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REPLICA SYMMETRY BREAKING AND EXPONENTIAL INEQUALITIES FOR THE SHERRINGTON–KIRKPATRICK MODEL[1]

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We provide an extremely accurate picture of the Sherrington–Kirkpatrick model in three cases: for high temperature, for large external field and for any temperature greater than or equal to 1 and sufficiently small external field. We describe the system at the level of the central limit theorem, or as physicists would say, at the level of fluctuations around the mean field. We also obtain much more detailed information, in the form of exponential inequalities that express a uniform control over higher order moments. We give a complete, rigorous proof that at the generic point of the predicted low temperature region there is "replica symmetry breaking," in the sense that the system is unstable with respect to an infinitesimal coupling between two replicas.

1. Introduction. We study the famous Sherrington–Kirkpatrick (SK) mean field model for spin glasses. This model is well understood at the physical level [3], but despite considerable efforts, remains rather mysterious from the point of view of mathematics. (Readers who are not experts are urged to consult [8], [9]. They will find there a more detailed description of the main issues than is possible in the present introduction.) We will denote by $\Sigma_N = \{-1, 1\}^N$ the space of configurations. Given $\boldsymbol{\sigma} = (\sigma_i)_{i \le N}$ in Σ_N, we consider the random Hamiltonian,

$$H_N(\boldsymbol{\sigma}) = -\frac{1}{\sqrt{N}} \sum_{1 \le i < j \le N} g_{ij}\sigma_i\sigma_j - \sum_{i \le N} h'\sigma_i, \tag{1.1}$$

where $h' \in \mathbb{R}^+$ represents an external field and where $(g_{ij})_{1 \le i < j \le N}$ are independent $N(0, 1)$ random variables that represent the disorder of the interaction between the sites. Given a number $\beta \ge 0$ (that represents the inverse of the temperature), we are interested in Gibbs measure G_N on Σ_N, given by

$$G_N(\boldsymbol{\sigma}) = \frac{1}{Z}\exp(-\beta H_N(\boldsymbol{\sigma})), \tag{1.2}$$

where $Z = Z_N = Z_N(\beta, h')$ is the normalization factor,

$$Z_N = \sum_{\boldsymbol{\sigma} \in \Sigma_N} \exp(-\beta H_N(\boldsymbol{\sigma})).$$

The Gibbs measure is a random measure. We try to understand its structure for N large and for the typical value of the disorder. In physical terms

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this means that the disorder is "frozen in" (quenched) and that the system is allowed to evolve according to thermal fluctuations [physically, $G_N(\boldsymbol{\sigma})$ is the probability to observe configuration $\boldsymbol{\sigma}$ when the system is in thermal equilibrium]. The reason for the name h' for the external field is that we reserve the simplest name for the quantity $h = \beta h'$. For mathematicians, the system is best viewed as depending upon the two independent parameters β, h.

Throughout the paper, thermal averages, that is, averages with respect to Gibbs measure, will be denoted by $\langle \cdot \rangle$. A fundamental concept will be that of replicas, which are simply powers of the probability space (Σ_N, G_N). Averages with respect to the corresponding power of G_N will also be called "thermal averages."

It is plausible that the structure of the system depends upon the values of the parameters (β, h). We will call the high temperature region the region where either $h = 0$, $\beta < 1$, or where $h > 0$ and

$$E \frac{\beta^2}{\mathrm{ch}^4(\beta g \sqrt{q} + h)} < 1, \tag{1.3}$$

where g is $N(0,1)$ and where $q = q(\beta, h)$ is the solution of the equation

$$q = E\, \mathrm{th}^2(\beta g \sqrt{q} + h). \tag{1.4}$$

It is certainly not obvious that (1.4) has a unique solution. This is considered as self-evident in the physics literature. It has probably been checked numerically. I am not aware of any reference containing a rigorous proof.

The physicists have constructed a fascinating theory for the structure of the system in the low temperature region, the so-called "Parisi solution." The physicists predict a simpler behavior in the high temperature region. It has not yet been widely realized that this behavior is highly nontrivial when $h \neq 0$. This is because the best known case is the case $h = 0$ considered in [1]. In that case it is not difficult to show that when $\beta < 1$, we have

$$\lim_{N\to\infty} \frac{1}{N} E \log Z_N = \log 2 + \frac{\beta^2}{4} \left(= \lim_{N\to\infty} \frac{1}{N} \log EZ_N \right) \tag{1.5}$$

and the model is well understood ([2], [5], Section 2).

By contrast, when $h > 0$, under (1.3), physicists predict that

$$\begin{aligned} \lim_{N\to\infty} \frac{1}{N} E \log Z_N &= \frac{\beta^2}{4}(1-q)^2 + E \ln(2\cosh(\beta g \sqrt{q} + h)) \\ &= \frac{\beta^2}{4}(1-q)^2 \\ &\quad + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \ln(2\cosh(\beta t \sqrt{q} + h)) \exp(-t^2/2)\, dt, \end{aligned} \tag{1.6}$$

when q satisfies (1.4). For $h > 0$, the quantity (1.6) is strictly less than $N^{-1} \lim_{N\to\infty} \log EZ_N$.

The remarkable character of a formula such as (1.6) should be self-evident. Such a formula is of course not accidental but is a consequence of a rich underlying structure. The main goal of the present paper is to provide a considerably more accurate picture of this underlying structure than was done in [5]. There, (1.6) is proved in the domain $\{(\beta, h); \beta < \beta_0\}$ for a certain $\beta_0 > 0$.

Our approach also allows us to get information about the low temperature region. As low temperature results are more glamorous than high temperature results, they should be stated first.

A central concept of the physicists' picture of the low temperature region is that of "replica symmetry breaking," which is formulated in [3] using objects such as integers n, with $0 < n < 1$, functions of a negative number of variables, etc. Fortunately, there are formulations of this concept that are much more amenable to a mathematical description and I am very grateful to Marc Mézard for having shown to me such a formulation (which was the starting point of this paper). Let us consider a coupling $t\sum_{i\le N}\sigma_i\sigma_i'$ between two replicas. That is, we study the space Σ_N^2 with Hamiltonian

$$H_N(\boldsymbol{\sigma}, \boldsymbol{\sigma}') = H_N(\boldsymbol{\sigma}) + H_N(\boldsymbol{\sigma}') - t' \sum_{i\le N} \sigma_i\sigma_i',$$

where H_N is given by (1.1). One can then consider the corresponding Gibbs measure $G_{N,t}$ on Σ_N^2. Consider the quantity

$$\varphi_N(t) = E\left\langle \frac{\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}'}{N} \right\rangle_t, \tag{1.7}$$

where $\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}' = \sum_{i\le N}\sigma_i\sigma_i'$ and where $\langle\cdot\rangle_t$ denotes average with respect to $G_{N,t/\beta}$. There, as well as in the rest of the paper, E denotes expectation with respect to the disorder (i.e., the variables g_{ij}). It is self-evident for a physicist that $\varphi(t) = \lim_{N\to\infty}\varphi_N(t)$ exists (when no explicit mentions of β, h are made, we understand that they are fixed).

INFORMAL DEFINITION 1.1. We say that there is replica symmetry breaking if φ is discontinuous at zero.

The quantity $N^{-1}\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}'$ is called "overlap" of the two configurations. It measures how close they are. In words, we can define replica symmetry breaking by saying that a very little "push" of one configuration toward the other is sufficient to greatly increase the average overlap.

In order to give a precise meaning to Definition 1.1, when we do not know how to prove the existence of the limit, we simply replace continuity by asymptotic equicontinuity.

DEFINITION 1.2. For given values of the parameters β, h, we say that there is replica symmetry breaking if the following condition holds.

(1.8) There exists $\varepsilon_0 > 0$, such that for each $t_0 > 0$, and each N_0, we can find $|t| < t_0$ and $N \ge N_0$ with $|\varphi_N(t) - \varphi_N(0)| \ge \varepsilon_0$.

THEOREM 1.3. *There is replica symmetry breaking at the generic point* (*in the sense of Baire category*) *of the low temperature region*.

Of course, the fact that there is replica symmetry breaking does not give very precise information on what happens. It does not even rule out that (1.6) still holds. [Physicists think that in the low temperature region the left-hand side of (1.6) is strictly less than the right-hand side.] This is certainly disappointing but one should also see the bright side. Currently, almost nothing rigorous is known about the low temperature region, and it was somewhat unexpected that a clean statement could be proved. The most notable feature of Theorem 1.3 is that it concerns the entire low temperature region, not only "low enough temperature." Since it is unlikely that there is replica symmetry breaking on the high temperature region, Theorem 1.3 can be seen at the very least as a regorous identification of the line (1.3).

Before we comment upon the analysis behind Theorem 1.3 we have to understand better what replica symmetry breaking is. The reader will note that *absence* of replica symmetry breaking is equivalent to the following condition:

$$\text{Given } \varepsilon > 0, \text{ there is } t_0 > 0 \text{ and } N_0 \text{ such that}\\ |t| < t_0 \text{ and } N \geq N_0 \Rightarrow |\varphi_N(t) - \varphi_N(0)| \leq \varepsilon. \tag{1.9}$$

Our next result shows how to characterize absence of replica symmetry breaking by the validity of a certain exponential inequality.

PROPOSITION 1.4. *At a given value of the parameters* β, h, *the following are equivalent*:

(1.10) *There is absence of replica symmetry breaking*.

(1.11) *Given* $u > 0$, *there is* $a(u) > 0$ *and* N_0 *such that*

$$\forall N \geq N_0,\ EG_n^2(\{|\boldsymbol{\sigma} \cdot \boldsymbol{\sigma}' - \langle \boldsymbol{\sigma} \cdot \boldsymbol{\sigma}' \rangle| \geq Nu\}) \leq \exp(-Na(u)).$$

We will prove Proposition 1.4, which is (a little more than) a very simple fact in large deviation theory in Section 2.

To prove Theorem 1.3, we will proceed by contradiction. Assuming absence of replica symmetry breaking, we see from (1.11) that (in a strong sense) we have

$$\boldsymbol{\sigma} \cdot \boldsymbol{\sigma}' \simeq \langle \boldsymbol{\sigma} \cdot \boldsymbol{\sigma}' \rangle \tag{1.12}$$

for large N and any two generic configurations $\boldsymbol{\sigma}, \boldsymbol{\sigma}'$.

It is better to discuss the meaning and the usefulness of (1.12) as part of the discussion of the structure behind (1.6), so we turn to the description of the structure of the system at high temperature. The main feature is that one has

$$\lim_{N\to\infty} \frac{1}{N} E\langle|\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}' - E\langle\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}'\rangle|\rangle = 0, \tag{1.13}$$

that is, "the overlap of two generic configurations is $E\langle\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}'\rangle$."

Physicists seem to consider (1.13) as self-evident. As will be apparent later, once one has obtained (1.13), it is easy to compute with Gibbs measure. The only task facing physicists is to obtain that $N^{-1}E\langle\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}'\rangle$ is approximately the number of q of (1.4). This is easy, and from this (1.6) follows in a few lines (see [5]). Physicists thus consider the high temperature case as trivial. It is, however, another matter to prove (1.13). In particular, it seems at first very hard to say anything at all about Gibbs measure. Much of the author's work on the SK and other models has been driven by the simple idea that to prove (1.13) one should break it down into two statements,

$$\lim_{N\to\infty} \frac{1}{N} E\langle|\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}' - \langle\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}'\rangle|\rangle = 0, \tag{1.14}$$

$$\lim_{N\to\infty} \frac{1}{N} \big|\langle\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}'\rangle - E\langle\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}'\rangle\big| = 0. \tag{1.15}$$

Thus usefulness of these will become apparent in Section 3. In particular, knowledge of (1.14) gives control of the error terms of a certain "Taylor type" expansion. This is the idea of writing (1.14), (1.15) separately. When trying to prove (1.13), one first proves (1.14); the control gained through (1.14) is then instrumental in proving (1.15). This idea will be used here in (1.17), (1.18) below. Going back to the discussion of Theorem 1.3, we see that an a priori knowledge of (1.14) is provided by (1.12). The technical difficulty is that this information is somewhat insufficient to make precise computations. It will be completed by "thermodynamical arguments," that is, arguments ultimately relying upon the convexity of the function $\beta \to \log Z_N(\beta, h)$. We will then be able to do computations that are precise enough to reach a contradiction. This contradiction is a mathematically precise formulation of the statement by physicists that "the replica symmetric solution is unstable in the low temperature region."

We now go back to the discussion of the high temperature region. The major problem left is the proof of (1.6) in the entire region (1.3). This, unfortunately, seems harder than expected. A number of signs (such as the analysis of [5], Section 6) point to the fact that one should not only attempt to prove (1.13) but that one should also attempt to control higher moments, that is, prove exponential inequalities. The natural form of these inequalities is stronger than (1.11); so they imply in particular that there is no replica symmetry breaking. Even though this approach has yet to succeed in covering the entire region (1.3), it does provide in the region it succeeds a considerably more detailed picture than was obtained in [5]. Another important motivation for proving exponential inequalities is that the information they contain can be "transferred" to a slightly different value of the parameters β, h, as will be apparent in the proof of Theorem 1.7.

THEOREM 1.5 (Very high temperature). *There is a number L such that if $L\beta \le 1$, then for all h and all t,*

$$E\big\langle \exp t(\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}' - E\langle\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}'\rangle)\big\rangle \le \exp Lt^2 N. \tag{1.16}$$

To see that this is stronger than (1.11), let us observe [following the philosophy of (1.14), (1.15)] that an inequality such as (1.16) is equivalent to the following two inequalities:

$$E\exp t(\langle\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}'\rangle - E\langle\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}'\rangle) \le \exp L_1 t^2 N, \tag{1.17}$$

$$E\exp t(\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}' - \langle\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}'\rangle) \le \exp L_2 t^2 N. \tag{1.18}$$

That (1.16) implies (1.17) follows from Jensen's inequality. That (1.16) implies (1.18) follows from

$$\exp t(\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}' - \langle\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}'\rangle) = \exp t(\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}' - E\langle\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}'\rangle)\exp t(\langle\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}'\rangle) - E\langle\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}'\rangle),$$

(1.16), (1.17) and Cauchy–Schwarz. That (1.17), (1.18) imply (1.16) follows by a similar argument. Thus (1.16) implies (1.11) because (1.18) already implies (1.11) (using Chebyshev's inequality).

We will leave to the reader to show that under the condition of Theorem 1.5 we have

$$|E\langle\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}'\rangle - Nq| \le L \tag{1.19}$$

so that (1.16) implies

$$\forall\, t,\ E\langle\exp t(\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}' - Nq)\rangle \le 2\exp Lt^2 N. \tag{1.20}$$

One marginal benefit of our new approach is that now we can control bigger regions than we could in [5].

THEOREM 1.6 (Large external field). *Given any $\beta > 0$ there is a number $L(\beta)$ such that if $h \ge L(\beta)$ then, for all $t \ge 0$,*

$$E\big\langle\exp t(\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}' - E\langle\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}'\rangle)\big\rangle \le \exp L(\beta)t^2 N. \tag{1.21}$$

THEOREM 1.7 (Small external field). *Given $0 < \beta < 1$, there is a number $h(\beta) > 0$ such that if $h < h(\beta)$, then* (1.21) *holds for all $t > 0$.*

The proof of this theorem is somewhat different from the proof of Theorems 1.5, 1.6. It uses in an essential way the knowledge of the case $h = 0$ that has been obtained through special methods [5].

The method we have developed allows not only proving exponential inequalities, but also limit theorems, and we explain a general result in this direction. We consider k replicas $\boldsymbol{\sigma}^1, \dots, \boldsymbol{\sigma}^k$. For a subset $J = \{j_1, j_2\}$ of $\{1, \dots, k\}$, we write $f_J = \boldsymbol{\sigma}^{j_1}\cdot\boldsymbol{\sigma}^{j_2} - E\langle\boldsymbol{\sigma}^1\cdot\boldsymbol{\sigma}^2\rangle$; for a collection $\mathscr{J}$ of such subsets, we write $f_{\mathscr{J}} = \prod_{J\in\mathscr{J}} f_J$.

THEOREM 1.8. *In the domain of Theorems* 1.6 *to* 1.8, *the limit of*

$$N^{-(\mathrm{card}\,\mathscr{J})/2} E\langle f_{\mathscr{J}}\rangle \tag{1.22}$$

exists a $N \to \infty$.

The proof of the theorem also contains an explicit method for computing the limit. We will prove (in Corollary 6.2) a central limit theorem concerning a "symmetrized" version of the overlaps using this method. It is likely that there is a simple algebraic structure behind the limits of Theorem 1.8, but this remains to be found.

One feels that Theorems 1.5 and 1.8 open the door to results whose accuracy will be limited only by the energy one is willing to invest in them. For example, it seems almost certain that one can compute

$$\lim_{N\to\infty} (E\langle \boldsymbol{\sigma}\cdot\boldsymbol{\sigma}'\rangle - Nq) \tag{1.23}$$

and even that there is an asymptotic expansion for $E\langle \boldsymbol{\sigma}\cdot\boldsymbol{\sigma}'\rangle$. (Here, as below, "almost certain" means that the author has not completely checked that what looks like the obvious approach does work.) Whether the general term of this expansion can be explicitly computed is unclear. If we denote by $SK(\beta, h)$ the right-hand side of (1.6), it is not difficult to show that (1.6) can be improved into

$$\sup N|E\log Z_N - NSK(\beta,h)| < \infty \tag{1.24}$$

and again one can ask the question of an asymptotic expansion of $E\log Z_N - NSK(\beta,h)$. The existence seems almost certain, but can it be computed effectively? It is again almost certain (through the Martingale central limit theorem) that $N^{-1/2}(\log Z_N - E\log Z_N)$ is asymptotically normal, but how precisely can it be understood? All these questions are better left for future research.

The paper is organized as follows. The simple Proposition 1.4 is proved in Section 2. The purpose of this proposition is to motivate our other results, and Section 2 is independent of the rest of the paper. The serious work starts in Section 3, where we prove the estimates upon which our approach (by induction on N) relies. Similar estimates were used first in the more complicated case of the Hopfield model [6]. It seems well worth proving them in detail in the present setting, since the present paper demonstrates how useful they are. Section 3 then ends with the proof of Theorem 1.3. The exponential inequalities are proved in two stages. In Section 4, we prove inequalities of the type (1.17). In Section 5, we use these to reach the full result (1.16). Convergence results are then proved in Theorem 6.1.

2. Proof of Proposition 1.4.

We first prove Proposition 1.4. Consider the function $R = R(\boldsymbol{\sigma},\boldsymbol{\sigma}')$ on $\Sigma_N\times\Sigma_N$ given by $R(\boldsymbol{\sigma},\boldsymbol{\sigma}') = (1/N)\sum_{i\le N}\sigma_i\sigma_i'$. Then, if $\langle\cdot\rangle$ denotes thermal average in Σ_N^2, we have the identity

$$\langle R\rangle_t = \frac{\langle R\exp tNR\rangle}{\langle \exp tNR\rangle}$$

so that

$$\varphi(t) = E\frac{\langle R \exp tNR\rangle}{\langle \exp tNR\rangle}. \tag{2.1}$$

We prove that (1.11) implies (1.10). We write, since $|R| \le 1$,

$$\langle R \exp tNR\rangle \le (\langle R\rangle + u)\langle \exp tNR\rangle + \exp|t|N\big\langle 1_{\{R\ge \langle R\rangle + u\}}\big\rangle$$

so that

$$\varphi(t) \le E\langle R\rangle + u + \exp(2|t|N)E\big\langle 1_{\{R\ge\langle R\rangle+u\}}\big\rangle. \tag{2.2}$$

We read (1.11) as

$$E\big\langle 1_{\{R\ge\langle R\rangle+u\}}\big\rangle \le \exp(-Na(u))$$

so that, for $|t| \le a(u)/4$ and $N \ge N_0$, (2.2) implies

$$\varphi(t) \le \varphi(0) + u + \exp(-Na(u)/2).$$

A lower bound based on the same principle completes the proof that (1.11) implies (1.10), so we turn to the converse. We observe first that

$$\varphi(t) - \varphi(0) = E\frac{\langle R' \exp tNR'\rangle}{\langle \exp tNR'\rangle},$$

where $R' = R - \langle R\rangle$. Given $\varepsilon \ge 0$, we can find $t > 0$ such that (for N large enough) we have

$$\varphi(t) \le \varphi(0) + \varepsilon; \qquad \varphi(-t) \ge \varphi(0) - \varepsilon. \tag{2.3}$$

In particular, if we consider the event

$$\Omega_1 = \left\{\frac{\langle R' \exp tNR'\rangle}{\langle \exp tNR'\rangle} \le 4\varepsilon\right\}$$

we have $P(\Omega_1) \ge 3/4$. On Ω_1 we can write

$$\langle R' \exp tNR'\rangle \le 4\varepsilon\langle \exp tNR'\rangle \tag{2.4}$$

so that

$$\begin{aligned} 5\varepsilon\big\langle 1_{\{R'\ge 5\varepsilon\}} \exp tNR'\big\rangle &\le 4\varepsilon\langle \exp tNR'\rangle \\ &\le 4\varepsilon\big\langle 1_{\{R'\le 5\varepsilon\}} \exp tNR'\big\rangle \\ &\quad + 4\varepsilon\big\langle 1_{\{R'\ge 5\varepsilon\}} \exp tNR'\big\rangle \end{aligned}$$

and thus

$$\varepsilon\big\langle 1_{\{R'\ge 5\varepsilon\}} \exp tNR'\big\rangle \le 4\varepsilon \exp 5t\varepsilon N$$

so that

$$\big\langle 1_{\{R'\ge 6\varepsilon\}}\big\rangle \le 4\exp(-t\varepsilon N).$$

In a similar manner, we find an event Ω_2 with $P(\Omega_2) \geq 3/4$ such that on Ω_2 we have

$$\langle 1_{\{-R' \geq 6\varepsilon\}} \rangle \leq 4 \exp(-t\varepsilon N)$$

so that on $\Omega_1 \cap \Omega_2$ we have

$$\langle 1_{\{|R'| \geq 6\varepsilon\}} \rangle \leq 8 \exp(-t\varepsilon N). \tag{2.5}$$

This inequality is ill adapted to the concentration of measure argument that will follow because R' depends upon G_N through $\langle R \rangle$. To remove this dependence, we use the classical device of symmetrization. We consider the function $\widetilde{R}$ on Σ_N^4 given by

$$\tilde{R}(\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2, \boldsymbol{\sigma}^3, \boldsymbol{\sigma}^4) = R(\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2) - R(\boldsymbol{\sigma}^3, \boldsymbol{\sigma}^4)$$

so that (2.5) implies that on $\Omega_1 \cap \Omega_2$,

$$\langle 1_{|\tilde{R}| \geq 12\varepsilon} \rangle \leq 16 \exp(-t\varepsilon N). \tag{2.6}$$

So, since $P(\Omega_1 \cap \Omega_2) \geq 1/2$, we have shown that the median M of the random variable

$$X = \log\langle 1_{|\tilde{R}| \geq 12\varepsilon} \rangle$$

is at most $\log 16 - t\varepsilon N$. If we think of X as a function of the Gaussian r.v. g_{ij}, it is elementary to see that the Lipschitz constant L of this function is at most $8\beta\sqrt{(N-1)/2}$. The Gaussian isoperimetric inequality then implies that

$$P(X \geq M + u) \leq \exp\left(-\frac{u^2}{2L^2}\right) \leq \exp\left(-\frac{u^2}{64\beta^2 N}\right).$$

In particular,

$$P\left(X \geq \log 16 - \frac{t\varepsilon N}{2}\right) \leq \exp\bigl(-\varepsilon^2 t^2 N / K(\beta)\bigr),$$

where $K(\beta)$ denotes a quantity that depends upon β only, and that is not necessarily the same at each occurrence. Since $\langle 1_{|\tilde{R}| \geq 12\varepsilon} \rangle \leq 1$, we have, for N large enough,

$$E\langle 1_{|\tilde{R}| \geq 12\varepsilon} \rangle \leq 2 \exp\bigl(-\varepsilon^2 t^2 N / K(\beta)\bigr). \tag{2.7}$$

Since $|\tilde{R}| \leq 2$, for $a = \varepsilon^2 t^2 / 2K(\beta)$ we have

$$E\langle \exp a|\tilde{R}| \rangle \leq 3 \exp(12\varepsilon a N)$$

and thus by Jensen's inequality,

$$E\langle \exp a|R'| \rangle \leq 3 \exp 12\varepsilon a N$$

so that

$$E\langle 1_{\{|R'| \geq 13\varepsilon\}} \rangle \leq 3 \exp(-\varepsilon a N). \qquad \square$$

3. How to use the cavity method. The basis of the cavity method is the following simple result. In this result, as well as in the rest of the paper, we denote by $(\boldsymbol{\sigma}^1, \dots, \boldsymbol{\sigma}^k)$ a point in a replica Σ_N^k of order k.

PROPOSITION 3.1. *For a function $f(\boldsymbol{\sigma}^1, \dots, \boldsymbol{\sigma}^k)$, we have*

$$E\langle f(\boldsymbol{\sigma}^1, \dots, \boldsymbol{\sigma}^k)\rangle = E\frac{1}{Z}\mathrm{Av}\langle f\mathscr{E}\rangle_0 \tag{3.1}$$

where

$$\mathscr{E} = \exp\biggl(\frac{\beta}{\sqrt{N}}\sum_{l\le k}\varepsilon_l \mathbf{g}\cdot\boldsymbol{\eta}^l + h\sum_{l\le k}\varepsilon_l\biggr)$$

and $Z = \mathrm{Av}\langle\mathscr{E}\rangle_0$. In these formulas, Av *means average over all values $(\varepsilon_l)_{l\le k}$, $\varepsilon_l = \pm 1$; $(\boldsymbol{\eta}^1, \dots, \boldsymbol{\eta}^k)$ is the generic point in Σ_{N-1}^k; $\langle\cdot\rangle_0$ denotes integration in Σ_{N-1}^k with respect to the Gibbs measure G_{N-1}^k at inverse temperature $\beta' = \beta(1-1/N)^{1/2}$; $\mathbf{g}$ denotes an independent sequence of $N(0,1)$ r.v., independent of all the other sequences considered and f is shorthand for $f(\boldsymbol{\sigma}^1, \dots, \boldsymbol{\sigma}^k)$, where $\boldsymbol{\sigma}^l = (\eta_1^l, \dots, \eta_{N-1}^l, \varepsilon_l)$.*

PROOF. This is mere algebraic identity if one writes

$$\begin{aligned} -H(\boldsymbol{\sigma}^l) = {}& \frac{1}{\sqrt{N}}\sum_{i\le j\le N-1} g_{ij}\sigma_i^l\sigma_j^l + h\sum_{i\le N-1}\sigma_i^l \\ &+ \sigma_N^l\biggl(\frac{1}{\sqrt{N}}\sum_{i\le N-1} g_{iN-1}\sigma_i^l + h\biggr) \end{aligned}$$

and one sets $\varepsilon_l = \sigma_N^l$. The change of temperature is due to the change of normalization as one goes from N to $N-1$ sites. Once reason for introducing h is that this quantity does not change when going from $\langle\cdot\rangle$ to $\langle\cdot\rangle_0$. □

Proposition 3.1 brings to light the importance of being able to estimate the right-hand side of (3.1). The technique to do this has been developed in [T6], where, we study the more difficult and more complicated case of the Hopfield model. Even in the simpler case being considered here, there is a computational aspect that is apparently intrinsic. At a certain (mostly algebraic) level things *are* complicated. This contributes significantly to the difficulty of the problem. Thus, in order to avoid readers being scared by Theorem 3.2 below, we had better explain in words what is going on. This theorem is a kind of power expansion, as in

$$f(x) = f(x_0) + (x - x_0)f'(x_0) + S,$$

where $f(x_0)$ is the main term, $(x - x_0)f'(x_0)$ a first-order term, and S a higher-order term. In (3.2) below, the term I is the "main term", the "first-order term" is the sum of four pieces labeled II to V. The individual pieces II to V have no special meaning on their own. This is simply the way they come out of the computation. Readers probably feel, rather appropriately, that they should not care about the way Theorem 3.2 is proved before it has been demonstrated what a powerful tool it is. In that case, they should proceed as follows. After reading the definitions (3.2), (3.10), (3.11), they should read Proposition 3.6 (and the comment following it), the simplest case of application of Theorem 3.2. They should then move to Proposition 3.5, the second simplest case of application, and finally to Proposition 3.4. This proposition is the way the terms I to V of Theorems 3.2 will actually be used everywhere in the present paper. The formulation is somewhat simpler in Proposition 3.4 than in Theorem 3.2, and there is no need to understand the more complicated formulation of Theorem 3.2 until one is willing to understand its proof. Even in the formulation of Proposition 3.4, the “first-order” terms II to V are complicated. Again, they come out of computation, and it is not obvious how to interpret them. Rather, the task is to figure out ways through the jungle, such as in Proposition 3.6. After having gained a feeling for the terms I to V, readers might try to look at the error term (3.8), the control of which is, of course, the essential point.

Readers might wonder why Theorem 3.2 (or even Proposition 3.4) is dumped upon them at this stage, rather than working out first manageable cases like Proposition 3.6. The reason is simple. Proposition 3.6 looks simpler because there is algebraic cancellation, but the main point (control of the error term) is not any easier. Readers must realize that the author was driven to Theorem 3.2 against his will, as it is better to go once for all through this computation rather than repeating many difficult separate cases.

THEOREM 3.2. *With the notation of Proposition* 3.1, *there is a number* $K(k)$ *depending upon* k *only such that we have*

$$E\langle f(\boldsymbol{\sigma}^1,\dots,\boldsymbol{\sigma}^k)\rangle = \mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV}+\mathrm{V}+S, \tag{3.2}$$

where

$$\mathrm{I} = E\frac{1}{\mathrm{ch}^k X}\langle \mathrm{Av} f\mathscr{E}_0\rangle_0, \tag{3.3}$$

$$\mathrm{II} = E\frac{\beta^2}{\mathrm{ch}^k X}\left\langle \mathrm{Av} f\mathscr{E}_0\sum_{l<l'}\varepsilon_l\varepsilon_{l'}\frac{\dot{\boldsymbol{\eta}}^l\cdot\dot{\boldsymbol{\eta}}^{l'}}{N}\right\rangle_0, \tag{3.4}$$

$$\mathrm{III} = -E\frac{\beta^2}{\mathrm{ch}^k X}\left\langle \mathrm{Av} f\mathscr{E}_0\sum_{l\le k}\frac{\dot{\boldsymbol{\eta}}^l\cdot\mathbf{b}}{N}\right\rangle_0, \tag{3.5}$$

$$\mathrm{IV} = E\frac{\beta^2}{\mathrm{ch}^k X}\left\langle \mathrm{Av} f\mathscr{E}_0\Big(\sum_{l\le k}\varepsilon_l\Big)\Big(\sum_{l\le k}\varepsilon_l\frac{\dot{\boldsymbol{\eta}}^l\cdot\mathbf{b}}{N}\Big)\right\rangle_0, \tag{3.6}$$

$$\mathrm{V} = -k\beta^2 E\frac{\mathrm{th} X}{\mathrm{ch}^k X}\left\langle \mathrm{Av} f\mathscr{E}_0\Big(\sum_{l\le k}\varepsilon_l\frac{\dot{\boldsymbol{\eta}}^l\cdot\mathbf{b}}{N}\Big)\right\rangle_0, \tag{3.7}$$

$$|S| \le \beta^4 K(k) \exp 4k\beta^2 \left\langle \mathrm{Av}|f| \left(\sum_{l\le k+1} \left(\frac{\dot{\boldsymbol{\eta}}^l \cdot \mathbf{b}}{N}\right)^2 + \sum_{l,\, l'\le k+2} \left(\frac{\dot{\boldsymbol{\eta}}^l \cdot \dot{\boldsymbol{\eta}}^{l'}}{N}\right)^2 \right) \right\rangle_0. \tag{3.8}$$

There,

$$\mathscr{E}_0 = \exp X \sum_{l\le k} \varepsilon_l, \tag{3.9}$$

$$\mathbf{b} = \langle \boldsymbol{\eta} \rangle_0;\, \dot{\boldsymbol{\eta}}^l = \boldsymbol{\eta}^l - \mathbf{b} = \boldsymbol{\eta}^l - \langle \boldsymbol{\eta}^l \rangle_0, \tag{3.10}$$

$$X = \frac{\beta}{\sqrt{N}} \mathbf{g} \cdot \mathbf{b} + h, \tag{3.11}$$

and the thermal integral in (3.8) *is over* Σ_{N-1}^{k+2}, *with the natural identification of* $\mathrm{Av}|f|$ *as a function on* Σ_{N-1}^{k+2}. *Moreover*, *when* f *has the property that*

$$f \ne 0 \Longrightarrow \left| \sum_{l\le k} \varepsilon_l \right| \le k-1$$

and when $h \ge 4\beta^2$, *we can improve the estimate in* (3.8) *by a factor* $\exp(-h/4)$.

Since this result is the cornerstone of the present paper, it does not seem appropriate to ask the reader to struggle through the more complicated version of [6], and we will provide a complete proof. Fortunately, the part of the proof that was tricky to discover is simple to explain; the rest is mere computation. Let us consider the quantity

$$\widehat{Z} = \exp\left(k \frac{\beta'^2}{2} (1-b) \right) \mathrm{ch}^k X,$$

where

$$b = (N-1)^{-1} \|\mathbf{b}\|^2 = \frac{1}{N-1} \sum_{i\le N-1} b_i^2.$$

The basic idea is that $\widehat{Z}$ is the "main part" of Z. We use the identity

$$\frac{U}{Z} = \frac{2U}{\widehat{Z}} - \frac{UZ}{\widehat{Z}^2} + \frac{U(\widehat{Z}-Z)^2}{\widehat{Z}^2 Z} \tag{3.12}$$

so that

$$E\left| \frac{U}{Z} - \frac{2U}{\widehat{Z}} + \frac{UZ}{\widehat{Z}^2} \right| \le E \frac{|U|(\widehat{Z}-Z)^2}{\widehat{Z}^2 Z}. \tag{3.13}$$

We will use this for

$$U = \langle \mathrm{Av} f \mathscr{E} \rangle_0$$

so that

$$|U| \le U' = \langle \mathrm{Av}|f| \mathscr{E} \rangle_0.$$

Also, use of Jensen's inequality shows that $Z \geq \mathrm{ch}^k X$ so the right-hand side of (3.13) is at most

$$E\frac{U'(\widehat{Z}-Z)^2}{\widehat{Z}^2\mathrm{ch}^k X}. \tag{3.14}$$

Products of the type $UZ, U'Z$, etc. can be expressed as integrals over replicas of order $2k$, as will be detailed below. To compute expectations, it is natural first to integrate in $\mathbf{g}$, conditionally upon the quenched variables implicit in $\langle\cdot\rangle_0$. This will be denoted by E_g. It then appears that the heart of the matter is as follows. Consider an integer $m \geq 0$, a function $\bar{f}$ on Σ_{N-1}^m, numbers $(\varepsilon_l)_{l\leq m}$, $\varepsilon_l = \pm 1$. We would then like to estimate

$$E_g\frac{1}{\mathrm{ch}^m X}\Big\langle \bar{f}\exp\Big(\sum_{l\leq m}\varepsilon_l\Big(\frac{\beta}{\sqrt{N}}\mathbf{g}\cdot\boldsymbol{\eta}^l + h\Big)\Big)\Big\rangle_0, \tag{3.15}$$

where $\langle\cdot\rangle_0$ now denotes thermal average in Σ_{N-1}^m.

Proposition 3.3. *There is a number $K(m)$, depending upon m only such that the expression* (3.15) *is equal to*

$$A^m(\mathrm{VI}+\mathrm{VII}+\mathrm{VIII}+\mathrm{IX}+\mathrm{X}+S_1),$$

where $A = \exp(\beta'^2(1-b)/2)$, and

$$\mathrm{VI} = E_g\frac{\exp\Delta X}{\mathrm{ch}^m X}\langle\bar{f}\rangle_0, \tag{3.16}$$

$$\mathrm{VII} = E_g\frac{\beta^2\exp\Delta X}{\mathrm{ch}^m X}\Big\langle\bar{f}\sum_{l<l'}\varepsilon_l\varepsilon_{l'}\frac{\dot{\boldsymbol{\eta}}^l\cdot\dot{\boldsymbol{\eta}}^l}{N}\Big\rangle_0, \tag{3.17}$$

$$\mathrm{VIII} = -\beta^2 E\frac{\exp\Delta X}{\mathrm{ch}^m X}\Big\langle\bar{f}\sum_{l\leq m}\frac{\dot{\boldsymbol{\eta}}^l\cdot\mathbf{b}}{N}\Big\rangle_0, \tag{3.18}$$

$$\mathrm{IX} = \beta^2\Delta E\frac{\exp\Delta X}{\mathrm{ch}^m X}\Big\langle\bar{f}\sum_{l\leq m}\varepsilon_l\frac{\dot{\boldsymbol{\eta}}^l\cdot\mathbf{b}}{N}\Big\rangle_0, \tag{3.19}$$

$$\mathrm{X} = -m\beta^2 E\frac{\mathrm{th}X\exp\Delta X}{\mathrm{ch}^m X}\Big\langle\bar{f}\sum_{l\leq m}\varepsilon_l\frac{\dot{\boldsymbol{\eta}}^l\cdot\mathbf{b}}{N}\Big\rangle_0 \tag{3.20}$$

and

$$\begin{aligned}|S_1| \leq{}& \beta^4 K(m)\exp(2m\beta^2)\\ &\times E\Big\langle|\bar{f}|\Big(\sum_{l\leq m+2}\Big(\frac{\dot{\boldsymbol{\eta}}^l\cdot\mathbf{b}}{N}\Big)^2 + \sum_{l\leq l'\leq m+2}\Big(\frac{\dot{\boldsymbol{\eta}}^l\cdot\dot{\boldsymbol{\eta}}^{l'}}{N}\Big)^2\Big)\Big\rangle_0.\end{aligned} \tag{3.21}$$

There $\Delta = \sum_{l\leq m}\varepsilon_l$, and the thermal integral in (3.21) *is on Σ_{N-1}^{m+2}, with the natural identification of $|\bar{f}|$ as a function on this space. Moreover, if $h \geq 4\beta^2$, and if $|\Delta| \leq m-1$, the estimate* (3.21) *can be improved by a factor* $\exp(-h/4)$.

We will prove Proposition 3.3 later. Let us explain how it implies Theorem 3.2. First we will use it to evaluate $2EU/\widehat{Z} - E(UZ/\widehat{Z}^2)$. To evaluate $EU/\widehat{Z}$, we fix $\varepsilon_1, \dots, \varepsilon_k$ and we use Proposition 3.3 for $m = k$ and

$$\bar{f}(\boldsymbol{\eta}^1, \dots, \boldsymbol{\eta}^k) = f(\boldsymbol{\sigma}^1, \dots, \boldsymbol{\sigma}^k), \tag{3.22}$$

where $\boldsymbol{\sigma}^l = (\eta_1^l, \dots, \eta_{N-1}^l, \varepsilon_l)$. Since $\widehat{Z} = A^k \mathrm{ch}^k X$, we see after averaging over all choices $\varepsilon_1, \dots, \varepsilon_k$ that the terms VI to X create the contributions of the terms I to V, respectively, while the term S_1 is of the type (3.8). To evaluate UZ, we observe that

$$UZ = \left\langle \mathrm{Av}\, f \exp\left(\frac{\beta}{\sqrt{N}} \sum_{l \le 2k} \varepsilon_l \mathbf{g} \cdot \boldsymbol{\eta}^l + h \sum_{l \le 2k} \varepsilon_l \right) \right\rangle_0.$$

There Av means average over all values of $\varepsilon_1, \dots, \varepsilon_{2k} = \pm 1$, $\langle \cdot \rangle_0$ denotes a thermal integral in Σ_{N-1}^{2k}, and f has the same meaning as in Proposition 3.1. In particular f depends only upon $\boldsymbol{\sigma}^1, \dots, \boldsymbol{\sigma}^k$. Fixing the values of $\varepsilon_1, \dots, \varepsilon_{2k}$, we will use Proposition 3.3 for $m = 2k$, where the function.

$$\bar{f}(\boldsymbol{\eta}^1, \dots, \boldsymbol{\eta}^{2k}) = f(\boldsymbol{\sigma}^1, \dots, \boldsymbol{\sigma}^k) \tag{3.23}$$

is defined as in (3.22). The remainder (3.21) is again of the type (3.8).

CLAIM. *After averaging upon $\varepsilon_{k+1}, \dots, \varepsilon_{2k}$, the contributions of the terms* VI *to* X *relative to $UZ/\widehat{Z}^2$ are identical to the contributions of the terms* VI *to* X *relative to $U/\widehat{Z}$.*

PROOF. The basic observation is that the function $\bar{f}$ of (3.23) depends only upon $\boldsymbol{\eta}^1, \dots, \boldsymbol{\eta}^k$, so that

$$\langle \bar{f} \dot{\boldsymbol{\eta}}^l \cdot \dot{\boldsymbol{\eta}}^{l'} \rangle_0 \quad \text{and} \quad \langle \bar{f} \dot{\boldsymbol{\eta}}^{l'} \cdot \mathbf{b} \rangle_0$$

are zero unless $l, l' \le k$. This makes it obvious that the contributions of the terms VI to VIII are the same for $UZ/\widehat{Z}^2$ and $U/\widehat{Z}$, using simply the fact that

$$\operatorname*{Av}_{\varepsilon_{k+1},\, \varepsilon_{2k} = \pm 1} \exp X \sum_{l \le 2k} \varepsilon_l = (\mathrm{ch}\, X)^k \exp X \sum_{l \le k} \varepsilon_l. \tag{3.24}$$

As for the contribution of the term IX to $UZ/\widehat{Z}^2$, we split it in

$$\mathrm{XI} = \beta^2 \left(\sum_{l \le k} \varepsilon_l \right) E_g \frac{\exp X (\sum_{l \le 2k} \varepsilon_l)}{\mathrm{ch}^{2k} X} \left\langle \bar{f} \sum_{l \le k} \varepsilon_l \frac{\dot{\boldsymbol{\eta}}^l \cdot \mathbf{b}}{N} \right\rangle_0$$

and

$$\mathrm{XII} = \beta^2 \left(\sum_{k < l \le 2k} \varepsilon_l \right) E_g \frac{\exp X (\sum_{k \le l \le 2k} \varepsilon_l)}{\mathrm{ch}^{2k} X} \left\langle \bar{f} \sum_{l \le k} \varepsilon_l \frac{\dot{\boldsymbol{\eta}}^l \cdot \mathbf{b}}{N} \right\rangle_0.$$

Using (3.24), after averaging upon $\varepsilon_{k+1}, \dots, \varepsilon_{2k}$, the contribution of XI to $UZ/\widehat{Z}^2$ is the same as the contribution of the term IX relative to $U/\widehat{Z}$. After averaging over $\varepsilon_{k+1}, \dots, \varepsilon_{2k}$, the contribution of XII to $UZ/\widehat{Z}^2$ is

$$k\beta^2 E_g \frac{\exp X(\sum_{l\le k}\varepsilon_l)}{\mathrm{ch}^k X}\left\langle \bar{f}\sum_{l\le k}\varepsilon_l \frac{\dot{\boldsymbol{\eta}}^l\cdot\mathbf{b}}{N}\right\rangle_0 \tag{3.25}$$

and the contribution of X to $UZ/\widehat{Z}^2$ is

$$-2k\beta^2 E_g \frac{\exp X(\sum_{l\le k}\varepsilon_l)}{\mathrm{ch}^k X}\left\langle \bar{f}\sum_{l\le k}\varepsilon_l \frac{\dot{\boldsymbol{\eta}}^l\cdot\mathbf{b}}{N}\right\rangle_0. \tag{3.26}$$

Combining these two gives exactly the contribution of the term X relative to $U/\widehat{Z}$. $\square$

Let us rewrite (3.13) as

$$E\frac{U}{Z} = 2E\frac{U}{\widehat{Z}} - E\frac{UZ}{\widehat{Z}^2} + W, \tag{3.27}$$

where $W \ge 0$,

$$EW \le E\frac{U'}{\mathrm{ch}^k X} - 2E\frac{U'Z}{\widehat{Z}\mathrm{ch}^k X} + E\frac{U'Z^2}{\widehat{Z}^2\mathrm{ch}^k X}. \tag{3.28}$$

What we have shown at that stage is that combining the contributions of the terms VI to X relative to $2EU/\widehat{Z}$ and $EUZ/\widehat{Z}^2$ creates exactly the terms I to V of Theorem 1.3. To finish the proof, it suffices to show that the combination of the terms VI to X relative to the three terms on the right of (3.28) cancel out. But this is obvious, because the argument we use to prove the claim shows that when evaluating $E(U'Z^r/\widehat{Z}^r\mathrm{ch}^k X)$, after averaging over the ε's, these contributions do not depend upon r. Theorem 3.2 is proved. $\square$

PROOF OF PROPOSITION 3.3. We have to estimate

$$E_g \frac{\exp \Delta X}{\mathrm{ch}^m X}\left\langle \bar{f}\exp\frac{\beta}{\sqrt{N}}\sum_{l\le m}\varepsilon_l(\mathbf{g}.\dot{\boldsymbol{\eta}}^l)\right\rangle_0. \tag{3.29}$$

This is a Gaussian integral. We will first take the expectation conditionally upon X, that is, conditionally upon $\theta = (\beta/\sqrt{N})\mathbf{g}\cdot\mathbf{b}$. This expectation will be denoted by E_θ, and we first have to learn how to compute these. We will assume $\mathbf{b} \ne 0$, and we will leave the much easier case $\mathbf{b} = 0$ to the reader. Let us write

$$a(\mathbf{x}) = \frac{\mathbf{x}\cdot\mathbf{b}}{\|\mathbf{b}\|};$$

so that $\mathbf{x} - a(\mathbf{x})\mathbf{b}$ is orthogonal to $\mathbf{b}$, and thus

$$\begin{aligned} E_\theta \exp \frac{\beta}{\sqrt{N}}\mathbf{g}\cdot\mathbf{x} &= E_\theta \exp\biggl(\frac{\beta}{\sqrt{N}}\mathbf{g}\cdot(\mathbf{x} - a(\mathbf{x})\mathbf{b}) + \theta a(\mathbf{x})\biggr) \\ &= \exp\biggl(\frac{\beta^2}{2N}\|\mathbf{x} - a(\mathbf{x})\mathbf{b}\|^2 + \theta a(\mathbf{x})\biggr) \\ &= \exp\biggl(\frac{\beta^2}{2N}\|\mathbf{x}\|^2 + \theta a(\mathbf{x}) - \frac{a(\mathbf{x})^2}{2}E\theta^2\biggr), \end{aligned} \tag{3.30}$$

using the fact that $E\theta^2 = \beta^2\|\mathbf{b}\|^2/N$.

Next, we learn how to evaluate

$$E\biggl(\frac{\exp\Delta X}{\mathrm{ch}^m X}\exp\biggl(\theta a - \frac{a^2}{2}E\theta^2\biggr)\biggr). \tag{3.31}$$

It is very useful to observe the general fact (that is obvious writing the expectation as integrals) that for any function $W(\theta)$, we have

$$E\biggl(W(\theta)\exp\biggl(a\theta - \frac{a^2E\theta^2}{2}\biggr)\biggr) = EW(\theta + aE\theta^2). \tag{3.32}$$

We will use this for

$$W(x) = \frac{\exp\Delta(x+h)}{\mathrm{ch}^m(x+h)}. \tag{3.33}$$

Thus

$$W'(x) = \frac{\Delta\exp\Delta(x+h)}{\mathrm{ch}^m(x+h)} - m\frac{\exp\Delta(x+h)}{\mathrm{ch}^m(x+h)}\mathrm{th}(x+h)$$

and by elementary estimates,

$$|W''(x)| \le 5m^2\frac{\exp\Delta(x+h)}{\mathrm{ch}^m(x+h)}. \tag{3.34}$$

In particular, since $\exp\Delta(x+h) \le 2^m\mathrm{ch}(x+h)^{|\Delta|}$, we have

$$|W'(x)| \le \frac{5m^2 2^m}{\mathrm{ch}^{m-|\Delta|}(x+h)}. \tag{3.35}$$

For clarity, we will first treat the case where $|\Delta| = m$ (or, more generally when we do not attempt to use that $|\Delta| < m$). We will later indicate the necessary modifications when $|\Delta| < m$. In that case, (3.35) provides a uniform bound on W'. We then use Taylor's formula to see that the quantity (3.31) is

$$E\frac{\exp\Delta X}{\mathrm{ch}^m X} + aE\theta^2\biggl(E\Delta\frac{\exp\Delta X}{\mathrm{ch}^m X} - mE\frac{\exp\Delta X}{\mathrm{ch}^m X}\mathrm{th}\,X\biggr) + S(a), \tag{3.36}$$

where

$$|S(a)| \le 5m^2 2^m(aE\theta^2)^2. \tag{3.37}$$

We then rewrite the quantity (3.29) as

$$\left\langle \bar{f} E_g \frac{\exp \Delta X}{\operatorname{ch}^m X} \exp \frac{\beta}{\sqrt{N}} \mathbf{g} \cdot \mathbf{x} \right\rangle_0 \tag{3.38}$$

where $\mathbf{x} = \sum_{l \le m} \varepsilon_l \dot{\boldsymbol{\eta}}^l$. We write $E_g = E_g E_\theta$, so that (3.38) is

$$\left\langle \bar{f} E_g \frac{\exp \Delta X}{\operatorname{ch}^m X} E_\theta \exp \frac{\beta}{\sqrt{N}} \mathbf{g} \cdot \mathbf{x} \right\rangle_0. \tag{3.39}$$

We use (3.30) to evaluate E_θ and then the evaluation of (3.31) by (3.36), for $a = a(\mathbf{x})$, $E\theta^2 = \beta^2 \|\mathbf{b}\|^2/N$, so that

$$a(\mathbf{x}) E\theta^2 = \beta^2 \frac{\mathbf{x} \cdot \mathbf{b}}{N}. \tag{3.40}$$

Thus, the quantity (3.29) can be written as the sum

$$\mathrm{XII} + \mathrm{XIII} + \mathrm{XIV} + \mathrm{XV},$$

where

$$\mathrm{XII} = E_g \frac{\exp \Delta X}{\operatorname{ch}^m X} \left\langle \bar{f} \exp\left(\frac{\beta^2}{2N} \left\| \sum_{l \le m} \varepsilon_l \dot{\eta}^l \right\|^2 \right) \right\rangle_0, \tag{3.41}$$

$$\mathrm{XIII} = \beta^2 E_g \Delta \frac{\exp \Delta X}{\operatorname{ch}^m X} \left\langle \bar{f} \left(\sum_{l \le m} \varepsilon_l \frac{\dot{\boldsymbol{\eta}}^l \cdot \mathbf{b}}{N} \right) \exp\left(\frac{\beta^2}{2N} \left\| \sum_{l \le m} \varepsilon_l \dot{\boldsymbol{\eta}}^l \right\|^2 \right) \right\rangle_0, \tag{3.42}$$

$$\begin{aligned} \mathrm{XIV} = {} & -m\beta^2 E_g \frac{\exp \Delta X}{\operatorname{ch}^m X} \operatorname{th} X \\ & \times \left\langle \bar{f} \left(\sum_{l \le m} \varepsilon_l \frac{\dot{\boldsymbol{\eta}}^l \cdot \mathbf{b}}{N} \right) \exp\left(\frac{\beta^2}{2N} \left\| \sum_{l \le m} \varepsilon_l \dot{\boldsymbol{\eta}}^l \right\|^2 \right) \right\rangle_0, \end{aligned} \tag{3.43}$$

$$|\mathrm{XV}| \le 5\beta^4 m^2 2^m \left\langle |\bar{f}| \left| \sum_{l \le m} \varepsilon_l \frac{\dot{\boldsymbol{\eta}}^l \cdot \mathbf{b}}{N} \right|^2 \exp\left(\frac{\beta^2}{2N} \left\| \sum_{l \le m} \varepsilon_l \dot{\boldsymbol{\eta}}^l \right\|^2 \right) \right\rangle_0. \tag{3.44}$$

We have

$$\left\| \sum_{l \le m} \varepsilon_l \dot{\boldsymbol{\eta}}^l \right\| \le 2m\sqrt{N} \tag{3.45}$$

and thus

$$|\mathrm{XV}| \le 5\beta^4 m^3 2^m \exp(2m^2\beta^2) \left\langle |\bar{f}| \sum_{l \le m} \left(\frac{\dot{\boldsymbol{\eta}}^l \cdot \mathbf{b}}{N} \right)^2 \right\rangle_0. \tag{3.46}$$

To study XII to XIV, we first study

$$\left\| \sum_{l \le m} \dot{\varepsilon}_l \dot{\boldsymbol{\eta}}^l \right\|^2 = \sum_{l \le m} \|\dot{\boldsymbol{\eta}}^l\|^2 + \sum_{l < l'} 2\varepsilon_l \varepsilon_{l'} \dot{\boldsymbol{\eta}}^l \cdot \dot{\boldsymbol{\eta}}^{l'}. \tag{3.47}$$

Next,

$$\begin{aligned}\|\dot{\boldsymbol{\eta}}^{l'}\|^2 = \|\boldsymbol{\eta}^l - \mathbf{b}\|^2 &= \|\boldsymbol{\eta}^l\|^2 + \|\mathbf{b}\|^2 - 2\boldsymbol{\eta}^l \cdot \mathbf{b} \\ &= N - 1 - \|\mathbf{b}\|^2 - 2\dot{\boldsymbol{\eta}}^l \cdot \mathbf{b}\end{aligned}$$

so that

$$\exp\left(\frac{\beta^2}{2N}\left\|\sum_{l\le m} \dot{\varepsilon}_l \dot{\boldsymbol{\eta}}^l\right\|^2\right) = \exp\left(m\frac{\beta'^2}{2}(1-b)\right)e^u = A^m e^u \tag{3.48}$$

for

$$u = \frac{\beta^2}{N}\left(-\sum_{l\le m} \dot{\boldsymbol{\eta}}^l \cdot \mathbf{b} + \sum_{l<l'} \varepsilon_l \varepsilon_{l'} \dot{\boldsymbol{\eta}}^l \cdot \dot{\boldsymbol{\eta}}^{l'}\right).$$

We observe that

$$|u| \le \beta^2\left(2m + 4m\frac{(m-1)}{2}\right) = 2m^2\beta^2$$

so that, since

$$|e^u - 1 - u| \le u^2 e^u \le u^2 e^u \le u^2 2m^2 \exp(2m^2\beta^2),$$

we have

$$e^u = 1 + u + R(u) \tag{3.49}$$

for

$$|R(u)| \le 2u^2 m^2 \exp(2m^2\beta^2).$$

We use (3.49) in (3.48) and substitute in each of the terms XII to XIII. We get nine terms. Elementary estimates (such as $|ab| \le a^2 + b^2$, etc.) show that the contribution of $R(u)$ to XII, of u and $R(u)$ to XIII and XIV are swallowed by the error term (3.21). The contribution of u to XII is VII and VIII; the contributions of 1 to XIII and XIV are (3.19) and (3.20), respectively.

To finish the proof, it suffices now to show how to improve upon the error term when $|\Delta| < m$. Consider the function

$$\Phi(y) = EW(\theta + y).$$

Then, by (3.35),

$$\Phi''(y) \le 5m^2 2^m E\frac{1}{\operatorname{ch}(\theta + y + h)}.$$

Now, for $|y| \le h/2$, we have, since $E\theta^2 \le \beta^2$,

$$\begin{aligned}E\frac{1}{\operatorname{ch}(\theta + y + h)} &\le \frac{1}{\operatorname{ch}(h/4)} + P(\theta \ge h/4) \\ &\le \frac{1}{\operatorname{ch}(h/4)} + \exp\left(-\frac{h^2}{32\beta^2}\right) \le 3\exp(-h/4)\end{aligned}$$

if $h \geq 8\beta^2$. Since the quantity $aE\theta^2$ of (3.32) satisfies $|aE\theta^2| \leq 2m\beta^2$, it follows that if $2m\beta^2 \leq h/2$, then we can improve the right-hand side of (3.37) by a factor $3\exp(-h/4)$. To complete the proof, one then observes that, if $|\Delta| < m$,

$$E\frac{\exp \Delta X}{\operatorname{ch}^m X} \leq 2^{m+1}\exp\left(-\frac{h}{4}\right),$$

an inequality that is obtained by previous arguments. This completes the proof of Proposition 3.3, and hence of Theorem 3.2. □

In order to avoid repeating the same computation, let us state the following principle, which is the way Theorem 3.2 will be actually used.

PROPOSITION 3.4. *In the special case where the function* $f(\boldsymbol{\sigma}^1, \ldots, \boldsymbol{\sigma}^k)$ *of* (3.2) *is of the type*

$$f(\boldsymbol{\sigma}^1, \ldots, \boldsymbol{\sigma}^k) = \bar{f}(\boldsymbol{\eta}^1, \ldots, \boldsymbol{\eta}^k) \prod_{k \in I} \sigma_N^k, \tag{3.50}$$

where $\boldsymbol{\eta}^l = (\sigma_1^l, \ldots, \sigma_{N-1}^l)$, *and* $\bar{f}$ *is a function on* Σ_{N-1}^k, *the terms* I *to* V *of* (3.2) *as follows, where* $n = \operatorname{card} I$,

$$\mathrm{I} = E \operatorname{th}^n X \langle \bar{f} \rangle_0, \tag{3.51}$$

$$\begin{aligned}
\mathrm{II} = {} & \beta^2 E \operatorname{th}^{n-2} X \left\langle \bar{f} \sum_{l, l' \in I} \frac{\dot{\boldsymbol{\eta}}^l \cdot \dot{\boldsymbol{\eta}}^{l'}}{N} \right\rangle_0 \\
& + \beta^2 E \operatorname{th}^{n+2} X \left\langle \bar{f} \sum_{l, l' \notin I} \frac{\dot{\boldsymbol{\eta}}^l \cdot \dot{\boldsymbol{\eta}}^{l'}}{N} \right\rangle_0 \\
& + \beta^2 E \operatorname{th}^n X \left\langle \bar{f} \sum_{l < l'} \frac{\dot{\boldsymbol{\eta}}^l \cdot \dot{\boldsymbol{\eta}}^{l'}}{N} \right\rangle_0,
\end{aligned} \tag{3.52}$$

where all the sums are over $l < l'$, *and, in the third sum, we have either* $l \in I$, $l' \notin I$ *or* $l \notin I$, $l' \in I$,

$$\begin{aligned}
\mathrm{III} + \mathrm{IV} + \mathrm{V} = {} & -n\beta^2 E \operatorname{th}^n X \left\langle \bar{f} \sum_{l \in I} \frac{\dot{\boldsymbol{\eta}}^l \cdot \mathbf{b}}{N} \right\rangle_0 \\
& + (n-1)\beta^2 E \operatorname{th}^{n-2} X \left\langle \bar{f} \sum_{l \in I} \frac{\dot{\boldsymbol{\eta}}^l \cdot \mathbf{b}}{N} \right\rangle_0 \\
& - (n+1)\beta^2 E \operatorname{th}^{n+2} X \left\langle \bar{f} \sum_{l \notin I} \frac{\dot{\boldsymbol{\eta}}^l \cdot \mathbf{b}}{N} \right\rangle_0 \\
& + n\beta^2 E \operatorname{th}^n X \left\langle \bar{f} \sum_{l \notin I} \frac{\dot{\boldsymbol{\eta}}^l \cdot \mathbf{b}}{N} \right\rangle_0.
\end{aligned} \tag{3.53}$$

The proof is a tedious but perfectly straightforward computation.

PROPOSITION 3.5. *If $\bar{f}$ is a function on Σ_{N-1}^k and if*

$$f(\boldsymbol{\sigma}^1, \dots, \boldsymbol{\sigma}^k) = \bar{f}(\boldsymbol{\eta}^1, \dots, \boldsymbol{\eta}^k)(\sigma_N^1 - \sigma_N^2)\sigma_N^3, \tag{3.54}$$

then, writing $\tilde{\boldsymbol{\eta}} = \boldsymbol{\eta}^1 - \boldsymbol{\eta}^2$, *the contributions of the terms* I *to* V *of Theorem* 3.2 *are*

$$\begin{aligned} &\beta^2 E \frac{1}{\mathrm{ch}^2 X}\left\langle \bar{f}\frac{\tilde{\boldsymbol{\eta}}\cdot\boldsymbol{\eta}^3}{N}\right\rangle_0 - 3\beta^2 E\frac{\mathrm{th}^2 X}{\mathrm{ch}^2 X}\left\langle \bar{f}\frac{\tilde{\boldsymbol{\eta}}\cdot\mathbf{b}}{N}\right\rangle_0 \\ &\quad + \beta^2 E\frac{\mathrm{th}^2 X}{\mathrm{ch}^2 X}\left\langle \bar{f}\sum_{4\le l\le k}\frac{\tilde{\boldsymbol{\eta}}\cdot\dot{\boldsymbol{\eta}}^l}{N}\right\rangle_0. \end{aligned} \tag{3.55}$$

The proof is again a straightforward computation. We use Proposition 3.4 for $I = \{1, 3\}$, then for $I = \{2, 3\}$, and substract the corresponding results.

It is good to observe that the function f of (3.55) satisfies

$$f \neq 0 \Rightarrow \left|\sum_{l\le k}\sigma_N^l\right| \le k-2$$

because when $f \neq 0$, we have $\sigma_N^1 = -\sigma_N^2$. Thus, if needed in the estimation of the remainder term of Theorem 1.3, we can use the "improved version" with extra factor $\exp(-h/4)$. This will be used in the proof of Theorem 1.6.

The simplest case of application of Theorem 3.2 is as follows.

PROPOSITION 3.6. *If $\bar{f}$ is a function on Σ_{N-1}^k, and if*

$$f(\boldsymbol{\sigma}^1, \dots, \boldsymbol{\sigma}^k) = \bar{f}(\boldsymbol{\eta}^1, \dots, \boldsymbol{\eta}^k)(\sigma_N^1 - \sigma_N^2)(\sigma_N^3 - \sigma_N^4), \tag{3.56}$$

then the contribution of the terms I *to* V *of Theorem* 3.2 *is*

$$\beta^2 E\frac{1}{\mathrm{ch}^4 X}\left\langle \bar{f}\frac{(\boldsymbol{\eta}^1-\boldsymbol{\eta}^2)\cdot(\boldsymbol{\eta}^3-\boldsymbol{\eta}^4)}{N}\right\rangle_0. \tag{3.57}$$

For the proof, apply Proposition 3.5 twice.

We now turn to the proof of Theorem 1.3. To make the proof understandable, we will first prove a simpler and weaker result. All our proofs will be at h fixed.

DEFINITION 3.7. Given h, and given β_0, we say that there is uniform absence of symmetry breaking at (β_0, h) if (1.9) holds uniformly in a neighborhood of β_0; that is, if there exists $\delta > 0$ such that given ε_0, we can find $t_0 > 0$, N_0 such that

$$|\beta - \beta_0| \le \delta,\ |t| < t_0 \text{ and } N > N_0 \Rightarrow |\varphi_{N,\beta}(t) - \varphi_{N,\beta}(0)| \le \varepsilon_0,$$

where $\varphi_{N,\beta}(t)$ is given by (1.7), at inverse temperature β and external field $h' = h/\beta$.

We will prove the following.

PROPOSITION 3.8. *If there is uniform absence of replica symmetry breaking at (β_0, h), then (β_0, h) belongs to the high temperature region.*

Of course, this also means that if (β_0, h) belongs to the low temperature region there can never be uniform absence of replica symmetry breaking at (β_0, h). After proving Proposition 3.8, it will be a separate task to explain how to modify the proof so that the work "uniform" can be removed, at the expense of possibly losing control of a small (in the sense of Baire category) set of parameters. This will show that at the generic point of the low temperature region, there is no absence of replica symmetry breaking; that is, there is replica symmetry breaking.

We now start the proof of Proposition 3.8. As we assume that there is uniform absence of replica symmetry breaking at β_0, Proposition 1.4 shows that (1.11) holds uniformly for β in a certain neighborhood of β_0 (always at given h). This condition will be used as follows (where, as usual, $\tilde{\boldsymbol{\sigma}} = \boldsymbol{\sigma}^1 - \boldsymbol{\sigma}^2$). If f is any function of $\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2, \ldots$ then in estimating $E\langle(\tilde{\boldsymbol{\sigma}}\cdot\boldsymbol{\sigma}^3)f\rangle$, given $u > 0$ we can write this quantity as $E\langle(\tilde{\boldsymbol{\sigma}}\cdot\boldsymbol{\sigma}^3)1_{\{|\tilde{\boldsymbol{\sigma}}\cdot\boldsymbol{\sigma}^3|\le Nu\}}f\rangle$ with error at most $N\|f\|_\infty E\langle 1_{\{|\tilde{\boldsymbol{\sigma}}\cdot\boldsymbol{\sigma}^3|\ge Nu\}}\rangle$ so that, if, say, $\|f\|_\infty \le N^2$, this error is exponentially small in N. To simplify the exposition, we will not write these exponentially small terms. We will describe the above phenomenon by saying that eventually, only the small $(= o(1))$ values of $\tilde{\boldsymbol{\sigma}}\cdot\boldsymbol{\sigma}^3/N$ are relevant.

The principle of the proof is to establish relation (3.68) below. This is exactly how physicists derive the low temperature region using the cavity method; the difference is that we control regorously the error terms rather than pretending that we are "at the limit" and ignoring them. The main tool for this is Theorem 3.2. To use this theorem successfully, we need to know that the r.v. $N^{-1}\langle\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}'\rangle$ does not behave pathologically (see note added in proof). This will be proved using "thermodynamical" arguments resembling Proposition 5.1 of [5]. In order to make the proof readable, these are delayed until the end of the main computation. These arguments will first imply [as a consequence of (11)] that the variance of $N^{-1}\langle\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}'\rangle$ goes to zero, "for most β", in a sense that was made precise in [5]. For clarity, we first pretend that this is the case for all β in a neighborhood of β_0. Changing N into $N-1$, we see that the variance of $N^{-1}\|\mathbf{b}\|^2 = \langle\boldsymbol{\eta}\cdot\boldsymbol{\eta}'/N\rangle_0$ also goes to zero.

We set

$$q_N(\beta) = E\left\langle\frac{\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}'}{N}\right\rangle. \tag{3.58}$$

We evaluate $q_N(\beta)$ in two different ways. We write, using symmetry among the variables, that

$$q_N(\beta) = E\langle\sigma_N\sigma_N'\rangle = E\,\mathrm{th}^2X + o(1), \tag{3.59}$$

$$\begin{aligned} q_N(\beta) = E\langle\sigma_N\sigma_N'\rangle &= \frac{1}{N-1}E\langle\boldsymbol{\eta}\cdot\boldsymbol{\eta}'\rangle, \\ &= E\left\langle\frac{\boldsymbol{\eta}\cdot\boldsymbol{\eta}'}{N-1}\right\rangle_0 + o(1) \qquad (3.60)\\ &= q_{N-1}(\beta') + o(1). \end{aligned}$$

Each of these relations follows from Proposition 3.4, where it is now obvious that all the terms but (3.51) are $o(1)$, and that the remainder (3.8) is also $o(1)$ because, as we explained, only the $o(1)$ values of $\tilde{\boldsymbol{\sigma}} \cdot \boldsymbol{\sigma}^3$ are relevant. Now, since the variance of $N^{-1/2}\|\mathbf{b}\|$ vanishes asymptotically, (3.59) implies that

$$q_N(\beta) = E\,\mathrm{th}^2\Big(g\beta\sqrt{q_{N-1}(\beta')} + h\Big) + o(1)$$

and comparing with (3.60) we see that $q_{N-1}(\beta')$ [and hence $q_N(\beta)$] converges to a root $q(\beta)$ of the equation

$$q = E\,\mathrm{th}^2(g\beta\sqrt{q} + h).$$

Consider now the quantity

$$C_N = C_N(\beta) = E\langle(\tilde{\boldsymbol{\sigma}} \cdot (\boldsymbol{\sigma}^3 - \boldsymbol{\sigma}^4))^2\rangle.$$

By symmetry among the variables,

$$\begin{aligned} C_N &= NE\langle\tilde{\sigma}_N(\sigma_N^3 - \sigma_N^4)\tilde{\boldsymbol{\sigma}} \cdot (\boldsymbol{\sigma}^3 - \boldsymbol{\sigma}^4)\rangle \\ &= NE\big\langle\big(\tilde{\sigma}_N(\sigma_N^3 - \sigma_N^4)\big)^2\big\rangle \\ &\quad + NE\langle\tilde{\sigma}_N(\sigma_N^3 - \sigma_N^4)\tilde{\boldsymbol{\eta}} \cdot (\boldsymbol{\eta}^3 - \boldsymbol{\eta}^4)\rangle. \end{aligned} \tag{3.61}$$

The two terms on the right will be computed through Proposition 3.4. Using this proposition with $\bar{f} = 1$, a simple computation shows that

$$E\big\langle\big(\tilde{\sigma}_N(\sigma_N^3 - \sigma_N^4)\big)^2\big\rangle = 4E\,\mathrm{th}^4 Y + o(1),$$

where $Y = g\beta\sqrt{q(\beta)} + h$. To handle the other term, we use Proposition 3.6 to write

$$NE\langle\tilde{\sigma}_N(\sigma_N^3 - \sigma_N^4)\tilde{\boldsymbol{\eta}} \cdot (\boldsymbol{\eta}^3 - \boldsymbol{\eta}^4)\rangle = E\frac{\beta^2}{\mathrm{ch}^4 X}\big\langle\big((\tilde{\boldsymbol{\eta}} \cdot (\boldsymbol{\eta}^3 - \boldsymbol{\eta}^4)\big)^2\big\rangle_0 + S,$$

where

$$|S| \le N\bigg\langle\sum_{l,\,l' \le 6}\big|\tilde{\boldsymbol{\eta}} \cdot (\boldsymbol{\eta}^3 - \boldsymbol{\eta}^4)\big|\bigg(\frac{(\dot{\boldsymbol{\eta}}^l \cdot \dot{\boldsymbol{\eta}}^{l'})^2}{N} + \frac{(\dot{\boldsymbol{\eta}}^l \cdot \mathbf{b})^2}{N}\bigg)\bigg\rangle_0.$$

Use of Cauchy–Schwarz and Jensen's inequality shows that

$$|S| \le LN^2 E\bigg\langle\bigg|\frac{\tilde{\boldsymbol{\eta}} \cdot \boldsymbol{\eta}^3}{N}\bigg|^3\bigg\rangle_0. \tag{3.62}$$

As previously explained, it follows from (1.11) that the contribution to the right-hand side of (3.62) eventually comes from arbitrarily small values of $|(\boldsymbol{\eta}^1 - \boldsymbol{\eta}^2) \cdot \boldsymbol{\eta}^3|/N$, so that $|S| = o(D_{N-1})$, where

$$D_N = E\langle(\tilde{\boldsymbol{\sigma}} \cdot \boldsymbol{\sigma}^3)^2\rangle.$$

Consider the function

$$\xi(x) = E\frac{\beta^2}{\mathrm{ch}^4\beta(g\sqrt{x} + h)}$$

so that

$$E\frac{\beta^2}{\operatorname{ch}^4 X}\big\langle(\tilde{\boldsymbol{\eta}}\cdot(\boldsymbol{\eta}^3-\boldsymbol{\eta}^4))^2\big\rangle = E\bigg(\xi\bigg(\frac{\|\mathbf{b}\|}{\sqrt{N}}\bigg)\big\langle(\tilde{\boldsymbol{\eta}}\cdot(\boldsymbol{\eta}^3-\boldsymbol{\eta}^4))^2\big\rangle\bigg).$$

Even though we know that $\|\mathbf{b}\|^2/N = \langle \boldsymbol{\eta}\cdot\boldsymbol{\eta}'/N\rangle_0$ is in probability close to $q(\beta)$, it could in principle be the case that a fixed proportion of the contribution to $E\langle(\tilde{\boldsymbol{\eta}}\cdot(\boldsymbol{\eta}^3-\boldsymbol{\eta}^4))^2\rangle_0$ comes from the vanishing event where $|\langle\boldsymbol{\eta}\cdot\boldsymbol{\eta}'/N\rangle_0 - q(\beta)| \geq a$ for some $a > 0$ independent of N. It will be shown at the end of the proof that this pathology does not happen, so that we have

$$E\bigg(\frac{\beta^2}{\operatorname{ch}^4 X}\big\langle(\tilde{\boldsymbol{\eta}}\cdot(\boldsymbol{\eta}^3-\boldsymbol{\eta}^4))^2\big\rangle_0\bigg) = \bigg(E\frac{\beta^2}{\operatorname{ch}^4 Y}\bigg)C_{N-1} + o(C_{N-1}), \tag{3.63}$$

where $C_{N-1} = C_{N-1}(\beta')$.

It is obvious that $C_N \leq 4D_N$. Thus, we get the estimate

$$C_N = \bigg(E\frac{\beta^2}{\operatorname{ch}^4 Y}\bigg)C_{N-1} + o(D_{N-1}) + 4NE\operatorname{th}^4 Y + o(N). \tag{3.64}$$

If we knew that $o(D_{N-1})$ is also $o(C_{N-1})$, we could deduce from (3.64) that $E(\beta^2/\operatorname{ch}^4 Y) \leq 1$. As this holds in a neighborhood of β_0, we must have strict inequality at β_0, which is what we want to prove. Unfortunately, even though it will a posteriori be true that both C_N and D_N are of order N, I do not know how to prove a priori that they are of the same order, and the rest of the argument is devoted to pass this difficulty. This argument involves several relations such as (3.64), which will not be detailed. First, it will help to know that C_N and C_{N-1} are close. To see this, we write

$$C_N = E\big\langle(\tilde{\boldsymbol{\eta}}\cdot(\boldsymbol{\eta}^3-\boldsymbol{\eta}^4) + \tilde{\sigma}_N\cdot(\sigma_N^3-\sigma_N^4))^2\big\rangle.$$

We expand the square and we compute each term with Proposition 3.4. We get

$$C_N = C_{N-1} + o(D_{N-1}) + o(N)$$

and thus (3.64) yields

$$C_{N-1} = \theta C_{N-1} + o(D_{N-1}) + 4NE\frac{1}{\operatorname{ch}^4 Y} + o(N) \tag{3.65}$$

for $\theta = \theta(\beta) = E(\beta^2/\operatorname{ch}^4 Y)$.

Next, we compute

$$\begin{aligned} D_N &= NE\big\langle(\tilde{\sigma}_N\sigma_N^3)(\tilde{\boldsymbol{\sigma}}\cdot\boldsymbol{\sigma}^3)\big\rangle \\ &= NE\big\langle(\tilde{\sigma}_N\sigma_N^3)^2\big\rangle + NE\big\langle(\tilde{\sigma}_N\sigma_N^3)\tilde{\boldsymbol{\eta}}\cdot\boldsymbol{\eta}^3\big\rangle. \end{aligned}$$

This is handled as (3.61), except that we now use Proposition 3.5 rather than Proposition 3.6 to handle the last term. We get, using that $\langle(\tilde{\boldsymbol{\eta}}\cdot\boldsymbol{\eta}^3)(\tilde{\boldsymbol{\eta}}\cdot\mathbf{b})\rangle_0 = \langle(\tilde{\boldsymbol{\eta}}\cdot\boldsymbol{\eta}^3)^2\rangle - \frac{1}{2}\langle(\tilde{\boldsymbol{\eta}}\cdot(\boldsymbol{\eta}^3-\boldsymbol{\eta}^4))^2\rangle$,

$$D_N = \gamma D_{N-1} + \delta C_{N-1} + o(D_{N-1}) + 2NE\frac{1}{\operatorname{ch}^2 Y} + o(N), \tag{3.66}$$

where

$$\gamma = \gamma(\beta) = \beta^2 E\biggl(\frac{1}{\mathrm{ch}^2 Y} - \frac{3\,\mathrm{th}^2 Y}{\mathrm{ch}^2 Y}\biggr); \qquad \delta = \frac{3}{2}\beta^2 E \frac{\mathrm{th}^2 Y}{\mathrm{ch}^2 Y}.$$

As in the case of C_N, we show that $D_N \simeq D_{N-1}$, so that (3.66) implies

$$D_{N-1} = \gamma D_{N-1} + \delta C_{N-1} + o(D_{N-1}) + 2NE\frac{1}{\mathrm{ch}^2 Y} + o(N). \tag{3.67}$$

We now prove that $\theta(\beta_0) < 1$. Assuming for contradiction that $\theta(\beta) \geq 1$, we can find β_1 close enough to β_0 so that (1.11) holds, but that $\theta(\beta_1) > 1$, $\gamma(\beta_1) \neq 1$. Then (3.67) implies, since $\gamma(\beta_1) \neq 1$, that $D_{N-1} = O(C_{N-1} + N)$. Substitution of this in (3.65) shows that

$$C_{N-1} = \theta(\beta_1) C_{N-1} + o(C_{N-1}) + 4NE\frac{1}{\mathrm{ch}^4 Y} + o(N) \tag{3.68}$$

so that for large N we reach the contradiction that $C_{N-1} > C_{N-1}$. $\square$

Let us now address the technical points that were left aside, and in particular the issue of the behavior of $\langle \boldsymbol{\sigma} \cdot \boldsymbol{\sigma}' \rangle / N$. The reader will check that what was needed to make the previous proof complete is the following.

LEMMA 3.9. *Given an interval I such that* (1.11) *holds uniformly in I and $\varepsilon > 0$, we can find β in I and N such that the following holds*:

$$E\biggl(\biggl\langle \frac{\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}'}{N}\biggr\rangle - q_N(\beta)\biggr)^2 \leq \varepsilon, \tag{3.69}$$

$$E\biggl(\biggl(\biggl\langle \frac{\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}'}{N}\biggr\rangle - q_N(\beta)\biggr)^2 \biggl\langle \biggl(\frac{\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}^3}{N}\biggr)^2 \biggr\rangle\biggr) \leq \varepsilon E\biggl\langle \biggl(\frac{\tilde{\boldsymbol{\sigma}}\cdot\boldsymbol{\sigma}^3}{N}\biggr)^2 \biggr\rangle + \exp(-N/K). \tag{3.70}$$

It is useful to note [for obtaining (3.63)] that in the left-hand side of (3.70), one can replace $\tilde{\boldsymbol{\sigma}} \cdot \boldsymbol{\sigma}^3$ by $\tilde{\boldsymbol{\sigma}} \cdot (\boldsymbol{\sigma}^3 - \boldsymbol{\sigma}^4)$.

PROOF. We consider the random convex function

$$f_N(\beta) = \frac{1}{N} \log F_N(\beta)$$

so that

$$\frac{\partial f_N(\beta)}{\partial \beta} = \frac{1}{N^{3/2}} \sum_{i<j} g_{ij} \langle \sigma_i \sigma_j \rangle.$$

We note the important fact that, by integration by parts,

$$E g_{ij} \langle \sigma_i \sigma_j \rangle = E \frac{\partial}{\partial g_{ij}} \langle \sigma_i \sigma_j \rangle = \frac{\beta}{\sqrt{N}} (1 - E\langle \sigma_i \sigma_j \rangle^2)$$

so that if we set

$$r = \frac{\beta}{N^2} \sum_{i<j} \langle \sigma_i \sigma_j \rangle^2$$

we have

$$r - Er = -\left(\frac{\partial f_N(\beta)}{\partial \beta} - E \frac{\partial f_N(\beta)}{\partial \beta} \right) + \frac{1}{N^{3/2}} \sum_{i<j} \left(g_{ij} \langle \sigma_i \sigma_j \rangle - \frac{\partial}{\partial g_{ij}} \langle \sigma_i \sigma_j \rangle \right)$$

and thus

$$(3.71) \quad \begin{aligned} (r - Er)^2 &\le 2 \left(\frac{\partial f_N(\beta)}{\partial \beta} - E \frac{\partial f_N(\beta)}{\partial \beta} \right)^2 \\ &\quad + 2 \left(\frac{1}{N^{3/2}} \sum_{i<j} \left(g_{ij} \langle \sigma_i \sigma_j \rangle - \frac{\partial}{\partial g_{ij}} \langle \sigma_i \sigma_j \rangle \right) \right)^2 =: 2A^2 + 2B^2. \end{aligned}$$

We will prove only (3.70), leaving the similar and easier (3.69) to the reader. We will prove the following, given $\varepsilon' > 0$:

(3.72) For all β, if N is large enough, we have

$$E\left(B^2 \left\langle \left(\frac{\tilde{\boldsymbol{\sigma}} \cdot \boldsymbol{\sigma}^3}{N} \right)^2 \right\rangle \right) \le \varepsilon' E \left\langle \left(\frac{\tilde{\boldsymbol{\sigma}} \cdot \boldsymbol{\sigma}^3}{N} \right)^2 \right\rangle.$$

(3.73) If for some $t > 0$, we have

$$(3.73\text{a}) \qquad |Ef_N(\beta + t) - Ef_N(\beta - t) - 2Ef_N(\beta)| \le t\varepsilon'/4,$$

then if N is large enough, we have

$$E\left(A^2 \left\langle \left(\frac{\tilde{\boldsymbol{\sigma}} \cdot \boldsymbol{\sigma}^3}{N} \right)^2 \right\rangle \right) \le \varepsilon' E \left\langle \left(\frac{\tilde{\boldsymbol{\sigma}} \cdot \boldsymbol{\sigma}^3}{N} \right)^2 \right\rangle.$$

Together with (3.71) this proves that

$$(3.74) \qquad E\left((r - Er)^2 \left\langle \left(\frac{\tilde{\boldsymbol{\sigma}} \cdot \boldsymbol{\sigma}^3}{N} \right)^2 \right\rangle \right) \le 4\varepsilon' E \left\langle \left(\frac{\tilde{\boldsymbol{\sigma}} \cdot \boldsymbol{\sigma}^3}{N} \right)^2 \right\rangle.$$

Let us first explain why this will prove (3.70). We have

$$\begin{aligned} \frac{r}{\beta} = \frac{1}{N^2} \sum_{i<j} \langle \sigma_i \sigma_j \rangle^2 &= \left\langle \frac{1}{N^2} \sum_{i<j} \sigma_i \sigma_i' \sigma_j \sigma_j' \right\rangle \\ &= \frac{1}{2} \left\langle \left(\frac{\boldsymbol{\sigma} \cdot \boldsymbol{\sigma}'}{N} \right)^2 \right\rangle - \frac{1}{2N}. \end{aligned}$$

Now

$$X_N = \left\langle \left(\frac{\boldsymbol{\sigma} \cdot \boldsymbol{\sigma}'}{N} \right)^2 \right\rangle - \left\langle \frac{\boldsymbol{\sigma} \cdot \boldsymbol{\sigma}'}{N} \right\rangle^2 = \left\langle \left(\frac{\boldsymbol{\sigma} \cdot \boldsymbol{\sigma}'}{N} - \left\langle \frac{\boldsymbol{\sigma} \cdot \boldsymbol{\sigma}'}{N} \right\rangle \right)^2 \right\rangle$$

is controlled by (1.11) [For each $a > 0$, we have $P(X_N \geq a) \leq \exp(-N/K)$ for N large enough] and thus (3.74) implies that for large N we have

$$\begin{aligned}\frac{\beta^2}{4} E\biggl(\biggl(\biggl\langle\frac{\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}'}{N}\biggr\rangle^2 - E\biggl(\biggl\langle\frac{\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}'}{N}\biggr\rangle^2\biggr)\biggr)^2 \biggl\langle\biggl(\frac{\tilde{\boldsymbol{\sigma}}\cdot\boldsymbol{\sigma}^3}{N}\biggr)^2\biggr\rangle\biggr) \\ \leq 4\varepsilon'\biggl\langle\biggl(\frac{\tilde{\boldsymbol{\sigma}}\cdot\boldsymbol{\sigma}^3}{N}\biggr)^2\biggr\rangle.\end{aligned} \tag{3.75}$$

Since $|E(\langle\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}'/N\rangle^2) - q_N(\beta)^2|$ can be made small by (3.69), this implies (3.70), provided $q(\beta) > 0$ (which is true for $h > 0$).

So, it remains to prove (3.72), (3.73). The proof of (3.73) is easy but tedious (see [5], Proposition 4.3, for complete details of a similar result) so we just sketch it. We write

$$\frac{f_N(\beta) - f_N(\beta - t)}{t} \leq \frac{\partial}{\partial\beta} f_N(\beta) \leq \frac{f_N(\beta + t) - f_N(\beta)}{t}$$

and we use concentration of measure to argue that for each $\eta > 0$, the probability that $|f_N(\beta) - Ef_N(\beta)| \geq \eta$ is exponentially small, and after elementary manipulations, we get that $P(|\langle\partial/\partial\beta) f_N(\beta) - E(\partial f_N/\partial\beta)(\beta)| \geq \varepsilon' + \eta') \leq \exp(-N/K)$ for each $\eta' > 0$, which of course is sufficient.

The proof of (3.72) is also easy and tedious. One simply expands B^2, and one integrates by parts twice to eliminate all the factors g_{ij}. One finds a sum of terms of the type

$$\biggl\langle\biggl(\frac{\tilde{\boldsymbol{\sigma}}\cdot\boldsymbol{\sigma}^3}{N}\biggr)^2\biggl(\biggl\langle\frac{\boldsymbol{\sigma}^{l_1}\cdot\boldsymbol{\sigma}^{l_2}}{N}\biggr\rangle^2 - \biggl\langle\frac{\boldsymbol{\sigma}^{l_3}\cdot\boldsymbol{\sigma}^{l_4}}{N}\biggr\rangle^2\biggr)\biggr\rangle, \tag{3.76}$$

where l_1, l_2, l_3, l_4 are replica indices, and of terms that are at most (K/N) $E\langle(\tilde{\boldsymbol{\sigma}}\cdot\boldsymbol{\sigma}^3/N)^2\rangle$ for obvious reasons. But terms such as (3.76) are controlled by (1.11). □

We now turn to the proof of Theorem 1.3. We recall that a subset of $\mathbb{R}^2$ is said to be of first Baire category if it is contained in a countable union of closed sets with empty interior. We will prove the following, that is more precise than Theorem 1.3.

THEOREM 3.10. *If we fix $h > 0$, the set of β for which $\theta(\beta) > 1$, and there is absence of replica symmetry breaking at (β, h) and for which there is of first Baire category.*

Thus, given $h > 0$, there is replica symmetry breaking at the "generic" point β where $\theta(\beta) > 1$. We of course expect that this should be true at every point of the low temperature region.

PROOF. We fix h once and for all. Given integers k, l, N, we consider the set

$$U(k, l, N) = \{\beta; EG_N^2\{|\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}' - \langle\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}'\rangle| \geq 2^{-k}N\} \leq \exp(-2^{-l}N).$$

We observe that $U(k, l, N)$ is closed (because everything depends continuously upon β at N finite). Thus the set

$$V(k, l, N_0) = \bigcap_{N \geq N_0} U(k, l, N)$$

is closed. Consider the set W of those β such that $\theta(\beta) \geq 1$ and that there is absence of replica symmetry breaking at (β, h). Then

$$W \subset \bigcap_k \bigcup_l \bigcup_{N_0} V(k, l, N_0)$$

by definition. Let us assume for contradiction that W is not of first Baire category. We will appeal to an elementary topological argument. If a set W is not of first Baire category, and if $W \subset \bigcap_k \bigcup_n V(k, n)$, where the sets $V(k, n)$ are closed, then $W \cap \bigcap_k \bigcup_n \overset{\circ}{V}(k, n)$ is not of first Baire category, where $\overset{\circ}{V}(k, n)$ is the interior of $V(k, n)$. This is simply because

$$\bigcap_k \bigcup_n V(k, n) \Big\backslash \left(\bigcap_k \bigcup_n \overset{\circ}{V}(k, n) \right) \subset \bigcup_{k, n} \left(V(k, n) \backslash \overset{\circ}{V}(k, n) \right)$$

is of first Baire category, and because the union of two (or even countably many) sets of first Baire category is still of first Baire category. This shows that we can find β_1 with $\theta(\beta_1) > 1$, $\gamma(\beta_1) \neq 1$ with the property that for each k, there are integers $l(k)$, $N_0(k)$ such that β_1 is in the interior of $U(k, l(k), N_0(k))$; that is, for some $\delta = \delta(k) > 0$ we have

$$\begin{aligned} &|\beta - \beta_1| < \delta \Rightarrow \forall\, N \geq N_0(k), \\ &\qquad\qquad EG_N^2\{|\boldsymbol{\sigma} \cdot \boldsymbol{\sigma}' - \langle \boldsymbol{\sigma} \cdot \boldsymbol{\sigma}' \rangle| \geq 2^{-k} N\} \leq \exp(-2^{l(k)} N). \end{aligned} \tag{3.77}$$

The reader will check that this is sufficient to obtain a contradiction along the lines of the proof of Proposition 3.8. [Equation (3.77) gives us enough room to use (3.73).] $\square$

4. Exponential inequalities I.

As a first stage toward the control of

$$E\big\langle \exp t(\boldsymbol{\sigma} \cdot \boldsymbol{\sigma}' - E\langle \boldsymbol{\sigma} \cdot \boldsymbol{\sigma}' \rangle)\big\rangle$$

we will control

$$E\big\langle \exp t(\boldsymbol{\sigma} \cdot \boldsymbol{\sigma}' - \langle \boldsymbol{\sigma} \cdot \boldsymbol{\sigma}' \rangle)\big\rangle.$$

Rather than dealing with $\boldsymbol{\sigma} \cdot \boldsymbol{\sigma}' - \langle \boldsymbol{\sigma} \cdot \boldsymbol{\sigma}' \rangle$, we find it convenient to use a more symmetric expression, namely $(\boldsymbol{\sigma}^1 - \boldsymbol{\sigma}^2) \cdot \boldsymbol{\sigma}^3$. We will prove statements of the type

$$E\langle \exp t\tilde{\boldsymbol{\sigma}} \cdot \boldsymbol{\sigma}^3 \rangle \leq \exp t^2 NM, \tag{4.1}$$

where $\tilde{\boldsymbol{\sigma}} = \boldsymbol{\sigma}^1 - \boldsymbol{\sigma}^2$. To prove such a statement it suffices, by symmetry, to consider $t \geq 0$. We assume $t \geq 0$ in this section.

To understand how (4.1) relates to a statement of the type

$$E\big\langle \exp t(\boldsymbol{\sigma} \cdot \boldsymbol{\sigma}' - \langle \boldsymbol{\sigma} \cdot \boldsymbol{\sigma}' \rangle)\big\rangle \leq \exp t^2 NM', \tag{4.2}$$

we observe that by Jensen's inequality we have, setting $\mathbf{b}_1 = \langle \boldsymbol{\sigma} \rangle$,

$$\langle \exp t(\boldsymbol{\sigma}^1 - \mathbf{b}_1) \cdot \boldsymbol{\sigma}^3 \rangle \le \langle \exp t\tilde{\boldsymbol{\sigma}} \cdot \boldsymbol{\sigma}^4 \rangle, \tag{4.3}$$

$$\langle \exp t(\boldsymbol{\sigma}^1 - \mathbf{b}_1) \cdot \mathbf{b}_1 \rangle \le \langle \exp t\tilde{\boldsymbol{\sigma}} \cdot \boldsymbol{\sigma}^3 \rangle \tag{4.4}$$

so that, writing

$$\boldsymbol{\sigma} \cdot \boldsymbol{\sigma}' - \langle \boldsymbol{\sigma} \cdot \boldsymbol{\sigma}' \rangle = (\boldsymbol{\sigma} - \mathbf{b}_1) \cdot \boldsymbol{\sigma}' + \mathbf{b}_1 \cdot (\boldsymbol{\sigma}' - \mathbf{b}_1)$$

and using (4.3), (4.4) and Cauchy–Schwarz, (4.2) follows from (4.1).

THEOREM 4.1. *There exists a number L_0 such that if $\beta L_0 \le 1$, then*

$$\forall\ t, \forall\ N,\ E\langle \exp t\tilde{\boldsymbol{\sigma}} \cdot \boldsymbol{\sigma}^3 \rangle \le \exp t^2 N L_0. \tag{4.5}$$

PROOF. We will try to control the derivative of the left-hand side of (4.5); that is

$$U_n(t, \beta) = E\langle \tilde{\boldsymbol{\sigma}} \cdot \boldsymbol{\sigma}^3 \exp t\tilde{\boldsymbol{\sigma}} \cdot \boldsymbol{\sigma}^3 \rangle \tag{4.6}$$

By symmetry between the variables, we have

$$\begin{aligned} U_N(t, \beta) &= NE\langle \tilde{\sigma}_N \sigma_N^3 \exp t\tilde{\boldsymbol{\sigma}} \cdot \boldsymbol{\sigma}^3 \rangle \\ &= NE\langle \tilde{\sigma}_N \sigma_N^3 \exp t\tilde{\sigma}_N \sigma_N^3 \exp t\tilde{\boldsymbol{\eta}} \cdot \boldsymbol{\eta}^3 \rangle \end{aligned} \tag{4.7}$$

using the notation of Section 3. Now $\tilde{\sigma}_N \sigma_N^3 \in \{-2, 0, 2\}$ so that

$$\tilde{\sigma}_N \sigma_N^3 \exp t\tilde{\sigma}_N \sigma_N^3 \le \tilde{\sigma}_N \sigma_N^3 \operatorname{ch} 2t + \operatorname{sh} 2t$$

and, since $\operatorname{sh} x \le x \operatorname{ch} x$ for $x \ge 0$, we have

$$\begin{aligned} U_n(t, \beta) \le N \operatorname{ch} 2t \big(&E\langle \tilde{\sigma}_N \sigma_N^3 \exp t\tilde{\boldsymbol{\eta}} \cdot \boldsymbol{\eta}^3 \rangle \\ &+ 2tE\langle \exp t\tilde{\boldsymbol{\eta}} \cdot \boldsymbol{\eta}^3 \rangle \big). \end{aligned} \tag{4.8}$$

We use Theorem 3.2 and Proposition 3.5 to see that

$$\begin{aligned} NE\langle \tilde{\sigma}_N \sigma_N^3 \exp t\tilde{\boldsymbol{\eta}} \cdot \boldsymbol{\eta}^3 \rangle &= NE\langle \tilde{\sigma}_N \sigma_N^3 (\exp t\tilde{\boldsymbol{\eta}} \cdot \boldsymbol{\eta}^3 - 1) \rangle \\ &= E \frac{\beta^2}{\operatorname{ch}^2 X} \langle \tilde{\boldsymbol{\eta}} \cdot \boldsymbol{\eta}^3 \exp t\tilde{\boldsymbol{\eta}} \cdot \boldsymbol{\eta}^3 \rangle_0 \\ &\quad - 3E\beta^2 \frac{\operatorname{th}^2 X}{\operatorname{ch}^2 X} \langle \tilde{\boldsymbol{\eta}} \cdot \mathbf{b} \exp t\tilde{\boldsymbol{\eta}} \cdot \boldsymbol{\eta}^3 \rangle_0 + S, \end{aligned} \tag{4.9}$$

where

$$\begin{aligned} |S| \le\ &\frac{K}{N} \beta^4 \exp 16\beta^2 \\ &\times E\Big\langle |\exp t\tilde{\boldsymbol{\eta}} \cdot \boldsymbol{\eta}^3 - 1| \Big\langle \sum_{l \le 5} (\dot{\boldsymbol{\eta}}^l \cdot \mathbf{b})^2 + \sum_{l \le l' \le 6} (\dot{\boldsymbol{\eta}}^l \cdot \dot{\boldsymbol{\eta}}^{l'})^2 \Big\rangle \Big\rangle_0. \end{aligned} \tag{4.10}$$

Next, we need a bound for $E\langle \exp t\tilde{\boldsymbol\eta}\cdot\boldsymbol\eta^3\rangle$. For this, we simply go back to (3.1), use that $Z \geq 1$ and we get

$$E\langle \exp t\tilde{\boldsymbol\eta}\cdot\boldsymbol\eta^3\rangle \leq \exp 8\beta^2 E\langle \exp t\tilde{\boldsymbol\eta}\cdot\boldsymbol\eta^3\rangle_0. \tag{4.11}$$

We now observe the nice fact that

$$\langle \tilde{\boldsymbol\eta}\cdot\boldsymbol\eta^3 \exp t\tilde{\boldsymbol\eta}\cdot\boldsymbol\eta^3\rangle_0 \geq 0 \tag{4.12}$$

so that

$$\begin{aligned} E\frac{1}{\operatorname{ch}^2 X}\langle \tilde{\boldsymbol\eta}\cdot\boldsymbol\eta^3 \exp t\tilde{\boldsymbol\eta}\cdot\boldsymbol\eta^3\rangle_0 &\leq E\langle \tilde{\boldsymbol\eta}\cdot\boldsymbol\eta^3 \exp t\tilde{\boldsymbol\eta}\cdot\boldsymbol\eta^3\rangle_0 \\ &= U_{N-1}(t,\beta'). \end{aligned} \tag{4.13}$$

Combining these estimates, and assuming without loss of generality that $\beta \leq 1$, we have

$$\begin{aligned} U_N(t,\beta) \leq \operatorname{ch} 2t\Big(&\beta^2 U_{N-1}(t,\beta') + NKtE\langle \exp t\tilde{\boldsymbol\eta}\cdot\boldsymbol\eta^3\rangle_0 \\ &- 3E\beta^2\frac{\operatorname{th}^2 X}{\operatorname{ch}^2 X}E\langle \tilde{\boldsymbol\eta}\cdot\mathbf{b}\exp t\tilde{\boldsymbol\eta}\cdot\boldsymbol\eta^3\rangle_0 + |S|\Big). \end{aligned} \tag{4.14}$$

This statement will be used to show by induction upon N that if L_0 is large enough and $\beta \leq 1/L_0$, the following condition holds for $t \geq 0$:

$$U_N(t,\beta) \leq 2NL_0 t\exp Nt^2 L_0. \tag{S_N}$$

Certainly this holds for $N = 1$ (the system is then nonrandom). Since $\beta' \leq \beta$, we can use (S_N) to show that, for $t \geq 0$,

$$E\langle \exp t\tilde{\boldsymbol\eta}\cdot\boldsymbol\eta^3\rangle_0 = \int_0^t U_{N-1}(s,\beta')\,ds \leq \exp(N-1)t^2 L_0. \tag{4.15}$$

This also holds for $t \leq 0$ by symmetry. Using $\operatorname{ch} 2t \leq \exp 2t^2$, we then get from (4.14) that (since we can assume that $\beta^2 \leq 1/2$)

$$\begin{aligned} U_N(t,\beta) \leq{}& (L_0+K)Nt\exp t^2(2+(N-1)L_0) \\ &+ \operatorname{ch} 2t\Big(-3\beta^2 E\frac{\operatorname{th}^2 X}{\operatorname{ch}^2 X}E\langle \tilde{\boldsymbol\eta}\cdot\mathbf{b}\exp t\tilde{\boldsymbol\eta}\cdot\boldsymbol\eta^3\rangle_0 + S\Big). \end{aligned} \tag{4.16}$$

The first term looks good, but unfortunately we do not know that $\langle \tilde{\boldsymbol\eta}\cdot\mathbf{b} \exp \tilde{\boldsymbol\eta}\cdot\boldsymbol\eta^3\rangle_0 \geq 0$, and we must also control S.

To control S, we will use the following principle.

LEMMA 4.2. *Consider two random variables V, W, and assume*

$$\forall\ t\in\mathbb{R},\qquad E\exp tV \leq \exp At^2, \tag{4.17}$$

$$\forall\ t\in\mathbb{R},\qquad E\exp tW \leq \exp Bt^2. \tag{4.18}$$

Then,

$$E|V|\,|\exp tW - 1| \le Kt\sqrt{AB}\exp Bt^2, \tag{4.19}$$

$$EV^2|\exp tW - 1| \le KtA\sqrt{B}(1+t\sqrt{B})\exp Bt^2. \tag{4.20}$$

PROOF. We will distinguish the cases $Bt^2 \ge 1$ and $Bt^2 \le 1$.

CASE 1. $Bt^2 \ge 1$. We write, for $s \ge 0$, $n \in \{1,2\}$,

$$\begin{aligned} E|V|^n\,|\exp tW - 1| \le s^n E|\exp tW - 1| \\ + E(|V|^n|\exp tW - 1|\,1_{\{|V|\ge s\}}). \end{aligned}$$

Using Hölder's inequality, we see that the last term is at most

$$(E|V|^{4n})^{1/4}(E1_{\{|V|\ge s\}})^{1/4}(E(1+\exp tW)^2)^{1/2}. \tag{4.21}$$

Now, using (4.18),

$$\begin{aligned} E(1+\exp tW)^2 \le 2(1+E\exp 2tW) &\le 2(1+\exp 4Bt^2) \\ &\le 4\exp 4Bt^2. \end{aligned}$$

Using (4.17) and the Chebyshev inequality, we get

$$E1_{\{|V|\ge s\}} \le 2\exp\left(-\frac{s^2}{4A}\right).$$

Finally, (4.17) implies $E\,\mathrm{ch}\,tV \le \exp At^2$, so that

$$\sum_{r\ge 0}\frac{t^{2r}}{(2r)!}EV^{2r} \le \exp At^2.$$

Taking $t^2 = A^{-1}$, this implies $EV^{2r} \le KA^r$ for $r \le 8$. It then follows from (4.21) that

$$\begin{aligned} E|V|^n\,|\exp tW - 1| \le{}& 2s^n\exp t^2B \\ &+ KA^{n/2}\exp\left(2Bt^2 - \frac{s^2}{4A}\right). \end{aligned} \tag{4.22}$$

We take $s = 4t\sqrt{AB}$ to conclude. [The first term to the right of (4.22) then dominates, since $Bt^2 \ge 1$.]

CASE 2. $Bt^2 \le 1$. We then use Cauchy–Schwarz,

$$E|V|^n|\exp tW - 1| \le (EV^{2n})^{1/2}(E(\exp tW - 1)^2)^{1/2}.$$

We have seen that $EV^{2n} \le KA^n$. Also, we have $EW = 0$ by (4.18), so that $E\exp tW \ge 1$, and thus

$$\begin{aligned} E(\exp tW - 1)^2 &= E\exp 2tW - E2\exp tW + 1 \\ &\le E\exp 4t^2B - 1 \le Kt^2B \end{aligned}$$

since $t^2B \le 1$. The result follows. $\square$

We continue the proof of Theorem 4.1. We will use the fact that the two successive integrals $\langle\cdot\rangle$ and E can be viewed as one single expectation to which we can apply Lemma 4.2.

To prove Theorem 4.1, assuming that (S_{N-1}) holds we will show that (S_N) holds (provided L_0 is large enough). Condition (S_{N-1}) implies

$$E\big\langle \exp t\tilde{\boldsymbol{\eta}}\cdot\boldsymbol{\eta}^3\big\rangle_0 \le \exp t^2 L_0(N-1),$$

that is, (4.18) for $B = L_0(N-1)$, so that Lemma 4.2 implies

$$E\big\langle|\tilde{\boldsymbol{\eta}}\cdot\mathbf{b}|\exp t\tilde{\boldsymbol{\eta}}\cdot\boldsymbol{\eta}^3\big\rangle_0 \le KtL_0N\exp t^2(N-1)L_0. \tag{4.23}$$

To control the remainder S, we observe that we can use the trivial bounds

$$E\big\langle(\dot{\boldsymbol{\eta}}^l\cdot\mathbf{b})^2|\exp t\tilde{\boldsymbol{\eta}}\cdot\boldsymbol{\eta}^3-1|\big\rangle_0 \le 2NE\big\langle|\dot{\boldsymbol{\eta}}^l\cdot\mathbf{b}|\,|\exp t\tilde{\boldsymbol{\eta}}\cdot\boldsymbol{\eta}^3-1|\big\rangle_0,$$

$$E\big\langle(\dot{\boldsymbol{\eta}}^l\cdot\dot{\boldsymbol{\eta}}^{l'})^2|\exp t\tilde{\boldsymbol{\eta}}\cdot\boldsymbol{\eta}^3-1|\big\rangle_0 \le 4NE\big\langle|\dot{\boldsymbol{\eta}}^l\cdot\dot{\boldsymbol{\eta}}^{l'}|\,|\exp t\tilde{\boldsymbol{\eta}}\cdot\boldsymbol{\eta}^3-1|\big\rangle_0.$$

Now $\dot{\boldsymbol{\eta}}^l\cdot\dot{\boldsymbol{\eta}}^{l'}$ satisfies (using Cauchy–Schwarz)

$$E\big\langle\exp t\dot{\boldsymbol{\eta}}^l\cdot\dot{\boldsymbol{\eta}}^{l}\big\rangle_0 \le \exp 2t^2(N-1)L_0$$

so that, by Lemma 4.2,

$$E|S| \le K\beta^2NtL_0\exp t^2(N-1)L_0.$$

Using (4.16), we then obtain

$$U_N(t,\beta) \le (L_0+K+K\beta^2L_0)Nt\exp t^2(2+(N-1)L_0). \tag{4.24}$$

If we take $L_0\ge 2$, then $2+(N-1)L_0\le NL_0$. If we take $\beta\le 1/L_0$, then if $L_0\ge 3K$, we have $L_0+K+K\beta^2L_0\le 2L_0$, and (4.24) becomes

$$U_N(t,\beta)\le 2L_0Nt\exp t^2NL_0, \tag{4.25}$$

which is condition (S_N). This completes the proof of Theorem 4.1. □

THEOREM 4.3. *There exists L_0 such that given $\beta_0>0$, there exists h_0 such that if $h\ge h_0$, $\beta\le\beta_0$, then for each t,*

$$E\big\langle\exp t\tilde{\boldsymbol{\sigma}}\cdot\boldsymbol{\sigma}^3\big\rangle \le \exp t^2NL_0.$$

The proof is similar to that of Theorem 4.1. One has to take advantage of the fact that

$$\begin{aligned} E_g\frac{1}{\mathrm{ch}^2\,X} &\le P\bigg(\beta\frac{\mathbf{g}\cdot\mathbf{b}}{\sqrt{N}}\ge\frac{h}{2}\bigg)+\frac{1}{\mathrm{ch}^2\;h/2}\\ &\le \exp\bigg(-\frac{h^2}{8\beta^2}\bigg)+\frac{1}{\mathrm{ch}^2\,h/2}\end{aligned}$$

becomes small for $\beta\le\beta_0$ and large h; and one has to take advantage of the version of Proposition 3.5 with improved bounds for the error term at large h. The details are left to the reader.

We now turn to the proof of the following first stage of the proof of Theorem 1.7.

THEOREM 4.4. *Given* $\beta_0 < 1$, *there is a constant* $L(\beta_0)$ *and a number* $h(\beta_0) > 0$ *such that if* $\beta \le \beta_0$ *and* $h \le h(\beta_0)$ *we have*

$$\forall\, t \in \mathbb{R}, \qquad E\langle \exp t\tilde{\boldsymbol{\sigma}} \cdot \boldsymbol{\sigma}^3 \rangle \le \exp N t^2 L(\beta_0).$$

The proof differs in two important respects from the proof of the previous theorems. First, it will not be possible to start the induction at $N = 1$ because we will use estimates that become efficient only at large N. Second, the estimate we have are efficient only at small t (unless we are in the previous cases) and for larger values of t we will need another argument, the starting point of which is the following.

PROPOSITION 4.5. *If* $h = 0$ *and* $\beta \le \beta_0 < 1$, *then for each* t,

$$E\langle \exp t\tilde{\boldsymbol{\sigma}} \cdot \boldsymbol{\sigma}^3 \rangle \le \exp L(\beta_0) N t^2, \tag{4.26}$$

where $L(\beta_0)$ *depends upon* β_0 *only.*

This is an easy consequence of the fact, proved in [5], that if $\beta \le \beta_0$,

$$E\left\langle \exp \frac{1}{NL} (\boldsymbol{\sigma} \cdot \boldsymbol{\sigma}')^2 \right\rangle \le L.$$

There, as in the rest of this section, L denotes a number depending upon β_0 only, not necessarily the same at each occurrence.

COROLLARY 4.6. *If* $\beta \le \beta_0 < 1$, *then*

$$E\langle \exp t\tilde{\boldsymbol{\sigma}} \cdot \boldsymbol{\sigma}^3 \rangle \le \exp N(6h + Lt^2). \tag{4.27}$$

PROOF. Fixing β, let us consider

$$\psi(h) = E\langle \exp t\tilde{\boldsymbol{\sigma}} \cdot \boldsymbol{\sigma}^3 \rangle, \tag{4.28}$$

where the thermal bracket on the right is taken for the external field $h' = h/\beta$. Then a straightforward computation shows that

$$\begin{aligned} \psi'(h) = {} & E\left\langle \sum_{i \le N} (\sigma_i^1 + \sigma_i^2 + \sigma_i^3) \exp t\tilde{\boldsymbol{\sigma}} \cdot \boldsymbol{\sigma}^3 \right\rangle \\ & - 3E\langle \exp t\tilde{\boldsymbol{\sigma}} \cdot \boldsymbol{\sigma}^3 \rangle \left\langle \sum_{i \le N} \sigma_i \right\rangle \end{aligned} \tag{4.29}$$

so that, by a very brutal bound, since $|\sigma_i| = 1$,

$$\psi'(h) \le 4N\psi(h) \tag{4.30}$$

and $\psi(h) \le \psi(0) \exp 4hN$. $\square$

To prove Theorem 4.4, we will prove the following statement.

PROPOSITION 4.7. *Given $\beta_0 < 1$, there exists a number $L_1 = L_1(\beta_0)$ and a number $h = h(\beta_0)$ such that if $h \le h(\beta_0)$ and $\beta \le \beta_0$, for each $t \ge 0$, we have*

$$(S_N) \qquad U_N(t,\beta) = E\langle \tilde{\boldsymbol\sigma} \cdot \boldsymbol\sigma^3 \exp t\tilde{\boldsymbol\sigma} \cdot \boldsymbol\sigma^3 \rangle \le 2NL_1 t \exp Nt^2 L_1.$$

We observe that from (4.27), we have

$$t \ge \sqrt{h} \Rightarrow E\langle \exp t\tilde{\boldsymbol\sigma} \cdot \boldsymbol\sigma^3 \rangle \le \exp NLt^2. \tag{4.31}$$

The first step is to complement (4.31) by getting information for $t \le \sqrt{h}$. In that case we write

$$\begin{aligned} E\langle \exp t\tilde{\boldsymbol\sigma} \cdot \boldsymbol\sigma^3 \rangle &\le \big(E\langle \exp \sqrt{h}\tilde{\boldsymbol\sigma} \cdot \boldsymbol\sigma^3 \rangle\big)^{t/\sqrt{h}} \\ &\le \exp NLt\sqrt{h} \end{aligned}$$

so that, for all $t > 0$, we have

$$E\langle \exp t\tilde{\boldsymbol\sigma} \cdot \boldsymbol\sigma^3 \rangle \le \exp NL(t^2 + t\sqrt{h}). \tag{4.32}$$

Using that $|e^x - 1| \le |x|e^{|x|}$ we have, for each $b > 0$,

$$\begin{aligned} E\langle \tilde{\boldsymbol\sigma} \cdot \boldsymbol\sigma^3 \exp t\tilde{\boldsymbol\sigma} \cdot \boldsymbol\sigma^3 \rangle &= E\langle \tilde{\boldsymbol\sigma} \cdot \boldsymbol\sigma^3 (\exp t\tilde{\boldsymbol\sigma} \cdot \boldsymbol\sigma^3 - 1)\rangle \\ &\le tE\langle (\tilde{\boldsymbol\sigma} \cdot \boldsymbol\sigma^3)^2 \exp t|\tilde{\boldsymbol\sigma} \cdot \boldsymbol\sigma^3| \rangle \\ &\le t(E\langle (\tilde{\boldsymbol\sigma} \cdot \boldsymbol\sigma^3)^4 \rangle)^{1/2} \exp NL(t^2 + t\sqrt{h}) \end{aligned} \tag{4.33}$$

since $e^{|x|} \le e^x + e^{-x}$. Now,

$$\begin{aligned} \frac{t^4}{4!} E\langle (\tilde{\boldsymbol\sigma} \cdot \boldsymbol\sigma^3)^4 \rangle &\le \frac{1}{2}\big(E\langle \exp t\tilde{\boldsymbol\sigma} \cdot \boldsymbol\sigma^3 \rangle + \langle \exp(-t\tilde{\boldsymbol\sigma} \cdot \boldsymbol\sigma^3)\rangle\big) \\ &\le \exp NL(t^2 + t\sqrt{h}) \end{aligned}$$

and taking $t = \min(1/NL\sqrt{h}, 1/\sqrt{NL})$ we get

$$E\langle (\tilde{\boldsymbol\sigma} \cdot \boldsymbol\sigma^3)^4 \rangle \le K \max\big((NL)^2, (NL\sqrt{h})^4\big)$$

so that (4.33) gives

$$E\langle \tilde{\boldsymbol\sigma} \cdot \boldsymbol\sigma^3 \exp t\tilde{\boldsymbol\sigma} \cdot \boldsymbol\sigma^3 \rangle \le KtNL(1 + NLh) \exp NL\big(t^2 + t\sqrt{h}\big). \tag{4.34}$$

Since $1 + NLh \le 2\exp NLh$, this shows that (S_N) is automatically satisfied for $t \ge \sqrt{h}$, provided L_1 is larger than a certain number $L(\beta_0)$ (*independent* of h). Next we show that (S_N) is automatically satisfied for $N \le L_1/hL(\beta_0)$, where $L(\beta_0)$ depend only upon β_0. Indeed, if $L_1 \ge KNL^2h$, then we have

$$NL(1 + NLh) \le L_1 N; \exp NL\big(t^2 + t\sqrt{h}\big) \le 2\exp NL_1 t^2,$$

since

$$Lt\sqrt{h} \le \frac{L_1 t^2}{2} + \frac{hL^2}{2L_1}.$$

We now attack the main part of the proof, the proof of (S_N) when $t \le \sqrt{h}$ and $N \ge L_1/hL(\beta_0)$. Looking back at (4.14), we see that we can no longer use that β is small to make the term

$$-3E\frac{\beta^2 \operatorname{th}^2 X}{\operatorname{ch}^2 X}\big\langle \tilde{\boldsymbol{\eta}} \cdot \mathbf{b} \exp t\tilde{\boldsymbol{\eta}} \cdot \boldsymbol{\eta}^3 \big\rangle_0 \tag{4.35}$$

small. Rather, we will use the fact that $\operatorname{th}^2 X$ is small. Taking first the expectation in $\mathbf{g}$, we see that the term (4.35) is at most

$$\begin{aligned} &3E \operatorname{th}^2 X \big\langle |\tilde{\boldsymbol{\eta}} \cdot \mathbf{b}|\ |\exp t\tilde{\boldsymbol{\eta}} \cdot \boldsymbol{\eta}^3 - 1| \big\rangle_0 \\ &\quad \le 3E\varphi\bigg(\frac{||\mathbf{b}||}{\sqrt{N}}, h\bigg)\big\langle |\tilde{\boldsymbol{\eta}} \cdot \mathbf{b}|\ |\exp t\tilde{\boldsymbol{\eta}} \cdot \boldsymbol{\eta}^3 - 1| \big\rangle_0, \end{aligned} \tag{4.36}$$

where, for g standard normal,

$$\varphi(a, h) = E \operatorname{th}^2(\beta a g + h).$$

For h small, $E||\mathbf{b}||/\sqrt{N}$ will be small; the problem is that there might be an exceptional set of quenched variables over which $||\mathbf{b}||/\sqrt{N}$ is large, and we have to show that such a set is small enough to be irrelevant, a technical point better skipped by the reader.

Consider a parameter c to be determined later, and $\psi(c, h) = \sup_{a \le c} \varphi(a, h)$. Then the term on the right of (4.36) is at most

$$\begin{aligned} &3\psi(c, h)E\big\langle |\tilde{\boldsymbol{\eta}} \cdot \mathbf{b}|\ |\exp t\tilde{\boldsymbol{\eta}} \cdot \boldsymbol{\eta}^3 - 1| \big\rangle \\ &\quad + E1_{\{||\mathbf{b}|| \ge c\sqrt{N}\}}\big\langle |\tilde{\boldsymbol{\eta}} \cdot \mathbf{b}|\ |\exp t\tilde{\boldsymbol{\eta}} \cdot \boldsymbol{\eta}^3 - 1| \big\rangle. \end{aligned} \tag{4.37}$$

It follows from (4.27) that if $t \ge \sqrt{h}$ we have

$$E \exp t\langle \boldsymbol{\sigma} \cdot \boldsymbol{\sigma}' \rangle \le \exp NLt^2. \tag{4.38}$$

We have $\langle \boldsymbol{\sigma} \cdot \boldsymbol{\sigma}' \rangle = ||\langle \boldsymbol{\sigma} \rangle||^2$. Using (4.38) for $N-1$ rather than N, we have

$$E \exp t||\mathbf{b}||^2 \le \exp NLt^2$$

so that, if $c \ge L\sqrt{h}$, we have

$$P(||\mathbf{b}||^2 \ge c\sqrt{N}) \le \exp \frac{Nc^2}{L}. \tag{4.39}$$

Using Hölder's inequality, the last term of (4.37) is at most

$$\exp\bigg(-\frac{Nc^2}{L}\bigg)(E\big\langle (\tilde{\boldsymbol{\eta}} \cdot \mathbf{b})^4 \big\rangle)^{1/4}(E\big\langle (\exp t\tilde{\boldsymbol{\eta}} \cdot \boldsymbol{\eta}^3 - 1)^2 \big\rangle)^{1/2}$$

and, as shown in the proof of Lemma 4.2, under (S_{N-1}) this is at most

$$KL_1 tN \exp\bigg(-\frac{Nc^2}{4} + KNt^2L_1\bigg) \le KL_1 Nt \exp\bigg(-\frac{Nc^2}{K}\bigg)$$

under the conditions

$$t^2 \le h, \qquad c^2 \ge KhL_1; \qquad c^2 \ge Lh. \tag{4.40}$$

As for the remainder term S of (4.14), (4.20) shows that it is at most

$$KN\left(t^2L_1^2+\frac{tL_1^{3/2}}{\sqrt{N}}\right)\exp t^2(N-1)L_1 \le KNtL_1\left(\sqrt{h}L_1+\sqrt{\frac{L_1}{N}}\right)\exp t^2(N-1)L_1$$

since $t \le \sqrt{h}$. Regrouping these contributions, we get from (4.14) that

$$U_N(t,\beta) \le 2\theta L_1 Nt \exp t^2 NL_1, \tag{4.41}$$

where

$$\theta \le \beta^2 + K\left(\psi(c,h)+\frac{1}{L_1}+\exp\left(-\frac{Nc^2}{L}\right)+\sqrt{h}L_1+\sqrt{\frac{L_1}{N}}\right).$$

But we have seen that (S_N) is automatically satisfied for $N \le L_1/hL$, so we can assume $N \ge L_1/hL$, and thus

$$\theta \le \beta^2 K\left(\psi(c,h)+\frac{1}{L_1}+\exp\left(-\frac{L_1c^2}{hL}\right)+\sqrt{hL}\right).$$

To choose the parameters, we choose c small enough and $L_1 \ge L(\beta_0)$ large enough that

$$\beta_0^2 + K\left(\psi(c,0)+\frac{1}{L^1}\right) < 1.$$

We then choose $h(\beta_0)$ small enough to ensure that $\theta \le 1$ and that (4.40) hold provided $h \le h(\beta_0)$. The proof is complete. □

5. Exponential inequalities II. In this section, we prove Theorem 1.5. The proof of Theorem 1.5 resembles the proof of Theorem 4.1 and uses the results of this theorem in an essential way.

We use the notation $q_N = E\langle \boldsymbol{\sigma}\cdot\boldsymbol{\sigma}'/N\rangle$, so that $q_N = q_N(\beta,h)$. There is a considerable psychological obstacle to the proof of Theorem 1.5. Namely, when one tries to gather information about $E\langle \exp t(\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}' - Nq_N)\rangle$ by induction upon N, when going to $N-1$, one has to replace Nq_N by $(N-1)q_{N-1}$, where $q_{N-1} = q_{N-1}(\beta',h)$. One feels that to prove anything at all one will need to have very detailed information about

$$\delta_N = Nq_N - (N-1)q_{N-1} \tag{5.1}$$

and that this change of centering will cause enormous difficulty. This turns out not to be case, and this is indeed surprising. Theorem 1.5 follows from the following.

PROPOSITION 5.1. *There exists a number L_1 such that if $\beta L_1 \le 1$, then for all $t \ge 0$, we have*

$$E\big\langle(\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}' - Nq_N)\exp t(\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}' - Nq_N)\big\rangle \le 2NtL_1\exp t^2NL_1, \tag{5.2}$$

$$E\big\langle(Nq_N - \boldsymbol{\sigma}\cdot\boldsymbol{\sigma}')\exp t(Nq_N - \boldsymbol{\sigma}\cdot\boldsymbol{\sigma}')\big\rangle \le 2NtL_1\exp t^2NL_1. \tag{5.3}$$

The reason why we have to prove these two inequalities is that we would like to know that

$$E\langle \exp t(\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}' - Nq_N)\rangle \le \exp t^2 NL_1 \tag{5.4}$$

for *all* values of t, whether positive or negative (this is needed in Lemma 5.2) and that (in contrast with what happened in Section 4) we can no longer use symmetry to say that (5.4) is equivalent to the case $t \ge 0$. We will denote by $U_N(\beta, t)$ the right-hand side of (5.2). [The similar study of the right-hand side $V_N(\beta, t)$ of (5.3) is left to the reader.] We will use in an essential way that, according to Theorem 4.1, we have

$$E\langle \exp t\tilde{\boldsymbol{\sigma}}\cdot\boldsymbol{\sigma}^3\rangle \le \exp t^2 NL_0 \tag{5.5}$$

for a certain number L_0.

We start the study of $U_N(\beta, t)$ as usually by the relation

$$U_N(\beta, t) = NE\langle (\sigma_N\sigma'_N - q_N)\exp t(\sigma_N\sigma'_N - q_N)\bar{f}\rangle, \tag{5.6}$$

where

$$\bar{f} = \exp t(\boldsymbol{\eta}\cdot\boldsymbol{\eta}' - Nq_N). \tag{5.7}$$

LEMMA 5.2. *If* $|x| \le 2$ *and* $t \ge 0$, *we have*

$$xe^{tx} \le \operatorname{sh} 2t + x\operatorname{ch} 2t \le (x + 4t)\operatorname{ch} 2t. \tag{5.8}$$

PROOF. If $\varphi(x) = xe^{tx}$, then $\varphi'(x) = (1 + tx)e^{tx}$, $\varphi''(x) = t(2 + tx)e^{tx}$, so that either $\varphi'(x) < 0$ or $\varphi''(x) > 0$. It is then clearly enough to check that the left inequality in (5.8) is satisfied for $x = \pm 2$. □

We then deduce from (5.6) that

$$U_N(t, \beta) \le N\operatorname{ch} 2tE\left(\langle(\sigma_N\sigma'_N - q_N)\bar{f}\rangle + 4t\langle\bar{f}\rangle\right). \tag{5.9}$$

To prove Proposition 5.1, we argue by induction over N, so that we assume the induction hypothesis,

$$(S_{N-1})\qquad \begin{aligned} &\forall\, t \ge 0, \forall\, \beta \le \beta_0, U_{N-1}(\beta, t) \le 2L_1(N-1)t\exp(N-1)t^2L_1,\\ &\forall\, t \ge 0, \forall\, \beta \le \beta_0, V_{N-1}(\beta, t) \le 2L_1(N-1)t\exp(N-1)t^2L_1\end{aligned}$$

and thus

$$\begin{aligned}\forall\, t \in \mathbb{R}, \forall\, \beta \le \beta_0,\ & E\langle \exp t(\boldsymbol{\eta}\cdot\boldsymbol{\eta}' - (N-1)q_{N-1})\rangle_0\\ &\le \exp(N-1)t^2L_1.\end{aligned} \tag{5.10}$$

LEMMA 5.3. *We have*

$$|\delta_N| \le KL_1. \tag{5.11}$$

PROOF. We have

$$\begin{aligned}Nq_N &= E\langle \boldsymbol{\sigma}\cdot\boldsymbol{\sigma}'\rangle\\ &= E\langle \sigma_N\sigma'_N\rangle + E\langle \boldsymbol{\eta}\cdot\boldsymbol{\eta}'\rangle.\end{aligned}$$

To estimate $E\langle\boldsymbol{\eta}\cdot\boldsymbol{\eta}'\rangle$ we use Theorem 3.2 and Proposition 3.4 in the case $n=2$, $I=\varnothing$. We get

$$\begin{aligned}E\langle \boldsymbol{\eta}\cdot\boldsymbol{\eta}'\rangle &= E\langle \boldsymbol{\eta}\cdot\boldsymbol{\eta}'\rangle_0 + \beta^2 E\Big\langle \boldsymbol{\eta}\cdot\boldsymbol{\eta}'\frac{\dot{\boldsymbol{\eta}}\cdot\dot{\boldsymbol{\eta}}'}{N}\Big\rangle_0\\ &\quad+\beta^2E(1-2\operatorname{th}^2X)\Big\langle \boldsymbol{\eta}\cdot\boldsymbol{\eta}'\Big(\Big(\frac{\dot{\boldsymbol{\eta}}\cdot\mathbf{b}}{N}\Big)+\Big(\frac{\dot{\boldsymbol{\eta}}'\cdot\mathbf{b}}{N}\Big)\Big)\Big\rangle_0 + S_1,\end{aligned}\tag{5.12}$$

where (using the bound $|\boldsymbol{\eta}\cdot\boldsymbol{\eta}'|\le N$), we have

$$|S_1| \le K\beta^2E\Big\langle\frac{(\dot{\boldsymbol{\eta}}\cdot\mathbf{b})^2}{N}\Big\rangle + K\beta^2E\Big\langle\frac{(\dot{\boldsymbol{\eta}}\cdot\dot{\boldsymbol{\eta}}')^2}{N}\Big\rangle.$$

Now,

$$\begin{aligned}\Big\langle \boldsymbol{\eta}\cdot\boldsymbol{\eta}'\frac{\dot{\boldsymbol{\eta}}\cdot\dot{\boldsymbol{\eta}}'}{N}\Big\rangle_0 &= \Big\langle\frac{(\dot{\boldsymbol{\eta}}\cdot\dot{\boldsymbol{\eta}}')^2}{N}\Big\rangle_0,\\ \Big\langle \boldsymbol{\eta}\cdot\boldsymbol{\eta}'\frac{\boldsymbol{\eta}\cdot\mathbf{b}}{N}\Big\rangle_0 &= \Big\langle\frac{(\dot{\boldsymbol{\eta}}\cdot\mathbf{b})^2}{N}\Big\rangle_0\end{aligned}$$

and, by (S_{N-1}) the expectation of both these quantities are bounded by KL_1. The result follows, since $E\langle\boldsymbol{\eta}\cdot\boldsymbol{\eta}'\rangle_0=(N-1)q_{N-1}$. □

We can now handle the last term of (5.9). Using crude bounds as in the proof of Theorem 4.1, we get, using (5.10),

$$\begin{aligned}E\langle\bar f\rangle \le KE\langle\bar f\rangle_0 &= K\exp(-\delta_N t)E\langle\exp t(\boldsymbol{\eta}\cdot\boldsymbol{\eta}'-(N-1)q_{N-1})\rangle_0\\ &\le K\exp|\delta_N t|\exp(N-1)t^2L_1.\end{aligned}$$

Using (5.11), we see that

$$\exp|\delta_N t| \le K\exp\frac{t^2L_1}{2}\tag{5.13}$$

and thus

$$E\langle\bar f\rangle \le K\exp\frac{t^2L_1}{2}\exp(N-1)t^2L_1.\tag{5.14}$$

We now turn to the study of the most dangerous term of (5.9), that is, $E\langle(\sigma_N\sigma'_N-q_N)\bar f\rangle$.

The fact that $q_N=E\langle\sigma_N\sigma'_N\rangle$ is used through the observation that

$$E\langle(\sigma_N\sigma'_N-q_N)\bar f\rangle = E\langle(\sigma_N\sigma'_N-q_N)\hat f\rangle,\tag{5.15}$$

where $\hat{f} = \bar{f} - \exp(-\delta_N t)$ so that

$$
\begin{aligned}
|\hat{f}| &\le \exp(-\delta_N t)|\exp t(\boldsymbol{\eta}\cdot\boldsymbol{\eta}' - (N-1)q_{N-1}) - 1| \\
&\le K \exp\frac{t^2 L_1}{2} |\exp t(\boldsymbol{\eta}\cdot\boldsymbol{\eta}' - (N-1)q_{N-1}) - 1|.
\end{aligned}
\tag{5.16}
$$

We use Theorem 3.2 and Proposition 3.4 in the case $n = 2$ with $I = \{1, 2\}$ and $I = \varnothing$, respectively, to get

$$
\begin{aligned}
E\langle \sigma_N \sigma'_N \hat{f}\rangle &= E\,\mathrm{th}^2\,X\langle \hat{f}\rangle_0 + \beta^2 E\left\langle \hat{f}\frac{\dot{\boldsymbol{\eta}}\cdot\dot{\boldsymbol{\eta}}'}{N}\right\rangle_0 \\
&\quad + \beta^2 E(1 - 2\,\mathrm{th}^2\,X)\left\langle \hat{f}\left(\frac{\dot{\boldsymbol{\eta}}\cdot\mathbf{b}}{N} + \frac{\dot{\boldsymbol{\eta}}'\cdot\mathbf{b}}{N}\right)\right\rangle_0 + S_2,
\end{aligned}
\tag{5.17}
$$

$$
\begin{aligned}
E\langle \hat{f}\rangle &= E\langle \hat{f}\rangle_0 + \beta^2 E\,\mathrm{th}^4\,X\left\langle \hat{f}\frac{\dot{\boldsymbol{\eta}}\cdot\dot{\boldsymbol{\eta}}'}{N}\right\rangle_0 \\
&\quad + \beta^2 E(2\,\mathrm{th}^2\,X - 3\,\mathrm{th}^4\,X)\left\langle \hat{f}\left(\frac{\dot{\boldsymbol{\eta}}\cdot\mathbf{b}}{N} + \frac{\dot{\boldsymbol{\eta}}'\cdot\mathbf{b}}{N}\right)\right\rangle_0 + S_3,
\end{aligned}
\tag{5.18}
$$

where S_2, S_3 satisfy (using $|\dot{\boldsymbol{\eta}}\cdot\mathbf{b}| \le 2N$, $|\dot{\boldsymbol{\eta}}\cdot\dot{\boldsymbol{\eta}}'| \le 2N$)

$$
|S_2|, |S_3| \le \frac{K\beta^4}{N} E\left\langle |\hat{f}|\left(\sum_{l\le 4} |\dot{\boldsymbol{\eta}}^l\cdot\mathbf{b}| + \sum_{l<l'} |\dot{\boldsymbol{\eta}}^l\cdot\dot{\boldsymbol{\eta}}^{l'}|\right)\right\rangle_0
$$

where $\boldsymbol{\eta}^1 = \boldsymbol{\eta}$, $\boldsymbol{\eta}^2 = \boldsymbol{\eta}'$. Thus

$$
E\langle(\sigma_N\sigma'_N - q_N)\bar{f}\rangle = E(\mathrm{th}^2\,X - q_N)\langle \hat{f}\rangle_0 + S_4,
$$

where

$$
|S_4| \le K\beta^2\left\langle |\hat{f}|\left(\sum_{l\le 4} |\dot{\boldsymbol{\eta}}\cdot\mathbf{b}| + \sum_{l<l'} |\dot{\boldsymbol{\eta}}^l\cdot\dot{\boldsymbol{\eta}}^{l'}|\right)\right\rangle_0.
$$

We now use (5.16), (5.2), (5.3), (S_{N-1}) and Lemma 4.2 to get

$$
|S_4| \le K\beta^2 t\sqrt{L_0 L_1}\exp\frac{t^2 L_1}{2}\exp t^2(N-1)L_1.
\tag{5.19}
$$

We consider the function

$$
\varphi(x) = E\,\mathrm{th}^2(\beta g\sqrt{x} + h),
$$

where g is $N(0,1)$ so that

$$
E(\mathrm{th}^2\,X - q_N)\langle \hat{f}\rangle_0 = E\left(\left(\varphi\left(\frac{\|\mathbf{b}\|^2}{N}\right) - q_N\right)\langle \hat{f}\rangle_0\right).
\tag{5.20}
$$

We write

$$
\frac{\|\mathbf{b}\|^2}{N} = \frac{\langle \boldsymbol{\eta}\cdot\boldsymbol{\eta}'\rangle_0}{N} = \frac{\langle \boldsymbol{\eta}\cdot\boldsymbol{\eta}'\rangle_0 - (N-1)q_{N-1}}{N} + \frac{N-1}{N}q_{N-1}
$$

so that

$$\left|\varphi\left(\frac{\|\mathbf{b}\|^2}{N}\right)-\varphi\left(\frac{N-1}{N}q_{N-1}\right)\right|\le\frac{1}{N}\sup|\varphi'|\big|\langle\boldsymbol{\eta}\cdot\boldsymbol{\eta}'\rangle_0-(N-1)q_N\big|. \tag{5.21}$$

We observe (integration by parts) that

$$\varphi'(x)=\beta^2E\left(\frac{1}{\operatorname{ch}^2(\beta g\sqrt{x}+h)}-\frac{3\operatorname{th}^2(\beta g\sqrt{x}+h)}{\operatorname{ch}^2(\beta g\sqrt{x}+h)}\right) \tag{5.22}$$

so that $|\varphi'(x)|\le 4\beta^2$. Thus, from (5.20),

$$\begin{aligned}E((\operatorname{th}^2X-q_N)\langle\hat{f}\rangle_0)\le&\left|\varphi\left(\frac{N-1}{N}q_{N-1}\right)-q_N\right|E\langle|\hat{f}|\rangle_0\\&+4\beta^2E(|\langle\boldsymbol{\eta}\cdot\boldsymbol{\eta}'\rangle-(N-1)q_N|\langle|\hat{f}|\rangle_0)\end{aligned} \tag{5.23}$$

To control the second term, we use (5.16), (5.10) and Lemma 4.2, so that this second term can be bounded by

$$K\beta^2L_1tN\exp\frac{L_1t^2}{2}\exp(N-1)t^2L_1.$$

To control the first term, since $q_N=E\langle\sigma_N\sigma_N'\rangle$, using Theorem 3.2 and (5.3) to control the remainder, we see that

$$|q_N-E\operatorname{th}^2X|\le\frac{KL_0\beta^2}{N}$$

and thus, using Taylor's formula to estimate $E\operatorname{th}^2X$, we have

$$\begin{aligned}\left|q_N-\varphi\left(\frac{N-1}{N}q_{N-1}\right)\right|&\le\frac{1}{N^2}\sup|\varphi''|E\big(\langle\boldsymbol{\eta}\cdot\boldsymbol{\eta}'\rangle_0-(N-1)q_{N-1}\big)^2\\&\le\frac{K\beta^2L_1}{N}\end{aligned}$$

using (5.10). Distinguishing the cases $t^2L_1N\ge1$ and $t^2L_1N\le1$, the reader should have no problem showing, using (5.10) again, that we have

$$E\langle|\hat{f}|\rangle_0\le Kt\sqrt{L_1N}\exp\frac{t^2L_1}{2}\exp t^2L_1(N-1).$$

Combining all these estimates and using that $\operatorname{ch}t\le\exp t^2/2$, we have now shown from (5.23) that

$$\begin{aligned}U_N(\beta,t)\le&K\beta^2L_1tN\left(1+\frac{1}{L_1}+\sqrt{\frac{L_1}{N}}\sqrt{\frac{L_0}{L_1}}\right)\\&\times\exp\left(\frac{t^2L_1}{2}+\frac{t^2}{2}\right)\exp t^2L_1(N-1).\end{aligned} \tag{5.24}$$

It is now easy (based upon the fact that $|\langle\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}'\rangle-Nq_N|\le2N$, $E(\langle\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}'\rangle-Nq_N)=0$) to show that to prove (S_N), it suffices to assume that $N\ge L_1/K$

(a more delicate phenomenon of the same nature is handled in the proof of Proposition 4.7). Thus in (5.24) the term $\sqrt{L_1/N}$ can be removed. Then (S_N) follows from (5.24) if β is small enough. □

The proof of Theorem 1.6 requires only obvious changes from the proof of Theorem 1.5. The proof of Theorem 1.7 is more delicate. The main feature is again the use of a priori estimates to show that one has only the small values of t to control. How to do this was detailed at length in Proposition 4.7, so we will not give details. One has to replace (5.21) by

$$\begin{aligned}&\left|\varphi\left(\frac{\|\mathbf{b}\|^2}{N}\right)-\varphi\left(\frac{N-1}{N}q_{N-1}\right)-\frac{1}{N}\varphi'\left(\frac{N-1}{N}q_N\right)(\langle\boldsymbol{\eta}\cdot\boldsymbol{\eta}'\rangle_0-(N-1)q_{N-1})\right|\\ &\qquad\le\frac{1}{N^2}\sup|\varphi''|\left|\langle\boldsymbol{\eta}\cdot\boldsymbol{\eta}'\rangle_0-(N-1)q_{N-1}\right|^2\end{aligned}\tag{5.25}$$

and the crucial point is that $\varphi'\big(((N-1)/N)q_N\big) < 1$. The details are left to the reader.

6. Convergence of moments. In this section, we consider replicas $\boldsymbol{\sigma}^1,\dots,\boldsymbol{\sigma}^p$. We will consider ordered quadruples $I=\{i_1,i_2,i_3,i_4\}$ of distinct indices. These quadruples will be of three different types. It is linguistically convenient to use color names to distinguish these types. The quadruple will come in one of three *colors* white, black, red. If I is red, we set

$$f_I=f_I(\boldsymbol{\sigma}^I,\dots,\boldsymbol{\sigma}^p)=\boldsymbol{\sigma}^{i_1}\cdot\boldsymbol{\sigma}^{i_2}-Nq_N(\beta),$$

where $q_N(\beta)q_N=E\langle\boldsymbol{\sigma}^1\cdot\boldsymbol{\sigma}^2\rangle$. If I is black, we set

$$f_I=f_I(\boldsymbol{\sigma}^1,\dots,\boldsymbol{\sigma}^p)=(\boldsymbol{\sigma}^{i_1}-\boldsymbol{\sigma}^{i_2})\cdot\boldsymbol{\sigma}^{i_3}$$

and if I is white we set

$$f_I=f_I(\boldsymbol{\sigma}^1,\dots,\boldsymbol{\sigma}^p)=(\boldsymbol{\sigma}^{i_1}-\boldsymbol{\sigma}^{i_2})\cdot(\boldsymbol{\sigma}^{i_3}-\boldsymbol{\sigma}^{i_4}).$$

For a collection $\mathscr{I}$ of colored quadruples, we set $f_{\mathscr{I}}=\Pi_{I\in\mathscr{I}}f_I$.

THEOREM 6.1. *There exists β_0 with the following property. If $\beta\le\beta_0$ given any p and any collection $\mathscr{I}$ of colored quadruples of $\{1,\dots,p\}$, there is a function $\mathscr{L}_{\mathscr{I}}(\beta,h)$ such that*

$$\lim_{N\to\infty}E\langle f_{\mathscr{I}}\rangle N^{-\operatorname{card}\mathscr{I}/2}=\mathscr{L}_{\mathscr{I}}(\beta,h).\tag{6.1}$$

PROOF. Since $(\boldsymbol{\sigma}^{i_1}-\boldsymbol{\sigma}^{i_2})\cdot\boldsymbol{\sigma}^{i_3}=(\boldsymbol{\sigma}^{i_1}\cdot\boldsymbol{\sigma}^{i_3}-Nq_N)-(\boldsymbol{\sigma}^{i_2}\cdot\boldsymbol{\sigma}^{i_3}-Nq_N)$, it would suffice to consider the case where there are no black quadruples (and similarly no white quadruples). But this is not the way the proof works.

The proof will go by induction upon card $\mathscr{I}=r$. The result is obvious if card $\mathscr{I}=0$ or 1, since $f_{\mathscr{I}}=1$ for $\mathscr{I}=\varnothing$ and $E\langle f_{\mathscr{I}}\rangle=0$ if card $\mathscr{I}=1$.

Let us assume that the result has been proved for all p and all $\mathscr{I}$ with card $\mathscr{I}\le r$; we will prove it for all p when card $\mathscr{I}=r+1$. As a first step, we assume that $\mathscr{I}$ contains at least a white quadruple. Without loss of generality, we can assume that this is $\{1, 2, 3, 4\}$, and we write $\mathscr{I}'$ the collection of the other colored quadruples. Thus, writing as usual $\tilde{\boldsymbol{\sigma}} = \boldsymbol{\sigma}^1 - \boldsymbol{\sigma}^2$, we have

$$f_{\mathscr{I}} = \tilde{\boldsymbol{\sigma}} \cdot (\boldsymbol{\sigma}^3 - \boldsymbol{\sigma}^4) f_{\mathscr{I}'}$$

and, by symmetry,

$$E\langle f_{\mathscr{I}}\rangle = NE\big\langle \tilde{\sigma}_N(\sigma_N^3 - \sigma_N^4) f_{\mathscr{I}'}\big\rangle. \tag{6.2}$$

For $I \in \mathscr{I}'$, we write

$$f_I = M_I + f'_I(\boldsymbol{\eta}^1, \dots, \boldsymbol{\eta}^p),$$

where M_I contains the contribution of the last ($=N$th) coordinate, and $f'_I(\boldsymbol{\eta}^1, \dots, \boldsymbol{\eta}^p)$ the contributions of the $N-1$ first coordinates. For example, if I is red,

$$M_I = \sigma_N^{i_1}\sigma_N^{i_2} - q_N,$$
$$f'_I(\boldsymbol{\eta}^1, \dots, \boldsymbol{\eta}^p) = \sum_{i \le N-1} \sigma_i^{i_1}\sigma_i^{i_2} - (N-1)q_N = \boldsymbol{\eta}^{i_1} \cdot \boldsymbol{\eta}^{i_2} - (N-1)q_N.$$

The " $'$ " is to indicate that $f'_I(\boldsymbol{\eta}^1, \dots, \boldsymbol{\eta}^p)$ *need not* be the function $f_I(\boldsymbol{\eta}^1, \dots, \boldsymbol{\eta}^p)$ that is obtained from f_I by replacing N by $N-1$. This is the case if I is either white or black; but when I is red, $f_I(\boldsymbol{\eta}^1, \dots, \boldsymbol{\eta}^p) = \boldsymbol{\eta}^{i_1} \cdot \boldsymbol{\eta}^{i_2} - (N-1)q_{N-1}$ (where of course $q_{N-1} = q_{N-1}(\beta')$) so that

$$f'_I(\boldsymbol{\eta}^1, \dots, \boldsymbol{\eta}^p) = f_I(\boldsymbol{\eta}^1, \dots, \boldsymbol{\eta}^p) + (N-1)(q_{N-1} - q_N). \tag{6.3}$$

We write

$$f_{\mathscr{I}'} = \prod_{I \in \mathscr{I}'} (M_I + f'_I(\boldsymbol{\eta}^1, \dots, \boldsymbol{\eta}^p))$$

and we expand this as

$$f'_{\mathscr{I}'}(\boldsymbol{\eta}^1, \dots, \boldsymbol{\eta}^p) \sum_{I \in \mathscr{I}'} M_I \prod_{J \ne I} f'_J(\boldsymbol{\eta}^1, \dots, \boldsymbol{\eta}^p) + S,$$

where S consists of the products that contain at least two terms M_I and where $f'_{\mathscr{I}'}(\boldsymbol{\eta}^1, \dots, \boldsymbol{\eta}^p) = \Pi_{I \in \mathscr{I}'} f'_I(\boldsymbol{\eta}^1, \dots, \boldsymbol{\eta}^p)$. Thus we have

$$\begin{aligned} E\langle f_{\mathscr{I}}\rangle &= NE\big\langle \tilde{\sigma}_N(\sigma_N^3 - \sigma_N^4) f'_{\mathscr{I}'}(\boldsymbol{\eta}^1, \dots, \boldsymbol{\eta}^p)\big\rangle \\ &\quad + N \sum_{I \in \mathscr{I}'} \left\langle \tilde{\sigma}_N(\sigma_N^3 - \sigma_N^4) M_I \prod_{J \ne I,\, J \in \mathscr{I}'} f'_J(\boldsymbol{\eta}^1, \dots \boldsymbol{\eta}^p)\right\rangle \\ &\quad + N\langle \tilde{\sigma}_N(\sigma_N^3 - \sigma_N^4) S\rangle. \end{aligned} \tag{6.4}$$

To compute each of these terms, we use Theorem 3.2. What makes life easy is that now the remainder terms are at most $K(r)N^{r/2}$. This follows from Theorem 5.1 and from the fact that $(N-1)(q_{N-1} - q_N)$ is of order 1. Using

Proposition 3.6 we see that up to error $K(r)N^{r/2}$, the first term on the right of (6.4) is

$$E\frac{\beta^2}{\operatorname{ch}^4 X}\langle f'_{\mathscr{I}}(\boldsymbol{\eta}^1,\dots,\boldsymbol{\eta}^p)\rangle_0$$

and, up to the same error, this is

$$E\frac{\beta^2}{\operatorname{ch}^4 Y}E\langle f'_{\mathscr{I}}(\boldsymbol{\eta}^1,\dots,\boldsymbol{\eta}^p)\rangle_0 \tag{6.5}$$

for $Y = g\beta\sqrt{q_{N-1}} + h$.

As for the second term to the right of (6.4), it is a sum of terms

$$N\biggl\langle\prod_{i\in L}\sigma_N^i\prod_{J\neq I}f'_J(\boldsymbol{\eta}^1,\dots,\boldsymbol{\eta}^p)\biggr\rangle. \tag{6.6}$$

When we apply Proposition 3.4 to this term, only the contribution of the terms I is not of order almost $N^{r/2}$, and this contribution is

$$\begin{aligned} &NE\operatorname{th}^n X\biggl\langle\prod_{J\neq I}f'_J(\boldsymbol{\eta}^1,\dots,\boldsymbol{\eta}^p)\biggr\rangle_0\\ &\quad\simeq NE\operatorname{th}^n YE\biggl\langle\prod_{J\neq I}f'_J(\boldsymbol{\eta}^1,\dots,\boldsymbol{\eta}^p)\biggr\rangle_0, \end{aligned}\tag{6.7}$$

where $\simeq$ indicates an error at most $K(r)N^{r/2}$. We would like to apply the induction hypothesis to the last term of (6.7) (which involves $r-1$ quadruples only) but the obstacle is that we have f'_J rather than f_J. We simply use (6.3) and the fact that $(N-1)(q_N-q_{N-1})$ is of order 1 to see that replacing f'_J by f_J involves again error at most $K(r)N^{r/2}$, so that this induction hypothesis shows convergence of the terms (6.7). We do not make a larger error in replacing $f'_{\mathscr{I}}$ by $f_{\mathscr{I}}$ in (6.5). (For this reason, in the remainder of the proof, we will no longer distinguish between $f_{\mathscr{I}}$ and $f'_{\mathscr{I}}$.)

In summary, if, given $\mathscr{I}$, we write $\theta_N(\beta) = E\langle f_{\mathscr{I}}\rangle$, we have shown the existence of a function $\mathscr{E}(\beta)$ such that

$$\lim_{N\to\infty}N^{-\operatorname{card}\mathscr{I}/2}(\theta_N(\beta) - a_N(\beta)\theta_{N-1}(\beta')) = \mathscr{E}(\beta) \tag{6.8}$$

uniformly in $\beta \le \beta_0$, where

$$a_N(\beta) = E\frac{\beta^2}{\operatorname{ch}^4(\beta g\sqrt{q_{N-1}} + h)}.$$

Now, using either iteration or the method of (3.60), (3.61), we get convergence of $q_{N-1}(\beta')$ toward the root of the equation $q = E\operatorname{th}^2(\beta g\sqrt{q} + h)$, and thus $a_N(\beta)$ converges toward

$$a(\beta) = E\frac{\beta^2}{\operatorname{ch}^4(\beta g\sqrt{q} + h)} < 1. \tag{6.9}$$

Providing the convergence of $\theta_N(\beta)$ toward $\mathscr{E}(\beta)/(1-a(\beta))$ is then an easy exercise (using iteration).

We have finished the proof of the induction step when $\mathscr{I}$ contains at least a white quadruple. We now consider the case when there is at least a black quadruple. The difference with the previous case is that in (6.3) we now have $\tilde{\sigma}_N\sigma_N^3$ rather than $\tilde{\sigma}_N(\sigma_N^3-\sigma_N^4)$, so that we must use Proposition 3.5 rather than Proposition 3.6. We rewrite (3.55) as

$$\beta^2 E\left\langle \frac{1}{\operatorname{ch}^2 X} - 3\frac{\operatorname{th}^2 X}{\operatorname{ch}^2 X}\right\rangle \left\langle \bar{f}\frac{\tilde{\boldsymbol{\eta}}\cdot\boldsymbol{\eta}^3}{N}\right\rangle_0 + 3\beta^2 E\frac{\operatorname{th}^2 X}{\operatorname{ch}^2 X}\left\langle \bar{f}\frac{\tilde{\boldsymbol{\eta}}\cdot\dot{\boldsymbol{\eta}}^3}{N}\right\rangle_0$$
$$+ E\frac{\operatorname{th}^2 X}{\operatorname{ch}^2 X}\left\langle \bar{f}\sum_{l\ge 4}\frac{\tilde{\boldsymbol{\eta}}\cdot\dot{\boldsymbol{\eta}}^l}{N}\right\rangle_0.$$

The key observation is that the last two terms correspond to functions of the type $f_{\mathscr{I}_1}$ (where $\operatorname{card}\mathscr{I}_1 = \operatorname{card}\mathscr{I}$) which contain at least one white quadruple and for which convergence has already been proved. The argument is thus identical to the previous case, except that one replaces a_N by

$$b_N = \beta^2 E\left\langle \frac{1}{\operatorname{ch}^2 Y} - \frac{3\operatorname{th}^2 Y}{\operatorname{ch}^2 Y}\right\rangle. \tag{6.10}$$

We now turn to the case where there are no black or white quadruples. Let us say that a quadruple I of $\mathscr{I}$ is *isolated* if none of its indexes belong to another quadruple of $\mathscr{I}$. As a third stage in the proof of the induction step, we assume that there is at least one isolated red quadruple in $\mathscr{I}$. Without loss of generality, we can assume that this is $\{1,2,3,4\}$, and we write

$$f_{\mathscr{I}} = (\boldsymbol{\sigma}^1\cdot\boldsymbol{\sigma}^2 - Nq_N)f_{\mathscr{I}'},$$

where no quadruple of $\mathscr{I}'$ contains 1, 2, 3, 4. Thus we have

$$E\langle(\boldsymbol{\sigma}^1\cdot\boldsymbol{\sigma}^2 - Nq_N)f_{\mathscr{I}'}\rangle = NE\langle(\sigma_N^1\sigma_N^2 - q_N)f_{\mathscr{I}'}\rangle.$$

We proceed as in (6.7) to get

$$\begin{aligned} E\langle f_{\mathscr{I}}\rangle &= NE\big\langle(\sigma_N^1\sigma_N^2 - q_N)f_{\mathscr{I}'}(\boldsymbol{\eta}^1,\dots,\boldsymbol{\eta}^p)\big\rangle \\ &\quad + N\sum_{I\in\mathscr{I}'}\left\langle(\sigma_N^1\sigma_N^2 - q_N)M_I\prod_{J\ne I} f_J(\boldsymbol{\eta})\right\rangle \\ &\quad + N\langle(\sigma_N^1\sigma_N^2 - q_N)S\rangle. \end{aligned}$$

We then evaluate the terms on the right, using Proposition 3.4. The last term gives a lower order contribution for which convergence has already been proved. The convergence of the contributions of the second term is proved through the induction hypothesis. As for the first term, the contribution is

$$NE(\operatorname{th}^2 X - q_N)\langle f_{\mathscr{I}'}(\boldsymbol{\eta}^1,\dots,\boldsymbol{\eta}^p)\rangle_0 \tag{6.11}$$

plus other terms for which convergence has already been proved, since the corresponding function $f_{\mathscr{I}''}$ contains either a black or a white quadruple. Using (5.25), the leading contribution to (6.11) is

$$\varphi'\left(\frac{N-1}{N}q_{N-1}\right)E\langle \boldsymbol{\eta}\cdot\boldsymbol{\eta}' - (N-1)q_{N-1}\rangle_0 \langle f_{\mathscr{I}'}(\boldsymbol{\eta}^1,\dots,\boldsymbol{\eta}^p)\rangle_0. \tag{6.12}$$

Now, since no quadruple of $\mathscr{I}'$ contains 1 or 2, we have the essential fact that

$$\langle \boldsymbol{\eta}\cdot\boldsymbol{\eta}' - (N-1)q_{N-1}\rangle_0 \langle f_{\mathscr{I}'}(\boldsymbol{\eta}^1,\dots,\boldsymbol{\eta}^p)\rangle_0 = \langle f_{\mathscr{I}}(\boldsymbol{\eta}^1,\dots,\boldsymbol{\eta}^p)\rangle_0,$$

so that we can again use iteration to conclude.

Finally, we consider the general case. But in that case we observe that

$$\langle \boldsymbol{\eta}\cdot\boldsymbol{\eta}' - (N-1)q_{N-1}\rangle_0 \langle f_{\mathscr{I}'}(\boldsymbol{\eta}^1,\dots,\boldsymbol{\eta}^p)\rangle_0 = \langle f_{\mathscr{I}_1}(\boldsymbol{\eta}^1,\dots,\boldsymbol{\eta}^{p+4})\rangle_0,$$

where $\mathscr{I}_1 = \mathscr{I}' \cup I$ for $I = \{p+1,\dots,p+4\}$ and $f_I = \boldsymbol{\eta}^{p+1}\cdot\boldsymbol{\eta}^{p+2} - q(N-1)$, so that $\mathscr{I}_1$ has an isolated quadruple, and convergence has been proved in this case.

Theorem 6.1 is proved. $\square$

There is nothing to change in the proof to replace the condition $\beta \le \beta_0$ by the conditions of Theorems 1.6 or 1.7.

As we have mentioned, the global underlying algebraic structure behind Theorem 6.1 is unclear to us; computations in simple cases can, however, be carried out quite effectively. As an example, we mention the following. It is left to the reader, following the procedure of Theorem 6.1.

THEOREM 6.2. *If $k \ge 0$ and $n \in \{1,2\}$, we have*

$$\lim_{n\to\infty} E\left(N^{-kn}\langle((\boldsymbol{\sigma}^1-\boldsymbol{\sigma}^2)(\boldsymbol{\sigma}^3-\boldsymbol{\sigma}^4))^{2k}\rangle^n\right) = \left(\prod_{l\le k}(2l-1)\left(\frac{4b}{1-\beta^2 b}\right)^k\right)^n,$$

where

$$b = E\frac{1}{\operatorname{ch}^4(\beta g\sqrt{q}+h)}.$$

Of course this would also hold for any n. The values of $n=1,2$, are enough to show that for the typical disorder, the moments of $N^{-1}(\boldsymbol{\sigma}^1-\boldsymbol{\sigma}^2)(\boldsymbol{\sigma}^3-\boldsymbol{\sigma}^4)$ for Gibbs measure are approximately those of a Gaussian r.v. of variance $4b(1-\beta^2 b)^{-1}$, a statement that can of course be formulated as a central limit theorem.

Note added in Proof. The author must apologize for having overlooked that in condition (1.11) one can equivalently replace the centering term $\langle\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}'\rangle$ by ${}_{\varphi N}(0)$ and that the resulting condition ensures that the random variable $N_{-1}\langle\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}'\rangle$ does not behave pathologically. Thus the work of the end of Section 3, starting with Lemma 3.9, is not needed. Moreover, our argument then shows absence of replica symmetry breaking of each part of the low temperature region except possibly on the line $\gamma(\beta)=1$.

After this final version of this paper was accepted, the author made further progress on several fronts. He discovered a somewhat easier derivation of Proposition 3.4. He also found how to prove directly (1.20), rather than proving separately (1.17) and (1.18). This makes possible significantly shorter proofs of the results presented here. The author also discovered that there is indeed a simple structure behind Theorem 1.8. He proved the validity of (1.6) in a region that probably coincides with the region (1.3). Some of these results are already available from the author.

Finally, R. Latala recently proved that (1.4) has a unique solution for all values of β, h.

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