# AN ASYMPTOTIC FORMULA FOR THE DIFFERENCES OF THE POWERS AT ZERO

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**1. Introduction.** In this paper saddlepoint approximations will be obtained for the Stirling numbers. Most of the discussion will be concerned with Stirling numbers of the second kind, which are essentially the same thing as the differences of the powers of the integers at zero, $\Delta^t 0^r$. The work is a direct application of a saddlepoint theorem, Theorem 6.1 of Good [4], which was itself an extension of a result given by Daniels [2]. This theorem enables us to approximate the coefficients in a power of a power series in one variable having non-negative real coefficients.

**2. Differences of Powers at Zero.** If the sequence $0^r$, $1^r$, $2^r$, is differenced $t$ times, the result for argument 0 is commonly denoted by $\Delta^t 0^r$. For example, $\Delta^2 0^r = 2^r - 2 \cdot 1^r + 0^r$, and generally

$$\Delta^t 0^r = t^r \left\{ 1 - t \left( \frac{t-1}{t} \right)^r + \binom{t}{2} \left( \frac{t-2}{t} \right)^r - \cdots \right\}. \tag{1}$$

This formula is an immediate consequence of the binomial theorem, if $\Delta$ is written in the form $E - 1$, where $E$ is the "suffix-raising operator". See, for example, Riordan [6], p. 13.

The differences of the powers at zero are essentially the same thing as the Stirling numbers of the second kind, since $\Delta^t 0^r = t! S(r, t)$. (The notation is that used, for example, by Riordan [6], p. 91.) A table of $\Delta^t 0^r$ for $r \leqq 25$ was presented by Stevens [7], and republished by Fisher and Yates [3], Table XXII. When a power, $x^t$, is expressed as a linear combination of factorial powers, $S(r, t)$ is the coefficient of $x^{(r)} = x(x-1) \cdots (x-r+1)$.

When $r$ objects are thrown equiprobably into $N$ cells, the probability that precisely $t$ are occupied is

$$\frac{1}{N^r} \cdot \frac{N!}{(N-t)!\,t!} \Delta^t 0^r. \tag{1a}$$

In a sense this is true even if $t > r$, since $\Delta^t 0^r$ then vanishes. The question of calculating numerical values arises only if $r \geqq t$. In order to emphasise this fact I shall write $r = t + n$, where $n \geqq 0$.

The problem of testing for equiprobability of a multinomial distribution arises in various practical problems, some of which are listed in Good [4], p. 862. Various tests are discussed in this reference, together with conditions under which they are appropriate. Stevens [7] gives two examples, one from agriculture and one from genetics, in which it seems appropriate to use the number of empty

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cells as the criterion. In general, the number of empty cells is an appropriate criterion if, on the non-null hypothesis, an abundance of empty cells is to be expected.

Formula (1) is convenient for the calculation of $\Delta^t 0^{n+t}$ if $t$ is small or if

$$t \ll \exp(n/t).$$

Hsu [5] gave the following asymptotic expansion, which is convenient when $n = O(t^{\frac{1}{2}})$:

$$(2) \qquad \Delta^t 0^{t+n} \sim \frac{t!\,t^{2n}}{2^n n!}\left[1 + \frac{f_1(n)}{t} + \frac{f_2(n)}{t^2} + \cdots + \frac{f_\nu(n)}{t^\nu} + O\left(\frac{1}{t^{\nu+1}}\right)\right],$$

where

$$f_1(n) = \tfrac{1}{3}(2n^2 + n),$$
$$f_2(n) = \tfrac{1}{18}(4n^4 - n^2 - 3n),$$
$$f_3(n) = \tfrac{1}{810}(40n^6 - 60n^5 - 2n^4 - 63n^3 + 133n^2 - 48n).$$

I shall give an asymptotic formula that is convenient if $n/t$ is bounded above and below by positive constants; i.e., if $n/t$ is neither very small nor very large. In the numerical examples I select values of $t$ and $n$ for which the exact formula above can be easily applied, and which are in the published tables.

The new asymptotic formula is

$$(3) \qquad \Delta^t 0^{t+n} \sim \frac{(t+n)!(e^\rho - 1)^t}{\rho^{t+n}\{2\pi t[1 + \kappa - (1+\kappa)^2 e^{-\rho}]\}^{\frac{1}{2}}} \cdot \left[1 + \frac{g_1(\kappa)}{t} + \frac{g_2(\kappa)}{t^2} + \cdots + \frac{g_\nu(\kappa)}{t^\nu} + O\left(\frac{1}{t^{\nu+1}}\right)\right]$$

where $\kappa = n/t$, and $\rho$ is the unique positive root of the equation

$$(4) \qquad \rho = (1+\kappa)(1 - e^{-\rho}).$$

and rules for calculating $g_1$ and $g_2$ will be given below. A table of roots of equation (4) is given in the Appendix. All the functions $g_1, g_2, \cdots$ are rational functions of $\rho$ and $\kappa$, and they take transcendental values when $n > 0$.

For example,

$$(5) \qquad \Delta^t 0^{2t} \sim \frac{(2t)!(1\cdot 54414)^t}{2\cdot 73124(t^{\frac{1}{2}})}\left[1 - \frac{0\cdot 10774}{t} - \frac{0\cdot 00345}{t^2} + \cdots\right].$$

The following numerical illustration shows that formula (5) gives a very good approximation, even if only one, two, or three terms are taken.

*Numerical Illustration of Formulae (2) and (5)*

| $t$: | 2 | 4 | 8 |
|---|---|---|---|
| $\Delta^t 0^{2t}$: | 14 | 40824 | $8.63559 \times 10^{13}$ |
| First term of (2) | 4 | 4096 | $1.10 \times 10^{12}$ |
| Three terms of (2) | 14 | 38433 | $2.1 \times 10^{13}$ |
| First term of (5) | 14.815 | 41964 | $8.75389 \times 10^{13}$ |
| Two terms of (5) | 14.017 | 40834 | $8.63600 \times 10^{13}$ |
| Three terms of (5) | 14.004 | 40825 | $8.63553 \times 10^{13}$ |

**3. Derivation of New Formula.** The proof of formula (3) depends on the familiar fact that

$$\Delta^t 0^{t+n} = (t+n)!\, c(n, t),$$

where $c(n, t)$ is the coefficient of $x^n$ in $(f(x))^t$, where

$$f(x) = (e^x - 1)/x. \tag{6}$$

(See, for example, Riordan [6], p. 13.) We may now apply the saddlepoint method, or, more easily, quote Theorem 6.1 of Good [4]. We obtain:

$$\begin{aligned} c(n, t) \sim \frac{(f(\rho))^t}{\sigma \rho^n (2\pi t)^{\frac{1}{2}}} \Big\{ 1 &+ \frac{1}{24t} (3\lambda_4 - 5\lambda_3^2) \\ &+ \frac{1}{1152t^2} (168\lambda_3\lambda_5 + 385\lambda_3^4 - 630\lambda_3^2\lambda_4 - 24\lambda_6 + 105\lambda_4^2) + \cdots \Big\} \end{aligned} \tag{7}$$

where

$$\lambda_s = \lambda_s(\rho) = \kappa_s(\rho)/\sigma^s, \qquad \sigma = (\kappa_2(\rho))^{\frac{1}{2}}, \tag{8}$$

$$\kappa_s = \kappa_s(\rho) = (\partial/\partial u)^s (\log f(\rho e^u))|_{u=0} \qquad (s = 1, 2, 3, \cdots), \tag{9}$$

$$t\rho f'(\rho) = nf(\rho). \tag{10}$$

Equation (4) is (10) with $f(x) = (e^x - 1)/x$. Quite generally

$$\kappa_1 = \frac{\rho f'(\rho)}{f(\rho)}, \qquad \kappa_{s+1} = \rho \frac{d\kappa_s}{d\rho} \tag{11}$$

from which we may calculate $\kappa_1, \kappa_2, \kappa_3, \cdots$ in turn. (I am taking the liberty of regarding $\rho$ as a continuous variable in some contexts and as a constant in others.) In our problem,

$$\begin{cases} \kappa_1 = n/t = \kappa, \qquad \kappa_2 = (\kappa_1 + 1)(\rho - \kappa_1), \\ \kappa_2 = \rho(\kappa_1 + \kappa_2 + 1) - \kappa_2 - 2\kappa_1\kappa_2, \\ \kappa_4 = \rho(1 + \kappa_1 + 2\kappa_2 + \kappa_3) - \kappa_3 - 2\kappa_2^2 - 2\kappa_1\kappa_3, \\ \kappa_5 = \rho(1 + \kappa_1 + 3\kappa_2 + 3\kappa_3 + \kappa_4) - \kappa_4 - 6\kappa_2\kappa_3 - 2\kappa_1\kappa_4, \\ \kappa_6 = \rho(1 + \kappa_1 + 4\kappa_2 + 6\kappa_3 + 4\kappa_4 + \kappa_5) - \kappa_5 - 6\kappa_3^2 - 8\kappa_2\kappa_4 - 2\kappa_1\kappa_5. \end{cases} \tag{12}$$

Formula (3) is now established and it is clear that $g_1, g_2, \cdots$ are rational functions of $\rho$ and $n/t$, with rational coefficients that are absolute constants. Since, for any $n/t$, $\rho$ is transcendental, it follows that the $g$'s are too. In particular they are irrational, so that in this respect formula (3) differs from Hsu's formula, and from Stirling's formula for $n!$.

The table of numerical results given above makes formula (3) appear in too favourable a light compared with (2). There are two reasons: (i) the terms of (3) take longer to calculate, (ii) we took $n = t$, whereas formula (2) is designed more for cases where $n/t$ is small. In order to redress the balance, let us take

$t = 20$, $n = 2$. We obtain the following results:

| | |
|---|---|
| $\Delta^{20}0^{22}$ | $20! \times 23485.$ |
| One term of (2) | $20! \times 20000.$ |
| Two terms of (2) | $20! \times 23333.3.$ |
| Three terms of (2) | $20! \times 23483.3.$ |
| Four terms of (2) | $20! \times 23484.7.$ |
| One term of (3) | $20! \times 24605.$ |
| Two terms of (3) | $20! \times 23150.$ |

In this example, selected deliberately as likely to be unfavourable for formula (3), its first term is nevertheless better than the first term of formula (2). But, without heavy calculation, the first four terms of formula (2) give the answer correct to the nearest integer.

**4. Stirling Integers of the First Kind.** For the Stirling integers of the first kind, we have $(-1)^n t!\, s(n + t, t) = (n + t)!$ times the coefficient of $x^n$ in $[-x^{-1} \log (1 - x)]^t$. (See, for example, Riordan [6], p. 42.) Hence we can obtain an asymptotic formula from the saddlepoint theorem, valid if $n/t$ is not small or large. We here give the first term only, though the other terms could be worked out as above.

$$s(n + t, t) \sim \frac{(-1)^n (n + t)![-\log (1 - \rho)]^t}{t!\,\rho^{n+t}(2\pi t\kappa_2)^{\frac{1}{2}}}, \tag{13}$$

where

$$\kappa_2 = \left(\frac{1}{1 - \rho} \cdot \frac{t}{n + t} - 1\right) \frac{(n + t)^2}{t^2} \tag{14}$$

and $\rho$ is now the unique root between 0 and 1 of the equation

$$\frac{\rho}{-(1 - \rho) \log (1 - \rho)} = \frac{n + t}{t} = 1 + \kappa. \tag{15}$$

(For example, we get $s(8, 4) \doteq 7007$, the correct value being 6769. Here $\rho = 0.71534$.) It may be noted that $\zeta = \log (1 - \rho)$ is the unique negative root of the equation

$$\zeta = \frac{t}{n + t} (1 - e^{-\zeta}). \tag{16}$$

This equation is of almost exactly the same form as equation (4), and its solution is also tabulated in the Appendix.

Stirling numbers of the first kind are "inverse" to those of the second kind in the sense that if a factorial power, $x^{(r)}$ is expressed as a linear function of ordinary powers, $s(r, t)$ is the coefficient of $x^t$.

**5. A Related Occupancy Problem.** The above methods can be used in order to obtain an asymptotic formula for the following occupancy probability, which I mention here *en passant* because of its close relationship to Section 2. Suppose that $n$ objects are thrown "at random" (equiprobably) into the $tu$ cells of a rectangular board of $t$ rows and $u$ columns. Then the probability, $p_n$, that no row will contain more than one occupied cell is equal to the coefficient of $x^n$ in the "pseudo-generating function"

$$(17) \qquad \frac{n!}{(tu)^n}(ue^x - u + 1)^t.$$

This is only a pseudo probability generating function since it depends on $n$.

An asymptotic formula may therefore be obtained from the saddle-point theorem with $f(x) = ue^x - u + 1$.

The pseudo-generating function (17) may be deduced from the following joint pseudo-generating function:

$$(18) \qquad \frac{n!}{(tu)^n}\prod_{r=1}^{t}\left(1 + \sum_{s=1}^{u}\sum_{m=1}^{\infty}\frac{x_{r,s}^m}{m!}\right) = \frac{n!}{(tu)^n}\prod_{r=1}^{t}\left\{\left(\sum_{s=1}^{u} e^{x_{r,s}}\right) - u + 1\right\}.$$

The terms of degree $n$ in the expansion of (18) give the probabilities individually of the legal fillings of the rectangular board. The total probability of all legal fillings is therefore the coefficient of $x^n$ after putting all the $x_{r,s}$ equal to $x$. (Note the check that the probability is 1 if $u = 1$.)

The (true) exponential generating function is

$$(19) \qquad \sum_{n=0}^{\infty}\frac{p_n x^n}{n!} = (ue^{x/(tu)} - u + 1)^t.$$

**6. Appendix. Solution of equations (4) and (15).** For any fixed values of $n$ and $t$ it would be a straightforward matter to solve equations (4) and (15) by means of the Newton-Raphson iterative method. (See, for example, Buckingham [1], index.) But since a free half-hour of time was available on a Pegasus computer I used the logically simpler iteration $\rho_{m+1} = f(\rho_m)$, where, in equation (4), $f(\rho) = (1 + \kappa)(1 - e^{-\rho})$. (See Table I.) The difference $\rho_m - \rho_\infty$ is approximately a geometrical series, a fact that could be used, though it was not, to speed the convergence considerably. I used the crude stopping rule $|\rho_n - \rho_{n-1}| < 10^{-7}$. An improved estimate of the solution is then

$$\rho_n + \frac{10^{-7}}{1 - f'(\rho_n)} \quad \text{or} \quad \rho_n - \frac{10^{-7}}{1 - f'(\rho_n)},$$

depending on whether $\rho_n$ is an increasing or decreasing function of $n$. (It was always one or the other in the present application.) I have made use of this adjustment in Tables I and II, so that none of the results should be in error by more than $10^{-6}$.

This primitive iterative method would diverge if used for equation (16), with $f(\zeta) = (1 + \kappa)^{-1}(1 - e^{-\zeta})$ (since $|f'| > 1$ at the root). Actually I put

$$\eta = \rho/(1 - \rho),$$

which converted (15) into $\eta = (1 + \kappa)\log(1 + \eta)$. This equation can be

TABLE I

*Solutions of equation* (4), $\rho = (1 + \kappa)\ (1 - e)^{-\rho}$

(The maximum error is 1 in the sixth place of decimals)

| $\kappa$ | $\rho$ | $\kappa$ | $\rho$ | $\kappa$ | $\rho$ |
|---|---|---|---|---|---|
| | | 0.80 | 1.318374 | 4.00 | 4.965114 |
| 0.02 | .039737 | 0.82 | 1.346554 | 4.10 | 5.067892 |
| 0.04 | .078961 | 0.84 | 1.374580 | 4.20 | 5.170453 |
| 0.06 | .117692 | 0.86 | 1.402454 | 4.30 | 5.272815 |
| 0.08 | .155948 | 0.88 | 1.430180 | 4.40 | 5.374993 |
| 0.10 | .193747 | 0.90 | 1.457763 | 4.50 | 5.477000 |
| 0.12 | .231107 | 0.92 | 1.485204 | 4.60 | 5.578849 |
| 0.14 | .268041 | 0.94 | 1.512508 | 4.70 | 5.680553 |
| 0.16 | .304564 | 0.96 | 1.539678 | 4.80 | 5.782123 |
| 0.18 | .340693 | 0.98 | 1.566715 | 4.90 | 5.883569 |
| 0.20 | .376438 | 1.00 | 1.593624 | 5.00 | 5.984901 |
| 0.22 | .411815 | 1.10 | 1.726336 | 5.10 | 6.086127 |
| 0.24 | .446838 | 1.20 | 1.856225 | 5.20 | 6.187256 |
| 0.26 | .481507 | 1.30 | 1.983754 | 5.30 | 6.288295 |
| 0.28 | .515846 | 1.40 | 2.108630 | 5.40 | 6.389251 |
| 0.30 | .549861 | 1.50 | 2.231612 | 5.50 | 6.490131 |
| 0.32 | .583562 | 1.60 | 2.352712 | 5.60 | 6.590940 |
| 0.34 | .616959 | 1.70 | 2.472100 | 5.70 | 6.691684 |
| 0.36 | .650061 | 1.80 | 2.589929 | 5.80 | 6.792368 |
| 0.38 | .682877 | 1.90 | 2.706335 | 5.90 | 6.892997 |
| 0.40 | .715416 | 2.00 | 2.821439 | 6.00 | 6.993575 |
| 0.42 | .747685 | 2.10 | 2.935353 | 6.10 | 7.094107 |
| 0.44 | .779692 | 2.20 | 3.048175 | 6.20 | 7.194595 |
| 0.46 | .811445 | 2.30 | 3.159994 | 6.30 | 7.295044 |
| 0.48 | .842952 | 2.40 | 3.270894 | 6.40 | 7.395456 |
| 0.50 | .874218 | 2.50 | 3.380947 | 6.50 | 7.495834 |
| 0.52 | .905250 | 2.60 | 3.490221 | 6.60 | 7.596182 |
| 0.54 | .936056 | 2.70 | 3.598779 | 6.70 | 7.696501 |
| 0.56 | .966640 | 2.80 | 3.706676 | 6.80 | 7.796794 |
| 0.58 | .997010 | 2.90 | 3.813964 | 6.90 | 7.897062 |
| 0.60 | 1.027170 | 3.00 | 3.920690 | 7.00 | 7.997309 |
| 0.62 | 1.057127 | 3.10 | 4.026899 | 7.10 | 8.097535 |
| 0.64 | 1.086884 | 3.20 | 4.132629 | 7.20 | 8.197743 |
| 0.66 | 1.116449 | 3.30 | 4.237918 | 7.30 | 8.297933 |
| 0.68 | 1.145824 | 3.40 | 4.342800 | 7.40 | 8.398107 |
| 0.70 | 1.175016 | 3.50 | 4.447305 | 7.50 | 8.498267 |
| 0.72 | 1.204029 | 3.60 | 4.551462 | 7.60 | 8.598414 |
| 0.74 | 1.232867 | 3.70 | 4.655298 | 7.70 | 8.698548 |
| 0.76 | 1.261534 | 3.80 | 4.758837 | 7.80 | 8.798672 |
| 0.78 | 1.290035 | 3.90 | 4.862102 | | |

## TABLE II

*Solutions of equation* (15), $\rho = -(1 + \kappa)(1 - \rho)\log(1 - \rho)$

(Strictly, $1/(1 - \rho)$ is tabulated. It is just as convenient as $\rho$ for the calculaton of expressions (13) and (14). The maximum error is 1 in the sixth place of decimals.)

| $\kappa$ | $1/(1-\rho)$ | $\kappa$ | $1/(1-\rho)$ | $\kappa$ | $1/(1-\rho)$ |
|---|---|---|---|---|---|
| | | 3.50 | 12.289268 | 6.50 | 25.201892 |
| 0.10 | 1.206454 | 3.60 | 12.686463 | 6.60 | 25.662121 |
| 0.20 | 1.425039 | 3.70 | 13.086393 | 6.70 | 26.123969 |
| 0.30 | 1.654727 | 3.80 | 13.489996 | 6.80 | 26.587415 |
| 0.40 | 1.894642 | 3.90 | 13.894215 | 6.90 | 27.052438 |
| 0.50 | 2.144033 | 4.00 | 14.301995 | 7.00 | 27.519017 |
| 0.60 | 2.402248 | 4.10 | 14.712281 | 7.10 | 27.987132 |
| 0.70 | 2.668715 | 4.20 | 15.125022 | 7.20 | 28.456763 |
| 0.80 | 2.942931 | 4.30 | 15.540169 | 7.30 | 28.927890 |
| 0.90 | 3.224447 | 4.40 | 15.957675 | 7.40 | 29.400496 |
| 1.00 | 3.512863 | 4.50 | 16.377494 | 7.50 | 29.874562 |
| 1.10 | 3.807819 | 4.60 | 16.799582 | 7.60 | 30.350069 |
| 1.20 | 4.108990 | 4.70 | 17.223897 | 7.70 | 30.827001 |
| 1.30 | 4.416081 | 4.80 | 17.650398 | 7.80 | 31.305340 |
| 1.40 | 4.728824 | 4.90 | 18.079045 | 7.90 | 31.785070 |
| 1.50 | 5.046970 | 5.00 | 18.509802 | 8.00 | 32.266175 |
| 1.60 | 5.370296 | 5.10 | 18.942631 | 8.10 | 32.748638 |
| 1.70 | 5.698591 | 5.20 | 19.377496 | 8.20 | 33.232445 |
| 1.80 | 6.031664 | 5.30 | 19.814364 | 8.30 | 33.717580 |
| 1.90 | 6.369336 | 5.40 | 20.253202 | 8.40 | 34.204028 |
| 2.00 | 6.711441 | 5.50 | 20.693977 | 8.50 | 34.691775 |
| 2.10 | 7.057824 | 5.60 | 21.136658 | 8.60 | 35.180806 |
| 2.20 | 7.408341 | 5.70 | 21.581215 | 8.70 | 35.671109 |
| 2.30 | 7.762856 | 5.80 | 22.027620 | 8.80 | 36.162668 |
| 2.40 | 8.121243 | 5.90 | 22.475843 | 8.90 | 36.655471 |
| 2.50 | 8.483382 | 6.00 | 22.925857 | 9.00 | 37.149505 |
| 2.60 | 8.849162 | 6.10 | 23.377636 | 9.10 | 37.644757 |
| 2.70 | 9.218476 | 6.20 | 23.831154 | 9.20 | 38.141214 |
| 2.80 | 9.591224 | 6.30 | 24.286385 | 9.30 | 38.638865 |
| 2.90 | 9.967313 | 6.40 | 24.743306 | 9.40 | 39.137696 |
| 3.00 | 10.346652 | | | | |
| 3.10 | 10.729156 | | | | |
| 3.20 | 11.114745 | | | | |
| 3.30 | 11.503341 | | | | |
| 3.40 | 11.894872 | | | | |

solved by the above iterative method, thus $\eta_{m+1} = (1 + \kappa) \log (1 + \eta_m)$. Table II lists the values of $\eta + 1 = (1 - \rho)^{-1}$, where $\rho$ is the solution of equation (15).

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