NOTES

THE ESSENTIAL COMPLETENESS OF THE CLASS OF GENERALIZED SEQUENTIAL PROBABILITY RATIO TESTS[1]

By M. H. DeGroot

Carnegie Institute of Technology

1. Introduction and summary. Consider a sequential decision problem in which independent observations are to be taken on a random variable X whose distribution is of the form

$$dG_\theta(x) = \psi(\theta)e^{\theta x}\, d\mu(x), \tag{1}$$

where the parameter θ lies in a given interval Ω of the real line but is otherwise unknown, and the measure $\mu(x)$ is either absolutely continuous or discrete. The problem is to decide between the hypotheses

$$\begin{aligned} H_1&: \theta \leqq \theta^* \\ H_2&: \theta > \theta^*, \end{aligned} \tag{2}$$

where θ^* is a given point of Ω.

Under the assumptions stated in Section 2 the class A of generalized sequential probability ratio tests is shown to be essentially complete relative to the class D of decision functions with bounded risk. A decision function δ belongs to the class A if and only if after taking n observations,

(i) δ depends on the observations only through n and $v_n = \sum_{i=1}^n X_i$;

(ii) δ specifies a closed interval $J_n : [a_{1n}, a_{2n}]$ for each n and the following rule of action:

(a) Stop experimentation as soon as $v_n \not\in J_n$. If $v_n < a_{1n}$, accept H_1. If $v_n > a_{2n}$, accept H_2.

(b) If $a_{1n} < v_n < a_{2n}$, take another observation.

(c) If $a_{1n} < a_{2n}$ and $v_n = a_{in}$, accept H_i or take another observation or randomize between these two ($i = 1, 2$).

The problem considered here is the same as that treated by Sobel [1], and the foregoing statement of the problem and conclusion, as well as the assumptions to be given in the next section, follow his work very closely. The contribution of this paper is that the requirement of bounded loss functions made by Sobel is removed.

2. Assumptions. Let $W(\theta, j)$, $j = 1, 2$, denote the loss incurred in accepting H_j when θ is the value of the parameter. It is assumed that there exist values

Received August 12, 1959.

[1] This research was supported in part by the National Science Foundation under grant NSF-G9662.

$\theta_1 \leqq \theta^* \leqq \theta_2$ such that

$$
\begin{aligned}
W(\theta, 1) &= 0 \quad \text{for} \quad \theta \leqq \theta_2 \\
W(\theta, 1) &> 0 \quad \text{for} \quad \theta > \theta_2 \\
W(\theta, 2) &= 0 \quad \text{for} \quad \theta \geqq \theta_1 \\
W(\theta, 2) &> 0 \quad \text{for} \quad \theta < \theta_1 .
\end{aligned} \tag{3}
$$

Thus, the zone of indifference may have positive length or may contain only the point θ^*. It is assumed that $W(\theta, 1)$ and $W(\theta, 2)$ are finite for all θ and that $W(\theta, 1)$ is a nondecreasing and $W(\theta, 2)$ a non-increasing function on Ω.

Let $C(n)$ denote the cost of taking n observations. It is assumed that $C(0) = 0$ and that for $n \geqq 1$,

$$C(n) = c_1 + \cdots + c_n , \tag{4}$$

where c_n is the cost of taking the nth observation. It is also assumed that for some positive K,

$$\liminf_{n \to \infty} c_n \geqq K. \tag{5}$$

Let S denote the smallest interval containing all possible values of $(\sum_{i=1}^{n} x_i)/n$ for all n, $n = 1, 2 \cdots$. Define

$$g_t(\theta) = \psi(\theta)e^{t\theta} \tag{6}$$

for all $t \varepsilon S$.

It is assumed that corresponding to each $t \varepsilon S$, there exists $\hat{\theta}(t)$ such that $g_t(\theta)$ is strictly increasing for $\theta \leqq \hat{\theta}$ and strictly decreasing for $\theta \geqq \hat{\theta}$. It is also assumed that for every $\epsilon > 0$ there exists $t \varepsilon S$ such that $\theta^* < \hat{\theta}(t) < \theta^* + \epsilon$. (This assumption is needed in Corollary 5 of [1].) It is readily checked that these assumptions are satisfied for the family of normal, binomial, or Poisson distributions.

The risk function $r(\theta, \delta)$ of a decision function δ is

$$
\begin{aligned}
r(\theta, \delta) = {} & W(\theta, 1) \Pr(\text{Accepting } H_1 \mid \theta, \delta) \\
& + W(\theta, 2) \Pr(\text{Accepting } H_2 \mid \theta, \delta) \\
& + E[C(N) \mid \theta, \delta],
\end{aligned} \tag{7}
$$

where N is the total number of observations taken.

3. Proof of the essential completeness. It will now be shown that under the assumptions of Section 2, the class A of generalized sequential probability ratio tests is essentially complete relative to the class D of decision functions with bounded risk. Sobel [1] proved this theorem under the additional assumption that both $W(\theta, 1)$ and $W(\theta, 2)$ are bounded functions on Ω. The proof to be given here leans heavily on this result.

Let Ω be the interval from $\underline{\theta}$ to $\bar{\theta}$. If $\underline{\theta}$ and $\bar{\theta}$ are finite and included in Ω, then it follows from the assumptions of Section 2 that $W(\theta, 1)$ and $W(\theta, 2)$ are bounded by $W(\bar{\theta}, 1)$ and $W(\underline{\theta}, 2)$, respectively, and the desired result follows from Sobel's theorem. Hence, the proof will be given here for the situation where Ω is the open interval $\underline{\theta} < \theta < \bar{\theta}$ with either end point possibly infinite. The modifications necessary when Ω is a half-open interval will be obvious.

Consider the sequence of problems $P^{(i)}$, $i = 1, 2, \cdots$, defined by the loss functions

$$\begin{aligned} W^{(i)}(\theta, 1) &= W(\theta, 1) && \text{for} \quad \theta \leqq \theta_2^{(i)} \\ W^{(i)}(\theta, 1) &= W(\theta_2^{(i)}, 1) && \text{for} \quad \theta > \theta_2^{(i)} \\ W^{(i)}(\theta, 2) &= W(\theta, 2) && \text{for} \quad \theta \geqq \theta_1^{(i)} \\ W^{(i)}(\theta, 2) &= W(\theta_1^{(i)}, 2) && \text{for} \quad \theta < \theta_1^{(i)}, \end{aligned} \tag{8}$$

where

$$\begin{aligned} &\theta_1 > \theta_1^{(1)} > \theta_1^{(2)} > \theta_1^{(3)} > \cdots \\ &\theta_2 < \theta_2^{(1)} < \theta_2^{(2)} < \theta_2^{(3)} < \cdots \end{aligned} \tag{9}$$

and

$$\lim_{i\to\infty} \theta_1^{(i)} = \underline{\theta}, \qquad \lim_{i\to\infty} \theta_2^{(i)} = \bar{\theta}. \tag{10}$$

Thus, the functions $W^{(i)}(\theta, 1)$ and $W^{(i)}(\theta, 2)$ are bounded for each i, $i = 1, 2, \cdots,$. The sampling cost function $C(n)$ for each problem $P^{(i)}$ is assumed to be the same as that for the original problem.

Let δ be any decision function for the original problem, with risk $r(\theta, \delta)$. Then, for each i, δ is also a decision function for the problem $P^{(i)}$, with risk $r^{(i)}(\theta, \delta)$. Furthermore, for each $\theta \varepsilon \Omega$,

$$r^{(1)}(\theta, \delta) \leqq r^{(2)}(\theta, \delta) \leqq \cdots, \tag{11}$$

$$r^{(i)}(\theta, \delta) \leqq r(\theta, \delta) \qquad \text{for } i = 1, 2, \cdots, \tag{12}$$

and

$$\lim_{i\to\infty} r^{(i)}(\theta, \delta) = r(\theta, \delta). \tag{13}$$

A stronger statement than (13) can be made; namely, for each $\theta \varepsilon \Omega$ there exists an integer k_o such that $r^{(i)}(\theta, \delta) = r(\theta, \delta)$ for $i \geqq k_o$.

Now let $\bar{\delta} \varepsilon D$ be any decision function with bounded risk $r(\theta, \bar{\delta})$. By Sobel's theorem, for each $i (i = 1, 2, \cdots)$ there exists $\delta_i \varepsilon A$ such that

$$r^{(i)}(\theta, \delta_i) \leqq r^{(i)}(\theta, \bar{\delta}) \quad \text{for all } \theta \varepsilon \Omega. \tag{14}$$

It follows from Wald, [2], Theorem 3.2, and Sobel, [1], Theorem 1 and its proof, that there exists a subsequence $\{\delta_{i_j}\}$ of the sequence of decision functions $\{\delta_i\}$ that converges in the regular sense (see [1], p. 321, for definition) to a decision function δ_o, and that

$$\liminf_{j\to\infty} r^{(k)}(\theta, \delta_{i_j}) \geqq r^{(k)}(\theta, \delta_o) \tag{15}$$

for each $k(k = 1, 2, \cdots)$ and all $\theta \varepsilon \Omega$. Furthermore, the decision function δ_o can be taken to be in the class A. It will be shown that

$$r(\theta, \delta_o) \leqq r(\theta, \bar{\delta}) \quad \text{for all } \theta \varepsilon \Omega. \tag{16}$$

Choose and fix $\theta \varepsilon \Omega$. It follows from (13) that (16) will be proven if it can be shown that

$$r^{(k)}(\theta, \delta_o) \leqq r(\theta, \bar{\delta}) \quad \text{for} \quad k = 1, 2, \cdots. \tag{17}$$

Accordingly, let k be any positive integer and let $\epsilon > 0$ be an arbitrary positive number. By (15), an integer i can be chosen large enough so that $i \geqq k$ and

$$r^{(k)}(\theta, \delta_i) > r^{(k)}(\theta, \delta_o) - \epsilon. \tag{18}$$

Hence, from (18), (11), (14), and (12),

$$r^{(k)}(\theta, \delta_o) - \epsilon < r^{(k)}(\theta, \delta_i) \leqq r^{(i)}(\theta, \delta_i) \leqq r^{(i)}(\theta, \bar{\delta}) \leqq r(\theta, \bar{\delta}).$$

Since ϵ was arbitrary, $r^{(k)}(\theta, \delta_o) \leqq r(\theta, \bar{\delta})$. This completes the proof.

REFERENCES

[1] MILTON SOBEL, "An essentially complete class of decision functions for certain standard sequential problems," *Ann. Math. Stat.*, Vol. 24(1953), pp. 319–337.
[2] ABRAHAM WALD, *Statistical Decision Functions*, John Wiley and Sons, New York, 1950.

A PROBLEM IN SURVIVAL[1]

BY JAMES B. MACQUEEN

University of California, Los Angeles

1. Introduction. Suppose that at a given time an individual has certain resources. These are used up at a specified rate, but from time to time "opportunities" arrive; at an opportunity a decision is made and the resources are changed—increased or decreased—in a random manner depending on the decision. If the resources ever fall to zero, the individual "perishes." The problem is to make the decision at each opportunity which will minimize the probability of ultimately perishing.

Received September 19, 1960.

[1] The form of the problem considered here was suggested in substantial part by work of Lester E. Dubins and Leonard J. Savage on optimal gambling, presented by Dubins at the departmental seminar, Department of Statistics, University of California, Berkeley, in October, 1959.

This research was supported by a grant from the Western Management Science Institute, University of California, Los Angeles.