LOCALLY MOST POWERFUL RANK TESTS FOR TWO-SAMPLE PROBLEMS[1]

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1. Summary and Introduction. In order to solve nonparametric statistical problems, it is often found useful to apply those criteria of optimality which are employed in parametric problems. In the present paper, we are concerned with nonparametric two-sample problems of testing the null hypothesis that two populations have the same distribution against certain nonparametric alternative hypotheses, and generalize the parametric optimality conditions of locally most powerfulness. A rank test for a two-sample problem in this paper is called locally most powerful if it is locally most powerful against a one-parameter family of alternatives. A criterion is constructed by which it is possible to solve the problem whether or not a given rank test is locally most powerful for a two-sample problem in which the set of all possible pairs of cumulative distribution functions is convex and closed (in the weak* topology).

Let $X_1, \cdots, X_{n_1}$ be sample elements from the first population, $X_{n_1+1}, \cdots, X_n$ $(n = n_1 + n_2)$ from the second population, and the statistic[2] $Z_j = 0$ or 1, according to whether the jth smallest observation is from the first population or the second, $j = 1, \cdots, n$.

It will be shown that any locally most powerful rank test has the following form:

$$(*)\qquad \begin{cases} \text{Reject the null hypothesis if } \sum_j a_j Z_j > c \\ \text{Accept the null hypothesis if } \sum_j a_j Z_j < c \end{cases}$$

where $a_1, \cdots, a_n$ are constant numbers. For the two-sided two-sample problem, any rank test of the form (*) is locally most powerful. For the one-sided two-sample problem, a non-trivial rank test of the form (*) is locally most powerful if, and only if,

$$c_j = \left[1 \Big/ \binom{n-2}{j}\right] \sum_{s=j}^{n-2} \binom{s+1}{j+1} (a_{s+2} - \bar{a}), \qquad j = 0, 1, \cdots, n-2,$$

where

$$\bar{a} = \frac{1}{n}(a_1 + \cdots + a_n),$$

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[2] The whole argument of the present paper is very much simplified by the use of the Z-statistics which are defined by I. R. Savage [6].

have all non-negative Hankel determinants (the precise definition of the Hankel determinants is given in Section 8 below).

For the symmetric two-sided two-sample problem, a non-trivial rank test of the form (*) is locally most powerful if, and only if,

$$\sum_{s=j+1}^{n} \binom{s-1}{j} (a_{n+1-s} - a_s) = 0, \qquad \text{for } j = 1, \cdots, n.$$

Finally, it will be shown that for the two-sample problem, in which the alternative hypothesis is that the expectation of the first cumulative distribution function with respect to the second distribution is not less than $\frac{1}{2}$, a non-trivial rank test of the form (*) is locally most powerful if, and only if,

$$\sum_{s=1}^{[(n+1)/2]} \left(\frac{n+1}{2} - s\right) (a_{n-s} - a_s) \geqq 0.$$

2. Two-Sample Problems. Suppose that there are two statistical populations with cumulative distribution functions $F(x)$ and $G(x)$, $-\infty < x < +\infty$. A two-sample problem is concerned with testing a null hypothesis H_0 against a certain alternative hypothesis H_1 based upon the observation of finite random samples $X_1, \cdots, X_{n_1}$ and $X_{n_1+1}, \cdots, X_n$ taken from populations F and G, respectively. We will confine ourselves here to the cases in which the sizes n_1 and $n_2 = n - n_1$ of random samples are fixed.

In the present paper, our main interest will be in the following two-sample problems:

Problem (I): Two-sided two-sample problem. Test the null hypothesis H_0 that two populations F and G have the same distribution: $F = G$, against the alternatives H_1 that two populations have different distributions: $F \neq G$.

Problem (II): One-sided two-sample problem. Test the same null hypothesis H_0 against the alternatives H_2 that the first population F is statistically smaller than the second population G:

$$F \geq G.$$

Problem (III): Symmetric two-sided two-sample problem. Test the null hypothesis H_0 that F and G are symmetric and identical against the alternatives H_3 that the two populations are both symmetric with the same median but are different.

Problem (IV): Test the null hypothesis H_0 against the alternatives H_4 that two populations have different distributions and the mean of the first F with respect to G is not greater than $\frac{1}{2}$:

$$F \neq G \quad \textit{and} \quad \int F\, dG \geqq \frac{1}{2}.$$

For the sake of simplicity, we use the following notation: For two functions F and G,

$$F = G \quad \text{if} \quad F(x) = G(x) \quad \text{for all} \quad x,$$

$$F \geqq G \quad \text{if} \quad F(x) \geqq G(x) \quad \text{for all} \quad x,$$
$$F \geq G \quad \text{if} \quad F \geqq G \quad \text{but} \quad F \neq G.$$

It will always be assumed that $F(x)$ is strictly increasing and continuous on $-\infty < x < +\infty$.

3. Rank Tests. Let $X_1, \cdots, X_{n_1}$, and $X_{n_1+1}, \cdots, X_n$ be two random samples taken from populations F and G, respectively. We will confine our attention to rank tests which may be defined conveniently in terms of the following Z statistics

$$Z_j = \begin{cases} 0, \text{ if the } i\text{th smallest observation among} \\ \qquad X_1, \cdots, X_n \text{ comes from population } F \\ 1, \text{ otherwise,} \end{cases} \qquad j = 1, \cdots, n;$$

then any non-randomized rank test ϕ may be expressed by

$$\phi(X_1, \cdots, X_n) = \begin{cases} 0, & \text{if} \quad T(Z_1, \cdots, Z_n) < c \\ 1, & \text{if} \quad T(Z_1, \cdots, Z_n) > c, \end{cases} \tag{1}$$

where $T(z_1, \cdots, z_n)$ is a function defined on $z = (z_1, \cdots, z_n)$ with $z_j = 0$ or $1, j = 1, \cdots, n$, and c is a constant. $\phi(X_1, \cdots, X_n)$ is the probability of rejecting the null hypothesis H_0 under observation $X_1, \cdots, X_n$. We denote by ϕ_T the test ϕ defined by (1).

If $T(z)$ is a constant function of z, the test ϕ_T is trivial. Two rank statistics $T(z)$ and $T'(z)$ define the same rank test if

$$T'(z) = \lambda T(z) + \beta \qquad \text{for all} \quad z \tag{2}$$

with positive λ and arbitrary β.

In what follows, we are interested only in non-trivial rank tests, and two statistics T' and T satisfying (2) may be considered as identical.

The size α_{ϕ_T} of test ϕ_T is given by

$$\alpha_{\phi_T} = \sum_z \phi_T(z) P(z \mid F, F) \tag{3}$$

and the power function $\beta_T(F, G)$ may be expressed as

$$\beta_T(F, G) = \sum_z \phi_T(z) P(z \mid F, G), \tag{4}$$

where $P(z \mid F, G)$ represents the probability of $Z = z$ when F and G are true distributions, and the summation $\sum_z$ is over all $z = (z_1, \cdots, z_n)$ with $z_i = 0$ or 1 such that $\sum_{i=1}^n z_i = n_2$. $P(z \mid F, G)$ may be expressed as follows:

$$P(z \mid F, G) = n_1! n_2! \underset{-\infty < u_1 \leqq \cdots \leqq u_n < +\infty}{\int \cdots \int} \prod_{j=1}^n d[F(u_j)^{1-z_j} G(u_j)^{z_j}[. \tag{5}$$

Since F is assumed to be continuous, (5) may be written:

$$P(z \mid F, G) = P(z, H) = n_1! n_2! \underset{0 \leqq t_1 \leqq \cdots \leqq t_n \leqq 1}{\int \cdots \int} \prod_{j=1}^n d[t_j^{1-z_j} H(t_j)^{z_j}] \tag{6}$$

where

$$H(t) = G[F^{-1}(t)], \qquad 0 \leqq t \leqq 1. \tag{7}$$

$H(t)$ is a cumulative distribution function on [0, 1]. We shall denote by Ω the set of all possible H's associated with any given two-sample problem, i.e.,

$$\Omega = \{H; H = GF^{-1}, (F, G) \varepsilon H_0 \text{ or } H_1\}. \tag{8}$$

The sets Ω corresponding to the two-sample problems mentioned in Section 2 are as follows:

PROBLEM (I): Ω_1 is the set of all cumulative distribution functions H over [0, 1].

PROBLEM (II): Ω_2 is the set of all cumulative distribution functions H over [0, 1] such that $H(t) \leqq t$ for all $0 \leqq t \leqq 1$.

PROBLEM (III): Ω_3 is the set of all symmetric cumulative distribution functions H over [0, 1]:

$$H(t) + H(1 - t) = 1, \qquad \text{for all } 0 \leqq t \leqq 1.$$

PROBLEM (IV): Ω_4 is the set of all cumulative distributions H over [0, 1] with mean not smaller than $\frac{1}{2}$:

$$\int t\, dH(t) \geqq \tfrac{1}{2}.$$

The set Ω in any problem of the above type has the following properties: First, Ω is a convex set; i.e., $H_1, H_2 \varepsilon \Omega$, and $0 \leqq \lambda \leqq 1$ imply $\lambda H_1 + (1 - \lambda)H_2 \varepsilon \Omega$. Secondly, for a sequence $H_1, H_2, \cdots$ of distributions in Ω, the condition that

$$\int_0^1 f(t)\, dH(t) = \lim_{\nu \to \infty} \int_0^1 f(t)\, dH_\nu(t)$$

for any continuous function $f(t)$ on [0, 1], implies that the distribution H also belongs to the set Ω. This last property is sometimes stated that the set Ω is closed in the weak* topology.[3]

4. Locally Most Powerful Rank Tests. A set of cumulative distribution functions $\{(F(x, \theta), G(x, \theta)) : 0 \leqq \theta \leqq \bar{\theta}\}$, where $\bar{\theta} > 0$, is called a *one-parameter family of alternatives* if the following conditions are satisfied:

(a) $(F(x, \theta), G(x, \theta)) \varepsilon H_1$, for $0 < \theta \leqq \bar{\theta}$,

(b) $F(x, 0) = G(x, 0)$,

and

(c) $H(t, \theta) = G[F^{-1}(t, \theta), \theta]$ is uniformly differentiable with respect to θ at $\theta = 0$. Here $H(t, \theta)$ is called uniformly differentiable at $\theta = 0$ if the convergence, as θ tends to 0, of $[H(t, \theta) - H(t, 0)]/\theta$ to $[\partial H(t, \theta)/\partial\theta]_{\theta=0}$ is uniform with respect to t.

A rank test ϕ_T is said to be *locally most powerful* if there exists a one-parameter family of alternatives $(F(x, \theta), G(x, \theta))$ such that ϕ_T is most powerful against

[3] Cf., e.g., Bourbaki [1].

the alternatives $(F(x, \theta), G(x, \theta))$, $0 < \theta < \theta_0$, for some *positive* number θ_0, i.e.,

$$\beta_{\phi_T}(F(x, \theta), G(x, \theta)) \geqq \beta_{\phi}(F(x, \theta), G(x, \theta)), \qquad 0 < \theta < \theta_0, \tag{9}$$

for any rank test ϕ with size α.

The calculation of locally most powerful rank tests will be done by the following theorem:

THEOREM 1: *The locally most powerful rank test ϕ_T against a one-parameter family $(F(x, \theta), G(x, \theta))$ is determined uniquely for each size of test defined by*

$$T(z) = \sum_{j=1}^{n} a_j z_j, \tag{10}$$

where

$$a_j = \binom{n-1}{j-1} \int_0^1 t^{j-1}(1-t)^{n-j}\, dQ(t), \qquad j = 1, \cdots, n, \tag{11}$$

$$Q(t) = \left[\frac{\partial H(t, \theta)}{\partial \theta}\right]_{\theta=0}, \tag{12}$$

$$H(t, \theta) = G[F^{-1}(t, \theta), \theta]. \tag{13}$$

PROOF: Let us define $P(z, \theta)$ by

$$P(z, \theta) = P(z \mid F(\cdot, \theta), G(\cdot, \theta)). \tag{14}$$

Then, by the Neyman-Pearson Lemma, any rank test ϕ is locally most powerful against $(F(\cdot, \theta), G(\cdot, \theta))$ if, and only if, ϕ is defined by the statistic

$$T(z) = \left[\frac{\partial P(x, \theta)}{\partial \theta}\right]_{\theta=0}. \tag{15}$$

On the other hand, differentiating (6) with respect to θ, and noting that $H(t, \theta)$ is uniformly differentiable at $\theta = 0$, we have

$$\left[\frac{\partial P(x, \theta)}{\partial \theta}\right]_{\theta=0} = n_1!\, n_2! \underset{0 \leqq t_1 \leqq \cdots \leqq t_n \leqq 1}{\int \cdots \int} \sum_{j=1}^{n} z_j\, dt_1 \cdots dt_{j-1}\, dQ_j(t)\, dt_{j+1} \cdots dt_n \tag{16}$$

where $Q(t)$ is defined by (12). Since

$$\underset{0 \leqq t_1 \leqq \cdots \leqq t_{j-1} \leqq t_j}{\int \cdots \int} dt_1 \cdots dt_{j-1} = \frac{1}{(j-1)!}\, t_j^{j-1}$$

and

$$\underset{t_j \leqq t_{j+1} \leqq \cdots \leqq t_n \leqq 1}{\int \cdots \int} dt_{j+1} \cdots dt_n = \frac{1}{(n-j)!}\, (1 - t_j)^{n-j}$$

the integral in (16) may be further simplified and we have

$$\begin{aligned} &\left[\frac{dP(z, \theta)}{d\theta}\right]_{\theta=0} \\ &\quad = n_1!\, n_2! \sum_{j=1}^{n} z_j[1/(j-1)!(n-j)!] \int_0^1 t^{j-1}(1-t)^{n-j}\, dQ(t). \end{aligned} \tag{17}$$

The relation (17), together with (15), proves the theorem. Q.E.D.

Theorem 1 easily implies the following

COROLLARY: *Any locally most powerful rank test is admissible.*

In what follows, we shall first investigate locally most powerful rank tests for a general two-sample problem in which the set Ω is convex and closed in the weak* topology, and obtain a criterion for a rank test to be locally most powerful. We then consider the class of all locally most powerful rank tests for various two-sample problems mentioned above.

5. The Set A: A Special Class of Locally Most Powerful Rank Tests. Let us now consider a general two-sample problem for which the set Ω of all corresponding H functions is convex and closed in the weak* topology. In this section we shall introduce the class of rank tests which are locally most powerful with respect to a special class of one-parameter families of alternatives.

Let H be an arbitrary distribution function in Ω, and consider a one-parameter family $(F(x, \theta), G(x, \theta))$, $0 \leqq \theta \leqq 1$, satisfying the condition that[4]

$$H(t, \theta) = (1 - \theta)t + \theta H(t), \qquad 0 \leqq t \leqq 1, \quad 0 \leqq \theta \leqq 1 \tag{18}$$

where

$$H(t, \theta) = G[F^{-1}(t, \theta), \theta].$$

By Theorem 1, a rank test ϕ_T is locally most powerful against the one-parameter family satisfying (18) if, and only if,

$$T(z) = \lambda \sum_{j=1}^{n} a_j(H) z_j + \beta, \tag{19}$$

where λ is a positive number, β an arbitrary number, and

$$a_j(H) = \binom{n-1}{j-1} \int_0^1 t^{j-1}(1-t)^{n-j}\, dH(t), \qquad j = 1, \cdots, n. \tag{20}$$

We shall define the set A of n-dimensional vectors by

$$A = \{a = (a_1, \cdots, a_n);\ \ a_j = \lambda a_j(H) + \beta, j = 1, \cdots, n, H \varepsilon \Omega, \lambda \geqq 0, \text{ and } \beta \text{ is an arbitrary number}\}. \tag{21}$$

The set A, in other words, consists of all vectors that describe rank tests locally most powerful with respect to one-parameter families satisfying (18). It may be noted in particular that

$$a_j(H_0) = \frac{1}{n}, \qquad j = 0, 1, \cdots, n-1, \tag{22}$$

where

$$H_0(t) = t, 0 \leqq t \leqq 1.$$

[4] The case in which $H(t)$ is a polynomial was considered by Lehmann [5].

It will first be seen that the set A is a closed convex set. For any $H_1, H_2 \varepsilon \Omega$, and $0 \leqq \lambda \leqq 1$, we have $a[(1-\lambda)H_1 + \lambda H_2] = (1-\lambda)a(H_1) + \lambda a(H_2)$, which, together with the definition (21) of A, implies that the set A is a convex cone.

In order to prove the closedness of the set A, let $\{(a_1^\nu, \cdots, a_n^\nu)\}$ be a sequence of n-vectors in A which converges to an n-vector $a^\circ = (a_1^0, \cdots, a_n^0)$. Since a^ν is in A, there exist $H^\nu \varepsilon \Omega$, $\lambda^\nu \geqq 0$, and β^ν such that

$$(23) \qquad a_j^\nu = \lambda^\nu a_j(H^\nu) + \beta^\nu, j = 1, \cdots, n, \quad \nu = 1, 2, \cdots.$$

Taking a suitable subsequence of $\{H^\nu\}$, if necessary, we may without loss of generality suppose[5] that for any continuous function $f(t)$ on [0, 1],

$$(24) \qquad \lim_{\nu \to \infty} \int_0^1 f(t)\, dH^\nu(t) = \int_0^1 f(t)\, dH(t),$$

with some distribution function $H(t)$ over [0, 1]. Since Ω is closed in the weak* topology, the function H belongs to the set Ω.

If the sequences $\{\lambda^\nu\}$ and $\{\beta^\nu\}$ are bounded, then, for any limiting points λ^0 and β^0, we have, by (23) and (24), $a_j^0 = \lambda^0 a_j(H) + \beta^0, j = 1, \cdots, n$, which shows that vector a belongs to the set A. If both sequences $\{\lambda^\nu\}$ and $\{\beta^\nu\}$ are unbounded, then we have $H(t) = t$, for all t. Hence, vector a trivially belongs to the set A.

6. The Set B. In order to investigate further the structure of the set A, we now introduce linear transformation L which maps n-vector $a = (a_1, \cdots, a_n)$ to b-vector $b = (b_0, b_1, \cdots, b_{n-1})$ defined by

$$L(a) = (L_0(a), L_1(a), \cdots, L_{n-1}(a)),$$

where

$$(25) \quad L_j(a) = \sum_{s=j+1}^{n} \binom{n-j-1}{s-j-1} \Big/ \left[a_s \Big/ \binom{n-1}{s-1} \right], \quad j = 0, 1, \cdots, n-1.$$

The inverse linear transformation L^{-1} of L is defined by

$$L^{-1}(b) = (L_1^{-1}(b), \cdots, L_n^{-1}(b)),$$

where

$$(26) \qquad L_j^{-1}(b) = \sum_{s=j-1}^{n-1} (-1)^{s-j+1} \binom{n-j}{s-j+1} b_s, \qquad j = 1, \cdots, n.$$

The fact that the linear transformation L^{-1} defined by (26) is the inverse of L defined by (25) is easily seen from the following identities:

$$(27) \qquad t^j = \sum_{s=j+1}^{n} \binom{n-j-1}{s-j-1} t^{s-1}(1-t)^{n-s}, \qquad j = 0, 1, \cdots, n-1,$$

and

$$(28) \qquad t^{j-1}(1-t)^{n-j} = \sum_{s=j-1}^{n-1} (-1)^{s-j+1} \binom{n-j}{s-j+1} t^s, \qquad j = 1, \cdots, n.$$

[5] Cf., e.g., Bourbaki [2].

Let us now define the set B as the image of the set A by the linear transformation L:

$$B = \{b = (b_0, \cdots, b_{n-1}) : b = L(a) \quad \text{for some} \quad a \in A\}. \tag{29}$$

Since the set A is a closed convex cone and L is linear, the set B again is a closed convex cone in the n-vector space E^n. It is noted that we have, by the identities (27) and (28),

$$L_j(a(H)) = \int_0^1 t^j \, dH(t), \qquad j = 0, 1, \cdots, n-1, \tag{30}$$

for any $H \in \Omega$. We have, in particular, that

$$L_j(1, \cdots, 1) = n/(j+1), \qquad j = 0, 1, \cdots, n-1. \tag{31}$$

The definitions (21) and (29) of the sets A and B, together with (30) and (31), imply that an n-vector $b = (b_0, \cdots, b_{n-1})$ belongs to the set B if, and only if, there exist $H \in \Omega$, $\lambda \geqq 0$ and real number β such that

$$b_j = \lambda \int_0^1 t^j \, dH(t) + \beta/(j+1), \qquad j = 0, 1, \cdots, n-1. \tag{32}$$

We shall give a necessary and sufficient condition for an n-vector $b = (b_0, \cdots, b_{n-1})$ to be in the set B.

We first define the *polar cone* B^* of any set B of n-vectors $b = (b_0, \cdots, b_{n-1})$ as the set of all n-vectors $y = (y_0, \cdots, y_{n-1})$ whose inner product with any vector in B is non-negative:

$$B^* = \{y = (y_0, \cdots, y_{n-1}) : y \cdot b \geqq 0 \quad \text{for all} \quad b \in B\}, \tag{33}$$

where $y \cdot b$ denotes the inner product of two vectors y and b:

$$y \cdot b = \sum_{j=0}^{n-1} y_j b_j .$$

For any n-vector $y = (y_0, \cdots, y_{n-1})$, let us define the polynomial $y(t)$ by

$$y(t) = \sum_{j=0}^{n-1} y_j t^j. \tag{34}$$

By the definition (33), and the relation (32), an n vector $y = (y_0, \cdots, y_{n-1})$ belongs to the set B^* if, and only if,

$$\sum_{j=0}^{n-1} y_j \left[\lambda \int_0^1 t^j \, dH(t) + \beta/(j+1) \right] \geqq 0 \tag{35}$$

for all $H \in \Omega$, $\lambda \geqq 0$, and β real.

The relation (35), in view of (34), is equivalent to the following:

$$\int_0^1 y(t) \, dH(t) \geqq 0, \qquad \text{for all } H \in \Omega, \text{and} \int_0^1 y(t) \, dt = 0.$$

Therefore, we have

LEMMA 1. *An n-vector* $y = (y_0, \cdots, y_{n-1})$ *belongs to the set* B^* *if, and only if,*

$$\int_0^1 y(t)\, dH(t) \geqq 0, \qquad \textit{for all} \quad H \, \varepsilon \, \Omega, \tag{36}$$

and

$$\int_0^1 y(t)\, dt = 0. \tag{37}$$

On the other hand, since the set B is a closed convex cone in the n-vector space E^n, we have, by the duality theorem[6] on closed convex cones, that

$$B^{**} = B. \tag{38}$$

The relation (38) may be expressed as

LEMMA 2. *An n-vector* $b = (b_0, \cdots, b_{n-1})$ *belongs to the set* B *if, and only if,*

$$b \cdot y \geqq 0, \qquad \text{for all} \quad y \, \varepsilon \, B^*.$$

7. The Two-Sided Two-Sample Problem. We shall first consider Problem (I): Test the null hypothesis $H_0 : F = G$ against the alternatives that $H_1 : F \neq G$. In this case, the set Ω_1 consists of all cumulative distribution functions H over [0, 1].

The class of all locally most powerful rank tests is characterized by the following theorem:

THEOREM 2: *A non-trivial rank test* ϕ_T *is locally most powerful for Problem* (I) *if, and only if,*

$$T(z) = \sum_{j=1}^{n} a_j z_j, \tag{40}$$

where $a_1 \cdots, a_n$ *are arbitrary constants.*

PROOF: It will be shown first that the set B^* consists of the zero vector $0 = (0, \cdots, 0)$ alone. Indeed, let an n-vector $y = (y_0, \cdots, y_{n-1})$ belong to B^*. By Lemma 1, the conditions (36) and (37) must be satisfied, where Ω_1 is the set of all cumulative distribution functions H on [0, 1]. Then the condition (36) implies that

$$y(t) \geqq 0 \quad \text{for all} \quad 0 \leqq t \leqq 1. \tag{41}$$

But since $y(t)$ is a polynomial, the relation (36), together with (37), implies that $y(t) = 0$, for all $0 \leqq t \leqq 1$. Hence, $y_j = 0, j = 0, 1, \cdots, n-1$.

The polar cone B^{**} of $B^* = (0)$ is now the set E^n of all n-vectors. By Lemma 2, therefore, we have $B = E^n$. Hence, the set A also is equal to the set E^n of all n-vectors, and any test ϕ_T defined in terms of $T(z)$ of the form (40) is locally most powerful.

[6] Cf., e.g., Bourbaki [1] or Fenchel [3].

8. The One-Sided Two Sample Problem. In this section we will be concerned with Problem (II): Test the hypothesis $H_0 : F = G$ against the alternatives that $H_2 : F \geq G$. The space Ω_2 for this problem consists of all cumulative distribution functions H on [0, 1] such that

$$H(t) \leqq t, \qquad 0 \leqq t \leqq 1. \tag{42}$$

Before stating the characterization of the class of all locally most powerful rank tests for Problem (II), we introduce some concepts from the Hausdorff theory of moments.[7]

An m-vector $c = (c_0, c_1, \cdots, c_{m-1})$ may be called here a solution to the m-dimensional moment problem over [0, 1] if there exist a distribution function H over [0, 1] and a non-negative number λ such that

$$c_j = \lambda \int_0^1 t^j \, dH(t), \qquad j = 0, 1, \cdots, m-1. \tag{43}$$

Let C_m be the set of all solutions to the m-dimensional moment problem over [0, 1]:

$$C_m = \left\{ c = (c_0, c_1, \cdots, c_{m-1}) : c_j = \lambda \int t^j \, dH(t), \right. \\ \left. \begin{array}{r} j = 0, 1, \cdots, m-1, \text{for some distribution } H \text{ over } [0, 1] \\ \text{and non-negative number } \lambda \end{array} \right\}. \tag{44}$$

Similar to the set B, the set C_m here is also a closed convex cone in the m-vector space.

The polar cone C_m^* may be written as

$$\begin{aligned} C_m^* &= \left\{ z = (z_0, \cdots, z_{m-1}) : \int_0^1 z(t) \, dH(t) \geqq 0 \text{ for all distributions } H \right\} \\ &= \{ z = (z_0, \cdots, z_{m-1}) : z(t) \geqq 0, \text{ for all } 0 \leqq t \leqq 1 \}, \end{aligned} \tag{45}$$

where $z(t)$ is defined by $z(t) = \sum_{j=0}^{m-1} z_j t^j$. Hence, by the duality theorem on closed convex cones, we have that

LEMMA 3. *For an m-vector* $c = (c_0, \cdots, c_{m-1})$, $c \in C_m$ *if, and only if,* $c \cdot z \geqq 0$, *for all* $z = (z_0, \cdots, z_{m-1})$ *such that* $z(t) \geqq 0$, $0 \leqq t \leqq 1$.

By a theorem from the Hausdorff theory of moment problems,[8] we have, on the other hand, that an m-vector $c = (c_0, \cdots, c_{m-1})$ is a solution to the m-dimensional moment problem over [0, 1] if, and only if, the Hankel determinants $\Delta_s(c_0, \cdots, c_s)$ and $\bar{\Delta}_s(c_0, \cdots, c_s)$ are all non-negative, for all $s = 0, 1, \cdots, m-1$.

[7] For the Hausdorff theory of moments, the reader is referred to, e.g., Shohat and Tamarkin [7] or Karlin and Shapley [4].

[8] Cf. Karlin and Shapley [4], pp. 54–57.

The *Hankel determinants* $\Delta_s(c_0, \cdots, c_s)$ and $\bar{\Delta}_s(c_0, \cdots, c_s)$ are defined by

$$\Delta_{2r}(c_0, \cdots, c_{2r}) = \begin{vmatrix} c_0 & c_1 & \cdots & c_r \\ \vdots & & & \\ c_r & c_{r+1} & \cdots & c_{2r} \end{vmatrix}$$

$$\Delta_{2r+1}(c_0, \cdots, c_{2r+1}) = \begin{vmatrix} c_1 & c_2 & \cdots & c_{r+1} \\ \vdots & & & \\ c_{r+1} & c_{r+2} & \cdots & c_{2r+1} \end{vmatrix}$$

$$\bar{\Delta}_{2r}(c_0, \cdots, c_{2r}) = \begin{vmatrix} c_1 - c_2 & c_2 - c_3 & \cdots & c_r - c_{r+1} \\ \vdots & & & \\ c_r - c_{r+1} & c_{r+1} - c_{r+2} & \cdots & c_{2r-1} - c_{2r} \end{vmatrix}$$

$$\bar{\Delta}_{2r+1}(c_0, \cdots, c_{2r+1}) = \begin{vmatrix} c_0 - c_1 & c_1 - c_2 & \cdots & c_r - c_{r+1} \\ \vdots & & & \\ c_r - c_{r+1} & c_{r+1} - c_{r+2} & \cdots & c_{2r} - c_{2r+1} \end{vmatrix}$$

The class of all locally most powerful rank tests for Problem (II) may now be characterized by the following:

THEOREM 3. *A non-trivial rank test ϕ_T is locally most powerful for Problem* (II) *if, and only if, $T(z_1, \cdots, z_n) = \sum_{j=1}^n a_j z_j$, and $(c_0, c_1, \cdots, c_{n-2})$ has all non-negative Hankel determinants, where*

$$c_j = \left[1 \Big/ \binom{n-2}{j}\right] \sum_{s=j}^{n-2} \binom{s+1}{j+1} (a_{s+2} - \bar{a}), \quad j = 0, \cdots, n-2, \tag{46}$$
$$\bar{a} = \frac{1}{n} \sum_{s=1}^{n} a_s .$$

PROOF: In the present case, the set Ω_2 consists of all cumulative distribution functions $H(t)$ over $[0, 1]$ such that

$$H(t) \leqq t, \qquad \text{for all} \quad 0 \leqq t \leqq 1. \tag{47}$$

By Lemma 1, an $\bar{n}$-vector $y = (y_0, \cdots, y_{n-1})$ belongs to the polar cone B^* if, and only if,

$$\int_0^1 y(t)\, dH(t) \geqq 0, \text{for all } H \text{ such that } H(t) \leqq t, 0 \leqq t \leqq 1, \tag{48}$$

and

$$\int_0^1 y(t)\, dt = 0. \tag{49}$$

We shall show that in view of (49) condition (48) may be replaced by

$$y'(t) \geqq 0 \quad \text{for all} \quad 0 \leqq t \leqq 1. \tag{50}$$

In fact, for any polynomial $y(t)$ satisfying (49), we have, by a partial integration, that

$$\int_0^1 y(t)\,dH(t) = [y(t)H(t)]_0^1 - \int_0^1 y'(t)H(t)\,dt \tag{51}$$
$$= \int_0^1 y'(t)(t - H(t))\,dt.$$

Now suppose that there exists t_0 such that $y'(t_0) < 0, 0 \leqq t_0 \leqq 1$. Then, by the continuity of $y'(t)$, there is an interval I in [0, 1] containing t_0 such that $y'(t) < 0$, for all $t \varepsilon I$. It is then possible to construct a cumulative distribution function H_1 over [0, 1] such that

$$\begin{cases} t - H_1(t) = 0, & \text{for all } t \not\varepsilon I, \\ t - H_1(t) > 0, & \text{for } t \text{ interior to } I. \end{cases}$$

Then H_1 belongs to the set Ω, and by (50) and (51),

$$\int_0^1 y'(t)\,dH_1(t) = \int_I y'(t)\,dH_1(t) < 0,$$

which contradicts (48) and (51). Therefore, we have $y'(t) \geqq 0$, for all $0 \leqq t \leqq 1$.

The polar cone B^*, therefore, may be characterized by

$$B^* = \left\{ y = (y_0, \cdots, y_{n-1}) : y'(t) \geqq 0, \text{ for all } 0 \leqq t \leqq 1, \right.$$
$$\left. \text{and} \int_0^1 y(t)\,dt = 0 \right\}. \tag{52}$$

We may rewrite (52) as follows: $y \varepsilon B^*$ if, and only if,

$$y_j = \frac{1}{j} z_{j-1}, \qquad j = 1, \cdots, n-2, \tag{53}$$

and

$$y_0 = -\sum_{j=1}^{n-1} \frac{1}{j+1} y_j, \tag{54}$$

for some $z = (z_0, \cdots, z_{n-2}) \varepsilon C_{n-1}^*$.

By Lemma 2, (53) and (54) imply that $b = (b_0, b_1, \cdots, b_{n-1}) \varepsilon B$ if, and only if,

$$\sum_{j=1}^{n-1} \frac{1}{j}\left(b_j - \frac{1}{j+1} b_0\right) z_{j-1} \geqq 0 \quad \text{for all} \quad z = (z_0, \cdots, z_{n-2}) \varepsilon C_{n-1}^*. \tag{55}$$

By Lemma 3, the relation (55) is satisfied if, and only if,

$$c_j = \frac{1}{j+1}\left(b_{j+1} - \frac{1}{j+2} b_0\right), \quad j = 0, 1, \cdots, n-2, \tag{56}$$

have all non-negative Hankel determinants. Substituting (25) into (56), c_j are expressed by (46).

We now show that any locally most powerful rank test ϕ_T may be expressed in terms of $T(z) = \sum_{j=1}^{n} a_j z_j$ with $a = (a_1, \cdots, a_n) \varepsilon A$. In fact, let ϕ_T be locally most powerful against a one-parameter family $(F(x, \theta), G(x, \theta))$. By Theorem 1, we have $T(z) = \sum_{j=1}^{n} a_j z_j$, where a_j are defined by (11). Since

$$H(0, \theta) = 0, \qquad H(1, \theta) = 1,$$

$$H(t, \theta) \leqq t = H(t, 0), \qquad 0 \leqq t \leqq 1,$$

we have

$$Q(0) = Q(1) = 0, \tag{57}$$

$$Q(t) \leqq 0, \qquad 0 \leqq t \leqq 1, \tag{58}$$

where $Q(t)$ is defined by (12).

Let us first consider the case where $Q(t)$ is *continuously differentiable* on [0, 1]. Then, by (57) and (58), there exists a *positive* number λ such that $H_1(t) = t + \lambda Q(t)$ is a cumulative distribution function over [0, 1], for which we have

$$H_1(t) \leqq t, \qquad 0 \leqq t \leqq 1. \tag{59}$$

Consider a one-parameter family $(F_1(x, \theta), G(x, \theta))$ satisfying

$$H_1(t, \theta) = (1 - \theta)t + \theta H_1(t). \tag{60}$$

$H_1(t, \theta)$ belongs to the set Ω_2 for $0 \leqq \theta \leqq 1$, and

$$\left[\frac{\partial H_1(t, \theta)}{\partial \theta}\right]_{\theta=0} = \lambda Q(t). \tag{61}$$

Therefore, $\sum_j a_j z_j$ is locally most powerful against the one-parameter family $(F_1(x, \theta), G_1(x, \theta))$ satisfying (60), and $a = (a_1, \cdots, a_n)$ belongs to the set A for Problem (II).

Now consider the general case where $Q(t)$ is not necessarily differentiable. Let $\{H_\nu(t, \theta); \nu = 1, 2, \cdots\}$ be a sequence of cumulative distribution functions in Ω such that

$$\lim_{\nu \to \infty} Q_\nu(t) = Q(t), \qquad 0 \leqq t \leqq 1, \tag{62}$$

where

$$Q_\nu(t) = \left[\frac{\partial H_\nu(t, \theta)}{\partial \theta}\right]_{\theta=0} \tag{63}$$

is continuously differentiable with respect to t. The locally most powerful rank order test against one-parameter family $(1 - \theta)t + \theta H_\nu(t)$ is defined by

$$T^\nu(z) = \sum_{j=1}^{n} a_j^\nu z_j \tag{64}$$

where

$$a_j^\nu = \binom{n-1}{j-1} \int_0^1 t^{j-1}(1 - t)^{n-j}\, dQ_\nu(t). \tag{65}$$

Then the relations (62) and (65) imply that

$$\lim_{\nu\to\infty} a_j^\nu = a_j, \qquad j = 1, \cdots, b. \tag{66}$$

The vector $a^\nu = (a_1^\nu, \cdots, a_n^\nu)$ belongs to A which is a closed set. Hence, by (66), we have $a = (a_1, \cdots, a_n) \varepsilon A$.

9. The Symmetric Two-Sided Two-Sample Problem. In this section we investigate the structure of the class of all locally most powerful rank tests for Problem (III). Problem (III) is to test the null hypothesis $H_0 : F = G$, symmetric against the alternatives that $H_3 : F \neq G$, symmetric with the same median. For Problem (III), the set Ω_3 consists of all cumulative distributions H over [0, 1] such that

$$H(t) + H(1-t) = 1, \qquad \text{for all} \quad 0 \leqq t \leqq 1. \tag{67}$$

The class of all locally most powerful rank tests is characterized by the following

THEOREM 4. *A non-trivial rank test ϕ_T is locally most powerful for Problem* (III) *if, and only if,* $T(z_1, \cdots, z_n) = \sum_{j=1}^n a_j z_j$ *with*

$$\sum_{s=j+1}^{n} \binom{s-1}{j} (a_{n+1-s} - a_s) = 0, \qquad \textit{for all} \quad j = 1, \cdots, n. \tag{68}$$

PROOF: Since the set Ω_3 consists of all cumulative distribution functions H satisfying (67), Lemma 1 implies that an n-vector $y = (y_0, \cdots, y_{n-1})$ belongs to the polar cone B^* for Problem (III) if, and only if,

$$\int_0^1 y(t)\, dH(t) \geqq 0, \qquad \text{for all cumulative distribution functions } H(t) \text{ satisfying (67)}, \tag{69}$$

and

$$\int_0^1 y(t)\, dt = 0. \tag{70}$$

If H satisfies (67), we have

$$\int_0^1 y(t)\, dH(t) = \int_0^{1/2} [y(t) + y(1-t)]\, dH(t).$$

Relations (69) and (70) may now be replaced by

$$\int_0^{1/2} [y(t) + y(1-t)]\, dH(t) \geqq 0, \qquad \text{for any cumulative distribution functions } H \text{ over } [0, \tfrac{1}{2}], \tag{71}$$

and

$$\int_0^{1/2} [y(t) + y(1-t)]\, dt = 0. \tag{72}$$

But relations (71) and (72) are satisfied if, and only if, $y(t) + y(1-t) = 0$,

$0 \leqq t \leqq \frac{1}{2}$. Hence, an n-vector $y = (y_0, \cdots, y_{n-1})$ belongs to the polar cone B^* if, and only if,

$$y(t) + y(1-t) = 0 \qquad \text{for all} \quad 0 \leqq t \leqq 1. \tag{73}$$

The relation (73) may be written

$$y_j + \sum_{s=j}^{n-1} (-1)^j \binom{s}{j} y_s = 0, \qquad j = 0, 1, \cdots, n-1. \tag{74}$$

By Lemma 2, the set B is equal to the polar cone B^{**} of B^*. Therefore, by (74), an n-vector $b = (b_0, \cdots, b_{n-1})$ is in B if, and only if,

$$b_j = u_j + \sum_{s=0}^{j} (-1)^s \binom{s}{j} u_s, \qquad j = 0, 1, \cdots, n-1, \tag{75}$$

for some $u = (u_0, \cdots, u_{n-1})$.

It may be noted that, for an n-vector $b = (b_0, \cdots, b_{n-1})$, there exists an n-vector $u = (u_0, \cdots, u_{n-1})$ for which the relations (75) are satisfied if, and only if,

$$b_j = \sum_{s=0}^{j} (-1)^s \binom{j}{s} b_s, \qquad j = 0, 1, \cdots, n-1. \tag{76}$$

It is evident that the relation (75) implies (76). On the other hand, let the relation (76) be satisfied. In order to prove the existence of $u_0, \cdots, u_{n-1}$ satisfying (75), we use the mathematical induction on n. Let us assume that we have found $u_0, \cdots, u_{n-2}$ which satisfy the relations (75) for $j = 0, 1, \cdots, n-2$. If $n-1$ is an even number, u_{n-1} may be determined by

$$2u_{n-1} = b_{n-1} - \sum_{s=0}^{n-2} (-1)^s \binom{n-1}{s} u_s.$$

$u_0, \cdots, u_{n-2}$ and u_{n-1} then satisfy (75) for $s = 0, \cdots, n-1$.

If $n-1$ is an odd number, the relation (75) may be satisfied with $u_0, \cdots, u_{n-2}$, and arbitrary u_{n-1}. In fact, by (76),

$$b_{n-1} = \sum_{s=0}^{n-1} (-1)^s \binom{n-1}{s} b_s = -b_{n-1} + \sum_{s=0}^{n-2} (-1)^s \binom{n-1}{s} b_s.$$

Hence,

$$\begin{aligned} 2b_{n-1} &= \sum_{s=0}^{n-2} (-1)^s \binom{n-1}{s} \left[u_s + \sum_{r=0}^{s} (-1)^r \binom{s}{r} u_r \right] \\ &= \sum_{s=0}^{n-2} (-1)^s \binom{n-1}{s} u_s + \sum_{r=0}^{n-2} \sum_{s=r}^{n-2} (-1)^{r+s} \binom{n-1}{s} \binom{s}{r} u_r. \end{aligned} \tag{77}$$

But

$$\begin{aligned} &\sum_{s=r}^{n-2} (-1)^{r+s} \binom{n-1}{s} \binom{s}{r} \\ &\qquad = \binom{n-1}{r} \sum_{k=0}^{n-r-2} (-1)^k \binom{n-r-1}{k} = -(-1)^{n-r-1} \binom{n-1}{r}. \end{aligned}$$

Since $n - 1$ is odd, we may now write (77) as follows:

$$2b_{n-1} = 2\sum_{r=0}^{n-2} (-1)^r \binom{n-1}{r} u_r. \tag{78}$$

Dividing (78) by 2 implies the relation (75) for $j = n - 1$. Expressing (76) in terms of $a_1, \cdots, a_n$, we get the relation (68).

We now have to show that the set A actually exhausts the class of all locally most powerful rank tests for Problem (III). Let $T(z) = \sum_{j=1}^{n} a_j z_j$ define the rank test which is locally most powerful against one-parameter family $(F(x, \theta), G(x, \theta))$ such that

$$H(t, \theta) + H(1 - t, \theta) = 1, \qquad 0 \leqq t \leqq 1. \tag{79}$$

Since the set A is a closed set, it again suffices to consider the case in which $Q(t) = [\partial H(t, \theta)/\partial\theta]_{\theta=0}$ is continuously differentiable.

By (79) we have

$$Q(0) = Q(1) = 0, \qquad Q(t) + Q(1 - t) = 0, \qquad 0 \leqq t \leqq 1. \tag{80}$$

Since $Q(t)$ is continuously differentiable, there exists a positive number λ such that $H_1(t) = t + \lambda Q(t)$ is a symmetric cumulative distribution function over [0, 1]. Hence, for some constant β, we have $a_j = \lambda a_j(H_1) + \beta$, which shows that $a = (a_1, \cdots, a_n)$ belongs to the set A for Problem (III).

10. The case $\int_0^1 F\,dG \geqq \frac{1}{2}$. We shall finally consider Problem (IV): Test the null hypothesis $H_0 : F = G$ against the alternatives that $H_4 : F \neq G$, $\int_0^1 F\,dG \geqq \frac{1}{2}$. The set Ω_4 for Problem (IV) is the set of all cumulative distributions H over [0, 1] such that

$$\int_0^1 t\,dH(t) \geqq \tfrac{1}{2}. \tag{81}$$

THEOREM 5. *A non-trivial rank test ϕ_T is locally most powerful for Problem* (IV) *if, and only if,* $T(z) = \sum_{j=1}^{n} a_j z_j$ *with*

$$\sum_{s=1}^{[(n+1)/2]} ((n + 1)/2 - s)(a_{n+1-s} - a_s) \geqq 0, \tag{82}$$

where $[(n + 1)/2]$ *denote the greatest integer less than or equal to* $(n + 1)/2$.

PROOF: The polar cone B^* in the present case consists of all n-vectors $y = (y_0, \cdots, y_{n-1})$ such that

$$\int_0^1 y(t)\,dH(t) \geqq 0, \qquad \text{for all } H \text{ for which} \qquad \int_0^1 t\,dH(t) \geqq \tfrac{1}{2}, \tag{83}$$

and

$$\int_0^1 y(t)\,dt = 0. \tag{84}$$

The set Ω_4 in particular contains the set of all cumulative distribution functions H such that $H(t) \leqq t$, for $0 \leqq t \leqq 1$. Therefore, by an argument similar to the one in Problem (II), we have

(85) $$y'(t) \geqq 0, \qquad 0 \leqq t \leqq 1, \qquad \text{for all} \quad y \, \varepsilon \, B^*.$$

Similarly, since the set Ω_4 contains all cumulative distribution functions H over $[0, 1]$ such that $H(t) + H(1 - t) = 1, 0 \leqq t \leqq 1$, we have that

(86) $$y(t) + y(1 - t) = 0, \qquad 0 \leqq t \leqq 1, \qquad \text{for all} \quad y \, \varepsilon \, B^*.$$

We shall show furthermore that, for any $y \, \varepsilon \, B^*$,

(87) $$y(t) \quad \text{is linear in} \quad t, \qquad 0 \leqq t \leqq 1.$$

In fact, let σ and τ be arbitrary numbers between 0 and $\frac{1}{2}$, and let $H_{\sigma,\tau}$ be the cumulative distribution function corresponding to the following probability distribution:

$$\text{Prob.}\,\{t = \tfrac{1}{2} - b\} = \tau/(\sigma + \tau),$$
$$\text{Prob.}\,\{t = \tfrac{1}{2} + \tau\} = \sigma/(\sigma + \tau).$$

The cumulative distribution function $H_{\sigma,\tau}$ belongs to the set Ω_4 for Problem (IV), and

$$\int_0^1 y(t)\, dH_{\sigma,\tau}(t) = \frac{\tau}{\sigma + \tau}\, y\,(\tfrac{1}{2} - \sigma) + \frac{\sigma}{\sigma + \tau}\, y\,(\tfrac{1}{2} + \tau).$$

Therefore, if $y = (y_0, \cdots, y_{n-1}) \, \varepsilon \, B^*$, the relation (83) implies that

(88) $$-y(\tfrac{1}{2} - \sigma)/\sigma \leqq y(\tfrac{1}{2} + \tau)/\tau, \qquad \text{for all} \quad 0 < \sigma, \tau < \tfrac{1}{2}.$$

The relation (88), together with (86), implies that $y(t)$ be linear in t, $0 \leqq t \leqq 1$. Therefore, if y belongs to the polar set B^*, we have, by (85), (86), and (87),

(89) $$y_0 = -\tfrac{1}{2}\, y_1, \qquad y_1 \geqq 0, \qquad y_2 = \cdots = y_{n-1} = 0.$$

On the other hand, let an n-vector $y = (y_0, \cdots, y_{n-1})$ satisfy the relation (89). Then, for any $H \, \varepsilon \, \Omega_4$,

$$\int_0^1 y(t)\, dH(t) = y_0 + y_1 \int_0^1 t\, dH(t) \geqq y_0 + \tfrac{1}{2}\, y_1 = 0,$$

and

$$\int_0^1 y(t)\, dt = y_0 + \tfrac{1}{2}\, y_1 = 0.$$

The vector y, therefore, belongs to the polar cone B^*. Hence, the polar cone B^* consists of all n-vectors $y = (y_0, \cdots, y_{n-1})$ satisfying the relation (89). The set B is, by Lemma 2, equal to the polar cone B^{**} to B^*. Thus, we have

(90) $$b = (b_0, \cdots, b_{n-1}) \, \varepsilon \, B \quad \text{if, and only if,} \quad b_1 \geqq \tfrac{1}{2}\, b_0.$$

Writing the relation (90) in terms of a's, we have that $a = (a_1, \cdots, a_n) \, \varepsilon \, A$ if, and only if, (82) is satisfied.

Let ϕ_T be locally most powerful against a one-parameter family $(F(x, \theta), G(x, \theta))$. By Theorem 1, we have $T(z) = \sum_{j=1}^{n} a_j z_j$, where a_j are defined by (11). Since

$$H(0) = 0, \qquad H(1) = 1, \qquad \int_0^1 t\,dH(t, 0) \geqq \tfrac{1}{2} = \int_0^1 t\,dH(t, 0), \qquad 0 \leqq t \leqq 1,$$

we have

$$Q(0) = Q(1) = 0, \qquad \int_0^1 t\,dQ(t) \geqq 0. \tag{91}$$

It again suffices to consider the case where $Q(t)$ is continuously differentiable at every point t. By (91) there exists a positive number λ such that $H_1(t) = t + \lambda Q(t)$ is a cumulative distribution function over [0, 1] and

$$\int t\,dH_1(t) = \tfrac{1}{2} + \lambda \int t\,dQ(t) \geqq \tfrac{1}{2}.$$

Consider a one-parameter family $(F_1(x, \theta), G(x, \theta))$ satisfying $H_1(t, \theta) = (1 - \theta)t + \theta H_1(t)$. $H_1(t, \theta)$ belongs to the set Ω_A for $0 \leqq \theta \leqq 1$, and, for some number β,

$$\left[\frac{\partial H_1(t, \theta)}{\partial \theta}\right]_{\theta=0} = \lambda Q(t) + \beta.$$

The vector $a = (a_1, \cdots, a_n)$, therefore, belongs to the set A for Problem (IV).

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