DISTRIBUTION OF QUADRATIC FORMS AND RATIOS OF QUADRATIC FORMS

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1. Summary. Let the random variable $X = (X_1, X_2, \cdots, X_n)$ have the probability density

$$p(x) = \frac{\det^{\frac{1}{2}} \Omega}{(2\pi)^{n/2}} e^{-\frac{1}{2}x\Omega x'}$$

where $x\Omega x'$ is positive definite. The present article solves, by means of Laguerrian expansions, the problem of finding the distribution of any nonnegative quadratic form XPX'. If the semimoments (defined below) are known, it also solves, by means of Laguerrian expansions the problems of finding the distribution of any indefinite quadratic form, and the distribution of the ratio of any indefinite quadratic form to any nonnegative quadratic form. For an outline of the procedure, see Section 2. If the distribution of the indefinite form is symmetric, the semimoments are easily found, but often, especially for the technique described below for ratios, the semimoments are difficult to obtain. In view of this, a new system of orthogonal polynomials is proposed, which is analogous to the Laguerre system, but which obviates the need of semimoments.

2. Introduction. There are various distribution problems associated with a quadratic form or a ratio of quadratic forms. Suppose P and Q are arbitrary symmetric $n \times n$ matrices of known constants. The notation $(DQ)_i$, $(DR)_i$, $(DQ)_{i1}$ $(DR)_{i1}$ $(i = 1, 2)$ refers to the following distribution problems:

$(DQ)_1$: Find the distribution of XQX'.

$(DQ)_2$: Same as $(DQ)_1$, but with Q nonnegative.

$(DR)_1$: Find the distribution of XQX'/XPX', but with P nonnegative.

$(DR)_2$: Same as $(DR)_1$ but also with Q nonnegative.

$(DQ)_{11}$, $(DQ)_{21}$: Same as $(DQ)_1$, $(DQ)_2$ respectively, with the further restriction that $\Omega = I$ and Q is diagonal. (I denotes the unit $n \times n$ matrix).

$(DR)_{11}$, $(DR)_{21}$: Same as $(DR)_1$, $(DR)_2$ respectively, with the further restriction that $\Omega = I$ and both P and Q are diagonal.

It should be noted that although it is always possible by a linear transformation to reduce $(DQ)_i$ to $(DQ)_{i1}$, $(i = 1, 2)$, the same is not true for reducing $(DR)_i$ to $(DR)_{i1}$. In fact, a necessary and sufficient condition for the latter reduction is (cf. Weyl [11]) $PGQ = QGP$ where $G = TT'$, and T is a matrix such that $T'\Omega T = I$.

It is possible, however, to reduce $(DR)_1$ to $(DQ)_{11}$ by means of a linear trans-

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formation. This will be shown in the following section. It is relevant to add that Robbins [7], and Pitman and Robbins [6] have solved, by means of certain convergent series, $(DQ)_2$ and a certain subclass of $(DR)_{21}$. The present article differs mainly in two respects. First, the more general problems $(DQ)_1$ and $(DR)_1$ are considered here. Second, only Laguerrian expansions are used. It may be added here that the distribution of quadratic forms or ratios of quadratic forms for certain cases has been investigated by McCarthy [13], von Neumann [14], and Bhattacharyya [15]. It may be noted further that Bhattacharyya [16] and Hotelling [17] have employed Laguerrian expansions for the cases they consider.

For convenience in applying the results obtained in this paper, the procedure will be outlined, as follows.

I. $(DQ)_2$. Reduce to $(DQ)_{21}$, so that $XQX' = \sum \gamma_i Y_i^2$ where Y has density $f(y)$ given by (2). By a change of scale arrange that $0 < \gamma_i < 1$. Let $F(x) = P\{XQX' \leqq x\}$ and $K(x) = \int_{-\infty}^{x} g(t)\, dt$ with $g(t)$ assigned by (14). Compute the moments $\mu_k = E(XQX')^k$. The Laguerrian expansion (convergent, here) of $F(x) - K(x)$ is given by (10), where

(i) $L_n^{(\alpha)}(x)$ is defined by (4) or (5),

(ii) $A_n^{(\alpha)} = a_{n+1}^{(\alpha-1)} = \sum_{v=0}^{n} (-1)^v \binom{n+\alpha}{n-v} \frac{\mu_v}{v!}$.

II. $(DQ)_1$. Reduce to $(DQ)_{11}$ so that $XQX' = \sum_1^{n_1} \gamma_i Y_i^2 - \sum_{n_1+1}^{n_1+n_2} \gamma_i Y_i^2$ where $0 < \gamma_i < 1$ (arranged by change of scale) and Y has density $f(y)$ given by (2). $K(x)$ and $F(x)$ have the same meaning as in I. Compute the semimoments

$$\delta_k = \int \cdots \int \left[\sum_1^{n_1+n_2} \gamma_i y_i^2 \right]^k f(y)\, dy$$

where the region of integration is given by $\sum_1^{n_1+n_2} \gamma_i y_i^2 \geqq 0$. If the density of XQX' is symmetric, its characteristic function may be used to find the δ_k's, without requiring explicit knowledge of the γ_i's. The Laguerrian expansion (convergent here) of $F(x) - K(x)$ for $x > 0$ is given by (10), with

$$A_n^{(\alpha)} = a_{n+1}^{(\alpha-1)} - \binom{n+\alpha}{n+1} [\delta_0 - 1].$$

For $x = 0$, and $x < 0$, see Section 7.

III. $(DR)_1$. Reduce to $(DQ)_{11}$ by method of Section 3, so that

$$G(z) = P\left\{\frac{XQX'}{XPX'} \leqq z\right\} = P\left\{\sum_1^n \lambda_i(z) Y_i^2 \leqq 0\right\}$$

where Y has density $f(y)$ given by (2). By a change of scale, it is possible to write $|\lambda_i(z)| < 1$. The method then follows that of II., replacing γ_i there by $\lambda_i(z)$.

3. Reduction of the $(DR)_1$ problem. For any real z, let $R_z = Q - zP$ and $G(z) = P\{(XQX')/(XPX') \leqq z\}$. Setting $H_z(\xi) = P\{XR_zX' \leqq \xi\}$ it follows

(recalling that xPx' is nonnegative) that $H_z(0) = G(z)$. Let $\lambda_i(z)$ $(i = 1, 2, \cdots, n)$ be the roots of det $(R_z - \lambda\Omega) = 0$ and Λ_z be a diagonal matrix with these roots as the diagonal elements. The roots $\lambda_i(z)$ will all be real, and there exists a matrix T such that (cf. Bôcher [1]) $T'\Omega T = I$ and $T'R_zT = \Lambda_z$.
Hence

$$H_z(\xi) = P\{X\Lambda_z X' \leqq \xi\} \tag{1}$$

where X has the probability density

$$f(x) = \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} e^{-\frac{1}{2}xx'} \tag{2}$$

Now $X\Lambda_z X'$ is a linear combination of independent random variables each of which is distributed as χ^2 with one degree of freedom. The $(DR)_1$ problem of determining $G(z)$ thus reduces to the special $(DQ)_{11}$ problem of finding $H_z(0)$, that is, the distribution of this linear combination *at the single point* $\xi = 0$. Geometrically, this may be interpreted as the problem of finding the probability measure of the interior of the cone $x\Lambda_z x' = 0$ when the probability density of X is given by (2).

In anticipation of Section 7 it should be remarked here that in determining the semimoments of $X\Lambda_z X'$ it is sometimes not necessary to find the roots $\lambda_i(z)$. For, by the uniqueness of the characteristic function

$$\det{}^{-\frac{1}{2}}(\Omega - 2itR_z) = \left[\prod_{j=1}^{n}(1 - 2i\lambda_j(z)t)\right]^{-\frac{1}{2}}$$

the moments are found upon evaluating the derivatives at $t = 0$, and dividing by the appropriate power of the imaginary number i. If, for instance, the density of $X\Lambda_z X'$ is symmetric, the semimoments are just half the corresponding moments.

It is also of some interest to see how the reduction of the $(DR)_1$ problem is accomplished to yield directly the probability *density* of the ratio. Of course, the probability density could be obtained by differentiation of the distribution function, but this might not be advisable or feasible, depending on the convergence properties of the approximation used in determining the distribution function.

The following theorem will now be proved. This theorem applies generally, irrespective of quadratic forms, to any ratio (absolutely continuous random variable, positive denominator) and any probability density $p(x)$.

THEOREM 1. *Let X have probability density $p(x)$, and define K and $q(y)$ by*

$$K = \int_{-\infty}^{\infty} xPx'p(x)\,dx, \qquad q(y) = \frac{yPy'p(y)}{K}.$$

Then, the probability density $G'(z)$ is given by $Kr_z(0)$ where $r_z(\xi)$ is the probability density of the random variable YR_zY' when $q(y)$ is the density of Y.

PROOF. From the theory of inversion formulae (cf. Gurland [4])

$$G'(z) = \frac{1}{2\pi i} \oint \left[\frac{\partial \phi(t_1, t_2)}{\partial t_2}\right]_{t_2 = -t_1 z} dt_1$$

where

$$\phi(t_1, t_2) = E(e^{i t_1 X Q X' + i t_2 X P X'})$$

and the notation $\oint$ signifies $\lim\limits_{\substack{\epsilon \to 0 \\ T \to \infty}} \left(\int_{-T}^{-\epsilon} + \int_{\epsilon}^{T}\right)$. Now

$$\left[\frac{\partial \phi}{\partial t_2}\right]_{t_2 = -t_1 z} = Ki \int_{-\infty}^{\infty} e^{i t_1 x R_z x'} q(x)\, dx = Ki\theta_z(t_1)$$

where $\theta_z(t)$ is the characteristic function of XR_zX' when X has the probability density $q(x)$. Hence

$$G'(z) = \frac{K}{2\pi} \oint \theta_z(t)\, dt = Kr_z(0)$$

by inversion of Fourier transforms. This completes the proof.

Before applying some results of this section, we shall, at this point, state some theorems relating to Laguerrian series.

4. Laguerrian series. By a Laguerrian series is meant an expansion of the form

$$f(x) \sim \sum c_n^{(\alpha)} L_n^{(\alpha)}(x) \tag{3}$$

where

$$L_n^{(\alpha)}(x) = \frac{1}{n!} e^x x^{-\alpha} \left(\frac{d}{dx}\right)^n (x^{n+\alpha} e^{-x}) \qquad \alpha > -1. \tag{4}$$

The sign of equivalence in (3) indicates the coefficients $c_n^{(\alpha)}$ are determined by

$$c_n^{(\alpha)} = \frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)} \int_0^\infty e^{-t} t^\alpha L_n^{(\alpha)}(t) f(t)\, dt$$

in view of the orthogonality realtions

$$\int_0^\infty e^{-t} t^\alpha L_m^{(\alpha)}(t) L_n^{(\alpha)}(t)\, dt = \begin{cases} 0, & m \neq n \\ \dfrac{\Gamma(n+\alpha+1)}{\Gamma(n+1)}, & m = n. \end{cases}$$

From (4) it follows that

$$L_n^{(\alpha)}(x) = \sum_{v=0}^{n} \binom{n+\alpha}{n-v} \frac{(-x)^v}{!v}. \tag{5}$$

Before stating the following theorem, the notion of equiconvergence of series will be recalled. If the series $\sum_0^\infty (u_n - Av_n)$ is convergent, where A is a nonzero constant, then the series $\sum_0^\infty u_n$, $\sum_0^\infty v_n$ are said to be equiconvergent. It is also convenient to recall the definitions of $O(g(x))$ and $o(g(x))$

$$f(x) = O(g(x), x \to x_0 \text{ means } (f(x))/(g(x)) \text{ remains bounded as } x \to x_0$$

$$f(x) = O(g(x)), x \to x_0, \text{ means } (f(x))/(g(x)) \to 0 \text{ as } x \to x_0 .$$

THEOREM 2. (Szegö [9]). *Let $f(x)$ be Legesgue measurable, $0 \leqq x < \infty$, and let the integrals*

$$\int_0^1 x^\alpha |f(x)| \, dx, \qquad \int_0^1 x^{\alpha/2-1/4} |f(x)| \, dx \tag{6}$$

exist. If the condition

$$\int_n^\infty e^{-x/2} x^{\alpha/2-13/12} |f(x)| \, dx = \dot{o}(n^{-\frac{1}{2}}) \tag{7}$$

is satisfied, and if $s_n(x)$ denotes the nth partial sum of the Laguerre series (3), *we have, for $x > 0$*

$$\lim_{n=\infty} \left\{ s_n(x) - \frac{1}{\pi} \int_{x^{\frac{1}{2}}-\delta}^{x^{\frac{1}{2}}+\delta} f(\tau^2) \frac{\sin [2n^{\frac{1}{2}}(x^{\frac{1}{2}} - \tau)]}{x^{\frac{1}{2}} - \tau} \, d\tau \right\} = 0 \tag{8}$$

where δ is a fixed positive number, $\delta < x^{\frac{1}{2}}$. This holds uniformly for every fixed positive interval $\epsilon \leqq x \leqq \omega$, $\delta < \epsilon^{\frac{1}{2}}$.

The same equiconvergence theorem (8) is valid if the intergals (6) exist and (7) is replaced by the following:

$$\int_1^\infty e^{-x/2} x^{\alpha/2-3/4} |f(x)| \, dx$$

is convergent, and

$$\int_n^\infty e^{-x} x^{\alpha-2} f(x)^2 \, dx = o(n^{-3/2}).$$

The integral occurring in (8) is essentially the partial sum of order $[n^{\frac{1}{2}}]$ of a Fourier series, where as usual $[n^{\frac{1}{2}}]$ denotes the largest integer $\leqq n^{\frac{1}{2}}$. A sufficient condition for the validity of (8) is

$$f(x) = O(e^{x/2} x^{-\alpha/2-1/4-\delta}), \qquad \delta > 0, \quad x \to \infty.$$

Before quoting a second theorem of Szegö, which ensures summability of (3) at $x = 0$, we recall the definition of Cesaro summability. Let s_n denote the nth partial sum $\sum_0^n u_\nu$. The series $\sum_0^\infty u_\nu$ is said to be (C, k) summable (cf. Zygmund [12]), $k > -1$, to the sum s if $\lim_{n=\infty} s_n^{(k)}/C_n^{(k)} = s$ where

$$C_n^{(k)} = \frac{(n + k)(n + k - 1) \cdots (k + 1)}{n!},$$

$$s_n^{(k)} = \sum_{r=0}^n C_{n-r}^{(k-1)} s_r = \sum_{r=0}^n C_{n-r}^k u_r .$$

THEOREM 3. (Szegö [9]). Let $f(x)$ be measurable $0 \leqq x < \infty$, and continuous at $x = 0$. If we assume the existence of the integral

$$\int_1^\infty e^{-x} x^{\alpha-k-1/3} \,|\, f(x) \,|\, dx \tag{9}$$

the Laguerrian series (3) is summable (C, k) at $x = 0$ to the sum $f(0)$, provided $k > \alpha + \frac{1}{2}$. This statement is not true for $k \leqq \alpha + \frac{1}{2}$.

The condition regarding (9) is satisfied if

$$f(x) = O(e^{x/2}\, x^{k-\alpha-2/3-\delta}), \qquad \delta > 0, \quad x \to \infty.$$

It should be remarked that for the case $x = 0$, the kth Cesaro mean has the simplified form

$$\{C_n^{(k)}\, \Gamma(\alpha + 1)\}^{-1} \int_0^\infty e^{-t} t^\alpha f(t) L_n^{(\alpha+k+1)}(t)\, dt.$$

5. Laguerrian expansions for distribution functions. Let a random variable have the distribution function $F(x) = \int_{-\infty}^x p(t)\, dt$. By analogy with Gram-Charlier series, we may consider

$$\begin{aligned} p(x) &\sim e^{-x} x^\alpha \sum_{n=0}^\infty a_n^{(\alpha)} L_n^{(\alpha)}(x) \\ F(x) - K(x) &\sim e^{-x} x^\alpha \sum_{n=0}^\infty A_n^{(\alpha)} L_n^{(\alpha)}(x) \end{aligned} \tag{10}$$

where

$$\begin{aligned} a_n^{(\alpha)} &= \int_0^\infty p(t) L_n^{(\alpha)}(t)\, dt \\ A_n^{(\alpha)} &= \int_0^\infty [F(t) - K(t)]\, L_n^{(\alpha)}(t)\, dt \end{aligned} \tag{11}$$

and $K(x)$ is a conveniently chosen distribution function

$$K(x) = \int_{\infty}^x g(t)\, dt. \tag{12}$$

Note that $a_n^{(\alpha)}$, $A_n^{(\alpha)}$ are linear functions of the "moments" taken over the interval $(0, \infty)$ and not $(-\infty, \infty)$. We shall call such moments "semimoments."

It is in order at this point to remark why Laguerrian rather than Gram-Charlier series are being considered here for the aforementioned distribution problems. The main reason is that Cramér's condition ([3], p. 233) $\int_{-\infty}^\infty e^{x^2}/4\; dF(x) < \infty$ sufficient for convergence, is not satisfied for these problems; and the theorems which guarantee Cesaro or Abel summability (cf. Szegö [9], Hille [5]) do not relax Cramér's condition very much if at all.

In this paper, expansions of the type (10) will be considered. In order to simplify the formula (11) for $A_n^{(\alpha)}$, it necessary to refer to the following lemma.

LEMMA 1. If all the absolute moments corresponding to $F(x)$ and $K(x)$ are finite, then

$$F(x) - K(x) = o(x^{-r}), \qquad x \to \pm\infty$$

for all $r > 0$.

PROOF[1]. Let $M_r = \int_{-\infty}^{\infty} |t|^r p(t)\,dt$. Then

$$M_r \geqq \int_x^{\infty} t^r p(t)\,dt \geqq x^r \int_x^{\infty} p(t)\,dt.$$

Hence

$$\int_x^{\infty} p(t)\,dt = O(x^{-r}), \qquad x \to \infty.$$

Similarly for $\int_x^{\infty} g(t)\,dt$. By similar reasoning it can also be shown that

$$\int_{-\infty}^{x} p(t)\,dt = O(x^{-r}), \qquad \int_{-\infty}^{x} g(t)\,dt = O(x^{-r}) \qquad x \to -\infty.$$

Since we can write

$$| F(x) - K(x) | = | 1 - K(x) - (1 - F(x)) | \leqq \int_x^{\infty} g(t)\,dt + \int_x^{\infty} p(t)\,dt$$

or

$$| F(x) - K(x) | \leqq \int_{-\infty}^{x} p(t)\,dt + \int_{-\infty}^{x} g(t)\,dt$$

the required result follows.

To apply this lemma, let $M_n^{(\alpha)}(x)$ be such a polynomial that

$$\frac{d}{dx} M_n^{(\alpha)}(x) = L_n^{(\alpha)}(x)$$

and, for convenience, let $M_n^{(\alpha)}(0) = 0$. Since

$$\frac{d}{dx} L_{n+1}^{(\alpha-1)}(x) = -L_n^{(\alpha)}(x),$$

as can be seen from (5), it follows that

$$M_n^{(\alpha)}(x) = -\left[L_{n+1}^{(\alpha-1)}(x) - \binom{n+\alpha}{n+1}\right].$$

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[1] The author is grateful to Morton Slater for his assistance in greatly simplifying the above proof.

Integrating by parts the expression for $A_n^{(\alpha)}$ and applying Lemma 1, $A_n^{(\alpha)}$ becomes

$$(13) \qquad \begin{aligned} A_n^{(\alpha)} &= -\int_0^\infty M_n^{(\alpha)}(t)[p(t) - g(t)]\,dt \\ &= a_{n+1}^{(\alpha-1)} - b_{n+1}^{(\alpha-1)} - \binom{n+\alpha}{n+1}\int_0^\infty [p(t) - g(t)]\,dt \end{aligned}$$

where

$$a_n^{(\alpha)} = \int_0^\infty L_n^{(\alpha)}(t)p(t)\,dt, \qquad b_n^{(\alpha)} = \int_0^\infty L_n^{(\alpha)}(t)g(t)\,dt.$$

By choosing

$$(14) \qquad \begin{aligned} g(x) &= \frac{1}{\Gamma(\alpha+1)}x^\alpha e^{-x} \qquad && x > 0 \\ &= 0, && x \leqq 0 \end{aligned}$$

the expression (13) simplifies to

$$(15) \qquad A_n^{(\alpha)} = a_{n+1}^{(\alpha-1)} - \binom{n+\alpha}{n+1}\int_0^\infty [p(t) - g(t)]\,dt.$$

If, further, $p(t) = 0$ for $t \leqq 0$ (as, for instance, in the $(DQ)_2$ problem), the formula becomes

$$(16) \qquad A_n^{(\alpha)} = a_{n+1}^{(\alpha-1)}.$$

6. Solution to the $(DQ)_2$ problem. Before proceeding to the solution, it is necessary to establish the following lemma.

LEMMA 2.[2] *Let X have the probability density $f(x)$ given by (2). Define $U = \sum_1^n \gamma_i X_i^2$ where the constants γ_i satisfy $0 < \gamma_i < 1$. Denote the probability density of U by $p_U(u)$. Then*

$$\frac{u^{(n/2)-1}e^{-(u/2)(1+\epsilon')}}{2^{n/2}\Gamma\left(\frac{n}{2}\right)\left[\prod_1^n \gamma_i\right]^{\frac{1}{2}}} \leqq p_U(u) \leqq \frac{u^{(n/2)-1}e^{-(u/2)(1+\epsilon)}}{2^{n/2}\Gamma\left(\frac{n}{2}\right)\left[\prod_1^n \gamma_i\right]^{\frac{1}{2}}}, \qquad u > 0$$

where

$$1 + \epsilon' = \max_i \frac{1}{\gamma_i}, \qquad 1 + \epsilon = \min_i \frac{1}{\gamma_i}.$$

[2] The author is grateful to Ray Mickey for his assistance in greatly simplifying the formulation and proof of this lemma.

Proof. Apply to (2) the transformation

$$z_i = \sqrt{\gamma_i}\, x_i \qquad i = 1, 2, \cdots, n.$$

Then

$$p_Z(z) = \frac{1}{\left[\prod_1^n \gamma_i\right]^{\frac{1}{2}} (2\pi)^{\frac{n}{2}}} \exp\left[-\frac{1}{2}\sum_1^n \frac{z_i^2}{\gamma_i}\right]$$

and

$$\underset{u<\sum_1^n z_i^2<u+\Delta u}{\int\cdots\int} \frac{\exp\left[-\frac{1}{2}(1+\epsilon')\sum_1^n z_i^2\right]}{\left[\prod_1^n \gamma_i\right]^{\frac{1}{2}} (\sqrt{2\pi})^n}\, dz \leqq \underset{u<\sum_1^n z_i^2<u+\Delta u}{\int\cdots\int} p_Z(z)\, dz$$

$$\leqq \underset{u<\sum_1^n z_i^2<u+\Delta u}{\int\cdots\int} \frac{\exp\left[-\frac{1}{2}(1+\epsilon)\sum_1^n z_i^2\right]}{\left[\sum_1^n \gamma_i\right]^{\frac{1}{2}} (\sqrt{2\pi})^n}\, dz.$$

By applying the mean value theorem of integral calculus, and letting $u \to 0$, the required result follows.

To apply this lemma, suppose $(DQ)_2$ has been reduced to $(DQ)_{21}$. Let $XQX' = \sum_1^n \gamma_i X_i^2$, where $0 < \gamma_i < 1$ and X has the probability density (2). Hence, by Lemma 2,

$$\int_x^\infty p_U(u)\, du = O(e^{-(x/2)(1+\epsilon)} x^{n/2}), \qquad x \to \infty.$$

Now, with $K(x)$ defined by (12) and (14), it follows that

$$1 - K(x) = O(e^{-x} x^{\alpha+1}), \qquad x \to \infty.$$

Thus,

$$[F(x) - K(x)]\, e^x x^{-\alpha} = O\left(\max\{e^{x/2(1-\epsilon)} x^{n/2-\alpha}, x\}\right), \qquad x \to \infty.$$

Theorem 2 is now applicable to establish the convergence of the expansion (10), with $A_n^{(\alpha)}$ given by (16). By Theorem 2, the series for $F(x) - K(x)$ will converge at each point x if the Fourier series converges there. Since $F(x) - K(x)$ is of bounded variation, convergence is assured by Jordan's test (cf. Titchmarsh [10]).

7. Solutions to $(DQ)_1$ and $(DR)_1$ if the semimoments are known. As remarked in Section 1, there may be instances where the semimoments are easily found, as in the case of an indefinite quadratic form with a symmetric probability density. Before applying the convergence theorems of Section 4 it is necessary to establish the following lemma.

LEMMA 3. Let X have the probability density (2) and define

$$U_1 = \sum_1^{n_1} \gamma_i X_i^2, \qquad U_2 = \sum_{n_1+1}^{n_1+n_2} \gamma_i X_i^2$$

where $0 < \gamma_i < 1$, $n_1 + n_2 = n$, $(i = 1, 2, \cdots, n)$. Denote by $p_V(v)$ the probability density of $V = U_1 - U_2$. Then

$$\int_x^\infty p_V(v)\, dv = O(e^{-(x/2)(1+\epsilon)} x^{n_1/2}), \qquad x \to \infty \tag{17}$$

$$\int_{-\infty}^x p_V(v)\, dv = O(e^{-(|x|/2)(1+\epsilon)} |x|^{n_2/2}), \qquad x \to -\infty \tag{18}$$

where $1 + \epsilon = \min_i \gamma_i^{-1}$.

PROOF. By Lemma 2

$$p_{U_1,U_2}(u_1, u_2) \leqq C e^{-\frac{1}{2}(u_1+u_2)(1+\epsilon)} u_1^{(n_1/2)-1} u_2^{(n_2/2)-1}$$

where

$$C^{-1} = 2^{n/2}\Gamma\left(\frac{n}{2}\right)\left[\prod_1^n \gamma_i\right]^{\frac{1}{2}}.$$

Hence

$$\begin{aligned}\int_x^\infty p_V(v)\, dv &\leqq \iint_{u_1-u_2\geqq x} C e^{-\frac{1}{2}(u_1+u_2)(1+\epsilon)} u_1^{(n_1/2)-1} u_2^{(n_2/2)-1}\, du_1\, du_2 \\ &= C \int_{v=x}^\infty \int_{u_2=0}^\infty e^{-\frac{1}{2}(1+\epsilon)(2u_2+v)} (v + u_2)^{n_1/2} u_2^{n_2/2}\, du_2\, dv.\end{aligned}$$

Now

$$(v + u_2)^{n_1/2} = v^{n_1/2}\left(1 + \frac{u_2}{v}\right)^{n_1/2} < v^{n_1/2}(1 + u_2)^{n_1/2}$$

since $v > 1$ (because $x \to \infty$ in (17)). The validity of (17) follows, since

$$\int_0^\infty e^{-(1+\epsilon)u_2}(1 + u_2)^{n_1/2} u_2^{n_2/2}\, du_2 < \infty.$$

Also, since

$$\begin{aligned}\int_{-\infty}^{-x} p_V(v)\, dv &= \iint_{u_1-u_2\leqq -x} p_{U_1,U_2}(u_1, u_2)\, du_1\, du_2, \\ &= \iint_{u_2-u_1\geqq x} p_{U_1,U_2}(u_1, u_2)\, du_1\, du_2, \qquad x > 0\end{aligned}$$

the result (18) is established by an argument similar to that for (17).

In applying the result of Lemma 3 to $(DQ)_1$, it may be assumed XQX' is in the form $U_1 - U_2$ (as can be effected by a linear transformation). If, also, $K(x)$

is defined by means of (14), then $\{F(x) - K(x)\}e^x x^{-\alpha}$ will satisfy the conditions of Theorems 2 and 3. Consequently, the expansion (10), with $A_n^{(\alpha)}$ given by (15) will converge (See the last remark of Section 6) for $x > 0$, while for $x = 0$ it will be $(C, 1)$ summable if α is chosen to be zero in Theorem 3.

For $x < 0$, the result of Theorem 2 applies by considering the expansion of $F(-x) - K(-x)$.

Lemma 3 may also be applied in solving the $(DR)_1$ problem by using the reduction of Section 3, and employing the same type of argument as for the $(DQ)_1$ problem above, to show that Theorem 3 ensures the $(C, 1)$ summability of the Laguerrian expansion at $x = 0$.

8. Proposed system of polynomials for the general solution of $(DQ)_1$, $(DR)_1$. As mentioned above, the semimoments are often difficult to obtain. The convergence properties of Laguerrian expansions are most convenient, but the main shortcoming is that the weight function is zero over the range $(-\infty, 0)$. What is required is a nonzero weight function over $(-\infty, \infty)$ which would generate a system of orthogonal polynomials behaving asymptotically in a manner similar to the Laguerrian system. In such a case, ordinary moments rather than semimoments would be used in the determination of the coefficients of the expansion, and, these ordinary moments can be found without difficulty. An orthogonal polynominal system which seems to suggest itself naturally is that generated, according to the Gram-Schmidt process (cf. Courant-Hilbert [2]), by means of the weight function

$$w(x) = e^{-|x|}|x|^{\alpha}, \qquad -\infty < x < \infty.$$

Shohat [8] has shown that for weight functions similar to this, the resulting system of polynomials is complete, but there appears to be no treatment of the convergence properties of such a system in the literature on orthogonal polynomials. If, as conjectured, this system behaves similarly to the Laguerrian system, then a much larger class of distribution functions will be expansible in convergent (or summable) series than the class to which Gram-Charlier series apply.

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REFERENCES

[1] M. Bôcher, *Introduction to Higher Algebra*, Macmillan, 1924.
[2] R. Courant and D. Hilbert, *Methoden der Mathematischen Physik*, Julius Springer, Berlin, 1931.
[3] H. Cramér, *Mathematical Methods of Statistics*, Princeton University Press, 1946.
[4] J. Gurland, "Inversion formulae for the distribution of ratios," *Ann. Math. Stat.* Vol. 19 (1948), pp. 228-237.
[5] E. Hille, "A class of reciprocal functions," *Ann. Mathematics*, Vol. 27 (1925-1926), pp. 427-464.

[6] H. Robbins and E. J. G. Pitman, "Application of the method of mixtures to quadratic forms in normal variates," *Ann. Math. Stat.*, Vol. 20 (1949), pp. 552–560.
[7] H. Robbins, "The distribution of a definite quadratic form," *Ann. Math. Stat.*, Vol. 19 (1948), pp. 266–270.
[8] J. A. Shohat, "The theory of closure of Tchebycheff polynomials for an infinite interval," *Bull. Amer. Math. Soc.*, Vol. 30 (1924), pp. 505–510.
[9] G. Szegö, "Orthogonal Polynomials," American Mathematical Society Colloquium Publication, Vol. 23 (1939).
[10] E. C. Titchmarsh, *The Theory of Functions*, Oxford University Press, 1939.
[11] H. Weyl, *The Theory of Groups and Quantum Mechanics*, Methuen and Company, London, 1931.
[12] A. Zygmund, *Trigonometrical Series*, Varszawa-Lwow, 1935.
[13] M. D. McCarthy, "On the application of the z-test to randomized blocks, "*Ann. Math. Stat.*, Vol. 10 (1939), pp. 337–359.
[14] J. von Neumann, "Distribution of the ratio of the mean square successive difference to the variance," *Ann. Math. Stat.*, Vol. 12 (1941), pp. 367–395.
[15] B. C. Bhattacharyya, "On an aspect of the Pearson system of curves, and a few analogies," *Sankhya*, Vol. 6 (1943), pp. 415–418.
[16] A. Bhattacharyya, "A note on the distribution of the sum of chi-squares," *Sankhya*, Vol. 7 (1945), pp. 27–28.
[17] H. Hotelling, "Some new methods for distributions of quadratic forms," (Abstract) *Ann. Math. Stat.*, Vol. 19 (1948), p. 119.