the sample size required to come to a terminal decision, but only certain aspects of it (for example, its expected value), can be handled as above, using the proper $W_{ijk}(x)$ at each stage.

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ON A CHARACTERISATION OF THE GAMMA DISTRIBUTION

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An intrinsic property of the gamma distribution, as proved by Pitman [1], is that if $X_1, X_2 \cdots, X_n$ are n identically distributed independent gamma variates with the distribution function

$$dF(X) = \frac{1}{\Gamma(p)} e^{-X} X^{p-1}\, dX \qquad (0 \leqq X \leqq \infty)$$

then the sum $X_1 + X_2 + \cdots + X_n$ is distributed independently of any function $g(X_1, X_2, \cdots, X_n)$ satisfying $g(X_1, X_2, \cdots, X_n) = g(\lambda X_1, \lambda X_2, \cdots, \lambda X_n)$ for any nonzero real λ. That is, $g(X_1, X_2, \cdots, X_n)$ should be a function independent of scale. In the present paper the converse theorem is proved for a particular class of g-function.

THEOREM. *Let $X_1, X_2, \cdots, X_n$ be n identically distributed independent random variables with a finite second moment. If the conditional expectation of the ratio of two quadratic forms $(\sum a_{ij}X_iX_j)/(\sum X_i)^2$, (where the elements of the matrix (a_{ij}) satisfy the relation $\sum a_{ii} \neq \sum a_{ij}/n$) for fixed sum $X_1 + X_2 + \cdots + X_n$ be equal to its unconditional expectation, then each X follows the gamma distribution.*

For a matrix $A = (a_{ij})$ where the relation $\sum a_{ii} = \sum a_{ij}/n$ holds, the method suggested does not lead to any solution of the problem. It is also interesting to note in this connection that the stronger assumption of stochastic independence of the sum $X_1 + X_2 + \cdots + X_n$ and $g(X_1, X_2, \cdots X_n)$ is not necessary for this particular class of g-function.

The following lemma is required for the proof of the Theorem.

LEMMA. *If u and v are two random variables such that for fixed v, the conditional expectation of $u/f(v)$, where $f(v)$ is a function of v, is equal to its unconditional expectation (provided it exists), then*

$$E(ue^{itv}) = E\{u/f(v)\} \cdot E\{f(v)e^{itv}\}.$$

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The proof of this lemma is very simple. If x and y are two variates such that the conditional expectation of y for fixed x is equal to its unconditional expectation, then $E_x(y) = E(y)$. Multiplying both sides by $\varphi(x)\, e^{itx}$ and taking expectation with respect to x, we get very easily

$$E\{y\,\varphi(x)e^{itx}\} = E(y)\cdot E\{\varphi(x)\, e^{itx}\}.$$

Putting $y = u/f(v)$, $\varphi(x) = f(v)$, $x = v$, this becomes

$$E\{(u/f(v))f(v)e^{itv}\} = E\{u/f(v)\}\cdot E\{f(v)e^{itv}\}.$$

To prove the lemma, this may be written as

$$E\{ue^{itv}\} = E\{u/f(v)\}\cdot E\{f(v)e^{itv}\}.$$

Proof of theorem. Using this lemma with $u = \sum a_{ij}X_iX_j$, $v = \sum X_i$, and $f(v) = (\sum X_i)^2$,

$$\begin{aligned} &E\{(\textstyle\sum a_{ij}X_iX_j)e^{it(X_1+X_2+\cdots+X_n)}\} \\ &\qquad = E\{(\textstyle\sum a_{ij}X_iX_j)/(\sum X_i)^2\}\cdot E\{(\sum X_i)^2\cdot e^{it(X_1+\cdots+X_n)}\}. \end{aligned} \tag{1}$$

Let $\varphi(t) = E(e^{itX})$ represent the characteristic function of the distribution of X. After some algebraic simplifications, (1) will reduce to

$$\begin{aligned} &(\textstyle\sum a_{ii})\cdot \dfrac{d^2\varphi}{dt^2}\cdot\varphi^{n-1} + (\sum\limits_{i\neq j} a_{ij})\cdot\left(\dfrac{d\varphi}{dt}\right)^2\cdot\varphi^{n-2} \\ &\qquad = K\left\{n\cdot\dfrac{d^2\varphi}{dt^2}\cdot\varphi^{n-1} + n(n-1)\cdot\left(\dfrac{d\varphi}{dt}\right)^2\cdot\varphi^{n-2}\right\}, \end{aligned} \tag{2}$$

where $K = E\{(\sum a_{ij}X_iX_j)/(\sum X_i)^2\}$. Then, we have

$$\begin{aligned} &(\textstyle\sum a_{ii})\cdot\left(\dfrac{d^2\varphi}{dt^2}\Big/\varphi\right) + (\sum\limits_{i\neq j} a_{ij})\cdot\left(\dfrac{d\varphi}{dt}\Big/\varphi\right)^2 \\ &\qquad = K\left\{n\cdot\left(\dfrac{d^2\varphi}{dt^2}\Big/\varphi\right) + n(n-1)\cdot\left(\dfrac{d\varphi}{dt}\Big/\varphi\right)^2\right\}. \end{aligned} \tag{3}$$

Writing $\psi(t) = \ln\varphi(t)$, we have

$$\frac{d\psi}{dt} = \frac{d\varphi}{dt}\Big/\varphi, \qquad \frac{d^2\psi}{dt^2} = \frac{d^2\varphi}{dt^2}\Big/\varphi - \left(\frac{d\varphi}{dt}\Big/\varphi\right)^2.$$

Substituting these in (3), we obtain the following differential equation for $\psi(t)$,

$$A\cdot\frac{d^2\psi}{dt^2} + B\cdot\left(\frac{d\psi}{dt}\right)^2 = 0, \qquad \begin{cases} A = \sum a_{ii} - nK, \\ B = \sum a_{ij} - n^2K, \end{cases} \tag{4}$$

together with the initial conditions

$$\left.\frac{d\psi}{dt}\right|_{t=0} = im, \qquad \left.\frac{d^2\psi}{dt^2}\right|_{t=0} = -\sigma^2.$$

Here m and σ^2 are respectively the mean and variance of the distribution of X. In the solution of this differential equation (4), three cases must be distinguished.

I. $A \neq 0, \quad B \neq 0;$

II. $A \neq 0, \quad B = 0;$

III. $A = 0, \quad B \neq 0.$

For Case I, the differential equation may be written as

$$\frac{d^2\psi}{dt^2} = C \cdot \left(\frac{d\psi}{dt}\right)^2, \qquad C = -\frac{B}{A} = \frac{\sigma^2}{m^2}, \tag{5}$$

using the initial condition in (4). Writing $\xi(t) = d\psi/dt$ equation (5) reduces to

$$\frac{d}{dt}\left(\frac{1}{\xi(t)}\right) = -\frac{\sigma^2}{m^2}. \tag{6}$$

Integrating this differential equation (6) with respect to t, using the initial condition $\xi(0) = im$, we get

$$\frac{1}{\xi(t)} = \frac{1}{im} - \frac{\sigma^2}{m^2}t, \qquad \text{or} \qquad \xi(t) = \frac{im}{1 - (\sigma^2/m)it}. \tag{7}$$

From (7), with the initial condition $\psi(0) = 0$, we get very easily

$$\psi(t) = -(m^2/\sigma^2)\log[1 - (\sigma^2/m)it], \qquad \text{or} \qquad \varphi(t) = [1 - (\sigma^2/m)it]^{-(m^2/\sigma^2)}. \tag{8}$$

By applying the inversion theorem, it can be very easily shown that the characteristic function $\varphi(t)$ in (8) leads uniquely to the gamma distribution with parameters $\alpha = m/\sigma^2$ and $\beta = m^2/\sigma^2$, the frequency function being given by

$$\begin{cases} [1/\Gamma(\beta)]\alpha^\beta \cdot e^{-\alpha X}X^{\beta-1} & \left.\begin{matrix} X > 0 \\ X \leqq 0 \end{matrix}\right\} \quad m > 0; \\ 0 & \\ 0 & \left.\begin{matrix} X \geqq 0 \\ X < 0 \end{matrix}\right\} \quad m < 0. \\ [1/\Gamma(\beta)](-\alpha)^\beta e^{-\alpha X}(-X)^{\beta-1} & \end{cases} \tag{9}$$

For cases II and III, it follows from the conditions stated in the theorem that

$$E\left\{\frac{\sum a_{ij}X_iX_j}{(\sum X_i)^2}\right\} = \frac{\sigma^2\sum a_{ii} + m^2\sum a_{ij}}{n\sigma^2 + n^2m^2}. \tag{10}$$

Thus the condition $B = 0$ yields the relation

$$\frac{\sum a_{ij}}{n^2} = K = E\left\{\frac{\sum a_{ij}X_iX_j}{(\sum X_i)^2}\right\} = \frac{\sigma^2\sum a_{ii} + m^2\sum a_{ij}}{n\sigma^2 + n^2m^2}. \tag{11}$$

On simplification, this reduces to $\sum a_{ii} = \sum a_{ij}/n$. Similarly, in Case III, the condition $A = 0$ obviously leads to the relation

$$\frac{\sum a_{ii}}{n} = K = \frac{\sigma^2\sum a_{ii} + m^2\sum a_{ij}}{n\sigma^2 + n^2m^2}. \tag{12}$$

On simplification, this also reduces to $\sum a_{ii} = \sum a_{ij}/n$, the same as obtained from the condition $B = 0$. Thus an important conclusion is reached that whenever the matrix $A = (a_{ij})$ is such that its elements satisfy the relation $\sum a_{ii} = \sum a_{ij}/n$ both the coefficients A and B of the differential equation (4) vanish simultaneously, thus leading to no solution of the problem.

Since cases II and III are excluded by our assumption $\sum a_{ii} \neq \sum a_{ij}/n$, the problem leads uniquely to the solution obtained in (9). Obviously when the matrix $A = (a_{ij})$ is either positive definite or negative definite, the relation $\sum a_{ii} \neq \sum a_{ij}/n$ is always satisfied. Thus the equality $\sum a_{ii} = \sum a_{ij}/n$ may hold only for some indefinite matrices.

Corollary. *Let $X_1, X_2, \cdots, X_n$ be identically distributed independent random variables with a finite second moment. If the ratio of the linear functions of random variables given by $(a_1X_1 + \cdots + a_nX_n)/(X_1 + \cdots + X_n)$ is distributed independently of the sum $X_1 + X_2 + \cdots + X_n$ then each X will follow a gamma distribution.*

Proof. From the statement above, it follows that the conditional expectation of $(a_1X_1 + \cdots + a_nX_n)^2/(X_1 + \cdots + X_n)^2$ for the fixed sum $X_1 + \cdots + X_n$ is equal to its unconditional expectation. Here the elements of the matrix A are given by $a_{ij} = a_ia_j$ for $i,j = 1, 2, \cdots, n$ and they always satisfy the Schwartz's inequality $\sum a_i^2 > (\sum a_i)^2/n$, excluding the trivial case $\sum a_i^2 = (\sum a_i)^2/n$ which is possible when and only when all a_i's are equal, thus reducing the ratio of the linear functions to a constant. Hence the relation $\sum a_{ii} \neq \sum a_{ij}/n$ is always satisfied and the proof follows at once.

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MATCHING IN PAIRED COMPARISONS

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1. One of the simplest designs for testing the effect of a treatment is the method of paired comparisons: $2n$ subjects are divided into n pairs, and within each pair the treatment is assigned at random to one of the two subjects while the other is used as a control. This method has the reputation of being most effective if the subjects within each pair are as closely matched as possible. We shall show below that while this is true in the situations occurring most commonly in practice, it is not correct universally.

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