is minimized by the choice $c = c^*$. A design which minizes $d(x_0)$ can then be obtained easily from c^* in a manner described in [8]; for our present considerations, we need only mention that $d(x_0) = [m(c^*)]^{-2}$, which can be used to tell us whether or not $x_0 \varepsilon B$.

Finally, we remark that the Chebyshev approximation problem just described in terms of the g_i's can be rewritten as a "modified Chebyshev problem" in terms of the original f_i's, namely, to minimize

$$\max_x |[1 + \sum_2^k c_i f_i(x_0)] f_1(x)/f_1(x_0) - \sum_2^k c_i f_i(x)|.$$

For computational purposes, it is often convenient to solve this problem by first solving the restricted Chebyshev problem of minimizing

$$\max_x |f_1(x)/f_1(x_0) - r^{-1} \sum_2^k c_i f_i(x)|$$

subject to $\sum_2^k c_i f_i(x_0) = r - 1$, then multiplying the resulting minimum by r and minimizing with respect to r.

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A CONTOUR-INTEGRAL DERIVATION OF THE NON-CENTRAL CHI-SQUARE DISTRIBUTION

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The brief discussion which follows presents a contour-integral derivation of the non-central chi-square distribution. Although this distribution is well

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known, the particular mode of derivation in this paper may have interest pedagogically and may serve as an example of the utility of the contour integral approach.

Consider the expression

$$\rho^2 = x_1^2 + x_2^2 + x_3^2 + \cdots + x_k^2, \tag{1}$$

where the x_i are independent and each normally distributed, $x_i : n(a_i, \sigma)$.

The Fourier integral $\phi_j(t)$ of the random variable x_j^2 is

$$\phi_j(t) = \frac{1}{(1 - 2it\sigma^2)^{\frac{1}{2}}} \exp\left(\frac{ita_j^2}{1 - 2it\sigma^2}\right), \tag{2}$$

and the Fourier integral $\phi(t)$ of ρ^2 is, therefore,

$$\phi(t) = \frac{1}{(1 - 2it\sigma^2)^{k/2}} \exp\left(\frac{it}{1 - 2it\sigma^2}\right) \sum_{j=1}^{k} a_j^2. \tag{3}$$

The probability density function of ρ^2, $f(\rho^2)$, is then given by the Fourier transform of (3),

$$f(\rho^2) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-it_\rho^2) \frac{1}{(1 - 2it\sigma^2)^{k/2}} \exp\left(\frac{itr^2}{1 - 2it\sigma^2}\right) dt, \tag{4}$$

where $r^2 = \sum_{j=1}^{k} a_j^2$. In (4) let $z = (\rho/r)(1 - 2\,it\sigma^2)$, which yields

$$\begin{aligned} f(\rho^2) = \frac{1}{4\pi i\sigma^2} \left(\frac{\rho}{r}\right)^{\frac{1}{2}(k-2)} &\exp\left[-\frac{1}{2\sigma^2}(\rho^2 + r^2)\right] \\ &\cdot \int_{\rho/r - i\infty}^{\rho/r + i\infty} \exp\left[\frac{\rho r}{2\sigma^2}\left(\frac{z^2 + 1}{z}\right)\right] \frac{dz}{z^{k/2}}. \end{aligned} \tag{5}$$

At this point, it is convenient and interesting (but not necessary) to write (5) in the equivalent forms,

$$\begin{aligned} f(\rho^2) = \frac{1}{4\pi i\sigma^2} \left(\frac{\rho}{r}\right)^{\frac{1}{2}(k-2)} &\exp\left[-\frac{1}{2\sigma^2}(\rho^2 + r^2)\right] \\ &\cdot \sum_{n=0}^{\infty} \left(\frac{\rho r}{2\sigma^2}\right)^n \frac{1}{n!} \int_{\rho/r - i\infty}^{\rho/r + i\infty} \exp\left(\frac{\rho r}{2\sigma^2} z\right) \frac{dz}{z^{n+k/2}} \quad (k \text{ odd}), \end{aligned} \tag{6}$$

$$\begin{aligned} f(\rho^2) = \frac{1}{4\pi i\sigma^2} \left(\frac{\rho}{r}\right)^{\frac{1}{2}(k-2)} &\exp\left[-\frac{1}{2\sigma^2}(\rho^2 + r^2)\right] \\ &\cdot \sum_{n=-\infty}^{\infty} I_n\left(\frac{\rho r}{\sigma^2}\right) \int_{\rho/r - i\infty}^{\rho/r + i\infty} z^{n-k/2}\, dz \quad (k \text{ even}), \end{aligned} \tag{7}$$

where $I_n(\rho r/\sigma^2)$ is the modified Bessel function of the first kind, of order n.

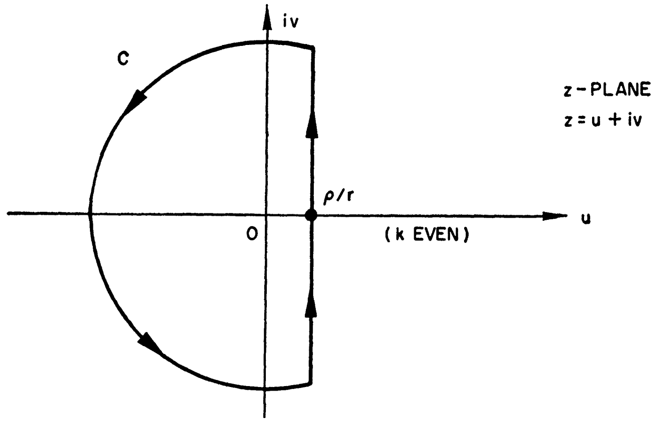

FIG. 1

Since the integrand in (7) has no branch points, the integration can be performed by considering the vertical path $(\rho/r) - i\infty$ to $(\rho/r) + i\infty$, to be part of a complex contour C in the z-plane as shown in Figure 1. Equation (7) is then easily evaluated as

$$f(\rho^2) = \frac{1}{2\sigma^2}\left(\frac{\rho}{r}\right)^{\frac{1}{2}k-1} \exp\left[-\frac{1}{2\sigma^2}(\rho^2 + r^2)\right] I_{(k/2-1)}\left(\frac{\rho r}{\sigma^2}\right) \tag{8}$$

$$= \frac{1}{2\sigma^2}\left(\frac{\rho}{r}\right)^{m-1} \exp\left[-\frac{1}{2\sigma^2}(\rho^2 + r^2)\right] \sum_{j=0}^{\infty} \frac{\left(\frac{\rho r}{2\sigma^2}\right)^{2j+m-1}}{\Gamma(j+1)\Gamma(j+m)}.$$

Since the integrand in expression (6) has a branch point at the origin, a somewhat different contour is required as shown in Figure 2. For reasons which will become evident later, the integral

$$\int_{\rho/r-i\infty}^{\rho/r+i\infty} \exp\left(\frac{\rho r}{2\sigma^2} z\right) \frac{dz}{z^{\lambda+1}} \tag{9}$$

will be considered, where $n + (k/2)$ has been parameterized in the form of $\lambda + 1$. Thus,

$$\lim_{\substack{R\to\infty \\ d\to 0}} \int_c \exp\left(\frac{\rho r}{2\sigma^2} z\right) \frac{dz}{z^{\lambda+1}} = \lim_{R\to\infty}\left\{\int_{c_1} + \int_{c_2} + \int_{c_3} + \int_{c_7} + \int_{c_8}\right\} \tag{10}$$

$$+ \lim_{d\to 0}\int_{c_5} + \lim_{\substack{R\to\infty \\ d\to 0}}\left\{\int_{c_4} + \int_{c_6}\right\} = 0.$$

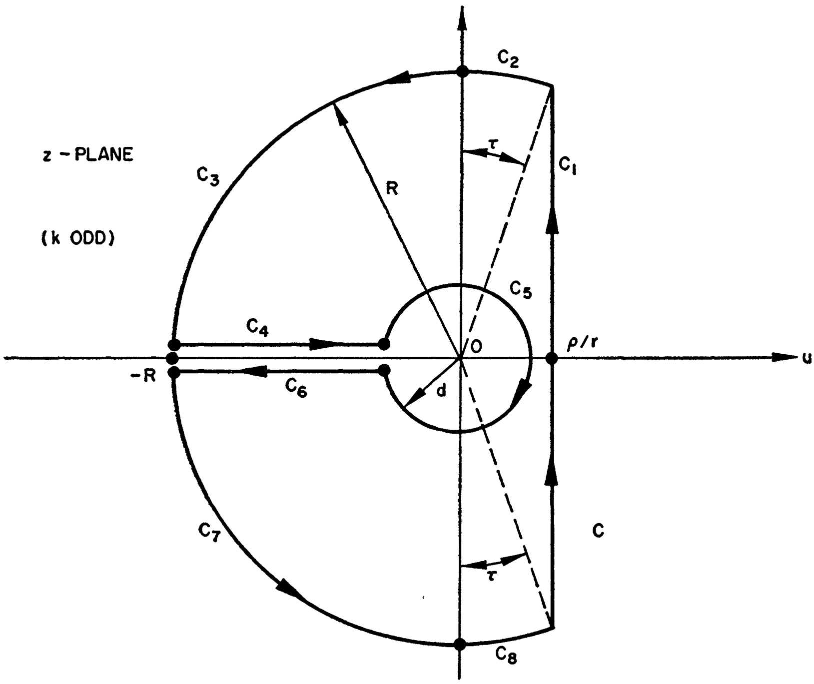

FIG. 2

It can be shown that

$$\lim_{R\to\infty}\int_{c_2} = \lim_{R\to\infty}\int_{c_8} = 0 \qquad \text{for } \lambda > -1, \tag{11}$$

$$\lim_{R\to\infty}\int_{c_3} = \lim_{R\to\infty}\int_{c_7} = 0 \qquad \text{for } \lambda > -1, \tag{12}$$

and

$$\lim_{d\to 0}\int_{c_5} = 0 \qquad \text{for } \lambda < 0. \tag{13}$$

From (10), (11), (12), and (13), the following can be written:

$$\begin{aligned}\int_{(\rho/r)-i\infty}^{(\rho/r)+i\infty} \exp\left(\frac{\rho r}{2\sigma^2} z\right)\frac{dz}{z^{\lambda+1}} &= -2i \sin(\lambda\pi)\int_0^\infty \exp\left(-\frac{\rho r}{2\sigma^2} s\right)\frac{ds}{s^{\lambda+1}} \\ &= \frac{2\pi i}{\Gamma(\lambda+1)}\left(\frac{\rho r}{2\sigma^2}\right)^\lambda.\end{aligned} \tag{14}$$

The equality in (14) is valid for $-1 < \lambda < 0$ and, in addition, both sides of (14) are holomorphic functions of λ throughout the region $-\infty < \lambda < \infty$. Therefore, by analytic continuation [see MacRobert [2], page 122], the equality in (14) holds for $-\infty < \lambda < \infty$ and namely for the $n + k/2$ under consideration in expression (6).

Expression (6) can now be written as

$$f(\rho^2) = \left(\frac{1}{2\sigma^2}\right)^{k/2} (\rho^2)^{\frac{1}{2}(k-2)} \exp\left[-\frac{1}{2\sigma^2}(\rho^2 + r^2)\right] \sum_{n=0}^{\infty} \frac{(\rho^2 r^2/2^2\sigma^4)^n}{\Gamma(n+1)\Gamma(n+\frac{1}{2}k)} \quad (k \text{ odd}). \tag{15}$$

But (8), with k even, can be written in exactly the same form as (15). Thus, (15) is the density function for ρ^2, with k even or odd.

Letting $\gamma = r^2/2\sigma^2$ and $\chi'^2 = \rho^2/\sigma^2$, (15) can also be written as

$$f\left(\frac{\rho^2}{\sigma^2}\right) d\rho^2 = f(\chi'^2)\sigma^2\, d\chi'^2 = \frac{1}{2}\left(\frac{\chi'^2}{2}\right)^{\frac{1}{2}(k-2)} \exp(-\gamma) \exp\left(-\frac{\chi'^2}{2}\right) \cdot \sum_{n=0}^{\infty} \frac{(\frac{1}{2}\chi'^2\gamma)^n\, d\chi'^2}{n!\Gamma(n+\frac{1}{2}k)}. \tag{16}$$

Equation (16) is the non-central chi-square distribution, and the Fourier integral derivation of Equation (16) is different from that usually found in the literature—see Mann [3], pages 65–68 and Anderson [1], pages 112 and 113.

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A CHARACTERIZATION OF THE INVERSE GAUSSIAN DISTRIBUTION

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1. Introduction and summary. M. C. K. Tweedie [2] defined the inverse Gaussian distributions via the density functions

$$\begin{aligned} f(x; m, \lambda) &= [\lambda/(2\pi x^3)]^{\frac{1}{2}} \exp[-\lambda(x-m)^2/(2m^2x)] && \text{for } x > 0 \\ &= 0 && \text{for } x \leqq 0. \end{aligned} \tag{1}$$

The parameters λ and m are positive. The corresponding densities reflected about the origin, and with λ and m negative, may also be considered as in the Inverse Gaussian family. The characteristic function of the Inverse Gaussian

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