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PALM MEASURE DUALITY AND CONDITIONING IN REGENERATIVE SETS

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For a simple point process Ξ on a suitable topological space, the associated Palm distribution at a point s may be approximated by the conditional distribution, given that Ξ hits a small neighborhood of s. To study the corresponding approximation problem for more general random sets, we develop a general duality theory, which allows the Palm distributions with respect to an associated random measure to be expressed in terms of conditional densities with suitable martingale and continuity properties. The stated approximation property then becomes equivalent to a certain asymptotic relation involving conditional hitting probabilities. As an application, we consider the Palm distributions of regenerative sets with respect to their local time random measures.

1. Introduction. Palm measures were originally devised to deal with problems involving conditioning in point processes [cf. Matthes, Kerstan and Mecke (1978), Daley and Vere-Jones (1988) and Kallenberg (1986)]. In that context, let Ξ be a locally finite random set on a suitable topological space S and introduce the elementary conditional distributions $Q_I = P[\Xi \in \cdot | \Xi \cap I \neq \varnothing]$ where I is a bounded Borel set in S. General results such as Theorem 12.8 of Kallenberg (1986) ensure that, as I shrinks toward a single point $s \in S$, the measures Q_I converge in a suitable average sense to the Palm distribution Q_s at s.

Our present aim is to study the corresponding approximation problem for random sets Ξ that are not necessarily locally finite. Since trivially $Q_I \to Q_{\{s\}}$ when $P\{s \in \Xi\} > 0$, we may restrict our attention to random sets Ξ such that $s \notin \Xi$ a.s. for all $s \in S$. The stated conditions are satisfied (outside the origin) for broad classes of regenerative sets—the archetype being the zero set of Brownian motion—and we shall use such sets as test objects for our general theory. [For an elementary introduction to regenerative sets, see Chapter 19 in Kallenberg (1997).]

The modern definition of Palm distributions is based on random measure theory. Assuming Ξ to be locally finite, we may introduce the associated random counting measure $\xi B = \mathrm{card}(\Xi \cap B)$, where $B \in \mathscr{B}(S)$, the class of Borel sets in S. If the corresponding intensity measure $E\xi$ is σ-finite, the Palm distributions of Ξ with respect to ξ are defined by the formula

$$Q_s A = \frac{E[\xi(ds); \Xi \in A]}{E\xi(ds)}, \qquad s \in S,\ A \in \mathscr{F},$$

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where $E[\xi; A] = \int_A \xi \, dP$ as usual. Here $\mathscr{F}$ denotes the σ-field in the space of closed subsets $F \subset S$ [cf. Chapter 14 in Kallenberg (1997)]. Since Ξ is measurably determined by ξ, we may prefer to consider the Palm distributions of the random measure ξ itself, which are defined in a similar way. Under suitable conditions on S, for example when S is locally compact, second-countable, and Hausdorff (henceforth abbreviated as lcscH), the associated spaces of sets and measures are Borel, which ensures the existence of regular versions of the set functions Q_s.

Though the definition of Palm measures continues to makes sense for any random measure ξ with σ-finite intensity $E\xi$, there is no reason in general why the mentioned approximation property should remain valid. Indeed, one may easily construct two random measures ξ and η with the same support Ξ but with different Palm distributions. If the approximation property holds for ξ, it will then necessarily fail for η. Often in applications, the random closed set Ξ is given, and it becomes a challenging problem to find an associated random measure ξ with appropriate approximation properties. Though there is often a natural candidate for ξ, no general approch to the approximation problem seems to be available, and it may be necessary to examine each case separately.

To provide a convenient framework for such studies, we shall develop a duality theory that applies under weak regularity conditions on ξ. In the simplest setting, let $\mathscr{F}$ be a sub-σ-field in the basic probability space Ω such that the conditional intensity measure $E[\xi|\mathscr{F}]$ is a.s. absolutely continuous with respect to the unconditional version $E\xi$. We may then choose the associated density X to be an $\mathscr{F}$-measurable process on S, in which case the Palm measures P_s on $\mathscr{F}$ may be represented as $P_s = X_s \cdot P$ or, equivalently, as $P_s A \equiv E[X_s; A]$.

Typically the condition $E[\xi|\mathscr{F}] \ll E\xi$ is not fulfilled for the entire σ-field $\hat{\mathscr{F}}$ of interest, so we may have to apply the previous construction to a suitable family of generating sub-σ-fields $\mathscr{F} \subset \hat{\mathscr{F}}$. For each $\mathscr{F}$ we obtain an associated set of Palm distributions $P_s^{\mathscr{F}} = X_s^{\mathscr{F}} P$ on $\mathscr{F}$, and to guarantee the existence of an extension to $\hat{\mathscr{F}}$, we need to impose a consistency constraint on the family $\{P_s^{\mathscr{F}}\}$. The required condition turns out to be equivalent to the martingale property of the density process $X_s^{\mathscr{F}}$, regarded as a function of $\mathscr{F}$ for fixed s. Thus, we see how martingales indexed by directed sets [previously studied extensively, for example by Krickeberg (1956), Kurtz (1980) and Ivanoff, Merzbach and Schiopu-Kratina (1993)] arise naturally in the present context.

With the indicated construction, the Palm measures P_s are still defined only up to the values on an $E\xi$-null set, which leaves us with some freedom to select convenient versions. An especially attractive case is when the densities $X_s^{\mathscr{F}}$ can be chosen to be L^1-continuous in s for fixed $\mathscr{F}$, since the associated Palm measures P_s will then be continuous in total variation on $\mathscr{F}$. Under those conditions, the previously described approximation property becomes equivalent to a certain asymptotic relation involving the conditional hitting probabilities $P[\xi I > 0|\mathscr{F}]$. Thus, the inherent duality enables us to study the basic continuity and approximation properties of Palm distributions in terms of associated properties of the more accessible conditional intensities and hitting probabilities. As a further benefit of the general theory, we shall see how the martingale property required for consistency holds automatically in the continuous case.

The basic principles of duality are developed in Section 3 below, following an auxiliary Section 2 that lists some technical preliminaries. In Sections 4 and 5 we shall see how the general theory applies to an extensive class of regenerative sets. For the latter it is natural to choose ξ as the associated local time random measure, the measure on $\mathbb{R}_+$ induced by the generating subordinator X. To ensure the required absolute continuity, we adopt from Kallenberg (1981) the exceedingly weak condition (C), the requirement that the characteristic functions $\hat{\mu}_t$ of X_t be integrable for all $t > 0$. This implies $\mu_t \equiv P \circ X_t^{-1} \ll \lambda$ for every $t > 0$ with a continuous density p^t, and we get $E\xi = p \cdot \lambda$ with $p = \int_0^\infty p^t \, dt$. (Here λ denotes Lebesgue measure on $\mathbb{R}_+$ and $p \cdot \lambda$ is the measure with λ-density p.)

Now let $\mathscr{X}_a$ denote the σ-field generated by $1_{[0,a]} \cdot \xi$, the restriction of ξ to $[0, a]$. In Proposition 4.5 we show that a.s. $E[\xi|\mathscr{X}_a] \ll \lambda$ on (a, ∞) and we provide an explicit representation of an associated density process M^a with required martingale and continuity properties. By duality we get a corresponding expression for the associated Palm distributions, and in Theorems 5.5 and 5.6 the latter are shown to possess desirable continuity and approximation properties at every continuity point of p.

Primitive versions of the quoted results for regenerative sets were stated more or less explicitly already in Section 4 of Kallenberg (1981). Unfortunately, the earlier proofs are flawed by some rather subtle gaps that are not easily filled. Though our present purpose is quite different: to exhibit the power of the basic duality theory in an important special case, we shall also, as a by-product of our general discussion, supply complete proofs of the basic theorems in the regenerative case. Other applications of interest will be considered elsewhere.

2. Some technical preliminaries.

In order to avoid subsequent interruption by technical details, we have collected in this section some elementary results on measures and densities that will be needed later. The reader might skip to Section 3 and return for reference when required.

The following result shows how an order relation between two absolutely continuous measures may carry over to the associated densities.

LEMMA 2.1. *Consider some σ-finite measures λ and $\mu \le \nu$ on a metric space S such that $\mu = p \cdot \lambda$ and $\nu = q \cdot \lambda$ where p is lower semicontinuous. Then $p \le \|q\|_\infty$ on* $\operatorname{supp} \lambda$.

PROOF. Since $\mu \le \nu$, we have $(q - p) \cdot \lambda \ge 0$, and so $p \le q$ a.e. λ. Fixing any $s \in \operatorname{supp} \lambda$, we may choose some $s_n \to s$ with $p(s_n) \le q(s_n)$ for each n.

Using the semicontinuity of p, we get

$$p(s) \le \liminf_{n\to\infty} p(s_n) \le \liminf_{n\to\infty} q(s_n) \le \|q\|_\infty. \qquad \square$$

The next result gives conditions for L^1-convergence of the integrals of a sequence of random processes.

LEMMA 2.2. *Let $X, X^1, X^2, \dots \ge 0$ be measurable processes on some σ-finite measure space (S, μ) such that $X_s^n \to_P X_s$ for μ-a.e. $s \in S$ and $E\int X^n\,d\mu \to E\int X\,d\mu$. Then $\int X^n\,d\mu \to \int X\,d\mu$ in $L^1(P)$.*

PROOF. Since μ is σ-finite, we may choose some measurable function $f > 0$ on S such that $\mu f < \infty$. By dominated convergence,

$$\int E(|X_s^n - X_s| \wedge 1) f_s\, \mu(ds) \to 0,$$

and so $X^n \to X$ in $P \otimes (f\cdot\mu)$-measure. Applying Lemma 1.32 in Kallenberg (1997) to any a.e. convergent subsequence, we get $X^n \to X$ in $L^1(P\otimes\mu)$. Hence,

$$E\left|\int X^n\,d\mu - \int X\,d\mu\right| \le E\int |X^n - X|\,d\mu \to 0. \qquad \square$$

We proceed to show that, under suitable conditions, the martingale property of an absolutely continuous, measure-valued process carries over to the process of densities.

LEMMA 2.3. *Fix a σ-finite measure μ on some metric space $(S, \mathscr{S})$ and an $x \in \operatorname{supp}\mu$. Let (ξ_t) be a measure-valued martingale on S with induced filtration $\mathscr{F}$ such that $\xi_t = M^t\cdot\mu$ a.s. for each t, where M^t is $\mathscr{F}_t\otimes\mathscr{S}$-measurable and L^1-continuous at x. Then M_x^t is a martingale in t.*

PROOF. Fix any times $s \le t$, and let $A \in \mathscr{F}_s$ and $B \in \mathscr{S}$. By Fubini's theorem and the martingale property of ξ, we have

$$\begin{aligned}
\int_B E[M_y^s; A]\,\mu(dy) &= E\left[\int_B M_y^s\,\mu(dy);\ A\right] \\
&= E[\xi_s B; A] = E[\xi_t B; A] \\
&= E\left[\int_B M_y^t\,\mu(dy);\ A\right] \\
&= \int_B E[M_y^t; A]\,\mu(dy).
\end{aligned}$$

Here B is arbitrary, and so $E[M_y^s; A] = E[M_y^t; A]$ for $y \in S$ a.e. μ. Since $x \in \operatorname{supp}\mu$, we may choose some $x_n \to x$ with $E[M_{x_n}^s; A] = E[M_{x_n}^t; A]$ for all n. By the L^1-continuity on each side, the relation extends to $E[M_x^s; A] = E[M_x^t; A]$, and since A is arbitrary we get $M_x^s = E[M_x^t|\mathscr{F}_s]$ a.s. $\square$

The following result gives the uniqueness of the density. Recall that L^0 denotes the space of finite random variables with the topology of convergence in probability.

LEMMA 2.4. *Fix a σ-finite measure μ on some metric space S and let X and Y be measurable and L^0-continuous processes on S with $X \cdot \mu = Y \cdot \mu$ a.s. Then $X_s = Y_s$ a.s. for all $s \in \operatorname{supp} \mu$.*

PROOF. Since $(X - Y) \cdot \mu = 0$ a.s., we have

$$\mu\{s \in S;\ X_s \neq Y_s\} = 0 \quad \text{a.s.},$$

and since X and Y are measurable, Fubini's theorem yields $X_s = Y_s$ a.s. for μ-a.e. $s \in S$. Fixing any $s \in \operatorname{supp} \mu$, we may choose some $s_n \to s$ with $X_{s_n} = Y_{s_n}$ a.s. for all n. Then $X_{s_n} \to_P X_s$ and $Y_{s_n} \to_P Y_s$, and so $X_s = Y_s$ a.s. $\square$

The next result expresses the total variation of a signed measure $Q = \alpha \cdot P$ in terms of the density α. Here $\|\mu\|_{\mathscr{F}}$ denotes the total variation of μ on the σ-field $\mathscr{F}$. Note that $\|\mu\|_{\mathscr{F}} = \sup_f |\mu f|$, where the supremum extends over all $\mathscr{F}$-measurable functions f with $|f| \leq 1$.

LEMMA 2.5. *If $Q = \alpha \cdot P$ with $\alpha \in L^1$, then for any sub-σ-field $\mathscr{F}$ in Ω,*

$$\|Q\|_{\mathscr{F}} = E|E[\alpha|\mathscr{F}]|.$$

PROOF. For any $\mathscr{F}$-measurable random variable ξ with $|\xi| \leq 1$, we have

$$\left|\int \xi\, dQ\right| = |E(\xi\alpha)| = |E(\xi\, E[\alpha|\mathscr{F}])| \leq E|E[\alpha|\mathscr{F}]|,$$

with equality when $\xi = \operatorname{sgn} E[\alpha|\mathscr{F}]$. $\square$

We conclude with a simple estimate of the total variation distance between two product measures.

LEMMA 2.6. *Let $\mu_1, \ldots, \mu_n$ and $\nu_1, \ldots, \nu_n$ be probability measures on some measurable spaces $S_1, \ldots, S_n$. Then*

$$\left\| \bigotimes_k \mu_k - \bigotimes_k \nu_k \right\| \leq \sum_k \|\mu_k - \nu_k\|.$$

PROOF. For any probability measures μ, ν on S and μ', ν' on T, we have

$$\begin{aligned} \|\mu \otimes \mu' - \nu \otimes \nu'\| &\leq \|(\mu - \nu) \otimes \mu'\| + \|\nu \otimes (\mu' - \nu')\| \\ &= \|\mu - \nu\| + \|\mu' - \nu'\|. \end{aligned}$$

Now continue by induction. $\square$

3. General duality theory. We are now ready to develop the general duality theory outlined in Section 1. The basic duality relation between Palm measures and conditional intensity measures is displayed in Proposition 3.1. In Proposition 3.2 we show how the fundamental continuity and approximation properties of Palm measures are equivalent to certain continuity and convergence properties of the corresponding density processes. Proposition 3.3 shows how the consistency of a family of Palm measures translates into a martingale property for the associated family of density processes; in addition, it provides the previously advertised relationship between continuity and martingale properties. The extension of a consistent family of Palm measures is examined in Proposition 3.4. Finally, a general criterion for tightness and approximation in the vague topology is provided in Proposition 3.5.

To introduce the basic notation, let ξ be a random measure with σ-finite intensity measure $E\xi$, defined on some measurable space $(S, \mathscr{S})$. For any random element η in some measurable space $(T, \mathscr{T})$, we define the *Palm probabilities* $Q_s(C) = Q_s C$ as the Radon–Nikodym derivatives

$$Q_s C = \frac{E[\xi(ds);\ \eta \in C]}{E\xi(ds)}, \qquad s \in S,\ C \in \mathscr{T}.$$

If the space T is Borel, we may choose the function $(s, C) \mapsto Q_s C$ to form a probability kernel (Q_s) from S to T, in which case we refer to the individual measures Q_s as *Palm distributions* of η with respect to ξ. In particular, we may fix a suitable sub-σ-field $\mathscr{F}$ in the basic probability space $(\Omega, \mathscr{A}, P)$ and let η be the identity mapping from $(\Omega, \mathscr{A})$ to $(\Omega, \mathscr{F})$. In that case, we may write P_s instead of Q_s and refer to (P_s) as the set of Palm distributions on $\mathscr{F}$ with respect to ξ.

For any ξ and $\mathscr{F}$ as above, we shall also consider the set of conditional expectations $E[\xi B|\mathscr{F}]$, $B \in \mathscr{S}$. If even S is Borel, we may choose the function $B \mapsto E[\xi B|\mathscr{F}]$ to be a random measure on S, the so-called *conditional intensity* $E[\xi|\mathscr{F}]$ of ξ given $\mathscr{F}$.

The following basic duality relationship between Palm measures and conditional intensities is suggested by the componentwise disintegrations of the so-called *Campbell measure* $E[\xi B;\ \eta \in C]$ on $S \times T$.

PROPOSITION 3.1. *Fix two Borel spaces $(S, \mathscr{S})$ and $(T, \mathscr{T})$, let ξ be a random measure on S with σ-finite intensity $E\xi$, and let η be a random element in T. Then the Palm distributions Q_s of η with respect to ξ and the conditional intensity $E[\xi|\eta]$ both exist, and the following two conditions are equivalent*:

(i) *$E[\xi|\eta] \ll E\xi$ on $\mathscr{S}$ a.s. P;*
(ii) *$Q_s \ll P \circ \eta^{-1}$ on $\mathscr{T}$ a.e. $E\xi$.*

Under (i) *and* (ii), *the Palm measures P_s exist on $\mathscr{F} = \sigma\{\eta\}$, and we have*:

(iii) *$P_s \ll P$ on $\mathscr{F}$ a.e. $E\xi$.*

Furthermore, product-measurable densities exist in all three cases, and any such density in (i) *or* (iii) *is a density for both.*

PROOF. Since S and T are Borel, the existence of the Palm distributions Q_s and the conditional intensity $E[\xi|\eta]$ may be proved as in Theorem 5.3 of Kallenberg (1997). Since $\mathscr{S}$, $\mathscr{T}$, and $\mathscr{F}$ are all countably generated, it is also clear from Theorem V.58 in Dellacherie and Meyer (1980) that product-measurable densities exist in all three cases.

Now assume (ii) with an $\mathscr{S}\otimes\mathscr{T}$-measurable density f. Define a kernel (P_s) from $(S,\mathscr{S})$ to $(\Omega,\mathscr{F})$ by

$$P_s A = E[f(s,\eta);A], \qquad s\in S,\ A\in\mathscr{F},$$

and note that (iii) holds with density $f(s,\eta)$. Fix any $A\in\mathscr{F}$ and $B\in\mathscr{S}$, and choose a $C\in\mathscr{T}$ with $A=\eta^{-1}C$. Using Fubini's theorem and the definitions of P_s and Q_s, we get

$$\begin{aligned}\int_B (P_sA)\,E\xi(ds) &= \int_B E[f(s,\eta);\ \eta\in C]\,E\xi(ds)\\ &= \int_B E\xi(ds)\int_C f(s,t)\,P\circ\eta^{-1}(dt)\\ &= \int_B (Q_sC)\,E\xi(ds)\\ &= E[\xi B;\ \eta\in C] = E[\xi B; A].\end{aligned}$$

Thus, the P_s are Palm measures on $\mathscr{F}$ with respect to ξ.

Next assume (iii) with an $\mathscr{F}\otimes\mathscr{S}$-measurable density X. Letting $A\in\mathscr{F}$ and $B\in\mathscr{S}$, we get by Fubini's theorem and the definitions of P_s and $E[\xi|\eta]$

$$\begin{aligned}E[E[\xi B|\eta];A] = E[\xi B;A] &= \int_B (P_sA)\,E\xi(ds)\\ &= \int_B E[X_s;A]\,E\xi(ds)\\ &= E\left[\int_B X_s E\xi(ds);A\right].\end{aligned}$$

Since A is arbitrary and both integrands are $\mathscr{F}$-measurable, we get

$$E[\xi B|\eta] = \int_B X_s E\xi(ds) \text{ a.s., } B\in\mathscr{S},$$

and S being Borel, we may choose the exceptional null set to be independent of B. Thus, (i) holds with the same density X.

Conversely, assume (i) with an $\mathscr{F}\otimes\mathscr{S}$-measurable density X. Define a kernel (P_s) from $(S,\mathscr{S})$ to $(\Omega,\mathscr{F})$ by

$$P_s A = E[X_s;A], \qquad s\in S,\ A\in\mathscr{F},$$

and note that (iii) holds with density X. Using Fubini's theorem and the definitions of X, P_s and $E[\xi|\eta]$, we get for any $A \in \mathscr{F}$ and $B \in \mathscr{S}$,

$$\begin{aligned}\int_B (P_s A)\, E\xi(ds) &= \int_B E[X_s; A]\, E\xi(ds) \\ &= E\left[\int_B X_s E\xi(ds); A\right] \\ &= E[E[\xi B|\eta]; A] = E[\xi B; A].\end{aligned}$$

Thus, the P_s are Palm measures on $\mathscr{F}$ with respect to ξ. To deduce (ii), we may write

$$Q_s B = P_s\{\eta \in B\} = E[X_s;\, \eta \in B], \qquad s \in S,\ B \in \mathscr{T},$$

and note that the right-hand side vanishes when $P\{\eta \in B\} = 0$. □

Now assume that S is lcscH. In the context of Proposition 3.1, we shall establish a basic relationship between continuity properties of the Palm measures and the conditional density process. Write $\|\mu\|_{\mathscr{F}}$ for the total variation of the signed measure μ on the σ-field $\mathscr{F}$ and denote the corresponding convergence $\|\mu_n - \mu\|_{\mathscr{F}} \to 0$ by $\mu_n \to_{tv} \mu$. For any sets $I \in \mathscr{S}$ with $E\xi I \in (0, \infty)$, we define

$$P_I = P[\,\cdot\,|\xi I > 0], \qquad \rho_I = \frac{P[\xi I > 0|\mathscr{F}]}{P\{\xi I > 0\}},$$

$$P'_I = \frac{E[\xi I;\,\cdot\,]}{E\xi I}, \qquad \rho'_I = \frac{E[\xi I|\mathscr{F}]}{E\xi I}.$$

By $I \downarrow s$ we mean that $s \in I$ and that I is ultimately contained in any fixed neighborhood of s.

PROPOSITION 3.2. *Let $P_s = X_s \cdot P$ on $\mathscr{F}$ a.e. $E\xi$, where X is $\mathscr{F} \otimes \mathscr{S}$-measurable with $EX \equiv 1$. Then for any points $r, s \in S$ and sets $I \in \mathscr{S}$ with $E\xi I \in (0, \infty)$, we have*:

(i) *$P_r \to_{tv} P_s$ on $\mathscr{F}$ iff $X_r \to_P X_s$;*
(ii) *$P_I \to_{tv} P_s$ on $\mathscr{F}$ iff $\rho_I \to_P X_s$;*
(iii) *if X is L^0-continuous at s, then $P'_I \to_{tv} P_s$ on $\mathscr{F}$ and $\rho'_I \to_P X_s$ as $I \downarrow s$.*

PROOF. (i) By Lemma 2.5 we have

$$\|P_r - P_s\|_{\mathscr{F}} = E|X_r - X_s|, \qquad r, s \in S.$$

Furthermore, $E|X_r - X_s| \to 0$ iff $X_r \to_P X_s$ since $EX_r = EX_s = 1$.

(ii) Noting that

$$P[A;\, \xi I > 0] = E[P[\xi I > 0|\mathscr{F}]; A], \qquad A \in \mathscr{F},$$

we get $P_I = \rho_I \cdot P$ on $\mathscr{F}$. Hence, Lemma 2.5 yields

$$\|P_I - P_s\|_{\mathscr{F}} = E|\rho_I - X_s|, \qquad s \in S,\ I \in \mathscr{S} \text{ with } E\xi I \in (0,\infty).$$

Since $E\rho_I = EX_s = 1$, the assertion follows as before.

(iii) Writing

$$E[\xi I; A] = E[E[\xi I|\mathscr{F}]; A], \qquad A \in \mathscr{F},$$

we get $P'_I = \rho'_I \cdot P$ on $\mathscr{F}$, and so as before

$$\|P'_I - P_s\|_{\mathscr{F}} = E|\rho'_I - X_s|, \qquad s \in S,\ I \in \mathscr{S} \text{ with } E\xi I \in (0,\infty).$$

Since $E\rho'_I = EX_s = 1$, we conclude that $P'_I \to_{tv} P_s$ on $\mathscr{F}$ iff $\rho'_I \to_P X_s$. Now Proposition 3.1 yields $E[\xi|\mathscr{F}] = X \cdot E\xi$, and hence

$$\rho'_I E\xi I = E[\xi I|\mathscr{F}] = \int_I X_r E\xi(dr).$$

Therefore,

$$E|\rho'_I - X_s| \le (E\xi I)^{-1} \int_I E|X_r - X_s|\, E\xi(dr) \le \sup_{r\in I} E|X_r - X_s|.$$

If X is L^0-continuous at s, then the right-hand side tends to 0 as $I \downarrow s$, and we get $E|\rho'_I - X_s| \to 0$. $\square$

The last result shows that, if X is L^0-continuous at some point $s \in \operatorname{supp} E\xi$, then the associated Palm measure P_s is well defined by continuity and may be approximated as $I \downarrow s$ by the probabilities P'_I. We may even approximate by the elementary conditional distributions P_I, provided that $\rho_I - \rho'_I \to_P 0$. It is suggestive to write the latter condition as

$$\frac{P[\xi I > 0|\mathscr{F}]}{E[\xi I|\mathscr{F}]} \sim_P \frac{P\{\xi I > 0\}}{E\xi I},$$

where $\sim_P$ denotes asymptotic equivalence in the sense of convergence in probability. This shows that the problems of approximating Palm measures and conditional hitting probabilities are essentially equivalent.

To examine the dependence on the σ-field $\mathscr{F}$, fix a measurable space $(\Omega, \mathscr{A})$ and a class $\mathscr{C}$ of sub-σ-fields $\mathscr{F} \subset \mathscr{A}$, and consider for each $\mathscr{F} \in \mathscr{C}$ a probability measure $\mu_{\mathscr{F}}$ on $\mathscr{F}$. Say that the measures $\mu_{\mathscr{F}}$ are *consistent* if

$$\mu_{\mathscr{F}} A = \mu_{\mathscr{G}} A, \qquad A \in \mathscr{F} \cap \mathscr{G},\ \mathscr{F},\ \mathscr{G} \in \mathscr{C}.$$

Writing $\hat{\mathscr{C}}$ for the class of sub-σ-fields $\mathscr{G} \subset \mathscr{F} \in \mathscr{C}$, it is clear that the collection $\{\mu_{\mathscr{F}}\}$ extends uniquely to a consistent family indexed by $\hat{\mathscr{C}}$. Without loss of generality, we may then assume that $\mathscr{C}$ is *ideal* in the sense that $\mathscr{G} \subset \mathscr{F} \in \mathscr{C}$ implies $\mathscr{G} \in \mathscr{C}$. In that case, we define a *martingale* on $\mathscr{C}$ as a process $M^{\mathscr{F}} \in L^1(\mathscr{F})$, $\mathscr{F} \in \mathscr{C}$, with

$$M^{\mathscr{F}} = E[M^{\mathscr{G}}|\mathscr{F}], \qquad \mathscr{F} \subset \mathscr{G} \text{ in } \mathscr{C}.$$

In the context of Palm measures, we may relate the consistency to suitable martingale and continuity properties.

PROPOSITION 3.3. *Fix a random measure ξ on S with σ-finite intensity $E\xi$ and an ideal class $\mathscr{C}$ of sub-σ-fields $\mathscr{F} \subset \mathscr{A}$. For each $\mathscr{F} \in \mathscr{C}$, consider an $\mathscr{F} \otimes \mathscr{S}$-measurable process $(M_s^{\mathscr{F}})$ such that the measures $P_s^{\mathscr{F}} = M_s^{\mathscr{F}} \cdot P$ are Palm distributions on $\mathscr{F}$ with respect to ξ. Then for fixed $s \in S$, the family $\{P_s^{\mathscr{F}}\}$ is consistent iff $M_s^{\mathscr{F}}$ is a martingale in $\mathscr{F}$. This holds in particular if $s \in \operatorname{supp} E\xi$ and $M^{\mathscr{F}}$ is L^1-continuous at s.*

PROOF. For any $A \in \mathscr{F} \subset \mathscr{G}$, we have

$$P_s^{\mathscr{G}} A = E[M_s^{\mathscr{G}}; A] = E[E[M_s^{\mathscr{G}}|\mathscr{F}]; A].$$

Thus, $P_s^{\mathscr{F}} = P_s^{\mathscr{G}}$ on $\mathscr{F}$ iff

$$E[E[M_s^{\mathscr{G}}|\mathscr{F}]; A] = E[M_s^{\mathscr{F}}; A], \qquad A \in \mathscr{F},$$

which is equivalent to the martingale property of M_s. Furthermore, it is seen as in Proposition 3.1 that $E[\xi|\mathscr{F}] = M_s^{\mathscr{F}} \cdot E\xi$ a.s. Since the kernels $E[\xi|\mathscr{F}]$ form a measure-valued martingale, the last assertion follows by Lemma 2.3. $\square$

We may now specialize to the case of Palm distributions Q_s of a random measure ξ with respect to itself. For $s \in I \subset S$ with I open, let Q_s^I denote the Palm distribution of the restriction of ξ to the complement I^c. The duality theory may often be exploited to yield specific versions of the measures Q_s^I, and we need to extend the latter to a set of Palm distributions Q_s on the whole space S. The extension clearly requires the family $\{Q_s^I\}$ to be consistent, which may be verified by means of the criteria in Proposition 3.3.

For precise statements, let $\pi_I \mu$ denote the restriction of the measure μ to the complement I^c. The measures Q_s^I are said to be *consistent* if

$$Q_s^J = Q_s^I \circ \pi_J^{-1}, \qquad s \in I \subset J \subset S. \tag{3.1}$$

In that case, we are looking for versions Q_s of the full Palm distributions satisfying

$$Q_s^I = Q_s \circ \pi_I^{-1}, \qquad s \in I \subset S. \tag{3.2}$$

PROPOSITION 3.4. *Let ξ be an a.s. diffuse random measure on S with locally finite intensity $E\xi$, and consider a consistent family of Palm distributions Q_s^I on I^c, where $s \in I \subset S$ with I open. Then there exist unique extensions of the measures Q_s^I to Palm distributions Q_s on S with $Q_s\{\mu;\, \mu\{s\} = 0\} = 1$.*

PROOF. Fix any $s \in S$ and choose a sequence of neighborhoods $I_n \downarrow s$. By the consistency in (3.1) and the Daniell–Kolmogorov theorem, there exists some probability measure Q_s satisfying (3.2) for all I_n. The general relation in (3.2) then follows by means of (3.1). The value $Q_s\{\mu;\, \mu\{s\} = 0\}$ remains arbitrary and may be chosen as 1, which determines Q_s completely as the distribution of a random measure on S.

Now fix any compact set K and a dissecting system of subsets $I_{nj} \subset K$, as defined in Chapter 10 of Kallenberg (1997). Next choose some open sets

$G_{nj} \supset I_{nj}$ with $G_n(s) \downarrow s$ for each $s \in K$, where $G_n(s)$ denotes the unique set $G_{nj} \supset I_{nj} \ni s$. If $B \in \mathscr{S}$ and $s \in K$, we have $\mu(B \setminus G_n(s)) \uparrow \mu B$ for Q_s-a.e. μ. Hence, for any $B_1, \dots, B_m \in \mathscr{S}$ and $r_1, \dots, r_m \geq 0$,

$$Q_s \bigcap_k \{\mu;\, \mu(B_k \setminus G_n(s)) \leq r_k\} \to Q_s \bigcap_k \{\mu;\, \mu B_k \leq r_k\}.$$

Here the left-hand side is clearly measurable in $s \in K$, and so the same thing is true for the expression on the right. The general measurability now follows by a monotone class argument.

Now fix any versions Q'_s of the Palm distributions of ξ. For any open set $I \subset S$, the Palm distributions on I^c are unique up to an $E\xi$-null set, and so Q'_s satisfies (3.2) for almost every $s \in I$. Here we may choose a common exceptional null set N for all sets I belonging to some countable base $\mathscr{I}$. Given any $s \notin N$, choose $I_1, I_2, \dots \in \mathscr{I}$ with $I_n \downarrow s$ and conclude by a monotone class argument that $Q_s = Q'_s$ on $\{s\}^c$. Moreover, $Q_s\{\mu;\, \mu\{s\} = 0\} = 1$ by definition, and for Q'_s the corresponding value is 1 a.e. $E\xi$ since ξ is diffuse. Thus, $Q_s = Q'_s$ for almost every s, and so even the measures Q_s are Palm distributions. □

Further conditions may be needed to ensure that the measures Q_s be *tight*, in the sense that each Q_s is the distribution of a locally finite random measure. We shall establish tightness along with a related continuity property, stated in terms of the weighted Palm distributions

$$Q_I B = \frac{E[\xi I;\, \xi \in B]}{E\xi I} = (E\xi)^{-1} \int_I (Q_s B)\, E\xi(ds),$$

where $I \subset S$ is measurable with $E\xi I \in (0, \infty)$. Say that the Palm measures Q_s are *mean continuous* at some $s \in \operatorname{supp} E\xi$ if $I \downarrow s$ implies $Q_I \to_w Q_s$ with respect to the vague topology on $\mathscr{M}(S)$, the space of locally finite measures on S. To ensure mean continuity $Q_{I_n} \to_w Q_s$ along a specific sequence $I_n \downarrow s$, we may impose the requirement

$$\lim_{m\to\infty} \limsup_{n\to\infty} \frac{E\, \xi I_n(\xi I_m \wedge 1)}{E\xi I_n} = 0. \tag{3.3}$$

PROPOSITION 3.5. *Let ξ be a diffuse random measure on S with Palm measures Q_s, whose restrictions are given by $Q_s^I = M_s^I \cdot (P \circ \xi^{-1})$ for all $s \in I \subset S$ with I open, where the processes M^I are L^1-continuous. Fix an $s \in \operatorname{supp} E\xi$ with bounded neighborhoods $I_n \downarrow s$ satisfying* (3.3). *Then Q_s is tight and $Q_{I_n} \to_w Q_s$ with respect to the vague topology on $\mathscr{M}(S)$.*

PROOF. By Lemma 2.5 we have $\|Q_{I_n}^I - Q_s^I\| \to 0$ for any neighborhood I of s. In particular, $Q_{I_n} \to_w Q_s$ with respect to the vague topology on $\mathscr{M}(S \setminus \{s\})$. Furthermore, (3.3) yields

$$\lim_{m\to\infty} \limsup_{n\to\infty} \int (\mu I_m \wedge 1)\, Q_{I_n}(d\mu) = 0.$$

We may now conclude, as in Theorem 4.9 in Kallenberg (1986), that $Q_s\{\mu;\, \mu I_n < \infty\} = 1$ and $Q_{I_n} \to_w Q_s$. □

Under the stronger hypothesis

$$\lim_{J\downarrow s} \limsup_{I\downarrow s} \frac{E\,\xi I\,(\xi J \wedge 1)}{E\xi I} = 0,$$

the same argument yields the unrestricted mean continuity at s. The convergence $Q_t \to_w Q_s$ as $t \to s$ requires even stronger conditions.

4. Conditional densities of regenerative sets. We now specialize to the case when Ξ is an unbounded regenerative set with associated local time random measure ξ. Recall that, in this case, there exists an a.s. unique subordinator X such that a.s. $\xi B = \int_0^\infty 1_B(X_t)\,dt$ for all $B \in \mathscr{B}(\mathbb{R}_+)$. In particular, X has the continuous inverse $L_x = \xi[0, x]$. Furthermore, the closure of Ξ agrees a.s. with the support of ξ as well as with the closed range of X. An elementary introduction to regenerative sets and processes appears in Chapter 19 of Kallenberg (1997). For more detailed information, we refer to Maisonneuve (1974) and to Dellacherie, Maisonneuve and Meyer (1987, 1992), Chapters XV and XX.

To avoid trivialities, we assume that X has unbounded Lévy measure ν and vanishing drift component. (Otherwise, Ξ is a renewal process or $P\{x \in \Xi\} > 0$ for all $x \ge 0$, respectively.) We need the fact that

$$P\{x \notin \Xi\} = P\{\sigma_x^- < x < \sigma_x^+\} = 1, \qquad x > 0, \tag{4.1}$$

where

$$\sigma_x^- = \sup(\Xi \cap [0, x]),$$
$$\sigma_x^+ = \inf(\Xi \cap [x, \infty)), \qquad x \ge 0.$$

The result in (4.1) is surprisingly hard in general. An equivalent statement was originally conjectured by K. L. Chung and subsequently proved, independently, by L. Carleson and H. Kesten [cf. Assouad (1971) and Bretagnolle (1971)]. Under condition (C), however, the result is elementary and may be easily deduced from Lemma 5.3 below.

The main result of this section is Proposition 4.5, which provides a conditional density with nice properties. Some auxiliary results are needed for the proof, and we begin with a basic formula that exhibits the joint distribution of σ_x^+ and σ_x^- and provides some pertinent information about the dependence on the past. Say that a process Y on $\mathbb{R}_+$ is *X-predictable* if it is predictable with respect to the right-continuous and complete filtration induced by X [cf. Kallenberg (1997), Chapter 22].

LEMMA 4.1. *Consider an X-predictable process $Y \ge 0$ and a measurable function $f\colon \mathbb{R}_+^2 \to \mathbb{R}_+$. Then for any $x > 0$,*

$$E\,f(\sigma_x^-, \sigma_x^+)Y(L_x) = E\int_0^x Y(L_u)\,\xi(du)\int_{x-u}^\infty f(u, u+v)\,\nu(dv). \tag{4.2}$$

PROOF. Let η denote the point process of jump times and sizes of X and note that η is a Poisson process on $(0, \infty)^2$ with compensator $\hat{\eta} = \lambda \otimes \nu$. First we assume that f is supported by $[0, x) \times (x, \infty)$. Noting that $L(X_s) \equiv s$ and $X_s = X_{s-}$ a.s., we get by dual predictable projection followed by the substitution $X_s = u$,

$$\begin{aligned} E\, f(\sigma_x^-, \sigma_x^+) Y_{L_x} &= E \iint Y_s f(X_{s-}, X_{s-} + v)\, \eta(ds\, dv) \\ &= E \iint Y_s f(X_{s-}, X_{s-} + v)\, ds\, \nu(dv) \\ &= E \iint Y_{L_u} f(u, u + v)\, \nu(dv)\, \xi(du). \end{aligned}$$

By (4.1) we may extend the relation to general f, in the form of (4.2). □

We need a simple application involving the measure-valued process,

$$\xi_t B = \xi([0, X_t] \cap B) = \int_0^t 1_B(X_s)\, ds, \qquad B \in \mathscr{B}(\mathbb{R}_+),\ t \geq 0.$$

LEMMA 4.2.

$$\int_0^x \nu(x - u, \infty)\, E\xi_t(du) = P\{X_t > x\}, \qquad x, t > 0.$$

PROOF. Using Lemma 4.1 with $f \equiv 1$ and $Y_s = 1_{[0,t]}(s)$, we get

$$\begin{aligned} P\{X_t > x\} = P\{X_t \geq x\} &= P\{L_x \leq t\} \\ &= E \int_0^x 1\{L_u \leq t\}\, \nu(x - u, \infty)\, \xi(du) \\ &= \int_0^x \nu(x - u, \infty)\, E\xi_t(du). \end{aligned}$$ □

Now define $\mathscr{X}_a = \sigma(1_{[0,a]}\xi)$. For convenience we record the fact that σ_a^+ and $\mathscr{X}_a$ are conditionally independent given σ_a^-, here suggestively expressed by means of the symbol $\perp\!\!\!\perp$.

LEMMA 4.3. $\sigma_a^+ \underset{\sigma_a^-}{\perp\!\!\!\perp} \mathscr{X}_a, \quad a > 0.$

PROOF. The result is an immediate consequence of the formula

$$P[\sigma_a^+ > b | \mathscr{X}_a] = \frac{\nu(b - \sigma_a^-, \infty)}{\nu(a - \sigma_a^-, \infty)} \text{ a.s., } a < b, \tag{4.3}$$

well known from excursion theory. [See, e.g., XV.88 or XX.70 in Dellacherie, Maisonneuve and Meyer (1987, 1992).] Note that (4.3) can also be derived by routine arguments from Lemma 4.1. □

Next let μ^t denote the distribution of X_t and write $\hat{\mu}^t$ for the corresponding characteristic function. We shall henceforth assume condition (C), the integrability of $\hat{\mu}^t$ for all $t > 0$. Then $\mu^t = p^t \cdot \lambda$ for each $t > 0$, where the densities p^t are continuous on $\mathbb{R}$ and vanish on $\mathbb{R}_-$. Put $p(x) = \int_0^\infty p^t(x)\,dt$ and note that $E\xi = p \cdot \lambda$. The following identity plays a crucial role for our construction of conditional densities and Palm measures.

LEMMA 4.4. $E\,p(x - \sigma_a^+) = p(x), \qquad a < x.$

A knowledgeable referee points out that if $g(x, y)$ is a "good" potential density of a Markov process X, then for any open set G we have

$$E_x g(X_{\tau_G}, y) = g(x, y), \qquad y \in G, \tag{4.4}$$

where $\tau_G = \inf\{t;\ X_t \in G\}$. The asserted relation follows from (4.4) if we take $g(x, y) = p(y - x)$, $G = (a, \infty)$ and $x = 0$. The choice of a good potential density is a classical problem [cf. Blumenthal and Getoor (1968)]. However, it is not clear (even to the expert referee) whether the general results apply here.

PROOF. The relation for $a \le 0$ being trivial, we may assume that $a > 0$. Noting that $X(L_a) = \sigma_a^+$ and using the strong Markov property at L_a together with the disintegration theorem [cf. Kallenberg (1997), Theorem 5.4], we get for any $B \in \mathscr{B}((a, \infty))$ and $t \ge 0$,

$$\begin{aligned} \mu^t B &= P\{X_t \in B\} = P\{X_t \in B,\ L_a \le t\} \\ &= P\{(\theta_{L_a} X)_{t - L_a} \in B - \sigma_a^+;\ L_a \le t\} \\ &= E[\mu^{t - L_a}(B - \sigma_a^+);\ L_a \le t]. \end{aligned}$$

Taking densities of both sides, we obtain for almost every $x > a$,

$$p^t(x) = E[p^{t - L_a}(x - \sigma_a^+);\ L_a \le t]. \tag{4.5}$$

To extend this to an identity, we may use Lemmas 4.1 and 4.2, Fubini's theorem, and the uniform boundedness of $p^s(u)$ for $s \ge t - \varepsilon$ to get

$$\begin{aligned} &E[p^{t - L_a}(x - \sigma_a^+);\ L_a \in (t - \varepsilon, t]] \\ &\quad = \int_{t-\varepsilon}^t ds \int_0^a p^s(u)\,du \int_{a-u}^{x-u} p^{t-s}(x - u - v)\,\nu(dv) \\ &\quad \lesssim \int_0^\varepsilon ds \int_{-\infty}^a du \int_{a-u}^\infty p^s(x - u - v)\,\nu(dv) \\ &\quad = \int_0^\varepsilon ds \int_0^\infty \nu(dv) \int_{a-v}^a p^s(x - u - v)\,du \end{aligned}$$

$$
\begin{aligned}
&= \int_0^\varepsilon ds \int_0^\infty \nu(dv) \int_0^v p^s(x-a-r)\,dr \\
&= \int_0^\varepsilon ds \int_0^{x-a} p^s(x-a-r)\, \nu(r,\infty)\,dr \\
&= \int_0^\varepsilon ds \int_0^{x-a} \nu(x-a-r,\,\infty) p^s(r)\,dr \\
&= \int_0^{x-a} \nu(x-a-r,\,\infty)\, E\xi_\varepsilon(dr) = P\{X_\varepsilon > x-a\},
\end{aligned}
$$

where $\underset{\frown}{<}$ denotes boundedness up to a constant factor. As $\varepsilon \to 0$, the right-hand side tends to 0, uniformly for $x > a$ bounded away from a, and so

$$
E[p^{t-L_a}(x-\sigma_a^+);\ L_a \le t-\varepsilon] \to E[p^{t-L_a}(x-\sigma_a^+);\ L_a \le t], \tag{4.6}
$$

uniformly on compacts in (a,∞). Since the family $\{p^s;\ s \ge \varepsilon\}$ is uniformly equicontinuous, the left-hand side of (4.6) is continuous in x, and the continuity extends to the limit because of the uniformity of the convergence. Thus, both sides of (4.5) are continuous, and the relation holds identically. Integrating with respect to t and using Fubini's theorem, we get

$$
\begin{aligned}
p(x) = \int_0^\infty p^t(x)\,dt &= \int_0^\infty E[p^{t-L_a}(x-\sigma_a^+);\ L_a \le t]\,dt \\
&= E\int_{L_a}^\infty p^{t-L_a}(x-\sigma_a^+)\,dt = E\,p(x-\sigma_a^+).
\end{aligned}
$$

□

We now introduce the processes

$$
M_x^a = E[p(x-\sigma_a^+)|\mathscr{X}_a], \qquad 0 \le a \le x. \tag{4.7}
$$

To ensure product measurability for each a, we may assume that all conditional expectations are computed from a common regular conditional distribution $P[\sigma_a^+ \in \cdot|\mathscr{X}_a]$. The following result clarifies the role of the processes M^a as conditional densities and provides some basic martingale and continuity properties.

PROPOSITION 4.5. *If the processes M^a in* (4.7) *are product-measurable, then*:

(i) $E[\xi|\mathscr{X}_a] = M^a \cdot \lambda$ *a.s. on* (a,∞) *for all* $a \ge 0$.
(ii) *Each M^a is L^1-continuous at all continuity points $x > a$ of p.*
(iii) *M_x^a is a martingale in $a < x$ for each $x > 0$ with $p_x < \infty$.*

PROOF. (i) Let $\mathscr{X}_a^+$ be the σ-field generated by $\mathscr{X}_a$ and σ_a^+. Using the regenerative property at σ_a^+ and the disintegration theorem, we get for any $B \in \mathscr{B}((a,\infty))$,

$$
E[\xi B|\mathscr{X}_a^+] = E[(\theta_{\sigma_a^+}\xi)(B-\sigma_a^+)|\mathscr{X}_a^+] = (E\xi)(B-\sigma_a^+).
$$

Hence, Fubini's theorem yields, for $A \in \mathscr{X}_a$,

$$\begin{aligned} E\left[\int_B M_x^a\,dx;\, A\right] &= \int_B E[M_x^a; A]\,dx \\ &= \int_B E[p(x-\sigma_a^+); A]\,dx \\ &= E\left[\int_B p(x-\sigma_a^+)\,dx;\, A\right] \\ &= E[(E\xi)(B-\sigma_a^+); A] = E[\xi B; A]. \end{aligned}$$

Choosing a kernel version of $E[\xi|\mathscr{X}_a]$, we get a.s.

$$\int_B M_x^a\,dx = E[\xi B|\mathscr{X}_a] = E[\xi|\mathscr{X}_a](B).$$

Since both sides are measure-valued, we may choose the exceptional null set to be independent of B, and the assertion follows.

(ii) By Fubini's theorem,

$$M_x^a = \int_0^\infty E[p^t(x-\sigma_a^+)|\mathscr{X}_a]\,dt, \qquad x > a,$$

and by dominated convergence the integrand is continuous in x for each $t > 0$. Furthermore, by Lemma 4.4,

$$EM_x^a = Ep(x-\sigma_a^+) = p(x), \qquad x > a. \tag{4.8}$$

The assertion now follows by Lemma 2.2.

(iii) If $x > 0$ with $p(x) < \infty$, then (4.8) yields $EM_x^a < \infty$ for all $a < x$. Writing $\tau = \sigma_a^+$ and $\tilde{\xi} = \theta_\tau \xi$, we note that $\tau \perp\!\!\!\perp \tilde{\xi} =_d \xi$ by the regenerative property at τ. Now fix any $a < b < x$ and let $\tilde{\sigma}^+$ be the process σ^+ associated with $\tilde{\xi}$, so that $\sigma_b^+ = \tau + \tilde{\sigma}_{b-\tau}^+$. Using Lemma 4.4, Fubini's theorem, and the regenerative property at τ, we get

$$\begin{aligned} E[p(x-\sigma_b^+)|\mathscr{X}_a^+] &= E[p(x-\sigma_b^+)|\sigma_a^+] \\ &= E[p(x-\tau-\tilde{\sigma}_{b-\tau}^+)|\tau] \\ &= p(x-\sigma_a^+), \end{aligned}$$

and so

$$\begin{aligned} E[M_x^b|\mathscr{X}_a] &= E[p(x-\sigma_b^+)|\mathscr{X}_a] \\ &= E[p(x-\sigma_a^+)|\mathscr{X}_a] = M_x^a. \qquad \square \end{aligned}$$

Since $E\xi = p \cdot \lambda$, we get formally $E[\xi|\mathscr{X}_a] = (M^a/p) \cdot E\xi$. The division is justified by the following result.

LEMMA 4.6. *$p^t(x) > 0$ for all $t, x > 0$.*

PROOF. First we note that $\operatorname{supp} \mu^t = \mathbb{R}_+$ for each $t > 0$. An abstract version of this statement appears in Theorem 6.8 of Kallenberg (1986). To give a direct elementary proof, note that if μ and μ' are measures on $\mathbb{R}_+$ with supports A and B, then $\operatorname{supp}(\mu * \mu') = A + B$. Applying this result to suitable decompositions $\mu^t = \mu_n * \mu_n'$ with $\mu_n' \to \delta_0$, we may reduce the assertion to the case where μ^t is compound Poisson with a characteristic measure ν satisfying $0 \in \operatorname{supp} \nu$. Each μ^t is then the distribution of a finite sum $\xi_1 + \cdots + \xi_\kappa$, where $\xi_1, \xi_2, \ldots$ are i.i.d. ν and κ is an independent Poisson random variable. By the convolution property, $\operatorname{supp} \mu^t$ is then the closed additive semigroup generated by $\operatorname{supp} \nu$ and hence equals $\mathbb{R}_+$.

Now fix any $t, x > 0$ and put $s = t/2$. Since p^s is continuous with dense support in $\mathbb{R}_+$, we have $p^s \geq \varepsilon > 0$ on some interval $[a, b] \subset (0, x)$, and so

$$\begin{aligned} p^t(x) &= \int_0^x p^s(x-u)p^s(u)\,du \geq \varepsilon \int_a^b p^s(x-u)\,du \\ &= \varepsilon\, \mu^s(x-b, x-a) > 0, \end{aligned}$$

where the support property is used again in the last step. □

In view of Proposition 4.5 and the results of Section 3, it is useful to know where the density p is continuous. In Theorem 5.2 of Kallenberg (1981) we identified a subset of $\mathbb{R}_+$ where $E\xi$ has a continuous density. We proceed to show that the result remains true for the specific density $p(x) = \int p^t(x)\,dt$. For a precise statement, we introduce, as before, the index

$$\alpha = \sup\Big\{ r \geq 0;\ \lim_{u \to 0} u^{r-2} \nu_2(u) = \infty \Big\},$$

where $\nu_2(u) = \int_0^u x^2 \nu(dx)$. Put $d = [1/\alpha] - 1$, and note that $d \geq 0$ since $\alpha \in [0, 1]$. Let S consist of all points $x \geq 0$ such that ν has no bounded density in any neighborhood of x, and note in particular that $0 \in S$. Write S_d for the d-fold sum

$$S_d = S + \cdots + S = \{ s_1 + \cdots + s_d;\ s_1, \ldots, s_d \in S \}.$$

PROPOSITION 4.7. *If* $\alpha > 0$, *then* p *is continuous on* $[0, \infty] \setminus S_d$.

PROOF. For any $s, t > 0$ we have $\mu^{s+t} = \mu^s * \mu^t$, and so

$$p^{s+t}(x) = (p^s * \mu^t)(x) \equiv \int_0^x p^s(x-y)\mu^t(dy), \qquad x \geq 0 \text{ a.e. } \lambda. \tag{4.9}$$

Now p^s is bounded and continuous for each s, and by dominated convergence the continuity carries over to $p^s * \mu^t$. Thus, both sides of (4.9) are continuous, and the relation holds identically.

Now write $\overline{p} = \int_0^1 p^s\,ds$. By (4.9) and Fubini's theorem,

$$\int_n^{n+1} p^s\,ds = \int_0^1 p^{n+s}\,ds = \int_0^1 (p^s * \mu^n)\,ds = \overline{p} * \mu^n,$$

and so

$$p = \sum_{n=0}^{\infty} \int_n^{n+1} p^s \, ds = \sum_{n=0}^{\infty} (\overline{p} * \mu^n) = \overline{p} * \sum_{n=0}^{\infty} \mu^n = \overline{p} + \overline{p} * \sum_{n=1}^{\infty} \mu^n.$$

From Kallenberg (1981) we know that $\sum_{n \geq 1} \mu^n \ll \lambda$ with a bounded, continuous density f. Hence,

$$\overline{p} * \sum_{n \geq 1} \mu^n = \overline{p} * (f \cdot \lambda) = f * (\overline{p} \cdot \lambda),$$

where the right-hand side is continuous by dominated convergence.

By scaling, it follows that $\int_\varepsilon^\infty p^s \, ds$ is continuous for every $\varepsilon > 0$. To establish the continuity of p on some interval I, it is then enough to show that $\int_0^\varepsilon p^s \, ds \to 0$ as $\varepsilon \to 0$, uniformly on I. If $I \subset [0, \infty] \setminus S_d$, this holds by the arguments in Kallenberg (1981) combined with Lemma 2.1 above. □

5. Palm distributions of regenerative sets. Here we continue our discussion of regenerative sets Ξ and their local time random measures ξ, as specified in Section 4. For convenience we assume condition (C) throughout the section, although some statements may be true more generally.

Our present aim is to use the conditional densities in Proposition 4.5 to construct specific versions of the Palm distributions Q_s, which will be shown in Theorems 5.5 and 5.6 to possess desirable continuity and approximation properties. Again some auxiliary results are needed, and we begin with a representation of the Palm distributions of ξ in terms of ordinary conditional distributions for the generating process X. For convenience, we may write $p(s) = p_s$ and $p^t(s) = p_s^t$.

LEMMA 5.1. *There exists a kernel κ from $\mathbb{R}_+^2$ to $\mathscr{M}(\mathbb{R}_+)$ satisfying*

$$P[\xi \in A | X_t] = \kappa(X_t, t, A) \ a.s., \ t \geq 0,$$

and we have

$$Q_s A = (p_s)^{-1} \int \kappa(s, t, A) p_s^t \, dt, \qquad s > 0 \ a.e. \ \lambda. \tag{5.1}$$

PROOF. To construct κ, fix a nested array (I_{nk}) of partitions of $\mathbb{R}_+$ with $\sup_k |I_{nk}| \to 0$ as $n \to \infty$, and define

$$M_n(s, t, A) = \sum_k \frac{P\{\xi \in A, \ X_t \in I_{nk}\}}{P\{X_t \in I_{nk}\}} 1_{I_{nk}}(s), \qquad s, t \geq 0, \ n \in \mathbb{N}.$$

By the measurability of X and Fubini's theorem, the functions $M_n(s, t, A)$ are measurable in the pair (s, t) for each A. Introducing the product-measurable functions

$$m(s, t, A) = \limsup_{n \to \infty} M_n(s, t, A), \qquad s, t \geq 0,$$

we get by martingale theory

$$P[\xi \in A | X_t] = m(X_t, t, A) \text{ a.s., } t \geq 0.$$

We may finally proceed as in the proofs of Theorem 5.3 and Proposition 6.26 in Kallenberg (1997) to construct a regular version κ of m.

Using Fubini's theorem and the definitions of p, p^t, Q_s, ξ and κ, we get for any measurable subsets $A \subset \mathscr{M}(\mathbb{R}_+)$ and $B \subset \mathbb{R}_+$,

$$\begin{aligned}
\int_B (Q_s A) p_s \, ds &= \int_B (Q_s A)\, E\xi(ds) = E[\xi B;\ \xi \in A] \\
&= E\left[\int 1_B(X_t)\, dt;\ \xi \in A\right] \\
&= \int P\{X_t \in B,\ \xi \in A\}\, dt \\
&= \int E[P[\xi \in A | X_t];\ X_t \in B]\, dt \\
&= \int E[\kappa(X_t, t, A);\ X_t \in B]\, dt \\
&= \int dt \int_B \kappa(s, t, A) p_s^t \, ds \\
&= \int_B ds \int \kappa(s, t, A) p_s^t \, dt.
\end{aligned}$$

Relation (5.1) now follows since B is arbitrary and since $p > 0$ on $(0, \infty)$ by Lemma 4.6. $\square$

The last result may be used to derive a basic factorization of the Palm distributions Q_s, which justifies the notation $Q_0 = P \circ \xi^{-1}$. For measures μ on $\mathbb{R}_+$, the restriction and shift operators k_s and θ_s are given by

$$(k_s\mu)B = \mu(B \cap [0, s]),\ (\theta_s\mu)B = \mu(B + s),\ s \geq 0.$$

LEMMA 5.2. *The Palm distributions Q_s of ξ satisfy*

$$Q_s \circ (k_s, \theta_s)^{-1} = (Q_s \circ k_s^{-1}) \otimes Q_0, \qquad s \geq 0 \ a.e.\ E\xi.$$

PROOF. Fix any $t > 0$ and conclude from the Markov property of X that, for measurable $A, B \subset \mathscr{M}(\mathbb{R}_+)$,

$$P[k_{X_t}\xi \in A,\ \theta_{X_t}\xi \in B | X_t] = P[k_{X_t}\xi \in A | X_t]\, Q_0 B \quad \text{a.s.}$$

Using the disintegration theorem, we may express this in terms of the kernel κ in Lemma 5.1 as

$$\int 1_{A\times B}(k_{X_t}\mu, \theta_{X_t}\mu)\, \kappa(X_t, t, d\mu) = \int 1_A(k_{X_t}\mu)\, \kappa(X_t, t, d\mu)\, Q_0 B \quad \text{a.s.,}$$

or, equivalently, as

$$\kappa(s, t, k_s^{-1}A \cap \theta_s^{-1}B) = \kappa(s, t, k_s^{-1}A)\, Q_0 B, \qquad s > 0 \text{ a.e. } p^t \cdot \lambda.$$

Hence, by Fubini's theorem and Lemmas 4.6 and 5.1,

$$Q_s(k_s^{-1}A \cap \theta_s^{-1}B) = Q_s(k_s^{-1}A)\, Q_0 B, \qquad s > 0 \text{ a.e. } \lambda.$$

By a monotone class argument, we may choose a common exceptional null set for all A and B, and the assertion follows. □

The next result gives the shift continuity of the distribution Q_0 of ξ.

LEMMA 5.3. *The function* $s \mapsto Q_0 \circ \theta_s^{-1}$ *is continuous in total variation on* $(0, \infty)$.

PROOF. By the Markov property of X we have, for any $s, t, h > 0$,

$$\begin{aligned}\|Q_0 \circ \theta_s^{-1} - Q_0 \circ \theta_{s+h}^{-1}\| &\le P\{X_t > s\} + \|P \circ X_t^{-1} - P \circ (X_t - h)^{-1}\| \\ &\le 3P\{X_t > s\} + \int_0^s |p^t(x) - p^t(x+h)|\, dx.\end{aligned}$$

By the continuity of p^t and the right-continuity of X, the right-hand side tends to 0 as $h \to 0$ and then $t \to 0$, uniformly for s in any compact subset of $(0, \infty)$. □

Next we record a simple invariance property of the distribution of ξ. We say that a random measure η is *symmetric* on an interval $[0, a]$, if its restriction to $[0, a]$ has the same distribution as the reflected measure $\tilde{\eta}B = \eta(a - B)$.

LEMMA 5.4. *Conditionally on* σ_x^- *for a fixed* $x > 0$, *the random measure* ξ *is a.s. symmetric on* $[0, \sigma_x^-]$.

PROOF. Fix any $x > 0$, put $\tau = L_x$ and let $\tau_n = 2^{-n}[2^n\tau - 1] \vee 0$, $n \in \mathbb{Z}_+$. Define $X_t^n = X_{\tau_n} - X_{\tau_n - t-}$ for $t \le \tau_n$ and $X_t^n = X_t$ otherwise. Note that the random measure ξ_n generated by X^n is the reflection of ξ on $[0, X_{\tau_n}]$. Since $\tau_n \uparrow \tau$, we have $X_{\tau_n} \uparrow X_{\tau-} = \sigma_x^-$, and so $\xi_n \to_v \tilde{\xi}$ where $\tilde{\xi}$ is the reflection of ξ on $[0, \sigma_x^-]$.

For each $s > 0$, the process X is a.s. exchangeable on $[0, s]$, conditionally on $\theta_s X$. Putting $Y_t = X_s - X_{s-t}$ for $t \le s$ and $Y_t = X_t$ otherwise, it follows that Y has conditionally the same distribution as X. Noting that $\{\tau_n = s\} \in \sigma\{\theta_s X\}$ for all n and s, we get $X^n =_d X$, so $\xi_n =_d \xi$, and in the limit $\tilde{\xi} =_d \xi$. Since σ_x^- is preserved by the mapping $\xi \mapsto \tilde{\xi}$, we obtain $(\tilde{\xi}, \sigma_x^-) =_d (\xi, \sigma_x^-)$, and the assertion follows. □

A more elaborate argument shows that ξ is conditionally exchangeable on $[0, \sigma_x^-]$, in the sense of Kallenberg (1982).

We may now use the duality theory of Section 3 to construct specific versions of the Palm distributions Q_s of ξ. Propositions 3.1 and 4.5 together with Lemma 5.2 suggest that we choose

$$Q_s \circ (k_a, \theta_s)^{-1} = p_s^{-1} E[p(s - \sigma_a^+);\, k_a\xi \in \cdot\,] \otimes Q_0, \qquad 0 < a < s. \tag{5.2}$$

The following result shows that the Palm distributions are uniquely specified by (5.2); it also provides some basic continuity properties. Unless otherwise specified, the weak convergence of probability measures on $\mathscr{M}(\mathbb{R}_+)$ is defined with respect to the vague topology.

THEOREM 5.5. *The Palm distributions Q_s of ξ are consistently and tightly defined by* (5.2) *for all $s > 0$ with $p_s < \infty$. The resulting family (Q_s) is continuous at every continuity point x of p, both weakly on $\mathscr{M}(\mathbb{R}_+)$ and in total variation outside any neighborhood of x.*

PROOF. By Proposition 4.5(i) and Lemma 4.6, we have

$$E[\xi|\mathscr{X}_a^{\cdot}] = M^a \cdot \lambda = (M^a/p) \cdot E\xi \quad \text{a.s. on } (a, \infty).$$

Thus, by Proposition 3.1 the left Palm distributions satisfy (5.2) a.e. for every $a > 0$. The normalization and consistency for different values of a follow from Propositions 3.3 and 4.5(iii), whereas Proposition 3.4 ensures the existence of a unique extension to the interval $[0, s)$. By Lemma 5.2 we may then define the full Palm distributions on $\mathbb{R}_+$ by (5.2).

To prove the asserted tightness, we may fix any $s > 0$ with $p_s < \infty$ and let $a, \varepsilon > 0$ with $2\varepsilon \vee a < s < a + \varepsilon$. By (5.2) and Lemmas 4.3 and 5.4,

$$\begin{aligned}
\int \{\mu(s-\varepsilon, a) \wedge 1\}\, Q_s(d\mu) &= p_s^{-1} E\, p(s - \sigma_a^+)\{\xi(s-\varepsilon, a) \wedge 1\} \\
&\le p_s^{-1} E\, p(s - \sigma_a^+)\{\xi(\sigma_a^- - \varepsilon, \sigma_a^-) \wedge 1\} \\
&= p_s^{-1} E\, p(s - \sigma_a^+) E[\xi(\sigma_a^- - \varepsilon, \sigma_a^-) \wedge 1 | \sigma_a^-] \\
&= p_s^{-1} E\, p(s - \sigma_a^+) E[\xi(0, \varepsilon) \wedge 1 | \sigma_a^-] \\
&= p_s^{-1} E\, p(s - \sigma_a^+)(\xi(0, \varepsilon) \wedge 1) \\
&= \int \{\mu(0, \varepsilon) \wedge 1\}\, Q_s(d\mu).
\end{aligned}$$

As $a \to s$, it follows by (5.2) and monotone convergence that

$$\begin{aligned}
\int \{\mu(s-\varepsilon, s) \wedge 1\}\, Q_s(d\mu) &\le \int \{\mu(0, \varepsilon) \wedge 1\}\, Q_s(d\mu) \\
&= p_s^{-1} E\, p(s - \sigma_{s/2}^+)(\xi(0, \varepsilon) \wedge 1).
\end{aligned} \tag{5.3}$$

By Lemma 4.4 and dominated convergence, the right-hand side tends to 0 as $\varepsilon \to 0$, and the required tightness follows.

Now fix any continuity point $x > 0$ of p. For the left Palm distributions, the continuity in total variation at x follows from Propositions 3.2(i) and 4.5(ii); for the right Palm distributions it holds by Lemma 5.3. By Lemma 2.6 we may combine the two properties into the asserted continuity outside any neighborhood of x. In particular, $Q_s \to_w Q_x$ as $s \to x$, in the sense of the vague topology on $\mathscr{M}(\mathbb{R}_+ \setminus \{x\})$.

To extend the convergence to the vague topology on $\mathscr{M}(\mathbb{R}_+)$, we conclude from (5.3) that, as $s \to x$,

$$\int \{\mu(s-\varepsilon, s) \wedge 1\} \, Q_s(d\mu) \le \int \{\mu(0, \varepsilon) \wedge 1\} \, Q_s(d\mu)$$
$$\to \int \{\mu(0, \varepsilon) \wedge 1\} \, Q_x(d\mu),$$

which tends to 0 as $\varepsilon \to 0$. Since also

$$\int \{\mu(s, s+\varepsilon) \wedge 1\} \, Q_s(d\mu) = E[\xi(0, \varepsilon) \wedge 1] \to 0,$$

we obtain

$$\lim_{\varepsilon \to 0} \limsup_{s \to x} \int \{\mu(x-\varepsilon, x+\varepsilon) \wedge 1\} \, Q_s(d\mu) = 0.$$

The strengthened convergence now follows as in Theorem 4.9 of Kallenberg (1986). □

We proceed to show how the specific versions of Palm distributions Q_s given by (5.2) admit approximation by elementary conditional distributions. For any bounded interval $I \subset \mathbb{R}_+$, we put $Q_I = P[\xi \in \cdot | \xi I > 0]$; by $I \to \{x\}$ we mean that both endpoints of I tend to x. Write

$$m_h = \int_0^h \nu(u, \infty) \, du = \int_0^\infty (x \wedge h) \, \nu(dx) < \infty, \qquad h > 0,$$

where the second equality holds by Fubini's theorem.

THEOREM 5.6. *For any continuity point x of p and intervals $I \to \{x\}$ with $|I| = h$, we have*:

(i) $m_h^{-1} P\{\xi I > 0\} \to p_x$;
(ii) $Q_I \to Q_x$, *both weakly on $\mathscr{M}(\mathbb{R}_+)$ and in total variation outside any neighborhood of x.*

PROOF. (i) Assuming $I = (y-h, y)$, we get by Lemma 4.1

$$P\{\xi I > 0\} = \int_{y-h}^{y} p_u \nu(y-u, \infty) \, du$$
$$\sim p_x \int_{y-h}^{y} \nu(y-u, \infty) \, du = m_h p_x.$$

(ii) Let $a < y - h$ and put $\mathscr{X}_a^+ = \mathscr{X}_a \vee \sigma\{\sigma_a^+\}$. By Lemma 4.1 and the regenerative property at σ_a^+, we get

$$P[\xi I > 0 | \mathscr{X}_a^+] = \int_{y-h}^{y} p(u - \sigma_a^+) \, \nu(y-u, \infty) \, du.$$

Hence, by Fubini's theorem and the definition of M^a,

$$P[\xi I > 0|\mathscr{X}_a] = \int_{y-h}^{y} E[p(u-\sigma_a^+)|\mathscr{X}_a]\,\nu(y-u,\infty)\,du$$
$$= \int_{y-h}^{y} M_u^a \nu(y-u,\infty)\,du.$$

Using Proposition 4.5(ii), we obtain

$$E\big|m_h^{-1}P[\xi I > 0|\mathscr{X}_a] - M_x^a\big| \le m_h^{-1}\int_{y-h}^{y} E|M_u^a - M_x^a|\,\nu(y-u,\infty)\,du$$
$$\le \sup_{u\in I} E|M_u^a - M_x^a| \to 0.$$

Combining this with (i) and recalling that $p_x > 0$ by Lemma 4.6, we get

$$\frac{P[\xi I > 0|\mathscr{X}_a]}{P\{\xi I > 0\}} = \frac{P[\xi I > 0|\mathscr{X}_a]}{m_h}\,\frac{m_h}{P\{\xi I > 0\}} \to_P \frac{M_x^a}{p_x}.$$

By Proposition 3.2(ii), it follows that $Q_I \to Q_x$ in total variation on $\mathscr{X}_a$.

To deduce the corresponding two-sided convergence, fix any $a < x < b$ with $I \subset (a,b)$. Letting c be the left endpoint of I, we get by regeneration at σ_c^+ and the disintegration theorem,

$$\begin{aligned} Q_I \circ (k_a,\theta_b)^{-1} &= P[(k_a,\theta_b)\xi \in \cdot \,|\, \sigma_c^+ \in I] \\ &= E\big[P[(k_a,\theta_b)\xi \in \cdot|\sigma_c^+]\big|\sigma_c^+ \in I\big] \\ &= E\big[P[k_a\xi \in \cdot|\sigma_c^+] \otimes (Q_0 \circ \theta_{b-\sigma_c^+}^{-1})\big|\sigma_c^+ \in I\big]. \end{aligned}$$

In particular,

$$(Q_I \circ k_a^{-1}) \otimes Q_0 = E\big[P[k_a\xi \in \cdot|\sigma_c^+] \otimes Q_0\big|\sigma_c^+ \in I\big],$$

and by combination

$$\begin{aligned} &\big\|Q_I \circ (k_a,\theta_b)^{-1} - (Q_I \circ k_a^{-1}) \otimes (Q_0 \circ \theta_{b-x}^{-1})\big\| \\ &\quad = \big\|E\big[P[k_a\xi \in \cdot|\sigma_c^+] \otimes \big\{Q_0 \circ \theta_{b-\sigma_c^+}^{-1} - Q_0 \circ \theta_{b-x}^{-1}\big\}\big|\sigma_c^+ \in I\big]\big\| \\ &\quad \le E\big[\big\|Q_0 \circ \theta_{b-\sigma_c^+}^{-1} - Q_0 \circ \theta_{b-x}^{-1}\big\|\,\big|\sigma_c^+ \in I\big]. \end{aligned}$$

Hence,

$$\begin{aligned} &\big\|Q_I \circ (k_a,\theta_b)^{-1} - Q_x \circ (k_a,\theta_b)^{-1}\big\| \\ &\quad \le \sup_{y\in I}\big\|Q_0 \circ \theta_{b-y}^{-1} - Q_0 \circ \theta_{b-x}^{-1}\big\| + \big\|Q_I \circ k_a^{-1} - Q_x \circ k_a^{-1}\big\|, \end{aligned}$$

which tends to 0 as $I \to \{x\}$, by the preceding paragraph and Lemma 5.3. In particular, $Q_I \to_w Q_x$ for the vague topology on $\mathscr{M}(\mathbb{R}_+ \setminus \{x\})$.

To extend the convergence to the vague topology on $\mathscr{M}(\mathbb{R}_+)$, we may use Lemmas 4.3 and 5.4 as before to get, as $I \to \{x\}$,

$$\begin{aligned}\int\{\mu(c-\varepsilon, c)\wedge 1\}\, Q_I(d\mu) &= E[\xi(c-\varepsilon, c)\wedge 1 |\, \sigma_c^+ \in I]\\ &\le E[\xi(\sigma_c^- - \varepsilon, \sigma_c^-)\wedge 1 |\, \sigma_c^+ \in I]\\ &= E\big[E[\xi(\sigma_c^- - \varepsilon, \sigma_c^-)\wedge 1 | \sigma_c^-]\big|\, \sigma_c^+ \in I\big]\\ &= E\big[E[\xi(0,\varepsilon)\wedge 1|\sigma_c^-]\big|\sigma_c^+ \in I\big]\\ &= E[\xi(0,\varepsilon)\wedge 1|\sigma_c^+ \in I]\\ &= \int\{\mu(0,\varepsilon)\wedge 1\}\, Q_I(d\mu)\\ &\to \int\{\mu(0,\varepsilon)\wedge 1\}\, Q_x(d\mu),\end{aligned}$$

which tends to 0 as $\varepsilon \to 0$. Furthermore, by the regeneration at σ_c^+,

$$\begin{aligned}\int\{\mu(c, c+\varepsilon)\wedge 1\}\, Q_I(d\mu) &= E[\xi(c, c+\varepsilon)\wedge 1 |\, \sigma_c^+ \in I]\\ &\le E\big[\,\xi(\sigma_c^+, \sigma_c^+ + \varepsilon)\wedge 1 \big|\, \sigma_c^+ \in I\big]\\ &= E[\xi(0,\varepsilon)\wedge 1] \to 0.\end{aligned}$$

Thus, by combination,

$$\lim_{\varepsilon\to 0}\limsup_{I\to\{x\}}\int\{\mu(x-\varepsilon, x+\varepsilon)\wedge 1\}\, Q_I(d\mu) = 0,$$

and so the strengthened convergence follows again as in Theorem 4.9 of Kallenberg (1986). $\square$

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