LIMIT DISTRIBUTIONS IN THE THEORY OF COUNTERS

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1. Introduction. Let us suppose that particles that arrive on the counter be randomly spaced on the positive time axis. In an actual counter each particle arriving in the time interval $(0, \infty)$ independently of others gives rise to an impulse with probability p or 1 according to whether at this instant there is an impulse present or not. Hence the counter is in one or other of two mutually exclusive states which we denote by A and B. The counter is in state A when no impulse covers the instant and is in state B otherwise and it assumes the states A and B alternatively. A particle striking the counter is recorded if and only if the counter is in state A. If $p = 0$,we get the type I counter and if $p = 1$, we get the type II counter.

Let $t_1, t_2, t_3 \cdots$ be the instants at which particles arrive and $\chi_1, \chi_2, \chi_3, \cdots$ be the lengths of successive impulses. Let $\tau_1, \tau_2, \tau_3 \cdots$ be the instants of successive recordings. Let us assume that the time from an arbitrary point in the positive time axis to the time of arrival of the first particle that follows it is a random variable with distribution function $F(x)$. Hence the differences $\{t_{r+1} - t_r\}$, $r = 1, 2, 3, \cdots$, are identically distributed random variables independent of each other with a distribution function $F(x)$. Let the time durations of the impulses be independent and identically distributed random variables with the distribution function $H(x)$. Let these random variables be independent of the instants of arrivals and of the events of the realizations of the impulses. Let ν_t denote the number of registered particles in the time interval $(0, t)$. Let the process start in state A. Let us denote by $\xi_1, \eta_1, \xi_2, \eta_2, \cdots$ the times spent in states A and B respectively. Consequently $\{\xi_n\}$ and $\{\eta_n\}$ are independent sequences of identically distributed positive random variables. It can be seen that $\Pr[\xi_n \leqq x] = F(x), x > 0$. Let $\Pr[\eta_n \leqq x] = U(x), x > 0$. It can also be seen that the instants of transitions $A \to B$ coincide with the instants τ_n, $n = 1, 2, 3, \cdots$. Hence the time differences $\{\tau_{n+1} - \tau_n\}$ are identically distributed random variables whose distribution function $G(x)$ is given by

$$G(x) = F(x) * U(x) = \int_0^x U(x - y)\, dF(y). \tag{1.1}$$

In [8] Takács has shown that ν_t, suitably normalized, is asymptotically normal. In [9] Takács has considered the asymptotic behavior of ν_t/t. Here he has also applied the law of the iterated logarithm, as stated by P. Hartman and A. Wintner [5], to ν_t. In this paper we consider the asymptotic $(\sim)$ behaviour of ν_t when $F(x)$ and $H(x)$ follow the stable laws with suitable characteristic exponents and we show that ν_t, suitably normalized, tends to the Mittag-Leffler distribution for all counters of the types detailed above.

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2. Definitions and notations. Let

$$r(s) = \int_0^\infty e^{-sx}\, dF(x)$$

and

$$w(s) = \int_0^\infty e^{-sx}\, d\,U(x),$$

so that

$$\gamma(s) = \int_0^\infty e^{-sx}\, dG(x) = r(s)\cdot w(s).$$

Also

$$\Pr[\tau_1 \leqq x] = F(x).$$

Let

$$\begin{aligned} W(t, n) &= \Pr[\nu_t \leqq n] \\ &= 1 - \Pr[\tau_{n+1} \leqq t] \\ &= 1 - F(t) * G_n(t), \end{aligned} \tag{2.1}$$

where $G_n(t)$ is the n-fold convolution of $G(t)$ with itself and $G_0(t) = 1$ if $t \geqq 0$ and 0 otherwise. So

$$\Pr[\nu_t = n] = F(t) * G_{n-1}(t) - F(t) * G_n(t).$$

Let $m_k(t)$ be the kth moment of ν_t,

$$\begin{aligned} m_k(t) &= \sum_{n=1}^{\infty} n^k [F(t) * G_{n-1}(t) - F(t) * G_n(t)] \\ &= \sum_{n=0}^{\infty} [(n+1)^k - n^k] \cdot [1 - W(t, n)]. \end{aligned} \tag{2.2}$$

Also

$$\begin{aligned} \int_0^\infty e^{-st}\, dm_k(t) &= s\int_0^\infty e^{-st}\, m_k(t)\, dt \\ &= s\int_0^\infty e^{-st} \sum_{n=0}^{\infty} [(n+1)^k - n^k]\cdot[1 - W(t,n)]\, dt \\ &= s\sum_{n=0}^{\infty} [(n+1)^k - n^k] \int_0^\infty e^{-st}\, [1 - W(t, n)]\, dt \\ &= r(s) \sum_{n=0}^{\infty} [(n+1)^k - n^k]\, [\gamma(s)]^n . \end{aligned} \tag{2.3}$$

Interchange of summation and integration is valid since all the terms on the right side of (2.2) are positive. Let $N(t)$ be the number of particles arriving in the counter in the time interval $(0, t)$. Let

$$Q_n(t) = \Pr[N(t) = n].$$

So

$$\begin{aligned} \Pr[N(t) \leqq n] &= \Pr[t \leqq t_{n+1}] \\ &= 1 - F_{n+1}(t), \end{aligned} \tag{2.4}$$

where $F_{n+1}(t)$ is the $n + 1$-fold convolution of $F(t)$ with itself. So $\sum_0^n Q_r(t) = 1 - F_{n+1}(t)$ and hence

$$Q_n(t) = F_n(t) - F_{n+1}(t). \tag{2.5}$$

3. Type II counter. Consider the type II counter. Let $P_A(t)$ denote the probability that at the instant t, the system is in state A.

$$\begin{aligned} P_A(t) &= \Pr(t < \tau_1) + \Pr(\tau_2 - \xi_2 < t < \tau_2) + \Pr(\tau_3 - \xi_3 < t < \tau_3) + \cdots \\ &= \Pr(t < \tau_1) + [\Pr(t < \tau_2) - \Pr(t \leqq \xi_1 + \eta_1)] \\ &\quad + [\Pr(t < \tau_3) - \Pr(t \leqq \xi_1 + \eta_1 + \xi_2 + \eta_2)] + \cdots \\ &= \sum_{n=0}^{\infty} G_n(t) - \sum_{n=0}^{\infty} G_n(t)*F(t). \end{aligned} \tag{3.1}$$

$P_A(t)$ can also be got as follows. If we know that in the time interval $(0, t)$ exactly n particles arrive at the counter (the probability of which is $Q_n(t)$), then the occurrence points of these n events may be regarded as independent uniformly distributed points in time interval $(0, t)$. The probability that an impulse started at a random point will end before the instant t is $p_t = (1/t) \int_0^t H(x)\, dx$. Because of independence the probability that all the n impulses started at random points will end before the instant t is $(p_t)^n$. So

$$\begin{aligned} P_A(t) &= \sum_{n=0}^{\infty} Q_n(t)(p_t)^n \\ &= \sum_{n=0}^{\infty} [F_n(t) - F_{n+1}(t)](p_t)^n \\ &= 1 - (1 - p_t)\sum_{n=0}^{\infty} F_n(t)(p_t)^{n-1}. \end{aligned} \tag{3.2}$$

Also using (3.1), we see that

$$\begin{aligned} \int_0^\infty e^{-st} P_A(t)\, dt &= \int_0^\infty e^{-st} \left[\sum_{n=0}^{\infty} G_n(t) - \sum_{n=0}^{\infty} G_n(t)*F(t)\right] dt \\ &= (s^{-1}) \{(1 - \gamma(s))^{-1} - r(s)/(1 - \gamma(s))\} \\ &= [1 - r(s)]/s[1 - \gamma(s)]. \end{aligned} \tag{3.3}$$

Hence

$$\gamma(s) = 1 - (1 - r(s))\left[s \int_0^\infty e^{st} P_A(t)\, dt\right]^{-1}. \tag{3.4}$$

4. A lemma. We now state a lemma which can be got as the converse of a theorem [[4] Theorem 108, p. 168] by an obvious change of variables.

LEMMA. *Let $L(x)$ be such that $L(cx) \sim L(x)$ for every positive c (as $x \to \infty$). Then*

$$\sum_{n=0}^{\infty} a_n e^{-(n/\lambda)^\theta} \sim \lambda^{\theta\alpha} L(\lambda^\theta)\Gamma(\alpha + 1), \qquad \text{as } \lambda \to \infty, \theta > 0, \alpha > 0, \tag{4.1}$$

if

$$A(n) = a_0 + a_1 + a_2 + \cdots + a_n \sim (1/\theta) n^{\alpha\theta} L(n^\theta), \qquad \text{as } n \to \infty. \tag{4.2}$$

In particular, if $\theta = 1$ and $e^{-1/\lambda} = x$, we get

$$\sum_{n=0}^{\infty} a_n x^n \sim (\log 1/x)^{-\alpha} L\{[\log (1/x)]^{-1}\}\Gamma(\alpha + 1), \qquad \text{as } x \to 1, \tag{4.3}$$

if

$$A(n) = a_0 + a_1 + \cdots + a_n \sim n^\alpha L(n), \qquad \text{as } n \to \infty. \tag{4.4}$$

Using this lemma in (2.3), we get

$$\int_0^\infty e^{-st}\, dm_k(t) = r(s) \sum_{n=0}^{\infty} [(n + 1)^k - n^k](\gamma(s))^n$$

$$\sim r(s)[\log \{1/\gamma(s)\}]^{-k}\, \Gamma(k + 1), \qquad \text{as } s \to 0.$$

5. Asymptotic behavior of $P_A(t)$ in a type II counter. We prove the following theorem.

THEOREM 5.1. *In a type* II *counter, if $F(t)$ has a characteristic function $\phi(\lambda)$ of form*

$$\log \phi(\lambda) = c\Gamma(-\alpha)[\cos (\pi\alpha/2) - i(\lambda/|\lambda|) \sin (\pi\alpha/2)]|\lambda|^\alpha \tag{5.1}$$

with $c > 0$ and $0 < \alpha < 1$ and further if p_t be chosen such that

$$1 - p_t \sim \lambda t^{-\beta} \qquad \text{as } t \to \infty \tag{5.2}$$

where $\beta < \alpha$ and λ are constants, then

$$P_A(t) \sim (c/\lambda\alpha) t^{-(\alpha-\beta)}, \qquad \text{as } t \to \infty. \tag{5.3}$$

REMARK I. In [3] it is shown that the characteristic function $\phi(\lambda)$ of the stable law is of the form

$$\log \phi(\lambda) = i\gamma t + c\Gamma(-\alpha)[\cos ([\pi\alpha/2] + i\beta(\lambda/|\lambda|) \sin (\pi\alpha/2)]|\lambda|^\alpha \tag{5.4}$$

where $0 < \alpha < 1$, $c > 0$, $-1 \leqq \beta \leqq 1$, and γ is any real number. In particular, if $\gamma = 0$ and $\beta = -1$[[6] p. 200], $\phi(\lambda)$ represents the characteristic function of a positive random variable. In Theorem 5.1 we have taken a distribution function with this characteristic function. This stable law for $F(t)$ correspond to well known recurrent processes [10, 11] which describe the arrivals of particles to the counter.

REMARK II. If $H(x) \sim 1 - \lambda(1 - \beta)x^{-\beta}$, as $x \to \infty$, then p_t has the property

$$1 - p_t \lambda t^{-\beta}, \qquad \text{as } t \to \infty.$$

PROOF. In our case [6] $F(t)$ can be put in the form

$$F(t) = 1 - [c/\alpha + \epsilon(t)](1/t^{\alpha}), \qquad 0 < t < \infty. \tag{5.5}$$

where $\epsilon(t) \to 0$ as $t \to \infty$. Also

$$F_n(t) = F(t/n^{1/\alpha}) = 1 - [nc/\alpha + n\,\epsilon(t/n^{1/\alpha})](1/t^{\alpha}). \tag{5.6}$$

From (3.3)

$$\begin{aligned} P_A(t) &= 1 - (1 - p_t)\sum_{n=1}^{\infty} F_n(t)(p_t)^{n-1} \\ &= (1 - p_t)\sum_{n=1}^{\infty} [1 - F_n(t)](p_t)^{n-1} \\ &= c(1 - p_t)\alpha^{-1}t^{-\alpha}\sum_{n=1}^{\infty} n(p_t)^{n-1} + (1 - p_t)t^{-\alpha}\sum_{n=1}^{\infty} n\,\epsilon(t/n^{1/\alpha})(p_t)^{n-1} \\ &= \mathrm{I} + \mathrm{II} \end{aligned}$$

where

$$\begin{aligned} \mathrm{I} &= c(1 - p_t)\alpha^{-1}t^{-\alpha}\sum_{n=1}^{\infty} n(p_t)^{n-1} \\ &= c/\{\alpha t^{\alpha}(1 - p_t)\}, \end{aligned}$$

and

$$\mathrm{II} = (1 - p_t)t^{-\alpha}\sum_{n=1}^{\infty} n\,\epsilon(t/n^{1/\alpha})(p_t)^{n-1}.$$

Consider the sum

$$R(t) = (1 - p_t)^2 \sum_{n=1}^{\infty} n\,\epsilon(t/n^{1/\alpha})(p_t)^{n-1}. \tag{5.7}$$

Using the theorem in the appendix, when $\alpha \lessgtr \frac{1}{2}$

$$\begin{aligned} R(t) \leqq (1 - p_t)^2 \sum_{n=1}^{\infty} n(p_t)^{n-1}\{&K_1' n^{\alpha/(1-\alpha)} t^{-\alpha^2/(1-\alpha)} + \\ &K_2' n^{(1+\alpha)/(1-\alpha)} t^{-\alpha(1+\alpha)/(1-\alpha)} + \\ &K_3' n^{1/(1-\alpha)} t^{-\alpha/(1-\alpha)} + \\ &K_4' n t^{-\alpha} + K_5' n^{(2-\alpha)/(1-\alpha)} t^{-\alpha(2-\alpha)/(1-\alpha)}\} \end{aligned}$$

where K_i', $i = 1, 2, 3, 4, 5$ are constants independent of t. So

$$\begin{aligned} R(t) \leqq (1 - p_t)^2\{ & (K_1'/t^{\alpha^2/(1-\alpha)}) \sum_1^\infty n^{1/(1-\alpha)}(p_t)^{n-1} + \\ & (K_2'/t^{\alpha(1+\alpha)/(1-\alpha)}) \sum_1^\infty n^{2/(1-\alpha)}(p_t)^{n-1} + \\ & (K_3'/t^{\alpha/1-\alpha}) \sum_1^\infty n^{(2-\alpha)/(1-\alpha)}(p_t)^{n-1} + \\ & (K_4'/t^{\alpha}) \sum_1^\infty n^2(p_t)^{n-1} \\ & (K_5'/t^{\alpha(2-\alpha)/(1-\alpha)}) \sum_1^\infty n^{(3-2\alpha)/(1-\alpha)}(p_t)^{n-1}\} \\ \sim (1 - p_t)^2[& \{K_1'/t^{\alpha^2/(1-\alpha)}\}\cdot\{1/(1 - p_t)^{(2-\alpha)/(1-\alpha)}\} + \\ & \{K_2'/t^{\alpha(1+\alpha)/(1-\alpha)}\}\cdot\{1/(1 - p_t)^{(3-\alpha)/(1-\alpha)}\} + \\ & \{K_3'/t^{\alpha/(1-\alpha)}\}\cdot\{1/(1 - p_t)^{(3-2\alpha)/(1-\alpha)}\} + \\ & \{K_4'/t^{\alpha}\}\cdot\{1/(1 - p_t)^3\} + \\ & \{K_5'/t^{\alpha(2-\alpha)/(1-\alpha)}\}\cdot\{1/(1 - p_t)^{(4-3\alpha)/(1-\alpha)}\}], \quad t \to \infty \\ & \{K_1'/t^{\alpha^2/(1-\alpha)}\}\cdot\{1/(1 - p_t)^{\alpha/(1-\alpha)}\} + \\ & \{K_2'/t^{\alpha(1+\alpha)/(1-\alpha)}\}\cdot\{1/(1 - p_t)^{(1+\alpha)/(1-\alpha)}\} + \\ & \{K_3'/t^{\alpha/(1-\alpha)}\}\cdot\{1/(1 - p_t)^{1/(1-\alpha)}\} + \\ & \{K_4'/t^{\alpha}\}\cdot\{1/(1 - p_t)\} + \\ & \{K_5'/t^{\alpha(2-\alpha)/(1-\alpha)}\}\cdot\{1/(1 - p_t)^{(2-\alpha)/(1-\alpha)}\}, \qquad t \to \infty. \end{aligned}$$

Since $1 - p_t \sim \lambda t^{-\beta}$, $t \to \infty$,

$$\begin{aligned} R(t) \leqq & K_1'/\{\lambda^{\alpha/(1-\alpha)} t^{\alpha(\alpha-\beta)/(1-\alpha)}\} + \\ & K_2'/\{\lambda^{(1+\alpha)/(1-\alpha)} t^{(1+\alpha)(\alpha-\beta)/(1-\alpha)}\} + \\ & K_3'/\{\lambda^{1/(1-\alpha)} t^{(\alpha-\beta)/(1-\alpha)}\} + \\ & K_4'/\{\lambda t^{\alpha-\beta}\} + \\ & K_5'/\{\lambda^{(2-\alpha)/(1+\alpha)} t^{(2-\alpha)(\alpha-\beta)/(1-\alpha)}\}. \end{aligned} \tag{5.8}$$

Hence when $\beta < \alpha$,

$$R(t) \to 0, \qquad \text{as } t \to \infty.$$

When $\alpha = \frac{1}{2}$, using the theorem in the appendix,

$$R(t) \leqq (1 - p_t)^2 \sum_{n=1}^{\infty} n(p_t)^{n-1}$$

$$[K_1''(t/n^2)^{-\frac{1}{2}}) + K_2''(t/n^2)^{-\frac{3}{2}} + K_3''(t/n^2)^{-1} + K_4''(n/t^{\frac{1}{2}})(\log t - 2 \log n)]$$

$$\leqq (1 - p_t)^2 \sum_{n=1}^{\infty} n(p_t)^{n-1}$$

$$[K_1''(n/t^{\frac{1}{2}}) + K_2''(n^3/t^{\frac{3}{2}}) + K_3''(n^2/t) + K_4''(n \log t)/t^{\frac{1}{2}} - K_4''(2\, n \log n)/t^{\frac{1}{2}}]$$

where K_i'', $i = 1, 2, 3, 4$ are constants independent of t. So

$$R(t) \leqq (1 - p_t)^2[(K_1''/t^{\frac{1}{2}}) \sum_1^{\infty} n^2(p_t)^{n-1} + (K_2''/t^{\frac{3}{2}}) \sum_1^{\infty} n^4(p_t)^{n-1} +$$

$$(K_3''/t) \sum_1^{\infty} n^3(p_t)^{n-1} + (K_4'' \log t)t^{-\frac{1}{2}} \sum_1^{\infty} n^2(p_t)^{n-1} -$$

$$(2\, K_4''/t^{\frac{1}{2}}) \sum_1^{\infty} n^2 \log n(p_t)^{n-1}].$$

$$\sim (K_1''/t^{\frac{1}{2}})\cdot\{1/(1 - p_t)\} + (K_2''/t^{\frac{3}{2}})\cdot\{1/(1 - p_t)^3\} +$$

$$(K_3''/t)\cdot\{1/(1 - p_t)^2\} + (K_4''(\log t)/t^{\frac{1}{2}})\cdot\{1/(1 - p_t)\} -$$

$$(2\, K_4''/t^{\frac{1}{2}})\cdot\{1/(1 - p_t)^{1+\epsilon}\}.$$

So

$$R(t) \leqq K_1''/\{\lambda t^{(\frac{1}{2}-\beta)}\} + K_2''/\{\lambda^3 t^{3(\frac{1}{2}-\beta)}\} + K_3''/\{\lambda^2 t^{(1-2\beta)}\} + \tag{5.9}$$

$$K_4'' \ (\log t)/\{\lambda t^{(\frac{1}{2}-\beta)}\} - 2\, K_4''/\{\lambda^{1+\epsilon} t^{\frac{1}{2}-\beta(1+\epsilon)}\}.$$

Here ϵ is a small positive number which can be taken as small as we like so that $\epsilon < (1 - 2\beta)/2\beta$. So when $\beta < \frac{1}{2}$, $R(t) \to 0$ as $t \to \infty$. Hence what ever be the value of α in $0 < \alpha < 1$, when $\beta < \alpha$, $R(t) \to 0$ as $t \to \infty$. So when $\beta < \alpha$, II in (5.8) is negligible compared to I. Hence

$$P_A(t) \sim \{c/(\alpha t^{\alpha})\}\cdot\{1/(1 - p_t)\}, \qquad t \to \infty \tag{5.10}$$

$$\sim \{c/(\lambda\alpha)\}\cdot t^{-(\alpha-\beta)}, \qquad t \to \infty.$$

6. Asymptotic behavior of ν_t in a type II counter.

THEOREM 6.1. *With the same assumptions on $F(t)$ and p_t as in Theorem 5.1, we have,*

$$\lim_{t\to\infty} \Pr\left[\frac{\nu_t}{(1/\lambda)\cdot t^{\beta}\cdot K(\alpha, \beta)} \leqq x\right] = g_{\beta}(x) \tag{6.1}$$

where

$$g_{\beta}(x) = [1/(\pi\beta)] \int_0^x \left\{\sum_{j=1}^{\infty} [(-1)^{j-1}/j!] \sin(\pi\beta j)\Gamma(\beta j + 1)y^{j-1}\right\} dy \tag{6.2}$$

and

(6.3) $K(\alpha, \beta) = \Gamma(1 - \alpha + \beta)/\Gamma(1 - \alpha).$

PROOF. From (5.3), we have,

$$P_A(t) = \{c/(\lambda\alpha)\}t^{-(\alpha-\beta)}, \qquad t \geqq t_o. \tag{6.4}$$

Also $P_A(t) = 1$, when $t = 0$. So

$$\begin{aligned} s\int_0^\infty e^{-st}P_A(t)\,dt &= [(cs)/(\lambda\alpha)]\int_{t_o}^\infty e^{-st}t^{-(\alpha-\beta)}\,dt + s\int_0^{t_o} e^{-st}P_A(t)\,dt \\ &= [(cs)/(\lambda\alpha)]\int_0^\infty e^{-st}t^{-(\alpha-\beta)}\,dt \\ &\quad - [(cs)/(\lambda\alpha)]\int_0^{t_o} e^{-st}t^{-(\alpha-\beta)}\,dt + s\int_0^{t_o} e^{-st}P_A(t)\,dt. \end{aligned} \tag{6.5}$$

In (6.5), the second integral is less than $t_0^{(1-\alpha+\beta)}/(1 - \alpha + \beta)$ and the third is less than $M(1 - e^{-st_o})\,s^{-1}$ where M is a constant. So

$$s\int_0^\infty e^{-st}P_A(t)\,dt \sim \frac{c\Gamma(1 - \alpha + \beta)}{\lambda\alpha s^{-(\alpha-\beta)}}, \qquad s \to 0. \tag{6.6}$$

It follows from (5.1) that

$$r(s) = \int_0^\infty e^{-st}\,dF(t) = \exp\left[-\frac{c\Gamma(1 - \alpha)}{\alpha}\,s^\alpha\right].$$

Using (3.4),

$$\begin{aligned} \gamma(s) &\sim 1 - \frac{[1 - \{1 - (1/\alpha)c\Gamma(1 - \alpha)s^\alpha\}]}{[c\Gamma(1 - \alpha + \beta)]/[\lambda\alpha s^{-(\alpha-\beta)}]}, \qquad s \to 0 \\ &\sim 1 - \frac{\lambda\Gamma(1 - \alpha)s^\beta}{\Gamma(1 - \alpha + \beta)}, \qquad s \to 0. \end{aligned}$$

Using (4.5) we have

$$\begin{aligned} \int_0^\infty e^{-st}\,dm_k(t) &\sim \frac{[1 - (1/\alpha)c\Gamma(1 - \alpha)s^\alpha]\Gamma(k + 1)}{\left\{-\log\left[1 - \frac{\lambda\Gamma(1 - \alpha)}{\Gamma(1 - \alpha + \beta)}\,s^\beta\right]\right\}^k}, \qquad s \to 0 \\ &\sim \frac{\Gamma(k + 1)}{\left[\frac{\lambda\Gamma(1 - \alpha)s^\beta}{\Gamma(1 - \alpha + \beta)}\right]^k}. \end{aligned} \tag{6.8}$$

Using Karamata's Tauberian theorem [12], we get

$$\text{(6.9)}\qquad \begin{aligned} m_k(t) &\sim \frac{\Gamma(k+1)t^{k\beta}}{\{[\lambda\Gamma(1-\alpha)]/\Gamma(1-\alpha+\beta)\}^k\Gamma[\beta k+1]}, && t\to\infty \\ &\sim \frac{\Gamma(k+1)}{\Gamma(\beta k+1)}\left[\frac{t^\beta K(\alpha,\beta)}{\lambda}\right]^k, && t\to\infty \end{aligned}$$

where $K(\alpha, \beta)$ and β are given by (6.3) and (5.2). That is

$$\text{(6.10)}\qquad E\left[\frac{\nu_t}{[t^\beta K(\alpha,\beta)]/\lambda}\right]^k \sim \frac{\Gamma(k+1)}{\Gamma(\beta k+1)}, \qquad t\to\infty.$$

Hence

$$\text{(6.11)}\qquad \lim_{t\to\infty} \Pr\left[\frac{\nu_t}{[t^\beta K(\alpha,\beta)]/\lambda} \leqq x\right] = g\beta(x)$$

where $g\beta(x)$ is defined by (6.2).

7. Type I counter. For a type I counter, $U(x) = H(x)$. So $\gamma(s) = r(s)\, W'(s)$ where $W'(s) = \int_0^\infty e^{-sx} dH(x)$. Assuming that $m = \int_0^\infty x dH(x)$ exists and $F(t)$ has the same form as in Theorem 5.1, we find that

$$\text{(7.1)}\qquad \begin{aligned} \gamma(s) &= r(s)\, W'(s) \\ &= \exp\,[(-c/\alpha)\Gamma(1-\alpha)s^\alpha]\cdot\{1 - sm + \cdots\} \\ &\sim 1 - (c/\alpha)\Gamma(1-\alpha)s^\alpha, && \text{as } s\to\infty. \end{aligned}$$

Hence in this case,

$$\text{(7.2)}\qquad \int_0^\infty e^{-st}\, dm_k(t) \sim \frac{\Gamma(k+1)}{[(c/\alpha)\Gamma(1-\alpha)s^\alpha]^k}, \qquad s\to 0$$

Using Karamata's theorem [12]

$$\text{(7.3)}\qquad m_k(t) \sim \frac{\Gamma(k+1)t^{\alpha k}}{[(c/\alpha)\Gamma(1-\alpha)]^k\Gamma(\alpha k+1)}, \qquad t\to\infty.$$

Hence we have

THEOREM 7.1. *For a type* I *counter with $F(t)$ as in Theorem* 5.1. *and an $H(x)$ having the first moment,*

$$\text{(7.4)}\qquad \lim_{t\to\infty} \Pr\left[\frac{\nu_t}{(\alpha t^\alpha)/[c\Gamma(1-\alpha)]} \leqq x\right] = g_\alpha(x)$$

where $g_\alpha(x)$ is defined by (6.2).

8. Actual counter. In the actual counter described in Section 1,

$$\text{(8.1)}\qquad P_A(t) = \sum_{j=0}^\infty Q_j'(t)(p_t)^j$$

where

$$Q'_j(t) = \sum_{n=0}^{\infty} Q_n(t) \binom{n}{j} p^j q^{n-j}. \tag{8.2}$$

We can write $P_A(t)$ in the form

$$\begin{aligned} P_A(t) &= \sum_{n=0}^{\infty} Q_n(t) q^n + \frac{(pp_t)}{1!} \sum_{n=1}^{\infty} Q_n(t) \frac{d}{dq}(q^n) \\ &\qquad + \frac{(pp_t)^2}{2!} \sum_{n=2}^{\infty} Q_n(t) \frac{d^2}{dq^2}(q^n) + \cdots \\ &= \left(1 + \frac{pp_t}{1!}\frac{d}{dq} + \frac{(pp_t)^2}{2!}\frac{d^2}{dq^2} + \cdots\right)\left(\sum_{n=0}^{\infty} Q_n(t) q^n\right) \qquad (8.3) \\ &= \left(1 + \frac{pp_t}{1!}\frac{d}{dq} + \frac{(pp_t)^2}{2!}\frac{d^2}{dq^2} + \cdots\right)\left(1 - (1-q)\sum_{n=1}^{\infty} F_n(t) q^{n-1}\right) \\ &= 1 - [1 - (q + pp_t)] \sum_{n=1}^{\infty} F_n(t)(q + pp_t)^{n-1}. \end{aligned}$$

Hence we have the following theorem.

THEOREM 8.1. *With the same assumptions on $F(t)$ and p_t as in Theorem* 5.1, *in the actual counter,*

$$P_A(t) \sim [c/(\alpha\lambda p)] t^{-(\alpha-\beta)}, \qquad t \to \infty. \tag{8.4}$$

9. Asymptotic behavior of ν_t in the actual counter. By using the same method as the one used by Takács in [9], we can deduce that for the actual counter,

$$\gamma(s) = r(s) \left[\frac{1}{r'(s)} - \frac{(1 - r'(s))}{r'(s)} \left\{ s \int_0^{\infty} e^{-st} P_A(t)\, dt \right\}^{-1}\right] \tag{9.1}$$

where $r'(s) = (pr(s))/(1 - qr(s))$, is the Laplace Transform of

$$K(t, q) = (1 - q) \sum_{r=1}^{\infty} F_r(t) q^{r-1}$$

which can be verified to be a distribution. It can also be seen that

$$\sum_{j=0}^{m} Q'_j(t) = 1 - K_{m+1}(t, q)$$

where $K_{m+1}(t, q)$ is the $(m + 1)$th convolution of $K(t, q)$ with itself. Taking $F(t)$ and p_t as in Theorem 5.1, with $\alpha = \beta$ and $\alpha\lambda p > qc$, we find that

$$\gamma(s) \sim 1 - [(\alpha\lambda p - qc)/(\alpha p)]\Gamma(1 - \alpha) s^{\alpha}. \tag{9.2}$$

Hence we have the following theorem.

THEOREM 9.1. *With the same assumptions on $F(t)$ and p_t as in Theorem* 6.1, *with $\alpha = \beta$ and $\lambda\alpha p > qc$, in the actual counter,*

$$\lim_{t\to\infty} \Pr\left[\frac{\nu_t}{(\alpha p t^{\alpha})/[(\alpha\lambda p - qc)\Gamma(1 - \alpha)]} \leq x\right] = g_\alpha(x). \tag{9.3}$$

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APPENDIX

Here we give a brief sketch of the proof leading to the results in 5.7. At the outset we state a Lemma (the proof of which is fairly simple) which we use in the proof of the accompanying theorem.

LEMMA. *Consider the integral*

$$I = \int_0^\delta \phi^{-\beta} e^{-\alpha T \phi^{-\gamma}} \, d\phi, \qquad \alpha > 0, \qquad \gamma > 0, \qquad T > 0. \tag{1}$$

If $\beta > 1$,

$$I = [(\alpha T)^{(1-\beta)/\gamma}/\gamma] \cdot [\Gamma(\beta - 1)/\gamma)] + K_1(T, \delta)T],$$

If $\beta < 1$,

$$I = K_2(T, \delta),$$

and if $\beta = 1$,

$$I = (1/\gamma)[\log (1/T) + K_3(T, \delta)],$$

where for all T *in the range* $0 < T < \infty$,

$$K_i(T, \delta) < K_i'(\delta), \qquad i = 1, 2, 3.$$

Note $K_i'(\delta)$ depends only on δ and not on T.

We now prove the following theorem concerning stable laws.

THEOREM. *Consider a distribution function* $F(t)$ *with Laplace transform*

$$r(s) = \exp \{[-c\Gamma(1 - \alpha)/\alpha]s^\alpha\}, \qquad c > 0, \quad 0 < \alpha < 1.$$

Then, $F(t)$ *can be put in the form* $1 - F(t) = t^{-\alpha}[(c/\alpha) + \epsilon(t)]$ *where*

$$\epsilon(t) \leqq [K_1' t^{-\alpha^2/(1-\alpha)} + K_2' t^{-\alpha(1+\alpha)/(1-\alpha)} + K_3' t^{-\alpha/(1-\alpha)} + K_4' t^{-\alpha} + K_5' t^{-\alpha(2-\alpha)/(1-\alpha)}] \tag{2}$$

where $\alpha > \frac{1}{2}$ *or* $\alpha < \frac{1}{2}$, *and*

$$\epsilon(t) \leqq [K_1'' t^{-\frac{1}{2}} + K_2'' t^{-\frac{3}{2}} + K_3'' t^{-1} + K_4'' (\log t) t^{-1}] \tag{3}$$

when $\alpha = \frac{1}{2} \cdot K_i'$ *and* K_i'', $i = 1, 2, 3 \cdots$ *are positive constants independent of* t.

PROOF. In [6] Mikusińsky has shown that

$$\begin{aligned} g(t) &= (1/2\pi i) \int_{-i\infty}^{i\infty} e^{ts - s^\alpha} \, ds, \qquad (0 < t < \infty; 0 < \alpha < 1) \\ &= (1/\pi)[\alpha/(1 - \alpha)](1/t) \int_0^\pi u e^{-u} \, d\phi, \end{aligned} \tag{4}$$

where

$$u = t^{-\alpha/(1-\alpha)} (\sin \alpha\phi / \sin \phi)^{\alpha/(1-\alpha)} \cdot [\sin [(1 - \alpha)\phi]/\sin \phi].$$

Put $T = t^{-\alpha/(1-\alpha)}$ so that $t = T^{-(1-\alpha)/\alpha}$ and

$$v_1(\phi) = \frac{(\sin \alpha\phi)^{\alpha/(1-\alpha)}(\sin \{(1-\alpha)\phi\})}{(\sin \phi)^{1/(1-\alpha)}}$$

so that $u = Tv_1(\phi)$. $v_1(\phi)$ and $v_1'(\phi)$ are continuous and bounded in $[0, \pi/2]$ and $v_1(\phi) > 0$ in $[0, \pi/2]$.

Let

$$v_2(\phi) = v_1(\pi - \phi) = \phi^{-1/(1-\alpha)}w(\phi).$$

$w(\phi)$ and $w'(\phi)$ are bounded and continuous in $[0, \pi/2]$ and $w(\phi) > 0$ in $[0, \pi/2]$. Let $G(t) = \int_0^\pi ue^{-u}d\phi$ so that $g(t) = \{\alpha/[\pi(1-\alpha)t]\}G(t)$.

We write $G(t) = G_1(t) + G_2(t)$ where

$$G_1(t) = T\int_0^{\pi/2} v_1(\phi)e^{-Tv_1(\phi)}\,d\phi.$$

Using the inequality $0 < 1 - e^{-x} < x$, $x > 0$, we have

$$G_1(t) = T\left[\int_0^{\pi/2} v_1(\phi)\,d\phi + K_4(T)T\right]$$
$$= T[A + K_4(T)T]$$

where $A = \int_0^{\pi/2} v_1(\phi)d\phi$ and $|K_4(T)| < A'$ for all T in $0 < T < \infty$, A and A' being constants independent of T, and

$$G_2(t) = T\int_0^{\pi/2} v_2(\phi)e^{-Tv_2(\phi)}\,d\phi.$$

In the following analysis we use frequently functions of the form $K_i(T, \delta)$, $A_i(T, \delta)$, $B_i(T, \delta)$, $C_i(t, \delta)$, $C_i'(t, \delta)$, $C_i''(t, \delta)$, $D_i(t, \delta)$, $D_i'(t, \delta)$, $D_i''(t, \delta)$, $i = 1, 2, 3 \cdots$. These functions which depend upon T (or t) and δ have the property that their absolute value will be less than a constant which is independent of T (or t), but may depend upon δ for all values of T(or t) in $0 < t$ (or T) $< \infty$.

We now write $G_2(t) = G_3(t) + G_4(t)$ where

$$G_3(t) = T\int_\delta^{\pi/2} \phi^{-1/(1-\alpha)}w(\phi)e^{-T\phi^{-1/(1-\alpha)}w(\phi)}\,d\phi$$
$$= K_5(T, \delta)T,$$

and

$$G_4(t) = T\int_0^\delta \phi^{-1/(1-\alpha)}w(\phi)e^{-T\phi^{-1/(1-\alpha)}w(\phi)}\,d\phi.$$

We now write $G_4(t) = G_5(t) + G_6(t) + G_7(t)$ where

$$G_5(t) = T\int_0^\delta \phi^{-1/(1-\alpha)}w(o)e^{-T\phi^{-1/(1-\alpha)}w(o)}\,d\phi$$
$$= (1-\alpha)[Tw(o)]^{1-\alpha}[\Gamma(\alpha) + K_6(T, \delta)T]$$

$$G_6(t) = Tw(o)\int_0^\delta [\phi^{-1/(1-\alpha)}e^{-T\phi^{-1/(1-\alpha)}w(o)}][e^{-T\phi^{-1/(1-\alpha)}\{w(\phi)-w(o)\}} - 1]\,d\phi.$$

Using the inequality $|1 - e^{-x}| \leqq xe^{|x|}$, $x > 0$ we get

$$| G_6(t) | = K_7(T, \delta)T^2 \int_0^\delta \phi^{1-(2/(1-\alpha))} e^{-Ta\phi^{-1/(1-\alpha)}} d\phi.$$

Here a can be equal to $w(o)/2$ and δ is chosen such that in the interval $0 < \phi < \delta$,

$$[w(o) - K\phi] \geqq w(o)/2,$$

K being the lowest upper bound of $|w'(\phi)|$ in $0 < \phi < \pi/2$.

Using the Lemma we have

$$| G_6(t) | = [K_8(T, \delta) + K_9(T, \delta)T]T^{2-2\alpha}$$

$$G_7(t) = T \int_0^\delta \phi^{1-(1/(1-\alpha))} w_1(\phi) e^{-T\phi^{-1/(1-\alpha)}w(\phi)} d\phi$$

where $w_1(\phi) = [w(\phi) - w(o)]/\phi$ is bounded. So

$$G_7(t) = K_{10}(T, \delta)T \int_0^\delta \phi^{1-(1/1-\alpha)} e^{-Ta'\phi^{-1/(1-\alpha)}} d\phi$$

where a' is the L.U.B of $w(\phi)$ in $[0, \pi/2]$, which is positive. Hence

$$G_7(t) = \begin{cases} K_{11}(T, \delta)T^{2-2\alpha} + K_{12}(T, \delta)T^2 & \text{if } 1/(1 - \alpha) > 2 \text{ or } \alpha > \frac{1}{2} \\ K_{11}(T, \delta)T & \text{if } 1/(1 - \alpha) < 2 \text{ or } \alpha < \frac{1}{2} \\ K_{11}(T, \delta)T[\log(1/T) + K_{12}(T, \delta)] & \text{if } 1/(1 - \alpha) = 2 \text{ or } \alpha = \frac{1}{2}. \end{cases}$$

Collecting the various terms we can express $G(t)$ in the form

$$\begin{aligned} G(t) = {} & (1 - \alpha)\Gamma(\alpha)[w(o)T]^{1-\alpha} + A_1(T, \delta)T + A_2(T, \delta)T^2 + \\ & A_3(T, \delta)T^{2-\alpha} + A_4(T, \delta)T^{2-2\alpha} + A_5(T, \delta)T^{3-2\alpha}, \\ & \qquad \text{if } \alpha > \tfrac{1}{2} \text{ or } \alpha < \tfrac{1}{2} \end{aligned} \tag{5}$$

and

$$\begin{aligned} G(t) = {} & (\tfrac{1}{2})\Gamma(\tfrac{1}{2})[w(o)T]^{\frac{1}{2}} + B_1(T, \delta)T + B_2(T, \delta)T^2 + B_3(T, \delta)T^{\frac{3}{2}} + \\ & B_4(T, \delta)T \log(1/T), \qquad \text{if } \alpha = \tfrac{1}{2}. \end{aligned} \tag{6}$$

That is

$$\begin{aligned} G(t) = {} & (1 - \alpha)\Gamma(\alpha)[w(o)]^{1-\alpha}t^{-\alpha} + C_1(t, \delta)t^{-\alpha/(1-\alpha)} \\ & C_2(t, \delta)t^{-2\alpha/(1-\alpha)} + C_3(t, \delta)t^{-\alpha(2-\alpha)/(1-\alpha)} \\ & C_4(t, \delta)t^{-\alpha(2-2\alpha)/(1-\alpha)} + C_5(t, \delta)t^{-(3-2\alpha)\alpha/(1-\alpha)} \qquad \text{if } \alpha \lessgtr \tfrac{1}{2}. \end{aligned} \tag{7}$$

And

$$\begin{aligned} G(t) = {} & (\sqrt{\pi}/2)[w(o)]^{\frac{1}{2}}t^{-\frac{1}{2}} + D_1(t, \delta)t^{-1} + D_2(t, \delta)t^{-2} + \\ & D_3(t, \delta)t^{-\frac{3}{2}} + D_4(t, \delta)(\log t)/t \qquad \text{if } \alpha = \tfrac{1}{2}. \end{aligned} \tag{8}$$

So

$$g(t) = (\alpha/\pi)t^{-\alpha-1}[w(o)]^{1-\alpha}\Gamma(\alpha) + C_1'(t, \delta)t^{-1/(1-\alpha)} + \tag{9}$$
$$C_2'(t, \delta)t^{-(1+\alpha)/(1-\alpha)} + C_3'(t, \delta)t^{(\alpha^2-\alpha-1)/(1-\alpha)}$$
$$C_4'(t, \delta)t^{(2\alpha^2-\alpha-1)/(1-\alpha)} + C_5'(t, \delta)t^{(2\alpha^2-2\alpha-1)/(1-\alpha)} \qquad \text{if } \alpha \gtrless \tfrac{1}{2}.$$

where

$$C_i'(t, \delta) = (\alpha/[\pi(1 - \alpha)])C_i(t, \delta),$$

and

$$g(t) = (\tfrac{1}{2}\sqrt{\pi})[w(o)]^{\frac{1}{2}}t^{-\frac{3}{2}} + D_1'(t, \delta)t^{-2} + D_2'(t, \delta)t^{-3} \tag{10}$$
$$+ D_3'(t, \delta)t^{-\frac{5}{2}} + D_4'(t, \delta)t^{-2} \log t \qquad \text{if } \alpha = \tfrac{1}{2}$$

where

$$D_i'(t, \delta) = (1/\pi)D_i(t, \delta)$$

If e^{-s^α} is the Laplace transform of $g(t)$, then

$$g(t) = (1/2\pi i) \int_{-i\alpha}^{i\alpha} e^{ts-s^\alpha}\, ds.$$

The frequency function whose Laplace transform is e^{-ms^α} is given by

$$f(t) = (1/m^{1/\alpha})g(t/m^{1/\alpha}) \tag{11}$$

In our case for the distribution considered in (5) $m = [c\Gamma(1 - \alpha)]/\alpha$.

We first consider the case when $\alpha < \frac{1}{2}$ or $> \frac{1}{2}$, $(0 < \alpha < 1)$. Using (9) and (11) $f(t)$ can be put in the form

$$f(t) = ct^{-\alpha-1} + C_1''(t, \delta)t^{-1/(1-\alpha)} + \tag{12}$$
$$C_2''(t, \delta)t^{-(1+\alpha)/(1-\alpha)} + C_3''(t,\delta)t^{(\alpha^2-\alpha-1)/(1-\alpha)}$$
$$C_4''(t, \delta)t^{(2\alpha^2-\alpha-1)/(1-\alpha)} + C_5''(t, \delta)t^{(2\alpha^2-2\alpha-1)/(1-\alpha)}$$

Now

$$1 - F(t) = \int_t^\infty f(t)\, dt.$$

Using (12), after integration, $F(t)$ can be put in the form

$$1 - F(t) = (c/\alpha)t^{-\alpha} + t^{-\alpha}\epsilon(t) \tag{13}$$

where

$$\epsilon(t) \leqq K_1't^{-\alpha^2/(1-\alpha)} + K_2't^{-\alpha(1+\alpha)/(1-\alpha)} + K_3't^{-\alpha/(1-\alpha)} + \tag{14}$$
$$K_4't^{-\alpha} + K_5't^{-\alpha(2-\alpha)/(1-\alpha)}$$

where K_i' are constants independent of t for $0 < t < \infty$.

In the same manner, where $\alpha = \frac{1}{2}$, using (10) and (11), $f(t)$ can be put in the form

$$f(t) = ct^{-\frac{3}{2}} + D_1''(t, \delta)t^{-2} + D_2''(t, \delta)t^{-3} + D_3''(t, \delta)t^{-\frac{5}{2}} + D_4''(t, \delta)t^{-2}[\log t - (\tfrac{1}{2}) \log m] \tag{15}$$

Using (15), after integration, $F(t)$ can be put in the form

$$1 - F(t) = 2ct^{-\frac{1}{2}} + t^{-\frac{1}{2}}\epsilon(t)$$

where

$$\epsilon(t) < K_i''t^{-\frac{1}{2}} + K_2''t^{-\frac{3}{2}} + K_3''t^{-1} + K_4''(\log t)/t \tag{16}$$

where K_i'', $i = 1, 2, 3, 4$ are constants independent of t in $0 < t < \infty$. Hence the theorem.

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