using the inequalities

$$\mathbf{E}Z_1^2|Z_2| \ll \mathbf{E}Z_1^2\mathbf{E}|Z_2| \ll w^{-4}\mu,$$

[here we apply (2.5) and (2.4)] we obtain $f_{12.2} \ll |t|^3u^{-5}\mu \prec \mathscr{R}$. Finally,

$$|f_{12.1}| \ll t^2|A_1|\mathbf{E}H_{A_3}T_B^2Q_1^2 \ll t^2mpq\mathbf{E}Z_1^2\mathbf{E}^{(1)}H_{A_3}T_B^2.$$

Combining the inequalities $\mathbf{E}^{(1)}H_{A_3}T_B^2 \ll u^{-5}$ [cf. (2.41)] and $\mathbf{E}Z_1^2 \ll w^{-2}\mu$ [see (2.4)] we obtain $|f_{12.1}| \ll t^2u^{-5}z\mu \prec \mathscr{R}$, thus completing the proof of (2.39).

Let us show

$$\begin{aligned} &f_{11} \prec \mathscr{R} \quad \text{where} \\ &f_{11} = \mathbf{E}\,\mathrm{e}\{W_3\}iw_1, \\ &W_3 = V_A + V_B^* + w_2 + w_3. \end{aligned} \tag{2.42}$$

By the symmetry, $f_{11} = it|A_1|\mathbf{E}\,\mathrm{e}\{W_3\}T_Bg_1Q_1$. Expanding the exponent in powers $iv_1$ and using the fact that the conditional expectation of $Q_1$ given all the random variables but $\nu_1$ is zero, we get

$$f_{11} = i^2t|A_1|\mathbf{E}\,\mathrm{e}\{V_{A'} + V_B^* + w_2 + w_3\}T_Bg_1Q_1v_1r_1, \qquad A' = A\setminus\{1\},$$

where $r_1$ is a bounded function of $v_1$. Clearly,

$$|f_{11}| \ll |t|m\mathbf{E}|Q_1v_1T_B|H_{A_1'} \ll |t|m\mathbf{E}|Q_1\tilde{v}_1|\mathbf{E}^{(1)}|T_B|H_{A_1'}, \qquad A_1' = A_1\setminus\{1\}.$$

Combining the inequality $\mathbf{E}^{(1)}H_{A_1'}|T_B| \ll u^{-5}$ [cf. (2.41)] and the simple bound $\mathbf{E}|Q_1\tilde{v}_1| \ll pq(|t|+|s|)w^{-2}\mu$ we obtain $|f_{11}| \ll (|t|+|s|)u^{-5}\mu \prec \mu$, thus proving (2.42).

Let us show $f_{10} \sim f_3$. Write $w_4 := w_2 + w_3$. We have $W_3 = V_A + V_B^* + w_4$. Expanding the exponent in $f_{10}$ in powers of $iw_4$, we obtain

$$f_{10} = f_3 + f_{13} + f_{14}, \qquad f_{13} = \mathbf{E}\,\mathrm{e}\{V_A + V_B^*\}iw_4, \qquad f_{14} = \mathbf{E}\,\mathrm{e}\{V_A + V_B^*\}w_4^2r,$$

where $r$ is a bounded function of $w_4$.The proof of $f_{13} \prec \mathscr{R}$ (respectively, $f_{14} \prec \mathscr{R}$) is much the same as that of $f_{11} \prec \mathscr{R}$ (respectively, $f_{12.1} \prec \mathscr{R}$) above. Therefore, $f_{10} \sim f_3$. Now, invoking (2.39) and (2.42), we obtain (2.30). □

PROOF OF (2.31). Split $A = A_1 \cup A_2$ so that

$$A_1 \cap A_2 = \varnothing \quad \text{and} \quad |A_j| \approx m/2 \quad \text{and} \quad j \in A_j \text{ for } j = 1, 2. \tag{2.43}$$

Write $D_A = D_{A_1} + D_{A_2}$ [see (2.19)] and denote $w_j = -tD_{A_j} 2^{-1}$, for $j = 1, 2$. We have $f_3 = \mathbf{E}\, \mathrm{e}\{V^* + w_1 + w_2\}$. Expanding the exponent in powers of $iw_1$ and $iw_2$ we get

$$f_3 = f_4 + f_{15} + f_{16}, \qquad f_{15} = \mathbf{E}\, \mathrm{e}\{V^*\} w_1 r_1, \qquad f_{16} = \mathbf{E}\, \mathrm{e}\{V^* + w_1\} w_2 r_2,$$

where $r_j$ is a bounded function of $w_j$, $j = 1, 2$. By the symmetry,

$$|f_{15}| \ll |t| \mathbf{E} |D_{A_1}| H^*_{A_2} \le |t| |A_1| \mathbf{E} |T_1 Q_1| H^*_{A_2}.$$

Similarly, $|f_{16}| \le |t| |A_2| \mathbf{E} |T_2 Q_2| H_{A_1}$. Combining the inequalities $\mathbf{E}^{(1)} H^*_{A_2} \ll u^{-5}$ and $\mathbf{E}^{(2)} H_{A_1} \ll u^{-5}$ [see Lemma 2.3] and the simple bound $\mathbf{E}|T_i Q_i| \ll pqw^{-2}\mu$, we obtain $f_{15} \prec \mathscr{R}$, and $f_{16} \prec \mathscr{R}$, thus proving (2.31). □

PROOF OF (2.32). Split $V^* = V^*_A + V^*_B$ and $V^*_A = V^*_{A_1} + V^*_{A_2}$, where $A_1 \cup A_2 = A$ satisfy (2.43). In order to prove (2.32) we shall show

$$f_4 \sim f_{17}, \qquad f_{17} = \mathbf{E}\, \mathrm{e}\{W_4\}, \qquad W_4 = V^\star_{A_1} + V^*_{A_2 \cup B} \tag{2.44}$$

and $f_{17} \sim f_5$.

Let us prove (2.44). Expanding $g_0 = g(1 + Q_B) = 1 - Q_B/2 + Q_B^2 r$ we get

$$V^*_{A_1} = V^\star_{A_1} + w_1 + w_2 \qquad \text{with } w_1 = -tT_{A_1} Q_B/2, \qquad w_2 = tT_{A_1} Q_B^2 r,$$

where $r$ is a bounded function of $Q_B$. Furthermore, expanding the exponent in $f_4 = \mathbf{E}\, \mathrm{e}\{W_4 + w_1 + w_2\}$ and in powers of $iw_2$ and $iw_1$ to obtain

$$f_4 = f_{17} + f_{18} + f_{19} + f_{20}, \qquad f_{18} = \mathbf{E}\, \mathrm{e}\{W_4\} iw_1,$$
$$f_{19} = \mathbf{E}\, \mathrm{e}\{W_4\} iw_1^2 r_1, \qquad f_{20} = \mathbf{E}\, \mathrm{e}\{W_4 + w_1\} iw_2 r_2,$$

where $r_j$ is a bounded function of $w_j$, $j = 1, 2$.

To show $f_{19} \prec \mathscr{R}$ we use symmetry, and the fact that conditionally, given all the random variables but $\nu_i$, $i \in B$, the random variables $Q_i$, $i \in B$ are uncorrelated,

$$|f_{19}| \ll t^2 \mathbf{E} Q_B^2 T_{A_1}^2 H^*_{A_2} \le t^2 |B| pq \mathbf{E} Z_n^2 T_{A_1}^2 \zeta_{A_2}^{1/2}.$$

Combining the bounds $\mathbf{E} Z_n^2 \ll w^{-2}\mu$ and $\mathbf{E}^{(n)} T_{A_1}^2 \zeta_{A_2}^{1/2} \ll u^{-5}$ [cf. (2.41), (3.8), (3.9)] we obtain $f_{19} \prec \mathscr{R}$.The proof of $f_{20} \prec \mathscr{R}$ is much the same.

Let us show $f_{18} \prec \mathscr{R}$. By the symmetry,

$$f_{18} = -2^{-1} it |A_1| |B| \mathbf{E}\, \mathrm{e}\{W_4\} T_1 Q_n.$$

Write $V^\star_{A_1} = V^\star_{A_1'} + v_1^\star$, where $A_1' = A_1 \setminus \{1\}$ and $V^*_{A_2 \cup B} = V^*_{A_2 \cup B'} + v_n^*$, where $B' = B \setminus \{n\}$. Expanding $g_0 = g(1 + Q_{B'} + Q_n) = g(1 + Q_{B'}) + Q_n r_n$, we get $V^*_{A_2 \cup B'} = W_5 + w_3$, where

$$W_5 = tT_{A_2 \cup B'} g(1 + Q_{B'}) + sS_{A_2 \cup B'} \quad \text{and} \quad w_3 = v_n^* + tT_{A_2 \cup B'} Q_n r_n.$$

Here $r_n$ is a bounded function of $Q_n$. We have $W_4 = V^\star_{A_1'} + W_5 + v_1^\star + w_3$ and therefore,

$$f_{18} = -2^{-1} it|A_1||B|\mathbf{E}\,\mathrm{e}\{V^\star_{A_1'} + W_5 + v_1^\star + w_3\}T_1 Q_n.$$

Expanding the exponent in powers of $iv_1^\star$ and then in powers of $iw_3$ and using the fact that the conditional expectation of $T_1$ (respectively, $Q_n$) given all the random variables, but $\nu_1$ (respectively, $\nu_n$) is zero, we get

$$f_{18} = 2^{-1} it|A_1||B|\mathbf{E}\,\mathrm{e}\{V^\star_{A_1'} + W_5\}T_1 v_1^\star Q_n w_3 r_3,$$

where $r_3$ is a bounded function of $v_1^\star$ and $w_3$. Clearly,

$$|f_{18}| \ll |t||A_1||B|\mathbf{E}|T_1 v_1^\star\, Q_n|H^\star_{A_1'}(1 + |T_{A_2UB'}|)(|\tilde{v}n| + |tQn|).$$

Combining the bound $\mathbf{E}^{(1,n)}(1 + |T_{A_2UB'}|)H^\star_{A_1'} \ll u^{-5}$ [see (2.41)] and the simple inequality

$$\mathbf{E}|T_1 v_1^\star Q_n|\,(|\tilde{v}_n| + |tQ_n|) \ll p^2q^2uw^{-4}\mu$$

we obtain $f_{18} \prec \mathscr{R}$, thus completing the proof of (2.44). The proof of $f_{17} \sim f_5$ is much the same. We arrive at (2.32). □

PROOF OF (2.33). Expanding

$$g_0 = g(1 + Q_B) = 1 + Q_B g_2(Q_B), \qquad g_2(Q_B) = \mathbf{E}_{\theta_1} g'(1 + \theta_1 Q_B),$$

we obtain $V_B^* = V_B^\star + tT_B Q_B g_2(Q_B)$. Split $T_B Q_B = U_B + D_B$ and write

$$V_B^* = V_B^\star + w_1 + w_2, \qquad w_1 = tU_B g_2(Q_B), \qquad w_2 = tD_B g_2(Q_B).$$

We have $f_5 = \mathbf{E}\,\mathrm{e}\{V^\star + w_1 + w_2\}$. Expanding in powers of $iw_1$ and $iw_2$ we get

$$f_5 = f^* + f_{21} + f_{22} + f_{23}, \qquad f_{21} = \mathbf{E}\,\mathrm{e}\{V^\star\}iw_1,$$
$$f_{22} = \mathbf{E}\,\mathrm{e}\{V^\star\}w_1^2 r_1, \qquad f_{23} = \mathbf{E}\,\mathrm{e}\{V^\star + w_1\}w_2 r_2,$$

where $r_j$ is a bounded function of $w_j$, $j = 1, 2$.

Let us show $f_{22} \prec \mathscr{R}$ and $f_{23} \prec \mathscr{R}$. Using the fact that given $\mathbb{X}^*$, the random variables $T_{i_1}Q_{j_1}$ and $T_{i_2}Q_{j_2}$, for $i_1 \neq j_1$, $i_2 \neq j_2$, are conditionally uncorrelated unless the sets $\{i_1, j_1\}$ and $\{i_2, j_2\}$ coincide, we get

$$\mathbf{E}_B U_B^2 = \sum_{i,\, j \in B,\, i \neq j} \mathbf{E}_B \tilde{Z}_{i,j}, \qquad \tilde{Z}_{i,j} = T_i^2 Q_j^2 + T_i Q_j T_j Q_i. \tag{2.45}$$

Therefore, by the symmetry,

$$|f_{22}| \ll t^2 \mathbf{E} U_B^2 H_A^\star = t^2(|B|^2 - |B|)\mathbf{E}\tilde{Z}_{n,n-1} H_A^\star.$$

Furthermore,

$$|f_{23}| \ll |t|\mathbf{E}|D_B|H_A^\star \le |t||B|\mathbf{E}|T_n Q_n|H_A^\star.$$

Combining the bound $\mathbf{E}^{(1,2)} H_A^\star < u^{-5}$ [see (2.26)] and the inequalities $\mathbf{E}|T_n Q_n| \ll pqw^{-2}\mu$ and $\mathbf{E}|\tilde{Z}| \ll p^2q^2w^{-4}\mu$, we obtain $f_{22} \ll |t|u^{-5}\mu \prec \mathscr{R}$ and $f_{23} \ll t^2 u^{-5}\mu \prec \mathscr{R}$.

We complete the proof of (2.33) by showing $f_{21} \prec \mathscr{R}$. By the symmetry,

$$f_{21} = (|B|^2 - |B|)itf_{24}, \qquad f_{24} = \mathbf{E}\,\mathrm{e}\{V^\star\}T_n Q_{n-1} g_2(Q_B). \tag{2.46}$$

Write $Q_B = Q_{B'} + Q_n$, $B' = B \setminus \{n\}$. Expanding $g_2$ in powers of $Q_n$ we get

$$f_{24} = f_{25} + R_1, \qquad f_{25} = \mathbf{E}\,\mathrm{e}\{V^\star\}T_n Q_{n-1} g_2(Q_{B'}),$$
$$|R_1| \ll \mathbf{E}|T_n Q_n Q_{n-1}|H_A^\star.$$

Combining (2.26) and the simple bound $\mathbf{E}|T_n Q_n Q_{n-1}| \ll p^2 q^2 w^{-4}\mu$, we obtain $|R_1| \ll n^{-2}u^{-5}\mu$.

Expanding the exponent in powers of $v_n^\star$ and using the fact that the conditional expectation of $T_n$ given all the random variables, but $\nu_n$ is zero, we obtain

$$f_{25} = f_{26}, \qquad f_{26} = \mathbf{E}\,\mathrm{e}\{V^\star_{\Omega_{n-1}}\}T_n Q_{n-1} g_2(Q_{B'})v_n^\star r_n^\star,$$

where $r_n^\star$ is a bounded function of $v_n^\star$.

Write $B'' = B' \setminus \{n-1\}$. Expanding $g_2$ in powers of $Q_{n-1}$ we obtain $f_{26} = f_{27} + R_2$, where $f_{27}$ is defined in the same way as $f_{26}$, but with $g_2(Q_{B'})$ replaced by $g_2(Q_{B''})$ and

$$|R_2| \ll \mathbf{E}|T_n v_n^\star| Q_{n-1}^2 H_A^\star \ll u^{-5}(|t| + |s|)n^{-2}\mu.$$

In the last inequality we apply (2.26) and the simple bound $\mathbf{E}|T_n v_n^\star| Q_{n-1}^2 \ll (|t| + |s|)p^2 q^2 w^{-4}\mu$.

Finally, expanding the exponent in $f_{27}$ in powers of $v_{n-1}^\star$ and using the fact that the conditional expectation of $Q_{n-1}$ given all the random variables but $\nu_{n-1}$ is zero, we obtain

$$|f_{27}| \ll \mathbf{E}|T_n v_n^\star Q_{n-1} v_{n-1}^\star| H_A^\star \ll (|t| + |s|)^2 u^{-5} n^{-2}\mu, \tag{2.47}$$

by (2.26) and the simple bound $\mathbf{E}|T_n v_n^\star Q_{n-1} v_{n-1}^\star| \ll (|t| + |s|)^2 p^2 q^2 w^{-4}\mu$.

It follows from (2.47) and the bounds for $R_1$, $R_2$ that $|f_{24}| \ll u^{-4}n^{-2}\mu$. Now, by (2.46), $f_{21} \prec \mathscr{R}$, we obtain (2.33) and thus complete the proof of (2.27).

We arrive at (2.15). The proof of the inequality $I_{[0;\, c_4]} \ll \mathscr{R}$ is similar to the proof of (2.15), but simpler. We have $I_{[0;\, H]} \ll \mathscr{R}$ and this completes the proof of the theorem. □

**3. Auxiliary inequalities.** Denote, for brevity, $Y_j^\star = Y_j - \mathbf{E}Y_j$, $1 \le j \le n$.

Proof of Lemma 2.2. Let us prove (2.20). It follows from the inequalities $\mathbf{E}|Y_1|^3 \le 4\mathbf{E}|Y_1^\star|^3 + 4|\mathbf{E}Y_1|^3$ and $\mathbf{E}|Y_1^\star|^3 \ge (\mathbf{E}|Y_1^\star|^2)^{3/2} = w^{-3}b^3$ that $\mu \le 4\mu_0 + 4w^{-4}\alpha^3$ and $\mu_0 \ge w^{-1}b^3$. Therefore, $\mu_0^{-1}\mu \le 4 + 4w^{-3}b^{-3}\alpha^3$ and $\mu_0^{-2}\mathbf{E}|Y_1^\star|^2 \le b^{-4}$. Finally, by (2.13),

$$H\mu = c_3 b^2 \mu_0^{-1}\mu \le c \quad \text{and} \quad H^2\mathbf{E}|Y_1^\star|^2 = c_3^2 b^4 \mu_0^{-2}\mathbf{E}|Y_1^\star|^2 \le c_3^2.$$

Let us prove (2.21). We have [see (2.45)]

$$\mathbf{E}U_A^2 = (|A|^2 - |A|)\mathbf{E}(T_1^2 Q_2^2 + T_1 Q_2 T_2 Q_1). \tag{3.1}$$

Combining the bounds

$$\mathbf{E}(Y_i^\star)^2 \ll w^{-2}, \qquad \mathbf{E}Z_i^2 \ll \mathbf{E}|Z_i|^{3/2} \ll w^{-2}\mu, \qquad \mathbf{E}|Y_i^\star Z_i| \ll w^{-2}\mu \tag{3.2}$$

and (2.5) we obtain

$$\mathbf{E}T_1^2 Q_2^2 = p^2q^4\mathbf{E}(Y_1^\star)^2 Z_2^2 \ll n^{-2}\mu,$$
$$\mathbf{E}|T_1Q_2T_2Q_1| = p^2q^4\mathbf{E}|Y_1^\star Z_1 Y_2^\star Z_2| \ll n^{-2}\mu^2.$$

These inequalities in combination with (3.1) and (2.13) give $\mathbf{E}U_A^2 \ll z^2\mu$.

The second inequality in (2.21) follows from $\mathbf{E}U_A^2 \ll z^2\mu$ and $\mathbf{E}Q_A^2 \ll z\mu$, by Cauchy–Schwarz. To prove $\mathbf{E}Q_A^2 \ll z\mu$ we use the identity $\mathbf{E}_A Q_A^2 = \sum_{i\in A}\mathbf{E}_A Q_i^2$, the symmetry and (3.2),

$$\mathbf{E}Q_A^2 = \mathbf{E}(\mathbf{E}_A Q_A^2) = |A|\mathbf{E}Q_1^2 = mpq^3\mathbf{E}Z_1^2 \ll mpqw^{-2}\mu = z\mu. \tag{3.3}$$

Let us prove (2.22). An application of Marcinkiewicz–Zygmund inequality conditionally given all the random variables, but $\nu_i$, $i \in A$, gives $\mathbf{E}_A|Q_A|^{3/2} \ll \sum_{i\in A}\mathbf{E}_A|Q_i|^{3/2}$. Therefore, by the symmetry,

$$\mathbf{E}|T_B|^{3/4}|Q_A|^{3/2} \ll |A|\mathbf{E}|Q_1|^{3/2}|T_B|^{3/4} \ll mpq\mathbf{E}|Z_1|^{3/2}\mathbf{E}^{(1)}|T_B|^{3/4}.$$

Finally, combining (2.24) and (3.2), we obtain the first inequality of (2.22). The proof of the second one is much the same.

Let us prove (2.23). By the symmetry and (3.2),

$$\mathbf{E}\left|\sum_{j\in A} T_j Q_j^2\right|^{3/4} \le m\mathbf{E}|T_1Q_1^2|^{3/4} \ll mpq\mathbf{E}|Z_1|^{3/2} \ll z\mu,$$

$$\mathbf{E}\left|\sum_{j\in A} T_j Q_j Q_A^{(j)}\right| \le m\mathbf{E}|T_1Q_1||Q_A^{(1)}| = mpq^2\mathbf{E}|Y_1^\star Z_1|\mathbf{E}^{(1)}|Q_A^{(1)}| \ll z^{3/2}\mu^{3/2}.$$

In the last step we used the bound $\mathbf{E}^{(1)}|Q_A^{(1)}| \ll z^{1/2}\mu^{1/2}$, which follows from $\mathbf{E}^{(1)}(Q_A^{(1)})^2 \ll z\mu$ [cf. (3.3)] by Cauchy–Schwarz.

It remains to prove (2.24). The proof for $r = 6$ is straightforward. Using (2.24), with $r = 6$ and Lyapunov's inequality, we obtain (2.24) for $0 < r < 6$. $\square$

PROOF OF LEMMA 2.3. Inequalities (2.26) follow from (2.25), by Cauchy–Schwarz. Let us prove (2.25). We shall prove the first inequality only. The proof of the remaining two inequalities is similar, but simpler. Write

$$H_G^2 \le \prod_{k\in G}\xi_k, \qquad \xi_k = |\mathbf{E}_{\{k\}}\,\mathrm{e}\{v_k\}|^2. \tag{3.4}$$

We shall majorize $\xi_k$ by a random variable, say $\zeta_k$, which is a function of $X_k$, and apply Hoeffding [(1963), Theorem 4] to the expectation of the product of $\zeta_k$, $k \in G$.

Since $\nu_k^2 = \nu_k$, we can write $(\nu_k - p)^2 = \nu_k - 2\nu_k p + p^2$. Therefore,

$$T_k Q_k = (\nu_k - p)^2 Y_k^\star q Z_k = (\nu_k - p)(1-2p) Y_k^\star q Z_k + r, \qquad r = (p - p^2) Y_k^\star q Z_k,$$

and we write

$$v_k = (\nu_k - p) b_k - 2^{-1} t r, \qquad b_k = t a_k Y_k^\star + s w^{-1} \qquad a_k = g_0 - 2^{-1}(1-2p) q Z_k.$$

Since $r$ does not depend on $\nu_k$, we have

$$\xi_k \le |\beta(b_k)|^2 \quad \text{where } \beta(x) = \mathbf{E}\, \mathrm{e}\{x(\nu_1 - p)\}, \qquad x \in \mathbb{R}.$$

Höglund (1978) showed that, for any $z_0 \in [0, \pi)$ and $z$ satisfying $|z| \le \pi + z_0$,

$$|\beta(z)|^2 \le 1 - pq(z)^2 \Theta(z_0), \qquad \Theta(z_0) = \left( \frac{2}{\pi} \frac{\pi - z_0}{\pi + z_0} \right)^2.$$

We apply this inequality to those $b_k$ satisfying $|a_k Y_k^\star| \le H^{-1}$. We have $|b_k| \le \pi + 1$ and therefore $\xi_k \le 1 - pq b_k^2 \Theta(1)$. Combining this inequality with the obvious bound $\xi_k \le 1$, $k = 1, 2, \dots, n$, we obtain

$$\xi_k \le 1 - pq\, b_k^2 \Theta(1) \mathbb{I}_k, \qquad \mathbb{I}_k = \mathbb{I}_{|H a_k Y_k^\star| \le 1}, \; 1 \le k \le n. \tag{3.5}$$

Write $b_k^\star = t Y_k^\star + s w^{-1}$. The simple inequality $(x+y)^2 \ge x^2/2 - y^2$ gives

$$b_k^2 \ge (b_k^\star)^2/2 - (b_k - b_k^\star)^2 \ge (b_k^\star)^2/2 - d_k^2, \qquad d_k = |t Y_k^\star|(c_1 + |Z_k|). \tag{3.6}$$

Here we estimated $|b_k - b_k^\star| \le d_k$, using $|g_0 - 1| \le c_1$. Furthermore, since $|Z_k| \le 2$ and $|g_0| \le 1 + c_1 \le 2$, we have $|a_k| \le 3$, and therefore $\mathbb{I}_k \ge \mathbb{I}_k^\star := \mathbb{I}_{|3HY_k^\star| \le 1}$. This inequality in combination with (3.6) and (3.5) gives

$$\xi_k \le \zeta_k, \qquad \zeta_k = 1 - 2^{-1} pq \big( (b_k^\star)^2 - 2 d_k^2 \big) \Theta(1) \mathbb{I}_k^\star, \qquad 1 \le j \le n. \tag{3.7}$$

Assume without loss of generality that $1 \in G$. By Hoeffding [(1963), Theorem 4],

$$\mathbf{E}^{(i,j)} \prod_{k \in G} \zeta_k \le \prod_{k \in G} \mathbf{E}^{(i,j)} \zeta_k = \big( \mathbf{E}^{(i,j)} \zeta_1 \big)^{|G|}. \tag{3.8}$$

In the last step we used the symmetry. Next we show that, for some $c_5 > 0$,

$$\mathbf{E}^{(i,j)} \zeta_1 < 1 - c_5 n^{-1} u, \qquad u = t^2 + s^2. \tag{3.9}$$

Note that by (3.9) and (2.17), the right-hand side of (3.8) is less than

$$(1 - c_5 n^{-1} u)^{m/4} \le \exp\left\{ -\frac{c_5}{4} \frac{m}{n} u \right\} \le \exp\left\{ -\frac{1}{8} c_5 c_4 \ln u \right\} < u^{-10},$$

provided that the constant $c_4$ in the definition of $m$ is sufficiently large. This bound in combination with (3.7) and (3.4) implies $\mathbf{E}^{(i,j)} H_G^2 < u^{-10}$.

In order to prove (3.9) we show that

$$I_1 := \mathbf{E}^{(i,j)} (b_1^\star)^2 \mathbb{I}_1^\star \ge 2^{-1} u w^{-2} \quad \text{and} \quad \mathbf{E}^{(i,j)} d_1^2 \le t^2 8^{-1} w^{-2}. \tag{3.10}$$

The second inequality follows from the crude bound $\mathbf{E}d_1^2 \le 32t^2w^{-2}(c_1^2+\mu)$ and (2.13), provided that $c_1$ and $c_2$ are sufficiently small. To prove the first inequality, write

$$I_1 = \frac{n}{n-2}I_2 - \frac{1}{n-2}I_3, \qquad I_2 = \mathbf{E}(b_1^\star)^2\mathbb{I}_1^\star, \qquad I_3 = (b_i^\star)^2\mathbb{I}_i^\star + (b_j^\star)^2\mathbb{I}_j^\star,$$
$$I_2 = I_4 - I_5, \qquad I_4 = \mathbf{E}(b_1^\star)^2 = uw^{-2} - t^2w^{-4}\alpha^2, \qquad I_5 = \mathbf{E}(b_1^\star)^2\mathbb{I}_{|3HY_1^\star|>1}.$$

Now it is easy so see that the first inequality of (3.10) follows from

$$I_3 \le 20^{-1}u(pq)^{-1}, \qquad I_5 \le 20^{-1}uw^{-2} \tag{3.11}$$

and the inequality $t^2w^{-4}\alpha^2 \le t^2w^{-4}c_2^2$, provided that $c_2$ is sufficiently small.

Let us prove the bound for $I_3$. It follows from the inequalities

$$\begin{aligned} (b_k^\star)^2 &\le 2t^2(Y_k^\star)^2 + 2s^2w^{-2}, \\ (Y_i^\star)^2 + (Y_j^\star)^2 &\le 2^{1/3}\big(|Y_i^\star|^3 + |Y_j^\star|^3\big)^{2/3} \\ &\le 2^{1/3}(n\mathbf{E}|Y_1^\star|^3)^{2/3} \le 8\left(\frac{\mu}{pq}\right)^{2/3} \end{aligned} \tag{3.12}$$

that $I_3 \le 16u(\mu^{2/3}(pq)^{-2/3} + w^{-2})$. This bound in combination with (2.13) yields the first inequality of (3.11) provided that $c_2$ is sufficiently small.

To prove the bound (3.11) for $I_5$, we combine (3.12) and Chebyshev's inequality,

$$I_5 \le 2\frac{t^2}{w^2}I_6 + 2\frac{s^2}{w^2}I_7, \qquad I_6 = w^2\mathbf{E}(Y_1^\star)^2|3HY_1^\star|, \qquad I_7 = \mathbf{E}|3HY_1^\star|^2.$$

By the definition of $H$ [see (2.14)] $I_6 = 3c_3b^2 \le 3c_3$. By (2.20), $I_7 \le 9c_3^2$. Choosing $c_3$ small enough, we obtain the second inequality of (3.11), thus completing the proof of the lemma. $\square$

**Acknowledgments.** I thank V. Bentkus for discussions and comments. A part of this work was done when I was visiting Leiden University. I thank J. Mijnheer for kind hospitality. I am most indebted to the anonymous referee whose remarks and suggestions have improved the presentation very much. The idea of the present version of the proof of the main result (Theorem 2.1) in the case $p > q$ was suggested by the referee.

## REFERENCES

Babu, J. G. and Singh, K. (1985). Edgeworth expansions for sampling without replacement from finite populations. *J. Multivariate Anal.* **17** 261–278.

Bentkus, V., Bloznelis, M. and Götze, F. (1996). The Berry–Esseen bound for Student's statistic in the non-i.i.d. case. *J. Theoret. Probab.* **9** 765–796.

Bentkus, V. and Götze, F. (1996). The Berry–Esseen bound for Student's statistic. *Ann. Probab.* **24** 491–503.

Bentkus, V., Götze, F. and van Zwet, W. R. (1997). An Edgeworth expansion for symmetric statistics. *Ann. Statist.* **25** 851–896.

Bhattacharya, R. N. and Ghosh, J. K. (1978). On the validity of the formal Edgeworth expansion. *Ann Statist.* **6** 434–451.

BLOZNELIS, M. and GÖTZE, F. (1997). An Edgeworth expansion for finite population $U$-statistics. Preprint 97-012, SFB 343, Univ. Bielefeld. Available at `http://www.mathematik.uni-bielefeld.de/sfb343/Welcome.html`.

BOLTHAUSEN, E. and GÖTZE, F. (1993). The rate of convergence for multivariate sampling statistics. *Ann Statist*. **21** 1692–1710.

CHIBISOV, D. M. (1980). Asymptotic expansion for the distribution of a statistic admitting a stochastic expansion I. *Theory Probab. Appl*. **25** 732–744.

CHUNG, K. L. (1946). The approximate distribution of Student's statistic. *Ann. Math. Statist*. **17** 447–465.

EFRON, B. (1969). Student's $t$-test under symmetry conditions. *J. Amer. Statist. Assoc*. 1278–1302.

ERDŐS, P. and RÉNYI, A. (1959). On the central limit theorem for samples from a finite population. *Publ. Math. Inst. Hungar. Acad. Sci*. **4** 49–61.

FELLER, W. (1971). *An Introduction to Probability Theory and Its Applications* **2**. Wiley, New York.

GINÉ, E., GÖTZE, F. and MASON, D. M. (1997). When is the Student $t$-statistic asymptotically standard normal? *Ann. Probab*. **25** 1514–1531.

GRIFFIN, P. S. and MASON, D. M. (1991). On the asymptotic normality of self-normalized sums. *Math. Proc. Cambridge Philos. Soc*. **109** 597–610.

HAJEK, J. (1960). Limiting distributions in simple random sampling from a finite population. *Publ. Math. Inst. Hungar. Acad. Sci*. **5** 361–374.

HALL, P. (1988). On the effect of random norming on the rate of convergence in the central limit theorem. *Ann. Probab*. **16** 1265–1280.

HELMERS, R. and VAN ZWET, W. R. The Berry–Esseen bound for $U$-statistics I. In *Statistical Decision Theory and Related Topics* (S. S. Gupta and J. O. Berger, eds.) **3** 497–512. Academic Press, New York.

HOEFFDING, W. (1963). Probability inequalities for sums of bounded random variables. *J. Amer. Statist. Assoc*. **58** 13–30.

HÖGLUND, T. (1978). Sampling from a finite population: a remainder term estimate. *Scand. J. Statist*. **5** 69–71.

KOKIC, P. N. and WEBER N. C. (1990). An Edgeworth expansion for $U$-statistics based on samples from finite populations. *Ann. Probab*. **18** 390–404.

LOGAN, B., MALLOWS, C., RICE, S. and SHEPP, L. (1973). Limit distributions of self-normalized sums. *Ann. Probab*. **1** 788–809.

PRAŠKOVA, Z. (1989). Sampling from a finite set of random variables: the Berry-Esseen bound for the Studentized mean. In *Proceedings of the Fourth Prague Symposium on Asymptotic Statistics* (P. Mandl and M. Hušková, eds.) 67–82. Charles Univ., Prague.

PUTTER, H. and VAN ZWET, W. R. (1998). Empirical Edgeworth expansions for symmetric statistics. *Ann. Statist*. **26** 1540–1569.

RAO, C. R. and ZHAO, L. C. (1994). Berry–Esseen bounds for finite-population $t$-statistics. *Statist. Probab. Lett*. **21** 409–416.

SHARAKHMETOV, S. (1995). The Berry-Esseen inequality for Student's statistic. *Uzbek. Mat. Zh*. **2** 101–112.

SLAVOVA, V. V. (1985). On the Berry-Esseen bound for Student's statistic. *Lecture Notes in Math*. **1155** 335–390. Springer, Berlin.

VAN ZWET, W. R. (1984). A Berry-Esseen bound for symmetric statistics. *Z. Wahrsch. Verw. Gebiete* **66** 425–440.

ZHAO, L. C. and CHEN, X. R. (1990). Normal approximation for finite-population $U$-statistics. *Acta Math. Appl. Sinica* **6** 263–272.

DEPARTMENT OF MATHEMATICS
VILNIUS UNIVERSITY
NAUGARDUKO 24
VILNIUS 2006
LITHUANIA
E-mail: mblozn@ieva.maf.vu.lt