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WEAK UNIQUENESS FOR THE HEAT EQUATION WITH NOISE[1]

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The uniqueness in law for the equation $\partial X_t/\partial t = \frac{1}{2}\Delta X_t + X_t^\gamma \dot{W}$ is established for $1/2 < \gamma < 1$. The proof uses a duality technique and requires the construction of an approximating sequence of dual processes.

1. Introduction.

In this article we will discuss the problem of uniqueness for the stochastic partial differential equation (SPDE)

$$\frac{\partial X_t}{\partial t} = \frac{1}{2}\Delta X_t + X_t^\gamma \dot{W}, \tag{1.1}$$

with $1/2 < \gamma < 1$, where $\dot{W}$ is two-parameter white noise on $(\mathbb{R}_+ \times \mathbb{R})$. The existence of a solution to (1.1) was proved in [7] by tightness arguments. By itself, (1.1) is a purely formal stochastic partial differential equation. More rigorously, we can consider the integral equation

$$X_t(x) = S_t X_0(x) + \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y) X_t^\gamma(y)\, W(ds, dy), \tag{1.2}$$

where $\{S_t\}$ is the semigroup with generator $\frac{1}{2}\Delta$, and p_t is the probability density function corresponding to S_t.

Before presenting our result, we need to introduce the following notation. Let M_F denote the finite measures on $\mathbb{R}$ with weak topology and let B (resp., $\overline{C}$) denote the bounded (resp., continuous bounded) Borel measurable functions on $\mathbb{R}$. In general, if F is a set of functions on $\mathbb{R}$, write F_+ or F^+ for nonnegative functions in F. For $\mu \in M_F$ and $f \in B$ let

$$\mu(f) = \langle \mu, f\rangle = \int f\, d\mu.$$

We will abbreviate "boundedly pointwise" by bp.

As in [7], we will consider solutions $X_t(x)$ to (1.2) starting from initial conditions rapidly decreasing in x. Therefore, define

$$C^+_{\text{rap}} = \Big\{ g \in \overline{C}\colon\ g \ge 0, |g|_p \equiv \sup_x e^{p|x|} g(x) < \infty\ \forall p > 0\Big\}.$$

Note that further we will consider $L^1 = L^1(\mathbb{R})$ (which obviously contains C^+_{rap}) as a subset of M_F using the correspondence $\phi(x) \mapsto \phi(x)\, dx$. Let $\mathscr{P}(C^+_{\text{rap}})$ be

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the set of probability measures on C^+_{rap} and define

$$\mathscr{P}_p(C^+_{\text{rap}}) \equiv \left\{\nu \in \mathscr{P}(C^+_{\text{rap}})\colon \sup_{x\in\mathbb{R}^1}\left\{\int_{C^+_{\text{rap}}} |\phi(x)|^p \nu(d\phi)\right\} < \infty\right\}, \qquad p > 0.$$

For any process X defined on some probability space $(\Omega, \mathscr{F}, P)$, let $\mathscr{F}_t^X \equiv \sigma(X_s, s \le t)$.

Now we are ready to present our main result. As our concern is with the proof of the weak uniqueness of the solution for (1.2), we intend to prove the following theorem.

THEOREM 1.1. *Assume that $\nu \equiv P(X_0)^{-1} \in \mathscr{P}_p(C^+_{\text{rap}})$ for some $p \ge 2$. Then any two solutions for the martingale problem*

$$M^\nu \begin{cases} \text{for all } \phi \in \mathscr{D}(\tfrac{1}{2}\Delta)\big[\mathscr{D}(\tfrac{1}{2}\Delta) \text{ is the domain of } \tfrac{1}{2}\Delta\big], \\ \qquad Z_t(\phi) = \langle X_t, \phi\rangle - \langle X_0, \phi\rangle - \displaystyle\int_0^t \langle X_s, \tfrac{1}{2}\Delta\phi\rangle\, ds \\ \text{is an } \mathscr{F}_t^X \text{ continuous square integrable martingale such that} \\ Z_0(\phi) = 0 \text{ and} \\ \qquad \langle Z(\phi)\rangle_t = \displaystyle\int_0^t \langle X_s^{2\gamma}, \phi^2\rangle\, ds \end{cases}$$

have the same finite-dimensional distributions, which means that M^ν has at most one solution.

REMARK 1.2. Since $\mathscr{D}(\frac{1}{2}\Delta)$ is bp-dense in B, the standard construction allows us to extend Z_t to an orthogonal martingale measure $\{Z_t(\phi)\colon t \ge 0, \psi \in B\}$. That is, for each $\psi \in B$, $Z_t(\phi)$ is a continuous square integrable martingale such that

$$\langle Z(\phi)\rangle_t = \int_0^t \langle X_s^{2\gamma}, \phi^2\rangle\, ds. \tag{1.3}$$

Let $\mathscr{D}(Z)$ denote the set of functions

$$\Big\{\phi\colon \Omega \times \mathbb{R}_+ \times \mathbb{R} \mapsto \mathbb{R} \text{ which is predictable (see [9], page 292) and}$$
$$E\left[\int_0^t \int_{\mathbb{R}} \phi(\cdot, s, y)^2 X_s(y)^{2\gamma}\, dy\, ds\right] < \infty\Big\}.$$

Proceeding as in [9], notice that for each $\phi \in \mathscr{D}(Z)$ we can define the stochastic integral

$$Z_t(\phi) = \int_0^t \int_{\mathbb{R}} \phi(s, y)\, dZ(s, y). \tag{1.4}$$

The term $Z_t(\phi)$ is a continuous square integrable martingale with quadratic variation

$$\int_0^t \int_{\mathbb{R}} \phi(s, y)^2 X_s(y)^{2\gamma}\, dy\, ds.$$

The idea behind the proof is based on a duality approach. This approach suggests proving the existence of a *dual* process Y with values in some space E and functions $f, g \in \mathscr{B}(C^+_{\text{rap}} \times E)$ such that

$$f(X_t, y) - \int_0^t g(X_s, y)\, ds \quad \text{is an } \mathscr{F}_t^X \text{ martingale for each } y \in E \tag{1.5}$$

and

$$f(\psi, Y_t) - \int_0^t g(\psi, Y_s)\, ds \quad \text{is an } \mathscr{F}_t^Y \text{ martingale for each } \psi \in C^+_{\text{rap}} \tag{1.6}$$

for any solution X to M^ν which is independent of Y. Then Theorem 4.4.11 in [2] (which assumes also some moment conditions) shows that

$$E\big[f(X_t, Y_0)\big] = E\big[f(X_0, Y_t)\big] \qquad \forall t \geq 0. \tag{1.7}$$

If $\{f(\cdot, y), y \in E\}$ is separating on $\mathscr{P}_p(C^+_{\text{rap}})$ and such a process Y can be constructed for any $Y_0 \in E$, the uniqueness of solutions to M^ν follows (see [2], Proposition 4.4.7, for this result in a more general setting).

Let us try to use the above method. By choosing $f(\phi, \psi) = e^{-\langle \phi, \psi \rangle}$ and applying Itô's formula, we easily obtain that

$$\exp\big(-\langle \phi, X_t\rangle\big) - \int_0^t \exp\big(-\langle \phi, X_s\rangle\big)\big(-\langle \tfrac{1}{2}\Delta\phi, X_s\rangle + \tfrac{1}{2}\langle \phi^2, X_s^{2\gamma}\rangle\big)\, ds$$

is an $\mathscr{F}_t^X$ martingale for each $\phi \in \mathscr{D}(\frac{1}{2}\Delta)_+$. This together with (1.5) and (1.6) suggests constructing the process Y such that

$$\exp\big(-\langle Y_t, \psi\rangle\big) - \int_0^t \exp\big(-\langle Y_s, \psi\rangle\big)\big(-\langle \tfrac{1}{2}\Delta Y_s, \psi\rangle + \tfrac{1}{2}\langle Y_s^2, \psi^{2\gamma}\rangle\big)\, ds \tag{1.8}$$

is an $\mathscr{F}_t^Y$ martingale for each $\psi \in C^+_{\text{rap}}$. If such a process Y exists and all the assumptions of Theorem 4.4.11 in [2] are satisfied, then we have

$$E\big[\exp\big(-\langle Y_0, X_t\rangle\big)\big] = E\big[\exp\big(-\langle Y_t, X_0\rangle\big)\big] \tag{1.9}$$

and the uniqueness for M^ν follows easily. Let us try to give another description [different from (1.8)] of the dual process Y we are looking for. Let Y be a solution of the stochastic partial differential equation

$$Y_t(x) = S_t\phi(x) + \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y) Y_s^{1/\gamma}(y) L(dy, ds), \qquad t \geq 0, \tag{1.10}$$

where $\dot{L}$ is a stable noise on $\mathbb{R} \times \mathbb{R}_+$ with nonnegative jumps and Laplace transform given by

$$\begin{aligned} &E\left[\exp\left(-\int_0^t \int_{\mathbb{R}} \phi(s, x)\, L(dx, ds)\right)\right] \\ &\quad = E\left[\exp\left(\int_0^t \int_{\mathbb{R}} \phi(s, x)^{2\gamma}\, dx\, ds\right)\right] \qquad \forall \phi \in L^{2\gamma}\big(\mathbb{R} \times [0, t]\big)_+. \end{aligned}$$

If such a process Y exists, then Itô's formula yields the result

$$\tilde{M}\begin{cases}\text{for all } \psi\in\mathscr{D}(\frac{1}{2}\Delta)_+\\[4pt] \qquad \tilde{Z}_t(\phi)=\exp(-\langle Y_t,\psi\rangle)-\int_0^t \exp(-\langle Y_s,\psi\rangle)\big(-\langle Y_s,\frac{1}{2}\Delta\psi\rangle+\frac{1}{2}\langle Y_s^2,\psi^{2\gamma}\rangle\big)\,ds\\[4pt] \text{is an } \mathscr{F}_t^Y \text{ local martingale.}\end{cases}$$

Observe that $\tilde{M}$ is just a weak form of (1.8), that is, in $\tilde{M}$ we do not require Y to be in $\mathscr{D}(\frac{1}{2}\Delta)_+$ and $\tilde{Z}$ to be a martingale. It can be conjectured that the existence of the process Y satisfying local martingale problem $\tilde{M}$ or SPDE (1.10) is sufficient to verify (1.9). Then, the "only" problem is the existence of Y solving SPDE (1.10). Note that if the exponent of the noise is less than 1, then (1.10) belongs to the class of SPDEs that was studied by Mueller [6]. However, in our case, the exponent of L equals $2\gamma>1$ and the existence of a solution to (1.10) is unresolved.

The approach of an approximating sequence of dual processes which was introduced in [8] allows us to avoid the proof of the existence of a solution to (1.10) and helps us in this case. The main problem is to choose the right approximating sequence of processes since when we treat convergence to processes driven by a stable noise, high moments may diverge. In Section 3 we will construct the sequence of processes $\{Y^{(n)}\}$ which satisfies the local martingale problem

$$\tilde{M}^{(n)}\begin{cases}\text{for all } \psi\in\mathscr{D}(\frac{1}{2}\Delta)_+,\\[4pt] \qquad \tilde{Z}_t^{(n)}(\phi)=\exp(-\langle Y_t^{(n)},\psi\rangle)-\int_0^t \exp(-\langle Y_s^{(n)},\psi\rangle)\\[4pt] \qquad\qquad \times\Big(-\langle Y_s,\frac{1}{2}\Delta\psi\rangle+\frac{1}{2}\eta\int_{\mathbb{R}^d}\int_{1/n}^{\infty}\big(\exp(-\lambda\phi(x))-1+\lambda\phi(x)\big)\\[4pt] \qquad\qquad\qquad \times\lambda^{-2\gamma-1}\,d\lambda\,Y_s^{(n)}(x)^2\,dx\Big)\,ds\\[4pt] \text{is an } \mathscr{F}_t^{Y^{(n)}} \text{ local martingale,}\end{cases}$$

where $\eta\equiv(2\gamma(2\gamma-1))/(\Gamma(2-2\gamma))$. Observe that $y^{2\gamma}=\eta\int_{0+}^{\infty}(e^{-\lambda y}-1+\lambda y)\lambda^{-2\gamma-1}\,d\lambda$. Hence we expect that the above local martingale problem should converge to the local martingale problem $\tilde{M}$ as $n\to\infty$. For each n, the process $\{Y^{(n)}\}$ which satisfies $\tilde{M}^{(n)}$ will be defined in Section 3 as a solution of some SPDE driven by a point process without jumps smaller than $1/n$. We do not give a precise definition of this SPDE here as this would require a significant amount of notation.

The rest of the paper is organized as follows. The basic tools needed for the implementation of the duality technique in our particular case are introduced in Section 2. Some simple properties of any solution to M^{ν} are also presented in Section 2. Section 3 contains the construction of the approximating sequence of dual processes and, by means of this sequence, we prove Theorem 1.1.

2. Properties of solutions to M^{ν} and duality tools. We start with a moment condition result.

LEMMA 2.1. *Assume that $p \geq 2$ and $\nu \in \mathscr{P}_p(C^+_{\mathrm{rap}})$. Let X be any solution of the martingale problem for M^{ν}. Then for each $T > 0$ we have*

$$\sup_{t \leq T} \sup_{x \in \mathbb{R}} E[X(t,x)^p] < \infty. \tag{2.1}$$

The proof of Lemma 2.1 is omitted, since it is standard (e.g., the reader may adapt the arguments from the proof of Proposition 4.2 in [7]).

Our goal is to prove that any two solutions to M^{ν} have the same finite-dimensional distributions. It is well known that as we deal with the solutions of the martingale problem, the problem can be transformed into the simpler one: to verify uniqueness of the one-dimensional distributions (see [2], Theorem 4.2). However, our attempt to use Theorem 4.2 in [2] directly met with some technical difficulties that we will try to describe below. Suppose that X is any solution to M^{ν} and

$$f(X_t) - \int_0^t g(X_s)\, ds$$

is an $\mathscr{F}^X_t$ martingale. Then g will be an unbounded function on C^+_{rap} for any usual function f. [For example, we can take $f_{\phi}(X) = \exp\{-\langle X, \phi\rangle\}$ for some $\phi \in \mathscr{D}(\frac{1}{2}\Delta)_+$, and then the corresponding g, found by Itô's formula, is unbounded.] Hence we cannot use Theorem 4.2 in ([2], Chapter 4) since it requires that all functions be bounded. However, a careful examination of the proof of this theorem leads us to the conclusion that this condition is not essential. Therefore, we can present the following lemma (which is just the reformulation of Theorem 4.2 in [2], Chapter 4, for our case):

LEMMA 2.2. *Let $p \geq 2$. Suppose that for each $\nu \in \mathscr{P}_p(C^+_{\mathrm{rap}})$ any two solutions X^1, X^2 of the martingale problem for M^{ν} have the same one-dimensional distributions. That is, for each $t > 0$,*

$$P\left\{X^1_t \in \Gamma\right\} = P\left\{X^2_t \in \Gamma\right\}, \qquad \Gamma \in \mathscr{B}\left(C^+_{\mathrm{rap}}\right). \tag{2.2}$$

Then any two solutions of the martingale problem for M^{ν} have the same finite-dimensional distributions (i.e., uniqueness holds).

PROOF. The proof is completely analogous to the proof of Ethier and Kurtz's Theorem 4.2 in ([2], Chapter 4). The only delicate point is that, at some point, we need the fact that if $P(X_0)^{-1} \in \mathscr{P}_p(C^+_{\mathrm{rap}})$, then for each time $t > 0$, $P(X_t)^{-1}$ is also in $\mathscr{P}_p(C^+_{\mathrm{rap}})$. Lemma 2.1 assures that this is the case. $\square$

As a consequence of the previous lemma, we need to verify that for each $\nu \in \mathscr{P}_p(C^+_{\mathrm{rap}})$ any two solutions to M^ν have the same one-dimensional distributions. One approach for doing this involves the notion of an approximating sequence of dual processes which was introduced in [8]. Section 3 is devoted to the proof of the following proposition, which establishes the existence of such a sequence of processes in our case.

PROPOSITION 2.3. *For each $\nu \in \mathscr{P}_p(C^+_{\mathrm{rap}})$ and each $\phi \in L^1(\mathbb{R})_+$, there exists a sequence of processes $\{Y^{(n)}\}$ taking values in M_F such that $Y_0^{(n)} = \phi$ and*

$$E\big[\exp\big[-\langle \phi, X_t\rangle\big]\big] = \lim_{n\to\infty} E\big[\exp\big[-\langle Y_t^{(n)}, X_0\rangle\big]\big] \tag{2.3}$$

for every $t \geq 0$ and each solution X to M^ν which is independent of $Y^{(n)}$.

REMARK 2.4. The motivation for the construction of $Y^{(n)}$ was briefly discussed in the Introduction.

REMARK 2.5. Note that Proposition 2.3 gives the unique characterization of one-dimensional distributions of solutions to M^ν via the Laplace transform. Therefore, Theorem 1.1 follows immediately from Proposition 2.3 (cf. [8], Theorem 1.7).

Define

$$\overline{C}^{1,2}(\mathbb{R}_+ \times \mathbb{R}) = \bigg\{\psi \in \overline{C}(\mathbb{R}_+ \times \mathbb{R})\colon \frac{\partial^k}{\partial t^k}\frac{\partial^i}{\partial x^i}\psi(t,x) \in \overline{C}(\mathbb{R}_+ \times \mathbb{R}),\ k = 0,1,\ i = 0,1,2\bigg\}.$$

The next lemma transforms the martingale problem M^ν into the martingale problem in the "exponential form."

LEMMA 2.6. *Assume that $p \geq 2$ and $\nu \in \mathscr{P}_p(C^+_{\mathrm{rap}})$. Let X be any solution of the martingale problem for M^ν. Let $\psi \in \overline{C}^{1,2}(\mathbb{R}_+ \times \mathbb{R})_+$ and*

$$\int_0^T \int_{\mathbb{R}} \psi(s,x)^2\, dx\, ds < \infty \qquad \forall T > 0. \tag{2.4}$$

Then

$$\begin{aligned} &\exp\big(-\langle X_t, \psi_t\rangle\big) \\ &\quad - \int_0^t \exp\big(-\langle X_s, \psi_s\rangle\big)\bigg(-\Big\langle X_s \frac{1}{2}\Delta\psi_s\Big\rangle + \frac{1}{2}\langle X_s^{2\gamma}, \psi_s^2\rangle - \Big\langle X_s \frac{\partial}{\partial s}\psi_s\Big\rangle\bigg)\, ds \end{aligned} \tag{2.5}$$

is an $\mathscr{F}_t^X$ martingale.

PROOF. By the usual application of Itô's formula we get that

$$
\begin{aligned}
&\exp(-\langle X_t, \psi_t\rangle)\\
&\quad = \exp(-\langle X_0, \psi_0\rangle)\\
&\qquad + \int_0^t \exp(-\langle X_s, \psi_s\rangle)\Big(-\Big\langle X_s, \frac{1}{2}\Delta\psi_s\Big\rangle + \frac{1}{2}\langle X_s^{2\gamma}, \psi_s^2\rangle - \Big\langle X_s \frac{\partial}{\partial s}\psi_s\Big\rangle\Big)\, ds \qquad (2.6)\\
&\qquad + \int_0^t \exp(-\langle X_s, \psi_s\rangle)\, dZ_s(\psi_s),
\end{aligned}
$$

where $Z_s(\psi_s) = \int_0^t \int_{\mathbb{R}} \psi(s, y)\, Z(ds, dy)$ is an $\mathscr{F}_t^X$ square integrable martingale [see Remark 1.2: the fact that $\psi \in \mathscr{D}(Z)$ follows from Lemma 2.1 combined with condition (2.4)].

The term $\exp(-\langle X_s, \psi_s\rangle)$ is bounded and $Z_\cdot(\psi_\cdot)$ is a square integrable martingale; therefore, we get that the last term in (2.6) is also a martingale. This completes the proof of the lemma. □

3. Dual approximation and proof of Theorem 1.1. As we have already mentioned above, to prove Theorem 1.1 it suffices to prove Proposition 2.3 so that we need to construct some approximating sequence of processes. The motivation for our construction was briefly discussed in the Introduction.

Let us introduce further notation. For each $m \in M_F$ and $n \geq 1$, $V_t^{(n)}(m)$ denotes the unique weak nonnegative solution of the nonlinear equation

$$
v_t = S_t m - \int_0^t S_{t-s}\big(\tfrac{1}{2} b_n v_s^2\big)\, ds, \qquad (3.1)
$$

where

$$
b_n = \frac{2\gamma}{\Gamma(2-2\gamma)} n^{2\gamma-1}.
$$

In the following we fix $n \geq 1$. Proposition A.2 in [3] shows that for each $m \in M_F$,

$$
\begin{aligned}
&V_t^{(n)}(m) \in L^2(\mathbb{R}) \qquad \forall t > 0,\\
&V_\cdot^{(n)}(m) \in L^2\big((0, T] \times \mathbb{R}\big) \qquad \forall T > 0,
\end{aligned}
$$

$$
w - \lim_{t\downarrow 0} V_t^{(n)}(m) = m.
$$

Integrating (3.1) over the space variable, we obtain

$$
\big\| V_t^{(n)}(m)\big\|_1 = m(1) - \tfrac{1}{2} b_n \int_0^t \big\| V_s^{(n)}(m)\big\|_2^2\, ds, \qquad (3.2)
$$

where $\|\cdot\|_1$ and $\|\cdot\|_2$ denote the norms in $L^1(\mathbb{R})$ and $L^2(\mathbb{R})$, respectively. Moreover, adapting the arguments used in the proof of Theorem 3.1 in [1], we can get

$$
\lim_{t\to\infty} \big\| V_t^{(n)}(m)\big\|_1 = 0 \qquad \forall m \in M_F. \qquad (3.3)
$$

Equations (3.2) and (3.3) imply that

$$\int_0^\infty \| V_t^{(n)}(m)\|_2^2\,dt = \frac{2}{b_n} m(1) \qquad \forall m \in M_F. \tag{3.4}$$

In the following text, we adopt the convention $V_0^{(n)}(m) = m$.

Let $(\Omega, \mathscr{F}, P)$ be a probability space which is sufficiently rich to contain all the processes and random variables defined below.

LEMMA 3.1. *Assume that $p \geq 2$ and $\nu \in \mathscr{P}_p(C_{\mathrm{rap}}^+)$. Let X be any solution of the martingale problem for M^ν defined on $(\Omega, \mathscr{F}, P)$. Then*

$$\begin{aligned} E_X\bigl[\exp\bigl(-\langle V_t^{(n)}(m), X_{T-t}\rangle\bigr)\bigr] &= E_X\bigl[\exp\bigl(-\langle V_0^{(n)}(m), X_T\rangle\bigr)\bigr] \\ &\quad + E_X\biggl[\int_{0+}^t \exp\bigl(-\langle V_s^{(n)}(m), X_{T-s}\rangle\bigr) \\ &\qquad \times \tfrac12\bigl(\langle b_n(V_s^{(n)}(m))^2, X_{T-s}\rangle - \langle (V_s^{(n)}(m))^2, X_{T-s}^{2\gamma}\rangle\bigr)\,ds\biggr] \end{aligned} \tag{3.5}$$

for each $m \in M_F$ and $0 \leq t \leq T$.

PROOF. For each $m \in M_F$, there exists $\{m_k\} \in M_F$ such that

$$\begin{aligned} &m_k(dx) = \phi_k(x)\,dx \quad (m_k \text{ is absolutely continuous with respect to Lebesgue measure}),\\ &\phi_k \in \mathscr{D}(\tfrac12\Delta) \cap C_{\mathrm{rap}}^+ \qquad \forall k \geq 1,\\ &m_k \Rightarrow m \quad \text{in } M_F, \text{as } k \to \infty. \end{aligned}$$

Theorem A in [5] implies that, in fact, $V_t^{(n)}(m_k)$ is the strong solution of

$$\begin{aligned} \frac{\partial v_t}{\partial t} &= \frac12 \Delta v_t - \frac12 b_n v_t^2, \qquad t > 0,\\ v_0 &= \phi_k. \end{aligned} \tag{3.6}$$

In this case the function $\psi(s,x) \equiv V_{T-s}^{(n)}(m_k)(x)$ satisfies the conditions of Lemma 2.6, which immediately yields

$$\begin{aligned} E_X\bigl[\exp\bigl(-\langle V_t^{(n)}(m_k), X_{T-t}\rangle\bigr)\bigr] &= E_X\bigl[\exp\bigl(-\langle V_0^{(n)}(m_k), X_T\rangle\bigr)\bigr] \\ &\quad + E_X\biggl[\int_0^t \exp\bigl(-\langle V_s^{(n)}(m_k), X_{T-s}\rangle\bigr) \\ &\qquad \times \tfrac12\bigl(\langle b_n(V_s^{(n)}(m_k))^2, X_{T-s}\rangle - \langle (V_s^{(n)}(m_k))^2, X_{T-s}^{2\gamma}\rangle\bigr)\,ds\biggr] \end{aligned} \tag{3.7}$$

for each $0 \le t \le T$. Let $k \to \infty$. From Proposition A.2 in [3] we know that $V_{\cdot}^{(n)}(m_k) \to V_{\cdot}^{(n)}(m)$ in $L^2((0,T]\times\mathbb{R})$. The variable X takes values in C_{rap}^{+}. Hence we get

$$\lim_{k\to\infty}\langle V_t^{(n)}(m_k), X_{T-t}^q\rangle = \langle V_t^{(n)}(m), X_{T-t}^q\rangle \tag{3.8}$$
$$\text{a.s.,}\quad \forall 0\le t\le T,\ \forall 1\le q<2.$$

Now, the result is immediate from (3.8), (3.7) and the uniform integrability condition from Lemma 2.1. □

Let us construct the approximating sequence of dual processes $\{Y^{(n)}\}$ as follows.

Let $\{\tilde{T}_{n,i}, i = 1,2,\ldots\}$ be independent identically distributed exponential random variables with parameter $\kappa_n = n^{2\gamma}(2\gamma-1)/(\Gamma(2-2\gamma))$ and let $\{Z_{n,i}, i=1,2,\ldots\}$ be independent $\mathbb{R}_+$-valued random variables with the distributions given by

$$P(Z_{n,i}\ge b) = \frac{\int_{b\vee 1/n}^{\infty}\lambda^{-2\gamma-1}\,d\lambda}{\int_{1/n}^{\infty}\lambda^{-2\gamma-1}\,d\lambda}\qquad \forall i\ge 1.$$

Note that

$$E[Z_{n,i}] = \frac{2\gamma}{n(2\gamma-1)}. \tag{3.9}$$

We suppose that $\{\tilde{T}_{n,i}, i = 1,2,\ldots\}$ and $\{Z_{n,i}, i = 1,2,\ldots\}$ are mutually independent.

Fix arbitrary $\phi \in L^1(\mathbb{R})_+$. Let $A_t^{(n)}$ be a Levy pure jump process with jumps $\{Z_{n,i},\ i=1,2,\ldots\}$ and corresponding times of jumps given by

$$T_{n,i} = \sum_{k=1}^{i}\tilde{T}_{n,k}\qquad \forall j\ge 1. \tag{3.10}$$

Assume $A_0^{(n)} = \langle\phi,1\rangle$ and define $\mathscr{F}_t^{A^{(n)}} \equiv \sigma\{A_s^{(n)}, s\le t\}$.

The random variables $\{T_{n,i},\ i=1,2,\ldots\}$ and $\{Z_{n,i},\ i=1,2,\ldots\}$ determine the times and heights of the jumps of the process $A^{(n)}$, and, as we will see later, they will also control the times and "masses" of the jumps of the desired process $Y^{(n)}$. However, for $Y^{(n)}$ we will also need to determine the spatial positions of the jumps. The positions of the jumps will be controlled by random variables $\{U_{n,i},\ i=1,2,\ldots\}$ defined later. For their definition the following notation is important.

Let $\{G(f,\cdot),\ f\in L_+^2(\mathbb{R})\}$ be the collection of probability measures on $\mathbb{R}$ such that

$$G(f,A) = \frac{\int_A f^2(x)\,dx}{\|f\|_2^2}\qquad \forall A\subset\mathscr{B}(\mathbb{R}).$$

Now we are ready to define the M_F-valued process $Y^{(n)}$ iteratively as follows. Let

$$Y_0^{(n)}(dx) = \phi(x)\,dx.$$

Define the time change $\gamma(t)$ and the σ-algebra $\tilde{\mathscr{F}}_1^n$:

$$\gamma(t) = \inf\left\{s\colon \int_{0+}^{s} \frac{1}{2}\left\| V_u^{(n)}(\phi)\right\|_2^2 du > t\right\}, \qquad 0 \le t \le T_{n,1},$$
$$\tilde{\mathscr{F}}_1^n = \mathscr{F}_{T_{n,1}}^{A^{(n)}} \vee \sigma(Y_0^{(n)}).$$

The time change $\gamma(t)$ depends on n, but to simplify the notation we do not make this dependence explicit. We use the convention $\inf \varnothing = \infty$. Hence, we can see that $\gamma(T_{n,1}) = \infty$ if and only if $\int_0^\infty \frac{1}{2}\|V_u^{(n)}(\phi)\|_2^2\,du = Y_0^{(n)}(1)/b_n \le T_{n,1}$.

Let

$$\begin{aligned} Y_t^{(n)} &= V_t^{(n)}(Y_0^{(n)}), \qquad 0 \le t < \gamma(T_{n,1}),\\ Y_{\gamma(T_{n,1})}^{(n)} &= Y_{\gamma(T_{n,1})-}^{(n)} + Z_{n,1}\delta_{U_{n,1}} \quad \text{if } \gamma(T_{n,1}) < \infty, \end{aligned} \tag{3.11}$$

where

$$P(U_{n,1} \in \cdot | \tilde{\mathscr{F}}_1^n) = P(U_{n,1} \in \cdot | Y_{\gamma(T_{n,1})-}^{(n)}) = G\big(Y_{\gamma(T_{n,1})-}^{(n)}, \cdot\big).$$

If $\gamma(T_{n,1}) = \infty$, the construction of $Y^{(n)}$ is finished since $Y^{(n)}$ is defined on the interval $[0, \infty)$. Otherwise, we proceed in the same way until the first time when $\gamma(T_{n,k})$ becomes infinite. To be more precise, for each $k \ge 1$, let

$$\tilde{\mathscr{F}}_{k+1}^n = \mathscr{F}_{T_{n,k+1}}^{A^{(n)}} \vee \sigma(Y_0^{(n)}) \vee \sigma(U_{n,j}, j \le k),$$
$$\gamma(t) = \inf\left\{s\colon \gamma(T_{n,k}) + \int_{0+}^{s-\gamma(T_{n,k})} \frac{1}{2}\left\| V_u^{(n)}\big(Y_{\gamma(T_{n,k})}^{(n)}\big)\right\|_2^2 du > t\right\},$$
$$T_{n,k} \le t \le T_{n,k+1},$$
$$Y_t^{(n)} = V_{t-\gamma(T_{n,k})}^{(n)}\big(Y_{\gamma(T_{n,k})}^{(n)}\big), \qquad \gamma(T_{n,k}) \le t < \gamma\big(T_{n,k+1}\big),$$
$$Y_{\gamma(T_{n,k+1})}^{(n)} = Y_{\gamma(T_{n,k+1})-}^{(n)} + Z_{n,k+1}\delta_{U_{n,k+1}} \quad \text{if} \gamma(T_{n,k+1}) < \infty,$$

where

$$P(U_{n,k+1} \in \cdot | \tilde{\mathscr{F}}_{k+1}^n) = P(U_{n,k+1} \in \cdot | Y_{\gamma(T_{n,k+1})-}^{(n)}) = G\big(Y_{\gamma(T_{n,k+1})-}^{(n)}, \cdot\big).$$

Note that, conditioning on $\tilde{\mathscr{F}}_k^n$, the random variables $U_{n,k+1}$ and $Z_{n,k+1}$ are independent. It is clear from the definition of $\gamma(t)$ and $Y^{(n)}$ that

$$\gamma(t) = \inf\left\{s\colon \int_0^s \left\| Y_{u-}^{(n)}\right\|_2^2 du > t\right\}.$$

Let $T_n^* \equiv \inf\{t\colon \gamma(t) = \infty\}$. From the construction and (3.3) it follows that $Y^{(n)}_{\gamma(T_n^*)}(1) = Y^{(n)}_\infty(1) = 0$. On the other hand, we can use our construction and (3.2) to see that

$$\left\langle Y^{(n)}_{\gamma(t)}, 1\right\rangle = A_t^{(n)} - b_n t, \qquad t \le T_n^*. \tag{3.12}$$

From (3.9) we obtain that $E\left[Z_{n,i}\right]\kappa_n = b_n$ and it follows easily that $b_n t$ is the compensator of $A_t^{(n)}$. Also $Y_t^{(n)}(1) > 0$ for all $t > 0$. This, together with (3.12), yields that, in fact,

$$T_n^* = \inf\left\{t\colon A_t^{(n)} - b_n t = 0\right\}, \tag{3.13}$$

$$P(T_n^* < \infty) = 1. \tag{3.14}$$

Define the point process ([4], I.9)

$$p^{(n)}\colon D_{p^{(n)}} \subset (0,\infty) \mapsto \mathbb{R}_+ \times \mathbb{R},$$

with countable domain $D_{p^{(n)}}$ given by

$$D_{p^{(n)}} \equiv \left\{T_{n,1}, T_{n,2}, \dots, T_{n,k}, \dots\right\} \cap [0, T_n^*]$$

and

$$p^{(n)}\left(T_{n,k}\right) = \left(Z_{n,k}, U_{n,k}\right), \qquad \forall k \ge 1,$$

where $(Z_{n,k}, U_{n,k})$ are defined above.

The corresponding counting measure is defined as

$$N^{(n)}(t,B) = N^{(n)}((0,t]\times B) \equiv \#\left\{s \in D_{p^{(n)}}; s \le t,\ p^{(n)}(s) \in B\right\}$$
$$\forall t > 0,\ \forall B \in \mathscr{B}(\mathbb{R}_+ \times \mathbb{R}).$$

Let $\mathscr{F}_t^n \equiv \bigcap_{\varepsilon>0} \sigma\{N^{(n)}(s,B);\ s \le t+\varepsilon,\ B \in \mathscr{B}(\mathbb{R}_+\times\mathbb{R})\}$ and recall that

$$\eta = \frac{2\gamma(2\gamma-1)}{\Gamma(2-2\gamma)}.$$

LEMMA 3.2. *The compensator of $N^{(n)}$ is*

$$\hat{N}^{(n)}(t, B_1\times B_2) = \eta \int_0^{t\wedge T_n^*} \frac{\int_{B_1}\int_{B_2} Y^{(n)}_{\gamma(s)-}(x)^2 1(\lambda > n^{-1})\lambda^{-2\gamma-1}\,dx\,d\lambda}{\left\|Y^{(n)}_{\gamma(s)-}\right\|_2^2}\,ds.$$

PROOF. For all $B \in \mathscr{B}(\mathbb{R}_+\times\mathbb{R})$ we have

$$E\left[N^{(n)}(t,B)\right] \le E\left[N^{(n)}(t,\mathbb{R}_+\times\mathbb{R})\right] = \kappa_n t < \infty, \tag{3.15}$$

which means that $N^{(n)}(t, B_1 \times B_2)$ is an adapted integrable increasing process. By the Doob–Meyer theorem, the compensator $\hat{N}^{(n)}(t, B_1 \times B_2)$ of this process exists, and it may be found as (see [2], Chapter 2 , proof of Theorem 5.1)

$$\begin{aligned}&\hat{N}^{(n)}(t, B_1 \times B_2)\\ &\quad = P\text{-}\lim_{\delta\downarrow 0} \int_0^t \frac{1}{\delta} E\big[N^{(n)}(s+\delta, B_1 \times B_2) - N^{(n)}(s, B_1 \times B_2) \,\big|\, \mathscr{F}_s^n\big]\, ds.\end{aligned} \tag{3.16}$$

Now, use the definition of the point process $p^{(n)}$ to handle the above limit and get the desired result. □

From the previous lemma and the construction of $Y^{(n)}$, it can be verified that $Y^{(n)}$, in fact, satisfies the local martingale problem $\tilde{M}$ from the Introduction. However, we omit the derivation of this fact since we can prove Proposition 2.3 without using it directly.

Define

$$\begin{aligned}\tau(t) &\equiv \tfrac{1}{2}\int_0^t \big\|Y_{s-}^{(n)}\big\|_2^2\, ds,\\ \tilde{\gamma}_k(t) &\equiv \gamma(k) \wedge t \qquad \forall k \geq 1,\\ g(u, y) &\equiv \int_{0+}^u \big(e^{-\lambda y} - 1 + \lambda y\big)\lambda^{-2\gamma-1}\, d\lambda \qquad \forall u, y \geq 0,\end{aligned}$$

and check that

$$\tau\big(\tilde{\gamma}_k(t)\big) = \tau(t) \wedge k \qquad \forall k \geq 1. \tag{3.17}$$

Lemma 3.3. *Assume that $p \geq 2$ and $\nu \in \mathscr{P}_p(C_{\mathrm{rap}}^+)$. Let X be any solution of the martingale problem for M^ν, independent of $Y^{(n)}$. Then*

$$\begin{aligned}&E\big[\exp\big(-\big\langle Y_{\tilde{\gamma}_k(t)}^{(n)}, X_{T-\tilde{\gamma}_k(t)}\big\rangle\big)\big]\\ &\quad = E\big[\exp\big(-\big\langle Y_0^{(n)}, X_T\big\rangle\big)\big]\\ &\qquad - \tfrac{1}{2}E\bigg[\eta \int_0^{\tilde{\gamma}_k(t)} \exp\big(-\big\langle Y_{s-}^{(n)}, X_{T-s}\big\rangle\big)\\ &\qquad\qquad \times \int_{\mathbb{R}} \big(Y_{s-}^{(n)}\big)^2(x) g\big(1/n, X_{T-s}(x)\big)\, dx\, ds\bigg] \qquad \forall 0 \leq t \leq T.\end{aligned} \tag{3.18}$$

Proof. By Lemma 3.1 we obtain that

$$\begin{aligned}&E_X\big[\exp\big(-\big\langle Y_{\gamma(T_{n,k+1})-}^{(n)}, X_{T-\gamma(T_{n,k+1})}\big\rangle\big)\big]\\ &\quad = E_X\big[\exp\big(-\big\langle Y_{\gamma(T_{n,k})}^{(n)}, X_{T-\gamma(T_{n,k})}\big\rangle\big)\big]\\ &\qquad + \tfrac{1}{2}E_X\bigg[\int_{\gamma(T_{n,k})+}^{\gamma(T_{n,k+1})} \exp\big(-\big\langle Y_{s-}^{(n)}, X_{T-s}\big\rangle\big)\\ &\qquad\qquad \times \big(\big\langle b_n(Y_{s-}^{(n)})^2, X_{T-s}\big\rangle - \big\langle (Y_{s-}^{(n)})^2, X_{T-s}^{2\gamma}\big\rangle\big)\, ds\bigg]\end{aligned}$$

for each $0 \le \gamma(T_{n,k+1}) \le T$. ($X_{\cdot}$ is continuous; hence $X_{u-} = X_u$ for all $u > 0$.) This together with the definition of $Y^{(n)}$ implies that

$$
\begin{aligned}
E_X&\big[\exp(-\langle Y_t^{(n)}, X_{T-t}\rangle)\big]\\
&= E_X\big[\exp(-\langle Y_0^{(n)}, X_T\rangle)\big]\\
&\quad + \tfrac{1}{2} E_X\bigg[\int_0^t \exp(-\langle Y_{s-}^{(n)}, X_{T-s}\rangle)\big(\langle b_n (Y_{s-}^{(n)})^2, X_{T-s}\rangle\\
&\qquad\qquad - \langle (Y_{s-}^{(n)})^2, X_{T-s}^{2\gamma}\rangle\big)\, ds\bigg]\\
&\quad + \int_0^{\tau(t)} \int_{\mathbb{R}} \int_{\mathbb{R}_+} E_X\big[\exp(-\langle Y_{\gamma(s)-}^{(n)}, X_{T-\gamma(s)}\rangle)\big]\big(\exp(-\lambda X_{T-\gamma(s)}(x)) - 1\big)\big]\\
&\qquad\qquad \times N^{(n)}(d\lambda\, dx\, ds)
\end{aligned}
\tag{3.19}
$$

for $0 \le t \le T$. Note that

$$
\begin{aligned}
y^{2\gamma} &= \eta \int_{0+}^{\infty} \big(e^{-\lambda y} - 1 + \lambda y\big)\, \lambda^{-2\gamma-1}\, d\lambda\\
&= \eta g(1/n, y) + \eta \int_{1/n}^{\infty} \big(e^{-\lambda y} - 1\big)\, \lambda^{-2\gamma-1}\, d\lambda + b_n y \qquad \forall y \ge 0.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
E_X&\big[\exp(-\langle Y_t^{(n)}, X_{T-t}\rangle)\big]\\
&= E_X\big[\exp(-\langle Y_0^{(n)}, X_T\rangle)\big]\\
&\quad - \tfrac{1}{2} E_X\bigg[\eta \int_0^t \exp(-\langle Y_{s-}^{(n)}, X_{T-s}\rangle) \int_{\mathbb{R}} \big(Y_{s-}^{(n)}\big)^2(x) g(1/n, X_{T-s}(x))\, dx\, ds\bigg]\\
&\quad + \int_0^{\tau(t)} \int_{\mathbb{R}} \int_{\mathbb{R}_+} E_X\big[\exp(-\langle Y_{\gamma(s)-}^{(n)}, X_{T-\gamma(s)}\rangle)\big]\big(\exp(-\lambda X_{T-\gamma(s)}(x)) - 1\big)\big]\\
&\qquad\qquad \times N^{(n)}(d\lambda\, dx\, ds)\\
&\quad - \tfrac{1}{2}\eta \int_0^t E_X\bigg[\exp(-\langle Y_{s-}^{(n)}, X_{T-s}\rangle)\\
&\qquad\qquad \times \int_{\mathbb{R}} \big(Y_{s-}^{(n)}\big)^2(x) \int_{1/n}^{\infty} \big(\exp(-\lambda X_{T-s}(x)) - 1\big)\lambda^{-2\gamma-1}\, d\lambda\, dx\bigg]\, ds.
\end{aligned}
\tag{3.20}
$$

It is easy to check (see, e.g., Exercise 12 in [2], Chapter 6) that

$$
\begin{aligned}
\frac{1}{2}\eta \int_0^t E_X&\bigg[\exp(-\langle Y_{s-}^{(n)}, X_{T-s}\rangle)\\
&\times \int_{\mathbb{R}} \big(Y_{s-}^{(n)}\big)^2(x) \int_{1/n}^{\infty} \big(\exp(-\lambda X_{T-s}(x)) - 1\big)\lambda^{-2\gamma-1}\, d\lambda\, dx\bigg]\, ds
\end{aligned}
$$

$$
\begin{aligned}
&= \eta \int_0^{\tau(t)} E_X\bigg[\exp\big(-\langle Y^{(n)}_{\gamma(s)-}, X_{T-\gamma(s)}\rangle\big) \int_{\mathbb{R}} \frac{1}{\|Y^{(n)}_{\gamma(s)-}\|_2^2} \big(Y^{(n)}_{\gamma(s)-}\big)^2(x) \\
&\qquad \times \int_{1/n}^{\infty} \big(\exp\big(-\lambda X_{T-\gamma(s)}(x)\big) - 1\big) \lambda^{-2\gamma-1}\, d\lambda\, dx\bigg]\, ds \\
&= \int_0^{\tau(t)} \int_{\mathbb{R}} \int_{\mathbb{R}_+} E_X\big[\exp\big(-\langle Y^{(n)}_{\gamma(s)-}, X_{T-\gamma(s)}\rangle\big) \\
&\qquad \times \big(\exp\big(-\lambda X_{T-\gamma(s)}(x)\big) - 1\big)\big] \hat{N}^{(n)}(d\lambda\, dx\, ds).
\end{aligned}
$$

Let

$$
\begin{aligned}
M_t \equiv & \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}_+} E_X\big[\exp\big(-\langle Y^{(n)}_{\gamma(s)-}, X_{(T-\gamma(s))\vee 0}\rangle\big) \\
&\qquad \times \big(\exp\big(-\lambda X_{(T-\gamma(s))\vee 0}(x)\big) - 1\big)\big] N^{(n)}(d\lambda\, dx\, ds) \\
& - \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}_+} E_X\big[\exp\big(-\langle Y^{(n)}_{\gamma(s)-}, X_{(T-\gamma(s))\vee 0}\rangle\big) \\
&\qquad \times \big(\exp\big(-\lambda X_{(T-\gamma(s))\vee 0}(x)\big) - 1\big)\big] \hat{N}^{(n)}(d\lambda\, dx\, ds).
\end{aligned}
$$

Then M_t is an $\mathscr{F}_t^n$-martingale (see [4], Chapter 2.3). Since $\tau(t)$ is not a bounded stopping time, we use truncation arguments. The definition of $\tilde{\gamma}_k(t)$ together with (3.17) and (3.20) implies that

$$
\begin{aligned}
& E_X\big[\exp\big(-\langle Y^{(n)}_{\tilde{\gamma}_k(t)}, X_{T-\tilde{\gamma}_k(t)}\rangle\big)\big] \\
&\quad = E_X\big[\exp\big(-\langle Y^{(n)}_0, X_T\rangle\big)\big] \\
&\quad\quad - E_X\bigg[\eta \int_0^{\tilde{\gamma}_k(t)} \exp\big(-\langle Y^{(n)}_{s-}, X_{T-s}\rangle\big) \\
&\quad\quad\quad \times \int_{\mathbb{R}} \big(Y^{(n)}_{s-}\big)^2(x) g\big(1/n, X_{T-s}(x)\big)\, dx\, ds\bigg] \\
&\quad\quad + M_{\tau(t)\wedge k}.
\end{aligned}
\tag{3.21}
$$

Whereas $\tau(t) \wedge k$ is a bounded stopping time, the optional sampling theorem implies that $M_{\tau(t)\wedge k}$ is an $\mathscr{F}^n_{\tau(t)\wedge k}$ martingale. Taking the expectation of both sides of (3.21), we get the desired result. □

LEMMA 3.4. *Let X be as in Lemma 3.3. Then*

$$
\lim_{n\to\infty} \big| E\big[\exp\big(-\langle Y^{(n)}_{\tilde{\gamma}_{k_n}(t)}, X_{T-\tilde{\gamma}_{k_n}(t)}\rangle\big)\big] - E_X\big[\exp\big(-\langle Y^{(n)}_0, X_T\rangle\big)\big]\big| = 0 \qquad \forall 0 \le t \le T,
\tag{3.22}
$$

where $k_n \equiv \ln n$.

PROOF. Let C_T denote a constant whose value depends on T and X_0. In the following text, C_T may change from line to line. Having in mind the simple inequality

$$0 \le e^{-\lambda} - 1 + \lambda \le \frac{\lambda^2}{2} \qquad \forall \lambda \ge 0,$$

we get

$$\begin{aligned}
&\biggl|E\biggl[\eta \int_0^{\tilde{\gamma}_k(t)} \exp\bigl(-\langle Y_{s-}^{(n)}, X_{T-s}\rangle\bigr) \int_{\mathbb{R}} \bigl(Y_{s-}^{(n)}\bigr)^2(x) \\
&\qquad \times \int_{0+}^{1/n} \bigl(\exp\bigl(-\lambda X_{T-s}(x)\bigr) - 1 + \lambda X_{T-s}(x)\bigr)\lambda^{-2\gamma-1}\, d\lambda\, dx\, ds\biggr]\biggr| \\
&\quad \le \eta \biggl|E\biggl[\int_0^{\tilde{\gamma}_k(t)} \int_{\mathbb{R}} \bigl(Y_{s-}^{(n)}\bigr)^2(x) \tfrac{1}{2} X_{T-s}(x)^2 \int_{0+}^{1/n} \lambda^{1-2\gamma}\, d\lambda\, dx\, ds\biggr]\biggr| \\
&\quad \le C_T E_Y\biggl[\int_0^{\tilde{\gamma}_k(t)} \int_{\mathbb{R}} \bigl(Y_{s-}^{(n)}\bigr)^2(x) n^{2\gamma-2}\, dx\, ds\biggr] \\
&\quad \le C_T k n^{2\gamma-2},
\end{aligned}$$

where the second inequality follows from (2.1), and the third one follows from the definition of $\tilde{\gamma}_k(t)$. We will assume subsequently that $k = k_n = \ln n$. Then we have

$$\begin{aligned}
&\bigl|E\bigl[\exp\bigl[-\langle Y_{\tilde{\gamma}_{k_n}(t)}^{(n)}, X_{T-\tilde{\gamma}_{k_n}(t)}\rangle\bigr)\bigr] - E_X\bigl[\exp\bigl(-\langle Y_0^{(n)}, X_T\rangle\bigr)\bigr]\bigr| \\
&\qquad \le C_T (\ln n) n^{2\gamma-2},
\end{aligned} \tag{3.23}$$

and letting $n \to \infty$, we are done, since $2\gamma - 2 < 0$. $\square$

LEMMA 3.5. *Let X be as in Lemma 3.3. Then*

$$\lim_{n\to\infty} \bigl|E\bigl[\exp\bigl(-\langle Y_{\tilde{\gamma}_{k_n}(t)}^{(n)}, X_0\rangle\bigr)\bigr] - E_X\bigl[\exp\bigl(-\langle \phi, X_t\rangle\bigr)\bigr]\bigr| = 0 \qquad \forall t \ge 0,$$

where $\phi = Y_0^{(n)}$.

PROOF. By (3.22) it is sufficient to show that

$$\lim_{n\to\infty} \bigl|E\bigl[\exp\bigl(-\langle Y_{\tilde{\gamma}_{k_n}(t)}^{(n)}, X_{t-\tilde{\gamma}_{k_n}(t)}\rangle\bigr)\bigr] - E\bigl[\exp\bigl(-\langle Y_{\tilde{\gamma}_{k_n}(t)}^{(n)}, X_0\rangle\bigr)\bigr]\bigr| = 0.$$

Since $\tilde{\gamma}_{k_n}(t) \le t$, it is obvious that

$$\begin{aligned}
&E\bigl[\exp\bigl(-\langle Y_{\tilde{\gamma}_{k_n}(t)}^{(n)}, X_{t-\tilde{\gamma}_{k_n}(t)}\rangle\bigr)\bigr] - E\bigl[\exp -\langle Y_{\tilde{\gamma}_{k_n}(t)}^{(n)}, X_0\rangle\bigr)\bigr] \\
&\qquad = E\bigl[\exp\bigl(-\langle Y_{\tilde{\gamma}_{k_n}(t)}^{(n)}, X_{t-\tilde{\gamma}_{k_n}(t)}\rangle\bigr) - \exp\bigl(-\langle Y_{\tilde{\gamma}_{k_n}(t)}^{(n)}, X_0\rangle\bigr); \tilde{\gamma}_{k_n}(t) < t\bigr].
\end{aligned}$$

Therefore,

$$\lim_{n\to\infty} \bigl|E\bigl(\exp\bigl(-\langle Y_{\tilde{\gamma}_{k_n}(t)}^{(n)}, X_{t-\tilde{\gamma}_{k_n}(t)}\rangle\bigr)\bigr] - E\bigl[\exp\bigl(-\langle Y_{\tilde{\gamma}_{k_n}(t)}^{(n)}, X_0\rangle\bigr)\bigr]\bigr| \le \lim_{n\to\infty} P(\tilde{\gamma}_{k_n}(t) < t).$$

However, $P(\tilde{\gamma}_{k_n}(t) < t) \le P(T_n^* > k_n)$, where, as we remember,

$$T_n^* = \inf\{t\colon A_t^{(n)} - b_n t = 0\}$$

and $P(T_n^* < \infty) = 1$. It is well known that, as n goes to infinity, $A_t^{(n)} - b_n t$ converges weakly to L, a stable process without negative jumps. It is easy to verify that $T_n^* \Rightarrow T^*$, where

$$T^* = \inf\{t\colon L_t = 0\},$$
$$P(T^* < \infty) = 1.$$

This implies that $\{T_n^*, n \ge 1\}$ is a tight set of $\mathbb{R}_+$-valued random variables. Therefore,

$$\lim_{n\to\infty} P(T_n^* > k_n) = 0.$$

This completes the proof of the lemma. □

Proof of Proposition 2.3. Let us define $\tilde{Y}_t^{(n)} \equiv Y_{\tilde{\gamma}_{k_n}(t)}^{(n)}$. Then Lemma 3.5 implies that $\{\tilde{Y}_t^{(n)}, n \ge 1\}$ is the sequence of the processes such that for any solution X to M^ν, independent of $\tilde{Y}_t^{(n)}$, we have

$$\lim_{n\to\infty} \left| E\left[\exp\left(-\langle \tilde{Y}_t^{(n)}, X_0\rangle\right)\right] - E\left[\exp\left(-\langle \phi, X_t\rangle\right)\right]\right| = 0.$$

Since $\nu \in \mathscr{P}_p(C_{\mathrm{rap}}^+)$ and $\phi \in L^1(\mathbb{R})_+$ were arbitrary, we are done. □

Remark 3.6. The proof of Theorem 1.1 is now complete (see Remark 2.5).

Conclusion. We believe the method of proving uniqueness in this case was interesting since it allowed us to avoid the difficulties associated with the nonexistence of high moments for the stable processes. In this paper, we did not deal with the weak convergence result for the dual processes, since our concern was only to prove uniqueness for the specific stochastic partial differential equation. We intend to consider the question of stochastic partial differential equations driven by stable noise in a forthcoming paper.

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